量子计算与机器学习作业

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3.1 The *fidelity* F of two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ is defined by $F \equiv |\langle\psi_1|\psi_2\rangle|^2$. It is a measure of the distance between the two quantum states: We have $0 \le F \le 1$, with $F = 1$ when $|\psi_1\rangle$ concides with $|\psi_2\rangle$ and $F = 0$ when $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal. Show that $F = \cos^2\frac{\alpha}{2}$, with α the angle between the Bloch vectors corresponding to the quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$.

假设两个量子态 $|\psi_1\rangle$ 和 $|\psi_2\rangle$ 在 Bloch 球面上的位置分别为 (θ_1, φ_1) 和 (θ_2, φ_2)，即

$$|\psi_1\rangle = \cos\frac{\theta_1}{2}|0\rangle + \mathrm{e}^{\mathrm{i}\varphi_1}\sin\frac{\theta_1}{2}|1\rangle = \begin{pmatrix}\cos\frac{\theta_1}{2}\\ \mathrm{e}^{\mathrm{i}\varphi_1}\sin\frac{\theta_1}{2}\end{pmatrix},$$

$$|\psi_1\rangle = \cos\frac{\theta_2}{2}|0\rangle + \mathrm{e}^{\mathrm{i}\varphi_2}\sin\frac{\theta_2}{2}|1\rangle = \begin{pmatrix}\cos\frac{\theta_2}{2}\\ \mathrm{e}^{\mathrm{i}\varphi_2}\sin\frac{\theta_2}{2}\end{pmatrix}.$$

则这两个量子态的内积为

$$\begin{aligned}\langle\psi_1|\psi_2\rangle &= \left(\cos\frac{\theta_1}{2}, \mathrm{e}^{-\mathrm{i}\varphi_1}\sin\frac{\theta_1}{2}\right)\begin{pmatrix}\cos\frac{\theta_2}{2}\\ \mathrm{e}^{\mathrm{i}\varphi_2}\sin\frac{\theta_2}{2}\end{pmatrix}\\ &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\mathrm{e}^{\mathrm{i}(\varphi_2-\varphi_1)}.\end{aligned}$$

因此两个量子态的 fidelity 为

$$\begin{aligned}F = |\langle\psi_1|\psi_2\rangle|^2 &= \left(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos(\varphi_2-\varphi_1)\right)^2 + \left(\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin(\varphi_2-\varphi_1)\right)^2\\ &= \cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} + 2\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos(\varphi_2-\varphi_1)\\ &= \frac{1}{2}(1+\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\varphi_2-\varphi_1)).\end{aligned}$$

而两个量子态在 Bloch 球面上的夹角 α 即为向量 $(\sin\theta_1\cos\varphi_1, \sin\theta_1\sin\varphi_1, \cos\theta_1)^\mathsf{T}$ 和 $(\sin\theta_2\cos\varphi_2, \sin\theta_2\sin\varphi_2, \cos\theta_2)^\mathsf{T}$ 之间的夹角，有

$$\begin{aligned}\cos\alpha &= (\sin\theta_1\cos\varphi_1, \sin\theta_1\sin\varphi_1, \cos\theta_1)\begin{pmatrix}\sin\theta_2\cos\varphi_2\\ \sin\theta_2\sin\varphi_2\\ \cos\theta_2\end{pmatrix}\\ &= \sin\theta_1\sin\theta_2(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)+\cos\theta_1\cos\theta_2\\ &= \cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos(\varphi_2-\varphi_1)\\ &= 2F-1.\end{aligned}$$

因此，$F = \frac{1}{2}(1+\cos\alpha) = \cos^2\frac{\alpha}{2}$。

3.2 Show that the unitary operator moving the state parametrized on the Bloch sphere by the angles (θ_1, φ_1) into the state (θ_2, φ_2) is given by

$$R_z\left(\frac{\pi}{2} + \varphi_2\right) H R_z(\theta_2 - \theta_1) H R_z\left(-\frac{\pi}{2} - \varphi_1\right).$$

The *phase-shift gate* is defined as

$$R_z(\delta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{bmatrix}.$$

角度 (θ_1, φ_1) 对应的量子态为

$$|\psi_1\rangle = \cos\frac{\theta_1}{2}|0\rangle + e^{i\varphi_1}\sin\frac{\theta_1}{2}|1\rangle = \begin{pmatrix} \cos\frac{\theta_1}{2} \\ e^{i\varphi_1}\sin\frac{\theta_1}{2} \end{pmatrix}.$$

经过给出的各个酉操作后，量子态依次变为：

$$|\psi_1\rangle = \begin{pmatrix} \cos\frac{\theta_1}{2} \\ e^{i\varphi_1}\sin\frac{\theta_1}{2} \end{pmatrix} \xrightarrow{R_z(-\frac{\pi}{2}-\varphi_1)} \begin{pmatrix} \cos\frac{\theta_1}{2} \\ -i\sin\frac{\theta_1}{2} \end{pmatrix} \xrightarrow{H} \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\frac{\theta_1}{2}} \\ e^{i\frac{\theta_1}{2}} \end{pmatrix} \xrightarrow{R_z(\theta_2-\theta_1)} \frac{1}{\sqrt{2}}e^{-i\frac{\theta_1}{2}}\begin{pmatrix} 1 \\ e^{i\theta_2} \end{pmatrix}$$
$$\xrightarrow{H} \frac{1}{2}e^{-i\frac{\theta_1}{2}}\begin{pmatrix} 1+e^{i\theta_2} \\ 1-e^{i\theta_2} \end{pmatrix} \xrightarrow{R_z(\frac{\pi}{2}+\varphi_2)} \frac{1}{2}e^{-i\frac{\theta_1}{2}}\begin{pmatrix} 1+e^{i\theta_2} \\ e^{\frac{\pi}{2}+\varphi_2}\left(1-e^{i\theta_2}\right) \end{pmatrix}$$

而

$$\begin{aligned}
\frac{1}{2}e^{-i\frac{\theta_1}{2}}\begin{pmatrix} 1+e^{i\theta_2} \\ e^{\frac{\pi}{2}+\varphi_2}\left(1-e^{i\theta_2}\right) \end{pmatrix} &= \frac{1}{2}e^{i(\theta_2-\theta_1)}\begin{pmatrix} e^{-i\frac{\theta_2}{2}} + e^{i\frac{\theta_2}{2}} \\ ie^{\varphi_2}\left(e^{-i\frac{\theta_2}{2}} - e^{i\frac{\theta_2}{2}}\right) \end{pmatrix} \\
&= e^{i\frac{\theta_2-\theta_1}{2}}\begin{pmatrix} \cos\frac{\theta_2}{2} \\ e^{i\varphi_2}\sin\frac{\theta_2}{2} \end{pmatrix} \\
&= e^{i\frac{\theta_2-\theta_1}{2}}|\psi_2\rangle \\
&= |\psi_2\rangle.
\end{aligned}$$

其中 $|\psi_2\rangle$ 是角度 (θ_2, φ_2) 对应的量子态。综上所述，$R_z\left(\frac{\pi}{2} + \varphi_2\right) H R_z(\theta_2 - \theta_1) H R_z\left(-\frac{\pi}{2} - \varphi_1\right)$ 是将角度 (θ_1, φ_1) 对应的量子态变为角度 (θ_2, φ_2) 的酉操作。

4.1 证明贝尔态 $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ 可以等效表达为 $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|aa\rangle + |bb\rangle)$，其中 $|a\rangle$ 和 $|b\rangle$ 是任意一组正交归一基。

因为 $|a\rangle$ 和 $|b\rangle$ 是任意一组正交归一基，所以可以将 $|0\rangle$ 和 $|1\rangle$ 表示为

$$\begin{aligned}
|0\rangle &= \alpha|a\rangle + \beta|b\rangle, \\
|1\rangle &= -\beta|a\rangle + \alpha|b\rangle.
\end{aligned}$$

其中 $\alpha^2 + \beta^2 = 1$。因此，$|\Phi^+\rangle$ 可以表示为

$$\begin{aligned}
\left|\Phi^+\right\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}}((\alpha|a\rangle + \beta|b\rangle) \otimes (\alpha|a\rangle + \beta|b\rangle) + (-\beta|a\rangle + \alpha|b\rangle) \otimes (-\beta|a\rangle + \alpha|b\rangle)) \\
&= \frac{1}{\sqrt{2}}\left(\left(\alpha^2 + \beta^2\right)|aa\rangle + \left(\alpha^2 + \beta^2\right)|bb\rangle\right) \\
&= \frac{1}{\sqrt{2}}(|aa\rangle + |bb\rangle).
\end{aligned}$$

命题得证。

5.1 Let $|x\rangle$ be a basis state of n qubits. Prove that

$$H^{\otimes n}|x\rangle = \frac{\sum_z (-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}$$

where $x \cdot z$ is the bitwise inner product of x and z, modulo 2, and the sum is over all $z \in \{0,1\}^n$.

如下所示，有

$$\begin{aligned}
H^{\otimes n}|x\rangle &= H|x_1\rangle \otimes H|x_2\rangle \otimes \cdots \otimes H|x_n\rangle \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{x_1}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{x_2}|1\rangle\right) \otimes \cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{x_n}|1\rangle\right) \otimes \\
&= \frac{1}{\sqrt{2^n}} \sum_{z_1\in\{0,1\}} (-1)^{x_1 z_1}|z\rangle \otimes \left(|0\rangle + (-1)^{x_2}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + (-1)^{x_n}|1\rangle\right) \\
&= \frac{1}{\sqrt{2^n}} \sum_{z_1\in\{0,1\}} \sum_{z_2\in\{0,1\}} (-1)^{x_1 z_1}|z_1\rangle \otimes (-1)^{x_2 z_2}|z_2\rangle \otimes \left(|0\rangle + (-1)^{x_3}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + (-1)^{x_n}|1\rangle\right) \otimes \\
&= \cdots \\
&= \frac{1}{\sqrt{2^n}} \sum_{z_1\in\{0,1\}} \sum_{z_2\in\{0,1\}} \cdots \sum_{z_n\in\{0,1\}} (-1)^{x_1 z_1}|z_1\rangle \otimes (-1)^{x_2 z_2}|z_2\rangle \otimes \cdots \otimes (-1)^{x_n z_n}|z_n\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{z_1, z_2, \cdots, z_n\in\{0,1\}} (-1)^{x_1 z_1 + x_2 z_2 + \cdots + x_n z_n}|z_1 z_2 \cdots z_n\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle.
\end{aligned}$$

即 $H^{\otimes n}|x\rangle = \frac{\sum_z (-1)^{x\cdot z}|z\rangle}{\sqrt{2^n}}$， 命题得证。