Mini-project - Numerical Approximation of PDEs MATH-451

A semilinear elliptic equation

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Contents

1 Introduction

This project focuses on the solution of the following problem:

$$\begin{cases} -\Delta u + \alpha u^3 = f & \text{over } \Omega \\ u = 0 & \text{along } \partial\Omega \end{cases} \tag{1}$$

with $f = 100$, $\Omega = [0,1]^2$ and $\alpha > 0$. The partial differential equation that defines the problem is an elliptic equation. In the following, a `Python` code for the solution of this problem using a finite element method (FEM) was implemented, based on the package available here. The convergence of two different schemes, *fixed point* iterations and *Newton* method, was studied for different values of α.

2 Weak formulation of the problem

In order to derive the weak formulation of the Problem 1, we multiply both sides of the PDE by a test function $v \in C^0_\infty(\Omega)$ and we integrate over the domain Ω:

$$\int_\Omega -\nabla \cdot (\nabla u) v d\Omega + \int_\Omega \alpha u^3 v d\Omega = \int_\Omega f v d\Omega \tag{2}$$

We now make use of the following identity:

$$\nabla \cdot (\nabla u) v = \nabla \cdot (\nabla(u) v) - \nabla u \cdot \nabla v \tag{3}$$

to rewrite Eq. 2 as:

$$\int_\Omega -\nabla \cdot ((\nabla u) v) d\Omega + \int_\Omega \nabla u \cdot \nabla v d\Omega + \int_\Omega \alpha u^3 v d\Omega = \int_\Omega f v d\Omega \tag{4}$$

It is now possible to apply the Gauss theorem (divergence theorem) to the first term of the LHS:

$$\int_{\partial\Omega} -(\nabla u) \cdot \mathbf{n} v d\Omega + \int_\Omega \nabla u \cdot \nabla v d\Omega + \int_\Omega \alpha u^3 v d\Omega = \int_\Omega f v d\Omega \tag{5}$$

Since by its definition $\gamma(v) = 0$, the first integral is equal to zero, and we get to the weak formulation:

$$\int_\Omega \nabla u \cdot \nabla v d\Omega + \int_\Omega \alpha u^3 v d\Omega = \int_\Omega f v d\Omega, \quad \forall v \in C^0_\infty(\Omega) \tag{6}$$

and by density of $C^0_\infty(\Omega)$ in $H^1_0(\Omega)$ this weak formulation holds true $\forall v \in H^1_0(\Omega)$ as well.

We can now move to the Galerking problem by defining a subspace of $H_0^1(\Omega)$ as $V_h = \{v \in C^0(\Omega) |\, \gamma(v) = 0,\, v_{|k}$ is linear, $\forall K \in \mathcal{T}_h\}$, where $\mathcal{T}_h$ is the triangulation of the domain Ω, characterized by the diameter h, and K is the generic triangle belonging to the triangulation $\mathcal{T}_h$. Let v be a generic element of V_h. The Galerkin problem is hence to find $u \in V_h$ such that Eq. 6 holds $\forall v \in V_h$. The mesh of the domain Ω used in this project with `meshsize` $= 0.1$ is shown in Fig. 1.

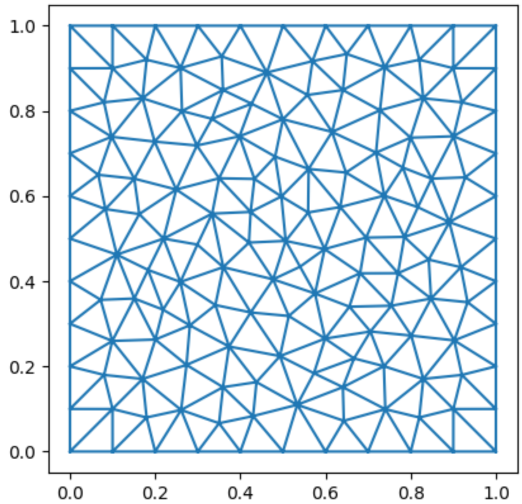


Figure 1: Mesh of the domain Ω, with `meshsize` $= 0.1$.

3 Fixed-point scheme

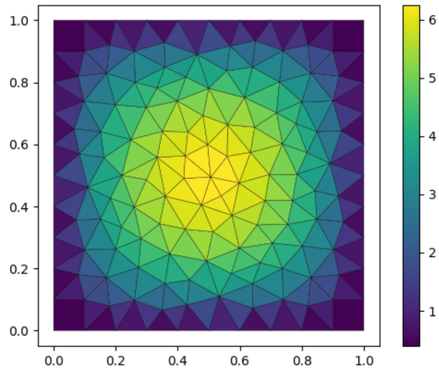
The fixed-point scheme uses a linearized version of the original problem:

$$\begin{cases} -\Delta u_{n+1} + \alpha u_n^2 u_{n+1} = f & \text{over } \Omega \\ u = 0 & \text{along } \partial\Omega \end{cases} \tag{7}$$

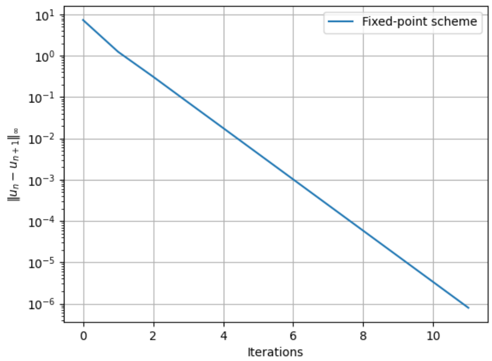
i.e. given some iterate u_n, the new iterate u_{n+1} is computed by solving Eq. 7 using a finite element method. According to the result found in Sec. 2, the weak formulation of Eq. 7 is:

$$\int_\Omega \nabla u_{n+1} \cdot \nabla v \, d\Omega + \int_\Omega \alpha u_n^2 u_{n+1} v \, d\Omega = \int_\Omega f v \, d\Omega \quad \forall v \in H_0^1(\Omega) \tag{8}$$

In the FEM framework, the solution to Eq. 8 is computed by solving a linear system of equations, with a suitable re-definition of the mass matrix. Consequently, the solution to Eq. 7 is obtained by solving the aforementioned linear system of equations iteratively, updating the mass matrix at each step. In the present work, $\mathbb{P}_1$ basis functions are exploited, together with a *Gauss quadrature scheme* of order 6. The solution of the problem was computed for $\alpha = 0.1$ and $\alpha = 2$, with initial guess $u_0 = 0$. In Fig. 2 and Fig. 3 the solutions are plotted, together with a `semilogy` plot showing the trend of the convergence criterion considered here: the solution was considered converged once $\|u_n - u_{n+1}\|_\infty < 1 \times 10^{-6}$.

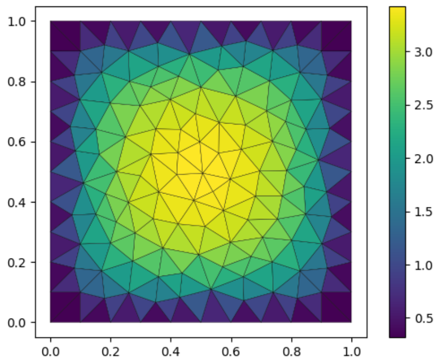


(a) Solution with $\alpha = 0.1$.

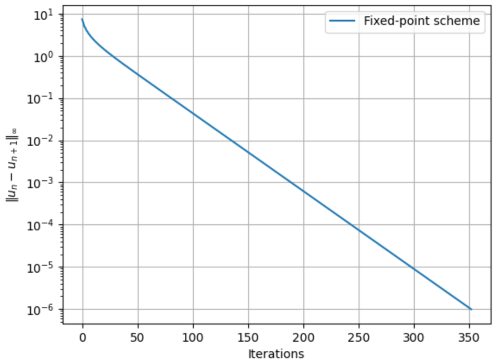


(b) Convergence of fixed-point scheme, $\alpha = 0.1$.

Figure 2: Fixed point scheme, $\alpha = 0.1$.



(a) Solution with $\alpha = 2$.



(b) Convergence of fixed-point scheme, $\alpha = 2$.

Figure 3: Fixed point scheme, $\alpha = 2$.

In particular, Fig. 3b shows that the computational cost required for the case $\alpha = 2$ case is significantly higher that that of $\alpha = 0.1$: roughly 350 iterations against 11. Besides, it can be noted that the solution reaches a lower maximum for a larger value of α.

4 Newton scheme

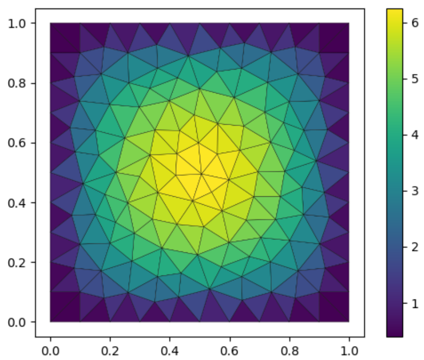
In this second case, the solution to the problem in Eq. 1 is found once again by analyzing the linearized version of the problem itself, i.e. Eq. 7. Here, a different iterative scheme is considered, namely a Newton scheme. In particular, the Newton scheme seeks the solution u as the limit $n \to \infty$ of the recursive Newton-sequence:

$$u_{n+1} = u_n + \partial u_n \tag{9}$$

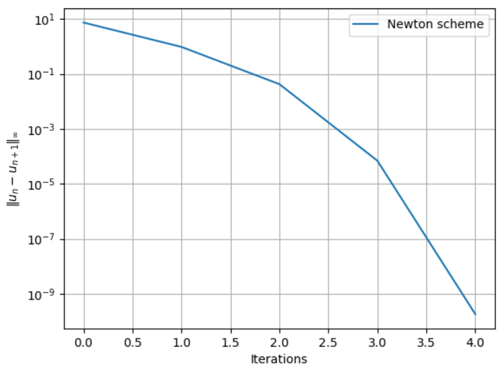
where ∂u_n satisfies:

$$\begin{cases} \int_\Omega (\nabla\phi \cdot \nabla\partial u_n + 3\alpha u_n^2 \phi \partial u_n)\, d\Omega = \int_\Omega (-\nabla\phi \cdot \nabla u_n - \phi(\alpha u_n^3 - f))\, d\Omega, & \forall \phi \in H_0^1(\Omega) \\ \partial u_n = 0, \quad \text{on } \partial\Omega \end{cases} \tag{10}$$

Again, the solution to Eq. 10 can be found in the FEM framework by solving a linear system of equations, with appropriate definitions of the RHS and the mass matrix. The initial guess considered is the same as the previous case. The solutions to the problem in Eq. 7 for $\alpha = 0.1$, $\alpha = 2$, and $\alpha = 5$ are then shown in Fig. 4, Fig. 5, and Fig. 6 respectively, together with the plot showing the trend of $\|u_n - u_{n+1}\|_\infty$. The same convergence criterion of the fixed-point scheme was adopted. It shall be observed that the Newton scheme fares substantially better than the fixed-point scheme also for higher values of the parameter α: the number of iterations required to reach convergence passes from 4 to 6 when moving from $\alpha = 0.1$ to $\alpha = 2$, and it is equal to only 7 for $\alpha = 5$.

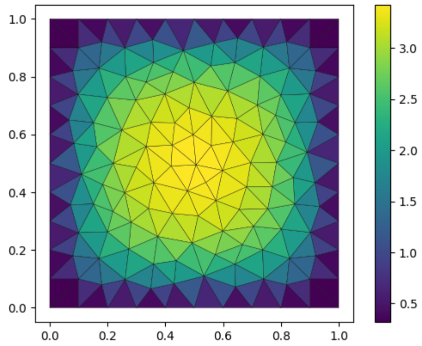


(a) Solution with $\alpha = 0.1$.

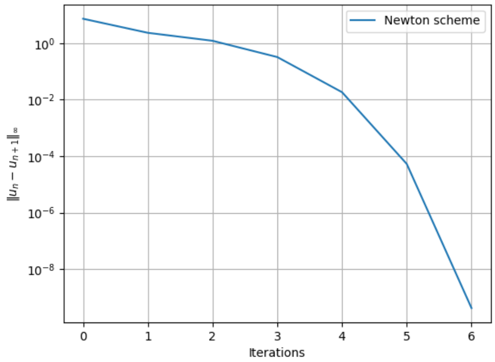


(b) Convergence of Newton scheme, $\alpha = 0.1$.

Figure 4: Newton scheme, $\alpha = 0.1$.

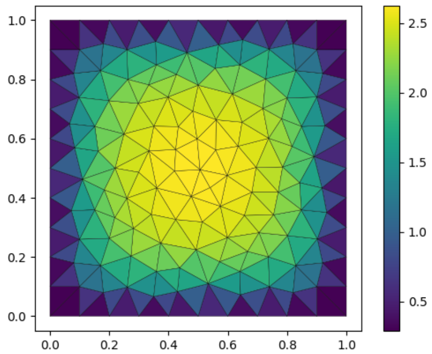


(a) Solution with $\alpha = 2$.

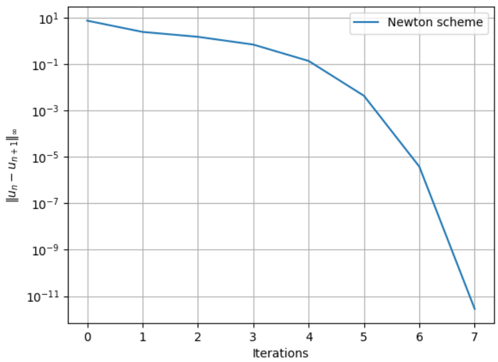


(b) Convergence of Newton scheme, $\alpha = 2$.

Figure 5: Newton scheme, $\alpha = 2$.



(a) Solution with $\alpha = 5$.



(b) Convergence of Newton scheme, $\alpha = 5$.

Figure 6: Newton scheme, $\alpha = 5$.

5 Summary

A `Python` code was implemented to solve a non-linear PDE through linearization of the non-linear term, both by means of a fixed-point and Newton scheme. The results show that the Newton method converges significantly faster than the fixed-point scheme, even for larger values of α. In particular, for $\alpha = 0.1$, the two schemes require roughly 10 and 4 iterations respectively to satisfy the condition $\|u_n - u_{n+1}\|_\infty < 1 \times 10^{-6}$. Conversely, for $\alpha = 2$, the Newton scheme converges in only 6 iterations while roughly 350 fixed-point iterations are required to meet the convergence criterion. Moreover, when moving to $\alpha = 5$, the Newton scheme requires only one additional iteration to converge. Regarding the solution, it is possible to see that, the higher α, the smaller the maximum value of u over Ω. Further work could focus on studying the convergence of the FE scheme, and to link the convergence rate of fixed point and Newton schemes to theoretical bounds.