This is the **renewal equation**

$$I(t) = R(t) \int_0^\infty \omega(\tau) I(t-\tau)\, d\tau$$

Assume that $R(t) = R$ in the time period of interest

Then, check that the number of infected grows exponentially by substituting $I(t) = ke^{rt}$ into it

$$ke^{rt} = R \int_0^\infty \omega(\tau) k e^{r(t-\tau)}\, d\tau \rightarrow R = \frac{1}{\int_0^\infty \omega(\tau) e^{-r\tau}\, d\tau}$$

Assume that $\omega(\tau) \sim \Gamma(a, b) = \frac{b^a}{\Gamma(a)} \tau^{a-1} e^{-b\tau}$, where $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t}$

So, $R = \frac{1}{\int_0^\infty \frac{b^a}{\Gamma(a)} t^{a-1} e^{-b\tau} e^{-r\tau} d\tau} = \frac{1}{\int_0^\infty \frac{b^a}{\Gamma(a)} \tau^{a-1} e^{-\tau(r+b)}\, d\tau} \overset{\tau'=\tau(r+b)}{=} \frac{1}{\frac{b^a}{\Gamma(a)} \frac{1}{(b+r)^a} \int_0^\infty \tau'^{a-1} e^{-\tau'} d\tau'}$

$$\longrightarrow \boxed{R = \frac{(b+r)^\alpha}{b^\alpha}}$$

Fixed point of SIR models and exponential growth

$$\begin{cases} \dfrac{dS(t)}{dt} = -\dfrac{\beta}{N}S(t)I(t) \\[2ex] \dfrac{dI(t)}{dt} = \dfrac{\beta}{N}S(t)I(t) - \gamma I(t) \\[2ex] \dfrac{dR(t)}{dt} = \gamma I(t) \end{cases} \longrightarrow \quad \frac{dS}{dR} = -\frac{\mathrm{R}_0}{N}S \quad \longrightarrow \quad S(R) = Ne^{-\frac{\mathrm{R}_0}{N}R(t)}$$

$$\Big\downarrow \lim_{t\to\infty}$$

$$\boxed{\ln\frac{S(\infty)}{N} = -\mathrm{R}_0\left(1 - \frac{S(\infty)}{N}\right)}$$

$$\Big\downarrow \lim_{S(t)\to N}$$

$$\frac{dI(t)}{dt} = \gamma I(t)\left(\mathrm{R}_0\frac{S(t)}{N} - 1\right) \sim \gamma I(t)\left(\mathrm{R}_0 - 1\right)$$

$$\Big\downarrow$$

$$\boxed{I(t) \sim I_0 e^{\gamma(\mathrm{R}_0 - 1)t}}$$