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February 11, 2026

Derivation of Hankel's Loop Integral for $1/\Gamma(z)$

1. The Integral Definition

We define the contour integral $I(z)$ along the Hankel contour C (starting at $-\infty$, circling the origin, and returning to $-\infty$):

$$I(z) = \int_C e^t t^{-z} dt$$

Assuming $Re(z) < 1$, the contribution from the small circle around the origin vanishes as the radius $\epsilon \to 0$.

2. Parameterizing the Paths

We split C into two linear branches along the negative real axis:

- **Upper Branch:** $t = re^{i\pi}$, where r goes from ∞ to 0. Thus, $dt = -dr$ and $t^{-z} = r^{-z}e^{-i\pi z}$.

$$\int_\infty^0 e^{-r}(r^{-z}e^{-i\pi z})(-dr) = e^{-i\pi z}\int_0^\infty e^{-r}r^{-z}dr$$

- **Lower Branch:** $t = re^{-i\pi}$, where r goes from 0 to ∞. Thus, $dt = dr$ and $t^{-z} = r^{-z}e^{i\pi z}$. (Note: the direction is 0 to $-\infty$, so we adjust the sign).

$$\int_0^\infty e^{-r}(r^{-z}e^{i\pi z})(-dr) = -e^{i\pi z}\int_0^\infty e^{-r}r^{-z}dr$$

3. Combining and Simplifying

Recognizing the definition $\Gamma(1-z) = \int_0^\infty e^{-r}r^{(1-z)-1}dr = \int_0^\infty e^{-r}r^{-z}dr$:

$$I(z) = (e^{-i\pi z} - e^{i\pi z})\Gamma(1-z)$$

Using the identity $\sin(\pi z) = \frac{e^{i\pi z}-e^{-i\pi z}}{2i}$, we have:

$$I(z) = -2i\sin(\pi z)\Gamma(1-z)$$

4. The Final Step: Euler's Reflection Formula

Using $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, we substitute for the sine term:

$$\sin(\pi z)\Gamma(1-z) = \frac{\pi}{\Gamma(z)}$$

Plugging this into our $I(z)$ equation:

$$I(z) = -2i\left(\frac{\pi}{\Gamma(z)}\right)$$

Rearranging gives the final identity:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C e^t t^{-z} dt$$