Differential Forms as Totally Antisymmetric Tensors of Type $(0, r)$

What Is a Tensor of Type $(0, r)$?

A tensor of type $(0, r)$ on a smooth manifold M is a multilinear map:

$$T : \underbrace{T_pM \times \cdots \times T_pM}_{r \text{ times}} \to \mathbb{R}$$

- Here, T_pM is the tangent space at a point $p \in M$.
- The tensor takes r tangent vectors and returns a real number.
- This is also called a **covariant tensor of rank** r.

What Does "Totally Antisymmetric" Mean?

A tensor ω is **totally antisymmetric** if swapping any two of its arguments changes the sign of the output:

$$\omega(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_r) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_r)$$

for all $i \neq j$. If two arguments are equal, the form evaluates to zero.

This antisymmetry is the defining feature of **differential forms**.

So What Is a Differential Form?

A **differential r-form** on a manifold M is a smooth assignment to each point $p \in M$ of a totally antisymmetric tensor of type $(0, r)$ on T_pM.

In other words:

- It's a smooth section of the **exterior power** $\Lambda^r T^*M$, the bundle of alternating r-covariant tensors.
- These forms can be added, multiplied via the **wedge product**, and differentiated using the **exterior derivative** d.

Why Antisymmetry Matters

Antisymmetry gives differential forms their geometric and topological power:

- It ensures that forms naturally integrate over oriented submanifolds.
- It leads to elegant identities like **Stokes' theorem**.
- It allows for coordinate-free expressions of physical laws (e.g., Maxwell's equations in electrodynamics).

Example: A 2-Form

Let ω be a 2-form on $\mathbb{R}^3$. In coordinates:

$$\omega = f(x,y,z)\,dx \wedge dy + g(x,y,z)\,dy \wedge dz + h(x,y,z)\,dz \wedge dx$$

This is a totally antisymmetric 2-tensor: swapping dx and dy flips the sign of the term.