From 0-Form to 2-Form via Exterior Derivative in $\mathbb{R}^2$

You started with a **0-form**:

$$f(x, y) = x^2 + y^2$$

1. Compute the 1-form (gradient as a differential)

The differential of f is:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 2x\, dx + 2y\, dy$$

This is your **1-form**:

$$\omega = 2x\, dx + 2y\, dy$$

2. Compute the exterior derivative $d\omega$ to get the 2-form

Now take the exterior derivative of the 1-form:

$$d\omega = d(2x\, dx + 2y\, dy)$$

Use the rule that $d(f\, dx^i) = df \wedge dx^i$, and remember that:

$$dx \wedge dx = 0, \quad dy \wedge dy = 0, \quad dx \wedge dy = -dy \wedge dx$$

So:

$$d(2x\, dx) = d(2x) \wedge dx = 2\, dx \wedge dx = 0$$

$$d(2y\, dy) = d(2y) \wedge dy = 2\, dy \wedge dy = 0$$

Thus:

$$d\omega = 0$$

Final Result

The **2-form** associated with your 1-form $\omega = 2x\, dx + 2y\, dy$ is:

$$d\omega = 0$$

This tells us that the 1-form is **closed** (its exterior derivative vanishes), which makes sense because it came from the differential of a 0-form (a scalar function), and the exterior derivative of an exact form is always zero.