Divergence as the Exterior Derivative of the Rotor via Differential Forms

Introduction

In differential geometry, vector calculus operations such as gradient, curl (rotor), and divergence can be elegantly expressed using differential forms and the exterior derivative. This document explains how the divergence of a vector field in $\mathbb{R}^3$ can be obtained from the exterior derivative of the rotor, using the Hodge star operator.

Vector Field and Associated Differential Form

Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$:

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$$

We associate to $\mathbf{F}$ a 1-form:

$$\omega = F_x\,dx + F_y\,dy + F_z\,dz$$

Exterior Derivative: Curl as a 2-Form

Taking the exterior derivative of ω yields a 2-form:

$$d\omega = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) dx \wedge dy + \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) dz \wedge dx$$

This 2-form corresponds to the curl (rotor) of $\mathbf{F}$.

Exterior Derivative of a 2-Form

Applying the exterior derivative again:

$$d(d\omega) = 0$$

This follows from the property of the exterior derivative:

$$d^2 = 0$$

Hence, we cannot directly obtain divergence by applying d again to $d\omega$.

Using the Hodge Star Operator

To obtain the divergence, we use the Hodge star operator $\star$, which maps k-forms to $(n-k)$-forms in an n-dimensional space.

Let $\eta = d\omega$ be the 2-form representing the curl. Then:

$$\star\eta \quad \text{is a 1-form}$$

Taking the exterior derivative:

$$d(\star\eta)$$

This is a 2-form, and applying the Hodge star again:

$$\star d \star \omega$$

This expression corresponds to the divergence of **F**:

$$\mathrm{div}(\mathbf{F}) = \star d \star \omega$$

Conclusion

Using differential forms and the Hodge star operator, we can express the divergence of a vector field as:

$$\mathrm{div}(\mathbf{F}) = \star d \star \omega$$

This formulation provides a powerful and coordinate-free way to understand classical vector calculus operations.