

Expansion of $(-z)^{s-1}$

We begin with the definition of complex exponentiation:

$$(-z)^{s-1} = e^{(s-1) \log(-z)}$$

Using the principal branch of the complex logarithm:

$$\log(-z) = \log z + i\pi$$

Substitute into the exponent:

$$(-z)^{s-1} = e^{(s-1)(\log z + i\pi)} = e^{(s-1) \log z} \cdot e^{i\pi(s-1)}$$

Since $e^{(s-1) \log z} = z^{s-1}$, we obtain:

$$(-z)^{s-1} = z^{s-1} \cdot e^{i\pi(s-1)}$$

This identity shows that raising a negative complex number to a power introduces a phase shift of $\pi(s-1)$, which connects to trigonometric functions via Euler's formula:

$$e^{i\pi(s-1)} = \cos(\pi(s-1)) + i \sin(\pi(s-1))$$