Analytic Continuation of the Zeta Function via the Hankel Contour

1. The Bose–Einstein Integral Representation

For $\Re(s) > 1$, the Riemann zeta function satisfies the well-known integral identity

$$\Gamma(s)\,\zeta(s) = \int_0^\infty \frac{x^{\,s-1}}{e^x - 1}\,dx.$$

This follows from expanding the denominator as a geometric series

$$\frac{1}{e^x - 1} = \sum_{n=1}^{\infty} e^{-nx},$$

and integrating term-by-term:

$$\Gamma(s)\zeta(s) = \sum_{n=1}^{\infty} \int_0^\infty x^{s-1} e^{-nx}\,dx = \sum_{n=1}^{\infty} n^{-s} \int_0^\infty u^{s-1} e^{-u}\,du.$$

2. Preparing for Analytic Continuation

To extend $\zeta(s)$ beyond $\Re(s) > 1$, rewrite the Bose integral as

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx = \int_0^\infty x^{s-1} e^{-x} \left(\frac{1}{1 - e^{-x}}\right) dx.$$

Introduce the substitution $z = x$ and consider deforming the path of integration into a contour $\mathcal{H}$ around the negative real axis (the Hankel contour).

The integrand can be analytically continued as

$$\frac{z^{\,s-1}}{e^z - 1},$$

3. The Hankel Contour

Let $\mathcal{H}$ be the classical Hankel contour:

- it starts at $+\infty$ just above the negative real axis,
- encircles the origin counterclockwise,
- and returns to $+\infty$ just below the negative real axis.

On this contour, take the principal branch of z^{s-1}:

$$z^{s-1} = e^{(s-1)(\log|z| + i\arg z)}, \qquad -\pi < \arg z < \pi.$$

4. Deforming the Real Integral into a Hankel Integral

The key identity is

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx = \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{(-z)^{s-1}}{e^z - 1}\, dz.$$

The factor $(-z)^{s-1}$ comes from crossing the branch cut along $(-\infty, 0]$.

Using

$$(-z)^{s-1} = e^{\pi i(s-1)} z^{s-1} \quad \text{(upper side)},$$
$$(-z)^{s-1} = e^{-\pi i(s-1)} z^{s-1} \quad \text{(lower side)},$$

the two sides of the cut contribute a difference

$$e^{\pi i(s-1)} - e^{-\pi i(s-1)} = 2i \sin(\pi(s-1)).$$

Thus,

$$\Gamma(s)\zeta(s) = \frac{1}{2\pi i} \int_{\mathcal{H}} \frac{(-z)^{s-1}}{e^z - 1}\, dz.$$

5. Hankel Representation of the Gamma Function

Recall the classical Hankel representation:

$$\Gamma(s) = \frac{1}{2\pi i} \int_{\mathcal{H}} (-z)^{s-1} e^{-z}\, dz.$$

Now write

$$\frac{1}{e^z - 1} = \frac{e^{-z}}{1 - e^{-z}} = e^{-z} \sum_{n=0}^{\infty} e^{-nz},$$

which is valid along the Hankel contour.

Hence,

$$\Gamma(s)\zeta(s) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{\mathcal{H}} (-z)^{s-1} e^{-(n+1)z}\, dz.$$

Factor the $(n+1)^{-s}$:

$$\int_{\mathcal{H}} (-z)^{s-1} e^{-(n+1)z}\, dz = (n+1)^{-s} \int_{\mathcal{H}} (-u)^{s-1} e^{-u}\, du.$$

Thus,

$$\Gamma(s)\zeta(s) = \left(\frac{1}{2\pi i} \int_{\mathcal{H}} (-u)^{s-1} e^{-u}\, du \right) \sum_{n=1}^{\infty} n^{-s}.$$

The term in parentheses is exactly $\Gamma(s)$, so the identity is consistent and holds for all s except at the poles of $\Gamma(s)$ and $\zeta(s)$.

6. Final Analytic Continuation Formula

We obtain the Hankel contour representation:

$$\boxed{\Gamma(s)\,\zeta(s) = \frac{1}{2\pi i}\int_{\mathcal{H}} \frac{(-z)^{s-1}}{e^z - 1}\,dz}.$$

Since the right-hand side is analytic for all $s \in \mathbb{C}$ except $s = 1$, this formula provides the analytic continuation of $\zeta(s)$ to the entire complex plane.