Why the 1-Form $\omega = -y\,dx + x\,dy$ Is Not Exact at the Origin

The 1-form

$$\omega = -y\,dx + x\,dy$$

is perfectly well-defined and smooth at the origin $(0,0)$. However, the subtlety arises when we ask whether this 1-form is **exact**, i.e., whether there exists a scalar function $f(x,y)$ such that $df = \omega$.

Topology and Exactness

In differential geometry, a key result is:

Every closed 1-form on a simply connected domain is exact.

Let us unpack this:

- A 1-form ω is **closed** if $d\omega = 0$.
- A 1-form is **exact** if $\omega = df$ for some scalar function f.
- A domain is **simply connected** if every loop can be continuously contracted to a point (i.e., the domain has no holes).

Now, for our 1-form:

$$\omega = -y\,dx + x\,dy$$

we computed:

$$d\omega = 2\,dx \wedge dy \neq 0$$

So ω is not even closed, and therefore not exact.

A More Subtle Example: The Normalized 1-Form

Consider the normalized version:

$$\tilde{\omega} = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

This 1-form is **closed** (you can verify that $d\tilde{\omega} = 0$), but it is **not exact** on all of $\mathbb{R}^2 \setminus \{(0,0)\}$. Why?

Because the domain $\mathbb{R}^2 \setminus \{(0,0)\}$ is **not simply connected**—it has a "hole" at the origin. This hole prevents us from defining a global potential function f such that $df = \tilde{\omega}$.

Summary

- The original 1-form $\omega = -y\,dx + x\,dy$ is defined everywhere, including the origin.
- But when we talk about **exactness**, the topology of the domain matters.
- The normalized version $\tilde{\omega} = \frac{-y\,dx + x\,dy}{x^2+y^2}$ is closed but not exact on $\mathbb{R}^2 \setminus \{(0,0)\}$ because the domain is not simply connected.
- This is a classic example in differential geometry illustrating how **topology controls the behavior of differential forms**.