Example of a Closed but Not Exact 1-Form in $\mathbb{R}^2 \setminus \{(0,0)\}$

In $\mathbb{R}^2$, every closed 1-form is also exact, provided the domain is simply connected. To find a 1-form that is closed but not exact, we consider a domain in $\mathbb{R}^2$ that is not simply connected—such as $\mathbb{R}^2 \setminus \{(0,0)\}$.

The 1-Form

Consider the 1-form:

$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

Why is it Closed?

Let:

$$M(x,y) = \frac{-y}{x^2+y^2}, \quad N(x,y) = \frac{x}{x^2+y^2}$$

Compute the partial derivatives:

$$\frac{\partial N}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial M}{\partial y} = \frac{-(x^2+y^2)(1) + y(2y)}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

So:

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

Hence, $d\omega = 0$, so ω is **closed**.

Why is it Not Exact?

If ω were exact, then there would exist a function f such that $df = \omega$. But integrating ω around a loop enclosing the origin (like the unit circle) gives:

$$\int_{S^1} \omega = 2\pi \neq 0$$

This violates the condition for exactness (since the integral of an exact form over a closed loop is always zero), so ω is **not exact**.