How Analytic Continuation Enters the Functional Equation of the Riemann Zeta Function

The Riemann zeta function is originally defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \Re(s) > 1.$$

The functional equation relates $\zeta(s)$ to $\zeta(1-s)$, but since

$$\Re(s) > 1 \quad \Rightarrow \quad \Re(1-s) < 0,$$

the Dirichlet series cannot be used to evaluate $\zeta(1-s)$. Therefore *analytic continuation is needed before the functional equation can even make sense.*

1. Mellin transform representation (valid only for $\Re(s) > 1$)

A classical identity gives

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx, \qquad \Re(s) > 1.$$

This integral representation still has the same limitation: it converges only in the half-plane $\Re(s) > 1$.

2. Splitting the integral: beginning analytic continuation

We split the integral:

$$\int_0^\infty = \int_0^1 + \int_1^\infty.$$

The second part,

$$\int_1^\infty \frac{x^{s-1}}{e^x - 1}\,dx,$$

converges for all s and defines an *entire function.*

The behaviour near $x = 0$ is responsible for divergence. The expansion

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + \frac{x}{12} - \frac{x^3}{720} + \cdots$$

allows us to subtract finitely many terms:

$$\frac{1}{e^x - 1} - \left(\frac{1}{x} - \frac{1}{2}\right),$$

which removes the singularity at $x = 0$.

The modified integrand is regular at $x = 0$, so

$$\int_0^1 \left[\frac{x^{s-1}}{e^x - 1} - x^{s-2} + \frac{1}{2}x^{s-1}\right] dx$$

converges for all complex s. This produces an analytic continuation of $\zeta(s)\Gamma(s)$ to the whole complex plane.

3. Introducing the theta function

Now consider the theta function

$$\theta(t) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t},$$

which satisfies the modular identity

$$\theta(t) = t^{-1/2}\,\theta(1/t).$$

Using the identity

$$\int_0^{\infty} x^{s/2-1}\big(\theta(x) - 1\big)\,dx = \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\,\zeta(s),$$

and applying the modular transformation, one obtains a relation between $\zeta(s)$ and $\zeta(1-s)$.

The key point: these theta-function integrals are valid for *all* s. This is why analytic continuation was needed—so that the $\zeta(s)$ which appears in this identity is well-defined outside $\Re(s) > 1$.

4. The functional equation emerges

One finally obtains the identity

$$\pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\,\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

Equivalently,

$$\boxed{\zeta(s) = 2^s\pi^{s-1}\sin\Big(\frac{\pi s}{2}\Big)\,\Gamma(1-s)\,\zeta(1-s)}.$$

Summary: where analytic continuation appears

- The Dirichlet series only defines $\zeta(s)$ for $\Re(s) > 1$.
- To write $\zeta(1-s)$ in the functional equation, we need values outside this region.
- Analytic continuation is achieved by rewriting the integral representation so that the $x = 0$ singularity is removed.
- Once $\zeta(s)$ is defined for all s, the modular transformation of the theta function produces a natural symmetry between s and $1-s$.

Thus *analytic continuation is a necessary prerequisite* for the functional equation to hold.