The Minkowski Metric in Special Relativity

What Is Minkowski Space?

Minkowski space is a four-dimensional spacetime combining three spatial dimensions (x, y, z) and one time dimension (t). It is named after Hermann Minkowski, who reformulated Einstein's special relativity using geometric principles.

Unlike Euclidean space, Minkowski space uses a pseudo-Euclidean metric that treats time differently from space.

The Minkowski Metric

The metric defines how distances (or intervals) are measured between two events in spacetime. In Minkowski space, the spacetime interval s^2 between two events is given by:

$$s^2 = -c^2t^2 + x^2 + y^2 + z^2$$

Or, in compact form using the metric tensor $\eta_{\mu\nu}$:

$$s^2 = \eta_{\mu\nu}x^\mu x^\nu$$

Where:

- x^μ are the coordinates, with $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$
- $\eta_{\mu\nu}$ is the Minkowski metric tensor, typically written as:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This signature $(-+++)$ is common, though some conventions use $(+---)$ instead.

Why Is This Important?

- **Invariant Interval**: The spacetime interval s^2 is invariant under Lorentz transformations. All observers, regardless of their motion, agree on this quantity.
- **Causal Structure**: Depending on the sign of s^2, events can be:
  - **Timelike** $(s^2 < 0)$: One event can causally influence the other.
  - **Spacelike** $(s^2 > 0)$: No causal connection is possible.
  - **Lightlike** $(s^2 = 0)$: Events are connected by a light signal.

A Quick Example

Consider two events:

- Event A: $(t = 0, x = 0, y = 0, z = 0)$
- Event B: $(t = 1\,\mathrm{s}, x = 3 \times 10^8\,\mathrm{m}, y = 0, z = 0)$

Then:

$$s^2 = -c^2(1)^2 + (3 \times 10^8)^2 = 0$$

This is a **lightlike interval**—a photon could travel from A to B.

Follows an example of Lorentz invariance:

Invariance of the Spacetime Interval under Lorentz Transformations

Goal

We aim to show that the spacetime interval between two events,

$$s^2 = -c^2t^2 + x^2 + y^2 + z^2,$$

remains unchanged under a Lorentz transformation.

Step-by-Step Sketch

1. Define the Interval in 4D Spacetime

Let the coordinates of an event be represented by a four-vector:

$$x^\mu = (ct, x, y, z).$$

The spacetime interval is given by:

$$s^2 = \eta_{\mu\nu}x^\mu x^\nu,$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. Apply a Lorentz Transformation

A Lorentz transformation relates coordinates in one inertial frame to another:

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu,$$

where $\Lambda^\mu{}_\nu$ is the Lorentz transformation matrix.

3. Compute the Interval in the New Frame

The interval in the primed frame is:

$$s'^2 = \eta_{\mu\nu}x'^\mu x'^\nu = \eta_{\mu\nu}\Lambda^\mu{}_\alpha x^\alpha \Lambda^\nu{}_\beta x^\beta.$$

Rewriting:

$$s'^2 = x^\alpha x^\beta \left(\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta\right).$$

4. Use Lorentz Invariance Condition

Lorentz transformations preserve the Minkowski metric:

$$\Lambda^{\mu}{}_{\alpha}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}.$$

Therefore:

$$s'^2 = x^{\alpha}x^{\beta}\eta_{\alpha\beta} = s^2.$$

Conclusion

Since $s'^2 = s^2$, the spacetime interval is invariant under Lorentz transformations. This means all inertial observers agree on the value of s^2, regardless of their relative motion.

Follows a worked out example of a Lorentz-boost along the x-axis

Lorentz Boost and Invariance of the Spacetime Interval

Setup

Consider two events in spacetime:

- Event A: (t_1, x_1)
- Event B: (t_2, x_2)

We ignore the y and z coordinates since the boost is along the x-axis.
The spacetime interval between these two events is:

$$s^2 = -c^2(t_2 - t_1)^2 + (x_2 - x_1)^2$$

Define:

$$\Delta t = t_2 - t_1, \quad \Delta x = x_2 - x_1$$

So:

$$s^2 = -c^2\Delta t^2 + \Delta x^2$$

Lorentz Transformation Along the x-axis

A Lorentz boost along the x-axis with velocity v transforms coordinates as:

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

where:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Apply the transformation to the differences:

$$\Delta x' = \gamma(\Delta x - v\Delta t)$$
$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)$$

Compute the Interval in the Primed Frame

$$\begin{aligned} s'^2 &= -c^2(\Delta t')^2 + (\Delta x')^2 \\ &= -c^2\gamma^2\left(\Delta t - \frac{v}{c^2}\Delta x\right)^2 + \gamma^2(\Delta x - v\Delta t)^2 \end{aligned}$$

Expand both terms:
Time term:

$$-c^2\gamma^2\left(\Delta t^2 - 2\frac{v}{c^2}\Delta t\Delta x + \frac{v^2}{c^4}\Delta x^2\right) = -\gamma^2\left(c^2\Delta t^2 - 2v\Delta t\Delta x + \frac{v^2}{c^2}\Delta x^2\right)$$

Space term:

$$\gamma^2\left(\Delta x^2 - 2v\Delta x\Delta t + v^2\Delta t^2\right)$$

Add both:

$$s'^2 = \gamma^2\left[-c^2\Delta t^2 + \Delta x^2 + v^2\Delta t^2 - \frac{v^2}{c^2}\Delta x^2\right]$$

Group terms:

$$s'^2 = \gamma^2\left[\Delta x^2\left(1 - \frac{v^2}{c^2}\right) - \Delta t^2\left(c^2 - v^2\right)\right]$$

Factor out:

$$s'^2 = \gamma^2\left(1 - \frac{v^2}{c^2}\right)\left[\Delta x^2 - c^2\Delta t^2\right]$$

But:

$$\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1$$

So:

$$s'^2 = \Delta x^2 - c^2\Delta t^2 = s^2$$

Conclusion

We have shown that:

$$s'^2 = s^2$$

Therefore, the spacetime interval is invariant under a Lorentz boost along the x-axis.