

# Sample LaTeX Document

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## Derivation of Hankel's Loop Integral for $1/\Gamma(z)$

### 1. The Integral Definition

We define the contour integral  $I(z)$  along the Hankel contour  $C$  (starting at  $-\infty$ , circling the origin, and returning to  $-\infty$ ):

$$I(z) = \int_C e^t t^{-z} dt$$

Assuming  $\operatorname{Re}(z) < 1$ , the contribution from the small circle around the origin vanishes as the radius  $\epsilon \rightarrow 0$ .

### 2. Parameterizing the Paths

We split  $C$  into two linear branches along the negative real axis:

- **Upper Branch:**  $t = re^{i\pi}$ , where  $r$  goes from  $\infty$  to 0. Thus,  $dt = -dr$  and  $t^{-z} = r^{-z}e^{-i\pi z}$ .

$$\int_{\infty}^0 e^{-r} (r^{-z} e^{-i\pi z})(-dr) = e^{-i\pi z} \int_0^{\infty} e^{-r} r^{-z} dr$$

- **Lower Branch:**  $t = re^{-i\pi}$ , where  $r$  goes from 0 to  $\infty$ . Thus,  $dt = dr$  and  $t^{-z} = r^{-z}e^{i\pi z}$ . (Note: the direction is 0 to  $-\infty$ , so we adjust the sign).

$$\int_0^{\infty} e^{-r} (r^{-z} e^{i\pi z})(-dr) = -e^{i\pi z} \int_0^{\infty} e^{-r} r^{-z} dr$$

### 3. Combining and Simplifying

Recognizing the definition  $\Gamma(1-z) = \int_0^{\infty} e^{-r} r^{(1-z)-1} dr = \int_0^{\infty} e^{-r} r^{-z} dr$ :

$$I(z) = (e^{-i\pi z} - e^{i\pi z}) \Gamma(1-z)$$

Using the identity  $\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}$ , we have:

$$I(z) = -2i \sin(\pi z) \Gamma(1-z)$$

#### 4. The Final Step: Euler's Reflection Formula

Using  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ , we substitute for the sine term:

$$\sin(\pi z)\Gamma(1-z) = \frac{\pi}{\Gamma(z)}$$

Plugging this into our  $I(z)$  equation:

$$I(z) = -2i \left( \frac{\pi}{\Gamma(z)} \right)$$

Rearranging gives the final identity:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C e^t t^{-z} dt$$