Grassmann Formula in $\mathbb{R}^4$ with Two Planes

Let U and V be two planes (i.e., 2-dimensional subspaces) in $\mathbb{R}^4$, both passing through the origin and not parallel. We apply the Grassmann formula:

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$$

Since both U and V are planes:

$$\dim(U) = \dim(V) = 2$$

Then:

$$\dim(U + V) = 2 + 2 - \dim(U \cap V) = 4 - \dim(U \cap V)$$

Possible Cases

- $\dim(U \cap V) = 0$: The planes intersect only at the origin. Then $\dim(U + V) = 4$, and their span fills all of $\mathbb{R}^4$.
- $\dim(U \cap V) = 1$: The planes intersect along a line. Then $\dim(U + V) = 3$, and their span is a 3D subspace of $\mathbb{R}^4$.
- $\dim(U \cap V) = 2$: The planes coincide. Then $\dim(U + V) = 2$, which means they are the same plane.

Since the planes are not parallel, we exclude the third case.

Example

Let:

$$U = \operatorname{span}\{(1, 0, 0, 0),\ (0, 1, 0, 0)\}$$
$$V = \operatorname{span}\{(0, 0, 1, 0),\ (0, 0, 0, 1)\}$$

Then:

$$U \cap V = \{0\}$$
$$\dim(U + V) = 4$$

These two planes intersect only at the origin and together span all of $\mathbb{R}^4$.