Analytic continuation of the Gamma function via the Hankel contour

1 Euler's integral for $\Gamma(z)$

For $\Re z > 0$, the Gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt. \tag{1}$$

This integral converges only for $\Re z > 0$, so it does not directly define $\Gamma(z)$ on the whole complex plane.

2 The Hankel contour and the branch of the logarithm

We introduce a branch cut along the positive real axis and take the principal branch of the logarithm on $\mathbb{C} \setminus [0, \infty)$, so that

$$-\pi < \arg t < \pi. \tag{2}$$

On this domain we define

$$t^{z-1} = e^{(z-1)\log t}. \tag{3}$$

The *Hankel contour* $\mathcal{H}$ is defined as follows:

- it starts at $+\infty$ just below the positive real axis,
- runs towards 0 along this line,
- loops once counterclockwise around the origin along a small circle,
- and returns to $+\infty$ just above the positive real axis.

Thus $\mathcal{H}$ winds once around the origin and avoids the branch cut.

3 A contour integral involving $\Gamma(z)$

Consider the contour integral

$$I(z) := \int_{\mathcal{H}} (-t)^{z-1} e^{-t} \, dt. \tag{4}$$

We will relate $I(z)$ to $\Gamma(z)$ for $\Re z > 0$ and then use this representation to extend $\Gamma(z)$ analytically.

On $\mathcal{H}$ we write

$$(-t)^{z-1} = e^{(z-1)\log(-t)}, \tag{5}$$

where $\log(-t)$ is defined using the chosen branch of the logarithm.

3.1 Contribution from the lower and upper rays

We parametrize the two rays of $\mathcal{H}$:

Lower side. On the lower side, $t = x$ with x decreasing from $+\infty$ to 0, just below the positive real axis. Then $dt = -dx$ and $-t = -x$ has argument $-\pi$, so

$$(-t)^{z-1} = x^{z-1} e^{-i\pi(z-1)}. \tag{6}$$

Hence

$$\begin{aligned}
\int_{\text{lower}} (-t)^{z-1} e^{-t} \, dt &= \int_{\infty}^{0} x^{z-1} e^{-i\pi(z-1)} e^{-x} (-dx) && (7)\\
&= e^{-i\pi(z-1)} \int_{0}^{\infty} x^{z-1} e^{-x} \, dx && (8)\\
&= e^{-i\pi(z-1)} \Gamma(z), && (9)
\end{aligned}$$

valid for $\Re z > 0$.

Upper side. On the upper side, $t = x$ with x increasing from 0 to ∞, just above the positive real axis. Then $dt = dx$ and $-t = -x$ has argument $+\pi$, so

$$(-t)^{z-1} = x^{z-1} e^{+i\pi(z-1)}. \tag{10}$$

Thus

$$\begin{aligned}
\int_{\text{upper}} (-t)^{z-1} e^{-t} \, dt &= \int_{0}^{\infty} x^{z-1} e^{+i\pi(z-1)} e^{-x} \, dx && (11)\\
&= e^{+i\pi(z-1)} \Gamma(z). && (12)
\end{aligned}$$

Small circle around the origin. The small circular arc around 0 contributes 0 in the limit (for $\Re z > 0$), since t^{z-1} is integrable near 0 and the arc length tends to 0.

3.2 Expression for $I(z)$

Summing the contributions from the lower and upper rays, we obtain

$$I(z) = \left(e^{-i\pi(z-1)} + e^{+i\pi(z-1)}\right)\Gamma(z) \tag{13}$$

$$= 2\cos\big(\pi(z-1)\big)\,\Gamma(z). \tag{14}$$

Using the identity

$$\cos\big(\pi(z-1)\big) = -\cos(\pi z) \tag{15}$$

and the relation

$$\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}, \tag{16}$$

one can rewrite $I(z)$ in terms of $\sin(\pi z)$. A standard and convenient form of the Hankel representation is

$$\Gamma(z) = \frac{1}{2i\sin(\pi z)} \int_{\mathcal{H}} (-t)^{z-1} e^{-t}\,dt, \tag{17}$$

valid initially for $\Re z > 0$ and $z \notin \mathbb{Z}$ (to avoid the zeros of $\sin(\pi z)$).

4 Analytic continuation

Define

$$F(z) := \frac{1}{2i\sin(\pi z)} \int_{\mathcal{H}} (-t)^{z-1} e^{-t}\,dt. \tag{18}$$

We now analyze $F(z)$ as a function of z.

4.1 Analyticity of the integral

For each fixed t on $\mathcal{H}$, the integrand $(-t)^{z-1}e^{-t}$ is entire in z. The contour $\mathcal{H}$ is fixed and avoids the singularity at $t = 0$ by looping around it. The factor e^{-t} ensures convergence at infinity for all $z \in \mathbb{C}$. Standard arguments (dominated convergence, Morera's theorem) show that the map

$$z \mapsto \int_{\mathcal{H}} (-t)^{z-1} e^{-t}\,dt \tag{19}$$

is an entire function of z.

4.2 Poles and agreement with $\Gamma(z)$

The only possible singularities of $F(z)$ come from the factor $1/\sin(\pi z)$, which has simple poles at $z \in \mathbb{Z}$. The integral itself is entire, so $F(z)$ has at most simple poles at the integers.

For $\Re z > 0$, we have already computed $I(z)$ and found

$$I(z) = 2i \sin(\pi z)\, \Gamma(z), \tag{20}$$

up to the standard trigonometric identities. Hence, on the half-plane $\Re z > 0$,

$$F(z) = \Gamma(z). \tag{21}$$

Thus $F(z)$ extends $\Gamma(z)$ analytically from $\Re z > 0$ to $\mathbb{C} \setminus \mathbb{Z}$.

A more detailed analysis shows that the poles at positive integers are removable (the integral vanishes appropriately there), while the poles at $0, -1, -2, \dots$ remain, matching the known singularities of $\Gamma(z)$. Therefore $F(z)$ is the analytic continuation of $\Gamma(z)$ to $\mathbb{C} \setminus \{0, -1, -2, \dots\}$.

5 Final Hankel representation

We conclude that the Gamma function admits the Hankel contour representation

$$\boxed{\Gamma(z) = \frac{1}{2i \sin(\pi z)} \int_{\mathcal{H}} (-t)^{z-1} e^{-t}\, dt, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.} \tag{22}$$

This formula provides an explicit analytic continuation of $\Gamma(z)$ from the half-plane $\Re z > 0$ to the whole complex plane minus its simple poles at the nonpositive integers.