Exterior Derivative of a Non-Exact 1-Form in $\mathbb{R}^2$

Let us consider the 1-form:

$$\omega = -y\,dx + x\,dy$$

This 1-form is **not exact** on all of $\mathbb{R}^2$, though it is closed on $\mathbb{R}^2 \setminus \{(0,0)\}$. We now compute its exterior derivative.

Compute the Exterior Derivative

Apply the exterior derivative:

$$d\omega = d(-y\,dx + x\,dy)$$

Break it into parts:

$$d(-y\,dx) = -dy \wedge dx, \quad d(x\,dy) = dx \wedge dy$$

So:

$$d\omega = -dy \wedge dx + dx \wedge dy$$

Using the antisymmetry of the wedge product:

$$dy \wedge dx = -dx \wedge dy$$

Therefore:

$$d\omega = -(-dx \wedge dy) + dx \wedge dy = dx \wedge dy + dx \wedge dy = 2\,dx \wedge dy$$

Final Result

The exterior derivative of the 1-form $\omega = -y\,dx + x\,dy$ is:

$$d\omega = 2\,dx \wedge dy$$

This is a **nonzero 2-form**, indicating that ω is not closed and hence not exact on all of $\mathbb{R}^2$.