Wave equation for medical imaging application

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1 Problem

Let Ω be some domain of size a few millimeters/centimeters, $T > 0$ be the final time (of the order of a few 10s of μs). The speed of sound is $c = 1540$ m/s in the background and takes different values (a few percent of difference) in the reflectors. Consider the wave problem, for the pressure $u = u(x,t)$ with $x \in \Omega$ and $t \in]0,T[$,

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \Delta u = fg, & \text{in } \Omega \times]0,T[\,, \\ \dfrac{\partial u}{\partial n} = 0, & \text{in } \partial\Omega \times]0,T[\,, \\ u(x,0) = \dfrac{\partial u}{\partial t}(x,0) = 0, & \text{in } \Omega. \end{cases}$$

The source term is a pulse, localized in space around some $x_0 \in \Omega$ and in time around $t_0 = 0.12$ μs, and is given by

$$f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\sin(\omega_0(t-t_0))\exp\left(-\frac{\omega_0^2(t-t_0)^2}{\tau^2}\right)\right] \qquad g(x) = \exp\left(-\kappa_0^2(x-x_0)^2\right), \tag{1.1}$$

for $\kappa_0 = 1/50$ (nm)$^{-1}$, $\omega_0 = 2\pi f_0$, $f_0 = 10^6$ Hz, $\tau = 0.8$.

2 Discretization in time

We instroduce a time discretization $t^n = n\Delta t$, and denote u^n the approximation of $u(x,t^n)$, f^n the approximation of $f(t^n)$. Then, using a finite difference approximation of the time derivative,

$$\begin{cases} u^{n+1} = 2u^n - u^{n-1} + (c\Delta t)^2 \Delta u^n + (\Delta t)^2 f^n g & \text{in } \Omega,\ \forall n > 1, \\ \dfrac{\partial u^n}{\partial n} = 0 & \text{in } \partial\Omega,\ \forall n > 1, \\ u^0 = u^1 = 0 & \text{in } \Omega. \end{cases}$$

or, in variational form, $\forall v \in V = H^1(\Omega)$:

$$\begin{cases} \displaystyle\int_\Omega u^{n+1} v = 2\int_\Omega u^n v - \int_\Omega u^{n-1} v - (c\Delta t)^2 \int_\Omega \nabla u^n \cdot \nabla v + (\Delta t)^2 \int_\Omega f^n g v \quad \forall n > 1, \\ u^0 = u^1 = 0 \text{ in } \Omega. \end{cases}$$

3 Discretization in space

We approximate the space V by a finite element space V_h, and denote U^n the approximation of u^n. The variational formulation is reinterpreted in terms of matrices and vectors as follows:

$$\begin{cases} U^{n+1} = 2U^n - U^{n-1} - \mathbb{M}^{-1}\left((c\Delta t)^2 \mathbb{K} U^n - (\Delta t)^2 B^n\right) \quad \forall n > 1, \\ U^0 = U^1 = 0 \text{ in } \Omega. \end{cases}$$

where $\mathbb{M}$ is the mass matrix, $\mathbb{K}$ the stiffness matrix and B^n is the right-hand-side defined as

$$\mathbb{M}_{ij} = \int_\Omega \varphi_i \varphi_j, \;\; \mathbb{K}_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j, \;\; (F^n)_i = \int_\Omega f^n g\, \varphi_i.$$

assuming φ_j are the basis functions of the finite element space.

The numerical scheme is not unconditionally stable. One needs to satisfy a CFL condition

$$\frac{(\Delta t)^2}{4} \sup_{V_h} \frac{(\mathbb{K}U, U)}{(\mathbb{M}U, U)} < 1, \tag{3.1}$$

which means in practice that one needs to choose $\Delta t \ll 2h$ if h is the typical mesh size.