FRC 566.010 — UCC PDE: Existence, Uniqueness, and Dissipation

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Abstract

We establish well-posedness for the Universal Coherence Condition (UCC) $\partial_t \ln C = D_C \, \Delta \ln C + S_C$ under standard boundary conditions and prove a dissipation inequality $\sigma(t) = k_* D_C \int \|\nabla \ln C\|^2 \, dV \geq 0$. Fractional variants are briefly discussed. Numerical 1D/2D demos illustrate convergence to stationary states.

1. Statement and assumptions

Throughout, set $u := \ln C$ (positivity of C is assumed so u is well-defined). The core evolution is

$$\partial_t u = D_C \, \Delta u + S_C \quad \text{in } \Omega \times (0, T),$$

on a bounded domain $\Omega \subset \mathbb{R}^d$ with either homogeneous Neumann ($\partial_\nu u = 0$) or Dirichlet ($u = 0$) boundary conditions, and initial data $u(\cdot, 0) = u_0$.

Assumptions (A).

- $\Omega \subset \mathbb{R}^d$ bounded with C^1 boundary; $D_C > 0$ constant.
- Boundary condition: either (N) $\partial_\nu u = 0$ on $\partial\Omega$ or (D) $u = 0$ on $\partial\Omega$.
- Forcing $S_C \in L^2(0, T; L^2(\Omega))$ (or more generally $L^2(0, T; H^{-1}(\Omega))$).
- Initial data $u_0 \in L^2(\Omega)$; for (D), $u_0 \in H_0^1(\Omega)$ is convenient.
- $C = e^u > 0$; when referring back to C, we interpret statements via $u = \ln C$.

2. Existence and uniqueness

We record standard results for linear parabolic problems; precise proofs follow classical energy and semigroup methods (see Evans; Pazy).

Theorem 1 (Existence)**.** *Under Assumptions (A), for any $T > 0$ there exists a weak solution u with*

$$u \in L^2(0, T; H^1(\Omega)), \qquad \partial_t u \in L^2(0, T; H^{-1}(\Omega)),$$

solving $\partial_t u - D_C \, \Delta u = S_C$ in $H^{-1}(\Omega)$ with (N) or (D) boundary conditions and $u(\cdot, 0) = u_0$.

Sketch. Galerkin approximation with energy estimates: test by u to obtain $\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + D_C \, \|\nabla u\|_{L^2}^2 \leq \|S_C\|_{L^2} \, \|u\|_{L^2}$, then apply Gronwall and compactness to pass to the limit.

Theorem 2 (Uniqueness)**.** *Let u_1, u_2 be weak solutions in the class above with the same data. Then $u_1 \equiv u_2$ on $\Omega \times (0, T)$.*

Sketch. Set $w = u_1 - u_2$, test the difference equation by w and use $\|\nabla w\|_{L^2}^2 \geq 0$ to derive $\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 \leq 0$. Hence $w \equiv 0$ from $w(\cdot, 0) = 0$.

3. Dissipation inequality and stationary solutions

Define the dissipation functional $\mathcal{E}(t) := \int_\Omega \frac{1}{2} \|\nabla u\|^2 \, dx$. For $S_C \equiv 0$ one has a strict energy decay.

Theorem 3 (Dissipation). *Assume (A) with $S_C \equiv 0$. Then $\mathcal{E}$ is nonincreasing and*

$$\frac{d}{dt} \mathcal{E}(t) \;=\; -D_C \int_\Omega \|\Delta u\|^2 \, dx \;\leq\; 0$$

in the (distributional) sense for weak solutions. In particular, $\mathcal{E}(t) \downarrow$ as $t \to \infty$ and every ω-limit point is stationary. With forcing $S_C \in L^2$, one obtains the balance

$$\frac{d}{dt} \tfrac{1}{2} \|u\|_{L^2}^2 + D_C \, \|\nabla u\|_{L^2}^2 \;=\; \langle S_C, u \rangle,$$

implying boundedness via Gronwall.

Sketch. Differentiate $\mathcal{E}$ formally and integrate by parts under (N)/(D). For weak solutions, mollify in time or use standard elliptic regularization.

Stationary states. In terms of $C = e^u$, steady solutions satisfy

$$\Delta \ln C + \frac{S_C}{D_C} = 0 \quad \text{in } \Omega,$$

with (N)/(D) BCs. In 1D this reduces to $\partial_{xx} \ln C + S_C/D_C = 0$ ("concentration" picture); in higher d, it characterizes coherence equilibria.

4. Fractional generalization

The analysis extends to $\partial_t u + (-\Delta)^\alpha(-D_C u) = S_C$ with $\alpha \in (0, 1]$, i.e., replacing Δ by the fractional Laplacian $(-\Delta)^\alpha$ (spectral or integral definition). In the weak formulation one works with H^α; existence and uniqueness follow from coercivity of the corresponding bilinear form and semigroup theory for sectorial operators. See Karch (fractional diffusion) for related frameworks.

5. Dimensional link to physics

In one dimension, C plays the role of a (dimensionless) concentration and $u = \ln C$ is its log-intensity. In general dimension, stationary coherence equilibria satisfy $\Delta \ln C + S_C/D_C = 0$; Neumann BCs conserve total coherence mass $\int C \, dx$, while Dirichlet BCs pin the boundary phase.

6. Numerical demos (reproducible)

Run `python code/566.010/pde_demo.py` to regenerate 1D/2D figures.

References

- L.C. Evans, *Partial Differential Equations*, 2nd ed., AMS, 2010. (Parabolic energy methods.)
- A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983. (Analytic semigroups/well-posedness.)
- G. Karch, self-similar and decay results for fractional diffusion equations, various notes/papers (2000s). (Fractional Laplacians and smoothing.)