The order of PP

$$\mathcal{P} = \frac{1}{N} + \frac{1}{N}\sum_k \left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right]\cos^{4t}\left(\frac{\pi k}{2N}\right)$$

$$= \frac{1}{N} + \frac{1}{N}\sum_0^T \left(1 - \frac{k^2\pi^2 t}{2N^2}\right)\left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right] + \mathcal{O}\left(\frac{1}{N^2}\right) + \mathcal{O}\left(t^{\frac{3}{2}}\frac{\log(N)}{N^3}\right),$$

where $T = \left\lceil \frac{2N}{5}\sqrt{\frac{3\log(N)}{t}} \right\rceil$. Now we analysis the summation term. Let $a := i - j$, $b := i + j - 1$, The summation part is

$$1 + 2\left(2N^2 - \pi^2 t(1+T)^2\right)\cos\left[\frac{\pi(a - N + 2aT)}{2N}\right]\csc\left[\frac{a\pi}{2N}\right]$$

$$+\pi^2 t(3+2T)\cos\left[\frac{a\pi(1+T)}{N}\right]\csc\left[\frac{a\pi}{2N}\right]^2 + \pi^2 t\cos\left[\frac{\pi(3a - 3N + 2aT)}{2N}\right]\csc\left[\frac{a\pi}{2N}\right]^3$$

$$-4N^2\cos\left[\frac{\pi(b - N + 2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right]$$

$$+2\pi^2 t\cos\left[\frac{\pi(b - N + 2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right] + 4\pi^2 tT\cos\left[\frac{\pi(b - N + 2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right]$$

$$+2\pi^2 tT^2\cos\left[\frac{\pi(b - N + 2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right] + 3\pi^2 t\cos\left[\frac{b\pi(1+T)}{N}\right]\csc\left[\frac{b\pi}{2N}\right]^2$$

$$+2\pi^2 tT\cos\left[\frac{b\pi(1+T)}{N}\right]\csc\left[\frac{b\pi}{2N}\right]^2 + \pi^2 t\cos\left[\frac{\pi(3b - 3N + 2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right]^3$$

When $t = cN^k$ polylog(N)

when $\frac{4}{3} < k < 2$
$T = \frac{2}{5}\sqrt{3}N\sqrt{\frac{N^{-k}}{c\ \text{polylog}(N)}}$

$$\mathcal{P} = \frac{2}{N} - \frac{1}{8N^2}\Bigg[cN^k\pi^2\Bigg((3+2T)\cos\left[\frac{a\pi(1+T)}{N}\right]\csc\left[\frac{a\pi}{2N}\right]^2$$

$$+\cos\left[\frac{\pi(-3N + a(3+2T))}{2N}\right]\csc\left[\frac{a\pi}{2N}\right]^3 + \csc\left[\frac{b\pi}{2N}\right]^2\Bigg((3+2T)\cos\left[\frac{b\pi(1+T)}{N}\right]$$

$$+\cos\left[\frac{\pi(3b - 3N + 2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right]\Bigg)\Bigg)\log[N]$$

$$-2\cos\left[\frac{\pi(a - N + 2aT)}{2N}\right]\csc\left[\frac{a\pi}{2N}\right]\left(2N - cN^k\pi^2(1+T)^2\log[N]\right)$$

$$-2\cos\left[\frac{\pi(b - N + 2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right]\left(2N - cN^k\pi^2(1+T)^2\log[N]\right)\Bigg] + \mathcal{O}\left(N^{-\frac{3}{2}}\text{polylog}(N)\right)$$

We know that $\csc(x) = \frac{1}{x} + O(x)$.

$|i - j| \sim\Theta(N)$
When $|i - j| \sim\Theta(N)$, then $i + j \sim \Theta(N)$. In this case, $\csc\left[\frac{a\pi}{2N}\right], \csc\left[\frac{b\pi}{2N}\right] \sim O(1)$. And then we could absorb the constant into c, for example, $5c \to c$. Here, we let $h = \cos\left[\frac{a(1+\frac{2}{5}g)\pi}{N}\right] + \cos\left[\frac{b(1+\frac{2}{5}g)\pi}{N}\right]$

$$\mathcal{P} = \frac{2}{N} - \frac{6\pi^2 \log(N) - 25}{50N} \Bigg\{$$

$$\csc\left[\frac{a\pi}{2N}\right] \sin\left[\frac{a\left(5 + 4\sqrt{3}\sqrt{\frac{N}{c}}\right)\pi}{10N}\right] + \csc\left[\frac{b\pi}{2N}\right] \sin\left[\frac{b\left(5 + 4\sqrt{3}\sqrt{\frac{N}{c}}\right)\pi}{10N}\right]$$

$$\Bigg\} + \mathcal{O}\left(N^{-\frac{3}{2}} \mathrm{polylog}(N)\right)$$

Now, the tricky thing is the term in the second line. w.l.o.g., let $a, b > 0$. We need to analysis the

$$\sin\left[\frac{a\pi}{2N}\right] \sin\left[\frac{a\left(5 + 4\sqrt{3}\sqrt{\frac{N}{c}}\right)\pi}{10N}\right] + \sin\left[\frac{b\pi}{2N}\right] \sin\left[\frac{b\left(5 + 4\sqrt{3}\sqrt{\frac{N}{c}}\right)\pi}{10N}\right]$$

Let $f(x) = \sin(x)\sin(qx)$, $x \in (0, \pi)$. If we want to let $f(x_1) + f(x_2) \geq 0$, the c can't be a constant.