Fermionic classical shallow shadows

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1. The variance of the fermionic classical shallow shadows

Let v be the estimation of $\mathrm{tr}(\rho\gamma_S)$ with shallow fermionic shadow $\mathcal{M}_d$The variance of the fermionic classical shallow shadows is determined by the following equation

$$\mathrm{Var}[v] = \frac{1}{\alpha_{S,d}}.$$

The variance of the fermionic classical shallow shadows is inversely proportional to the α, where α could be calculated by the tensor network contraction.

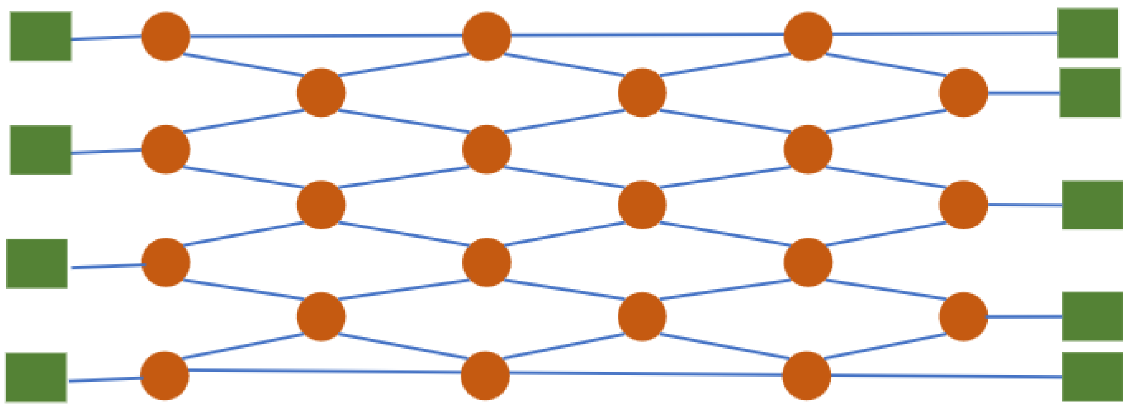

Figure 1: The green boxes are vectors with Pauli basis. The red circles are tensors, and the concrete form of these tensors are shown in overleaf.

After few layers of tensors acting on the initial vector, the result state could be writen as

$$\sum_{\{i \mid P_i \in \mathcal{P}_n\}} c_i P_i, \tag{1}$$

where $\mathcal{P}_n$ is the set of n bits Pauli group. Finally, the α could be given by the summation of the coefficients in Eqn. (1)

$$\alpha = \sum_{\{i \mid P_i \in \mathcal{Z}_n\}} c_i, \tag{2}$$

where $\mathcal{Z}_n$ is the set of n bits Pauli group with I or Z operator.

Thus, the key thing here is to calculate the distribution of the coefficients in Eqn. (1). And the calculation of the distribution could be mapped to a random walk problem, which is shown in `doc/deductions/CalculateAlp.pdf` (we will use `doc/deductions/*` as default path and use `*.pdf` as default format).

If we set the number of layers d as a odd number, and the number of qubits n as a even number, the random walk could be simplified to a neat form, which is a 2D-lattice.

Let $N = \frac{n}{2}$, $t = \frac{d-1}{2}$, $k := |S|$, and the vertice of the 2D-lattice is (μ, ν), $\mu, \nu = 1, 2, ..., N$. Denote the coefficient of (μ, ν) with d layers as $c^{i,j}(\mu, \nu, t)$, where (i, j) is the initial state. We will ignore the superscript (i, j) if it is not ambiguous.

Due to the note `CalculateAlp`, we could write down the transition equation of the random walk as

$$c^{i,j}(\mu, \nu, t) = P_i(\mu, t) P_j(\nu, t) + I^{i,j}(\mu, \nu, t) \tag{3}$$

Notice that the I is zero in most of place. It gets non-zero only when $\mu = \nu$ or $|\mu - \nu| = 1$. And it is a small term, too. Thus, the dominant term in Eqn. (3) is the first term.

Recall Eqn. (2), the α could be calculated by

$$\alpha_{2t+1} = \sum_{\mu,\nu} \delta_{\mu,\nu} c(\mu, \nu, t) = \sum_{\mu} c(\mu, \mu, t)$$

because only when $\mu = \nu$, the basis $P_{\mu,\nu}$ is in $\mathcal{Z}_n$. Substituting Eqn. (3) into the above equation, we have

$$\alpha_{2t+1} = \sum_{\mu} P(\mu, t) P(\mu, t) + \sum I(\mu, \mu, t)$$

From numerical results, we should see that the summation $\mathcal{P} = \sum_{\mu} P(\mu, t) P(\mu, t)$ reflects the trend of the α. And the α could be approximated by the $\mathcal{P}$ with a constant coefficient.

2. Analysis $\mathcal{P}$

See `CalculatePP`.