The order of PP

$$\mathcal{P} = \frac{1}{N} + \frac{1}{N}\sum_k \left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right]\cos^{4t}\left(\frac{\pi k}{2N}\right)$$
$$= \frac{1}{N} + \frac{1}{N}\sum_0^T \left(1 - \frac{k^2\pi^2 t}{2N^2}\right)\left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right] + \mathcal{O}\left(\frac{1}{N^2}\right) + \mathcal{O}\left(t^{\frac{3}{2}}\frac{\log(N)}{N^3}\right),$$

where $T = \left\lceil \frac{2N}{5}\sqrt{\frac{3\log(N)}{t}} \right\rceil$. Now we analysis the summation term. Let $a := i-j$, $b := i+j-1$, The summation part is

$$1 + 2\left(2N^2 - \pi^2 t(1+T)^2\right)\cos\left[\frac{\pi(a-N+2aT)}{2N}\right]\csc\left[\frac{a\pi}{2N}\right]$$
$$+\pi^2 t(3+2T)\cos\left[\frac{a\pi(1+T)}{N}\right]\csc\left[\frac{a\pi}{2N}\right]^2 + \pi^2 t\cos\left[\frac{\pi(3a-3N+2aT)}{2N}\right]\csc\left[\frac{a\pi}{2N}\right]^3$$
$$-4N^2\cos\left[\frac{\pi(b-N+2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right]$$
$$+2\pi^2 t\cos\left[\frac{\pi(b-N+2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right] + 4\pi^2 tT\cos\left[\frac{\pi(b-N+2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right]$$
$$+2\pi^2 tT^2\cos\left[\frac{\pi(b-N+2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right] + 3\pi^2 t\cos\left[\frac{b\pi(1+T)}{N}\right]\csc\left[\frac{b\pi}{2N}\right]^2$$
$$+2\pi^2 tT\cos\left[\frac{b\pi(1+T)}{N}\right]\csc\left[\frac{b\pi}{2N}\right]^2 + \pi^2 t\cos\left[\frac{\pi(3b-3N+2bT)}{2N}\right]\csc\left[\frac{b\pi}{2N}\right]^3$$

When $t = cN\log(N)$

Let $g = \sqrt{3}\sqrt{\frac{N}{c}}$,

$$\mathcal{P} = \frac{2}{N} - \frac{1}{200N^2}\Bigg[5c(15+4g)\pi^2\cos\left[\frac{a\left(1+\frac{2}{5}g\right)\pi}{N}\right]\csc\left[\frac{a\pi}{2N}\right]^2\log[N]$$
$$+75c\pi^2\cos\left[\frac{b\left(1+\frac{2}{5}g\right)\pi}{N}\right]\csc\left[\frac{b\pi}{2N}\right]^2\log[N] + \frac{20\sqrt{3}\pi^2\cos\left[\frac{b\left(1+\frac{2}{5}g\right)\pi}{N}\right]\csc\left[\frac{b\pi}{2N}\right]^2\log[N]}{\sqrt{\frac{1}{cN}}}$$
$$+2\csc\left[\frac{a\pi}{2N}\right](-50N + (12N + 5c(5+4g))\pi^2\log[N])\sin\left[\frac{a(5+4g)\pi}{10N}\right] - 100N\csc\left[\frac{b\pi}{2N}\right]\sin\left[\frac{b(5+4g)\pi}{10N}\right]$$
$$+50c\pi^2\csc\left[\frac{b\pi}{2N}\right]\log[N]\sin\left[\frac{b(5+4g)\pi}{10N}\right] + \frac{40\sqrt{3}\pi^2\csc\left[\frac{b\pi}{2N}\right]\log[N]\sin\left[\frac{b(5+4g)\pi}{10N}\right]}{\sqrt{\frac{1}{cN}}}$$
$$+24N\pi^2\csc\left[\frac{b\pi}{2N}\right]\log[N]\sin\left[\frac{b(5+4g)\pi}{10N}\right] - 25c\pi^2\csc\left[\frac{a\pi}{2N}\right]^3\log[N]\sin\left[\frac{a(15+4g)\pi}{10N}\right]$$
$$-25c\pi^2\csc\left[\frac{b\pi}{2N}\right]^3\log[N]\sin\left[\frac{b(15+4g)\pi}{10N}\right]\Bigg] + \mathcal{O}\left(N^{-\frac{3}{2}}\mathrm{polylog}(N)\right)$$

We know that $\csc(x) = \frac{1}{x} + O(x)$.

$|i-j| \sim\Theta(N)$

When $|i-j| \sim\Theta(N)$, then $i+j\sim\Theta(N)$. In this case, $\csc\left[\frac{a\pi}{2N}\right], \csc\left[\frac{b\pi}{2N}\right]\sim O(1)$. And then we could absorb the constant into c, for example, $5c \to c$. Here, we let $h = \cos\left[\frac{a(1+\frac{2}{5}g)\pi}{N}\right] + \cos\left[\frac{b(1+\frac{2}{5}g)\pi}{N}\right]$

$$\mathcal{P} = \frac{2}{N} - \frac{6\pi^2\log(N) - 25}{50N}\Bigg\{$$

$$\csc\left[\frac{a\pi}{2N}\right]\sin\left[\frac{a\left(5+4\sqrt{3}\sqrt{\frac{N}{c}}\right)\pi}{10N}\right] + \csc\left[\frac{b\pi}{2N}\right]\sin\left[\frac{b\left(5+4\sqrt{3}\sqrt{\frac{N}{c}}\right)\pi}{10N}\right]$$

$$\Bigg\} + \mathcal{O}\left(\frac{1}{N^2}\right) + \mathcal{O}\left(t^{\frac{3}{2}}\frac{\log(N)}{N^3}\right)$$

Now, the tricky thing is the term in the second line. w.l.o.g., let $a, b > 0$. We need to analysis the

$$\sin\left[\frac{a\pi}{2N}\right]\sin\left[\frac{a\left(5+4\sqrt{3}\sqrt{\frac{N}{c}}\right)\pi}{10N}\right] + \sin\left[\frac{b\pi}{2N}\right]\sin\left[\frac{b\left(5+4\sqrt{3}\sqrt{\frac{N}{c}}\right)\pi}{10N}\right]$$

Let $f(x) = \sin(px)\sin(qx)$