Estimate $\cos^{4t}(x)$

A good approximation of $\cos^{4t}(x)$ is e^{-2tx^2}. The difference between the two terms is

$$\begin{aligned} & e^{-2tx^2} - \cos^{4t}(x) \\ &= e^{-2tx^2} - e^{-2tx^2+O(tx^4)} \\ &= e^{-2tx^2}\left(1 - e^{O(tx^4)}\right) \\ &\sim O\left(tx^4 e^{-2tx^2}\right) \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathcal{P} &= \frac{1}{N} + \frac{1}{N}\sum_{k=1}^{N-1}\left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right]\cos^{4t}\left(\frac{\pi k}{2N}\right) \\ &= -\frac{1}{N} + \frac{1}{N}\sum_{k=0}^{N-1}\left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right]\cos^{4t}\left(\frac{\pi k}{2N}\right) \\ &= -\frac{1}{N} + \frac{1}{N}\sum_{k} e^{-\frac{k^2\pi^2 t}{2N^2}}\left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right] + O\left(t^{-\frac{3}{2}}\right) \\ &= -\frac{1}{N} + \frac{1}{N}\mathfrak{R}\left[\sum_{k}\left(\exp\left(-\frac{k^2\pi^2 t}{2N^2} + ia\frac{\pi k}{2N}\right) + \exp\left(-\frac{k^2\pi^2 t}{2N^2} + ib\frac{\pi k}{2N}\right)\right)\right] + O\left(t^{-\frac{3}{2}}\right), \end{aligned}$$

where $a = 2|i-j|$, $b = i+j-1$

Estimate $\sum_k \exp\left(-\frac{k^2\pi^2 t}{2N^2} + ia\frac{\pi k}{2N}\right)$

$$\begin{aligned} & \mathfrak{R}\left[\frac{\pi}{2N}\sum_{k}\exp\left(-\frac{k^2\pi^2 t}{2N^2} + ia\frac{\pi k}{2N}\right)\right] \\ &< \mathfrak{R}\left[\int_0^\infty \exp(-2tx^2 + iax)\,dx\right] \\ &= \frac{1}{2}\mathfrak{R}\left[\int_{-\infty}^\infty \exp(-2tx^2 + iax)\,dx\right] \end{aligned}$$

And we have

$$\int_{-\infty}^{\infty}\exp(-2tx^2 + iax)\,dx = \frac{e^{-\left(\frac{a^2}{8t}\right)}\sqrt{\frac{\pi}{2}}}{\sqrt{t}}$$

Now, we analysis how much we throw when we transfer the summation to integral.

$$\begin{aligned} & \mathfrak{R}\left[\int_0^\infty \exp(-2tx^2 + iax)\,dx\right] - \mathfrak{R}\left[\frac{\pi}{2N}\sum_{k}\exp\left(-\frac{k^2\pi^2 t}{2N^2} + ia\frac{\pi k}{2N}\right)\right] \\ &= \sum_{i=0}^{N-1}\left(\int_{x_i}^{x_{i+1}}\exp(-2tx^2)\cos(ax)\,dx - \frac{\pi}{2N}\exp\left(-2t\frac{i^2\pi^2 t}{2N^2}\right)\cos\left(a\frac{\pi i}{2N}\right)\right) + \int_{\frac{\pi}{2}}^{\infty}\exp(-2tx^2 + iax)\,dx \end{aligned}$$

The first term obey

$$
\begin{aligned}
&\left| \int_{x_k}^{x_{k+1}} \exp(-2tx^2)\cos(ax)\mathsf{d}x - \frac{\pi}{2N}\exp\left(-2t\frac{k^2\pi^2 t}{2N^2}\right)\cos\left(a\frac{\pi k}{2N}\right)\right| \\
&\leq \frac{\pi}{2N}\left(\max_{x\in[x_i,x_{i+1}]}(\exp(-2tx^2)\cos(ax)) - \exp\left(-2t\frac{k^2\pi^2 t}{2N^2}\right)\cos\left(a\frac{\pi k}{2N}\right)\right)
\end{aligned}
$$