Shallow fermionic shadow with error mitigation

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1 Open question

(1) Can we directly measure γ_S with a fermionic quantum device?

2 PPT for CS

$\mathcal{M} = \mathbb{E}_{\mathcal{U}}\left[\mathcal{U}^{-1}M_x\mathcal{U}\right] = \Pi_0 + f\Pi_1$ where $f = \frac{\mathrm{Tr}(M_x\Pi_1)}{\mathrm{Tr}(\Pi_1)} = \frac{1}{2^n+1}$.

$\mathcal{M}^{-1} = \Pi_0 + (2^n+1)\Pi_1$

$\widetilde{\mathcal{M}} = \mathbb{E}_{\mathcal{U}}\left[\mathcal{U}^{-1}M_x\Lambda\mathcal{U}\right] = \Pi_0 + f\Pi_1$ where $f = \frac{\mathrm{Tr}(M_x\Lambda\Pi_1)}{\mathrm{Tr}(\Pi_1)}$

$\mathcal{M}^{-1} = \Pi_0 + f^{-1}\Pi_1$

$$\mathbb{E}_{U\sim\mu_H(4)}U^{\otimes 2}\otimes\bar{U}^{\otimes 2} = \sum_{\sigma,\tau} Wg(\sigma^{-1}\tau)\left|\sigma\right\rangle\left\langle\tau\right|$$

3 Preliminaries

Let $\widetilde{\mathcal{M}}_d$ denote the noisy shallow classical shadow channel with depth d and noise Λ for each layer. Let $O_d \in \mathbb{R}^{n\times n}$ represent the orthogonal matrix.

4 Main results

Lemma 1 (Ref. [1]). *For $\mu \in \{0,1\}^{2n}$, let $\alpha_{\mu,d} \in [0,1]$ be defined as*

$$\alpha_{\mu,d} := \Pr_{\mathcal{U}\sim\mathbb{U}_d}\left[\mathcal{U}\left(P^{(\mu)}\right) \in \pm\mathcal{Z}\right] = \langle 0|\mathcal{U}(P^{(\mu)})|0\rangle^2, \tag{1}$$

then $\widetilde{\mathcal{M}}_d\left(P^{(\mu)}\right) = \frac{(2^n-1)(1-p)^k+1}{2^n}\alpha_{\mu,d}P^{(\mu)}$.

Theorem 1. *In the assumption that the noise for each layer forms a 2-design, the classical shadow for the qubit system can be denoted as*

$$\widetilde{\mathcal{M}} = \tag{2}$$

5 Shallow fermionic classical shadow

Let

$$\mathcal{M}_d := \int_{Q \in O_d} d\mu(Q) \mathcal{U}_Q^\dagger M_z \mathcal{U}_Q. \tag{3}$$

denote the fermionic shadow with depth d.

Lemma 2. *The shallow fermionic shadow has the property*

$$\mathcal{M}_d(\gamma_S) = \alpha_{S,d}\gamma_S, \tag{4}$$

where $\alpha_{S,d} = \int_{Q\sim O_d} d\mu(Q) \left| \langle 0|U_Q \gamma_S U_Q^\dagger|0\rangle \right|^2$.

Proof. Since Pauli-X is in the matchgate group, we can simplify the shallow fermionic channel as

$$\begin{aligned}
\mathcal{M}_d(\gamma_S) &= \int_{Q\sim O_d} d\mu(Q) \left[\sum_{b\in\{0,1\}^n} \langle b|U_Q \gamma_S U_Q^\dagger|b\rangle \, U_Q^\dagger \, |b\rangle \, \langle b| \, U_Q \right] &(5)\\
&= 2^n \int_{Q\sim O_d} d\mu(Q) \left[\langle 0|U_Q \gamma_S U_Q^\dagger|0\rangle \, U_Q^\dagger \, |0\rangle \, \langle 0| \, U_Q \right]. &(6)
\end{aligned}$$

If S' is not equal to S, then there exists a permutation matrix Q such that

$$Q|_{S,S} = \mathbb{I}, \quad Q|_{S',S'} = -\mathbb{I}, \tag{7}$$

and hence

$$[\gamma_S, U_Q] = 0, \quad \{\gamma_{S'}, U_{Q'}\} = 0. \tag{8}$$

It implies

$$\begin{aligned}
\frac{1}{2^n}\mathrm{Tr}\left(\gamma_{S'}\mathcal{M}_d(\gamma_S)\right) &= \int_{Q\sim O_d} d\mu(Q) \, \langle 0|U_Q\gamma_S U_Q^\dagger|0\rangle \, \langle 0|U_Q \gamma_{s'} U_Q^\dagger|0\rangle &(9)\\
&= \int_{Q\sim O_d} d\mu(Q) \, \langle 0|U_Q U_{Q'}\gamma_S U_{Q'}^\dagger U_Q^\dagger|0\rangle \, \langle 0|U_Q U_{Q'} \gamma_{s'} U_{Q'}^\dagger U_Q^\dagger|0\rangle &(10)\\
&= -\int_{Q\sim O_d} d\mu(Q) \, \langle 0|U_Q\gamma_S U_Q^\dagger|0\rangle \, \langle 0|U_Q \gamma_{s'} U_Q^\dagger|0\rangle. &(11)
\end{aligned}$$

Hence $\mathcal{M}_d(\gamma_S) = \alpha_{s,d}\gamma_S$. □

Lemma 3 (Wan et al. [2])**.** *Let Q be a matrix uniformly randomly sampled from orthogonal group $\mathcal{O}(n)$, then*

$$\begin{aligned}
\int_{Q\sim M_n} d\mu(Q) \left[\mathcal{U}_Q\right] &= |\mathbb{I}\rangle\rangle\langle\langle\mathbb{I}| &(12)\\
\int_{Q\sim M_n} d\mu(Q) \left[\mathcal{U}_Q^{\otimes 2}\right] &= \sum_{k=0}^{2n} |\mathcal{R}_k^{(2)}\rangle\rangle\langle\langle\mathcal{R}_k^{(2)}| &(13)\\
\int_{Q\sim M_n} d\mu(Q) \left[\mathcal{U}_Q^{\otimes 3}\right] &= \sum_{\substack{k_1,k_2,k_3\geq 0\\ k_1+k_2+k_3\leq 2n}} |\mathcal{R}_{k_1,k_2,k_3}^{(3)}\rangle\rangle\langle\langle\mathcal{R}_{k_1,k_2,k_3}^{(3)}| &(14)
\end{aligned}$$

where

$$\begin{aligned}
|\mathcal{R}_k^{(2)}\rangle\rangle &= \binom{2n}{k}^{-1/2} \sum_{S\subseteq[2n],|S|=k} |\gamma_S\rangle\rangle|\gamma_S\rangle\rangle &(15)\\
|\mathcal{R}_k^{(3)}\rangle\rangle &= \binom{2n}{k_1, k_2, k_3, 2n-k_1-k_2-k_3}^{-1/2} \sum_{\substack{S_1,S_2,S_3\subseteq[2n] disjoint\\ |S_j|=k_j, 1\leq j\leq 3}} |\gamma_{S_1}\gamma_{S_2}\rangle\rangle|\gamma_{S_2}\gamma_{S_3}\rangle\rangle|\gamma_{S_3}\gamma_{S_1}\rangle\rangle &(16)
\end{aligned}$$

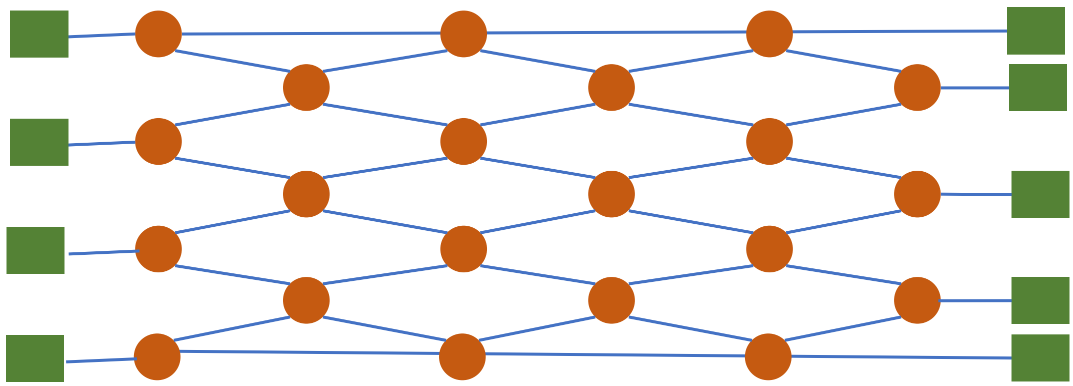


Figure 1: Tensor network representation for computing $\alpha_{S,d}$.

Lemma 4.

$$\int_{Q\sim M_n} d\mu(Q)\left[\mathcal{U}_Q^{\otimes 2}\right] = ??? \tag{17}$$

Lemma 5. *Given any observable γ_S and an unknown quantum state ρ, let v be the estimation of $Tr(\rho\gamma_S)$ with shallow fermionic shadow $\mathcal{M}_d$. Then the variance in the average of the input state can be bounded to $1/\alpha_{S,d}$.*

Proof.

$$\begin{aligned}
\mathrm{Var}(v) &\leq \mathbb{E}[|v|^2] &(18)\\
&= \int_{Q\sim O_d} d\mu Q \mathbb{E}_\rho \left[\sum_b \langle b|U_Q\rho U_Q^\dagger|b\rangle \left|\langle b|U_Q\mathcal{M}_d^{-1}(\gamma_S)U_Q^\dagger|b\rangle\right|^2\right] &(19)\\
&= \int_{Q\sim O_d} d\mu Q \left[\sum_b \left|\langle b|U_Q\mathcal{M}_d^{-1}(\gamma_S)U_Q^\dagger|b\rangle\right|^2\right] &(20)\\
&= \frac{2^n}{|\alpha_{S,d}|^2}\int_{Q\sim O_d} \langle 0|U_Q\gamma_S U_Q^\dagger|0\rangle \langle 0|U_Q\gamma_S^\dagger U_Q^\dagger|0\rangle &(21)\\
&= \frac{1}{\alpha_{S,d}}. &(22)
\end{aligned}$$

□

Lemma 6. *$\alpha_{S,d}$ can be bounded to...*

- Bound the complexity to calculate $\alpha_{S,d}$.

  We can utilize the MPS method to solve it.

  Let matrix T be the probability matrix and the (j,k)-th element $T_{j,k}$ denote the probability transforming P_j to P_k after performing a unitary U_Q with Q uniformly randomly picked from P_{2n}, where $j \in \{0,1\}^2$, and $j_l = 1$ iff the l-th qubit of γ_S be identity.

- Bound the variance.

6 Error mitigation for fermionic shallow CS

Theorem 2. *The noisy classical shadow with any noise model is the same as that with Pauli noise.*

From Ref. [3], we can get the robust version for shallow classical shadow.

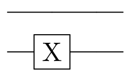
7 gamma to pauli

k = 0:

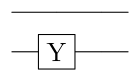
$\gamma_0 =$

k = 1:

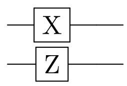
$\gamma_1 =$



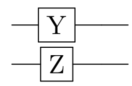
$\gamma_2 =$



$\gamma_3 =$

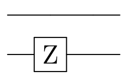


$\gamma_4 =$

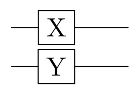


k = 2:

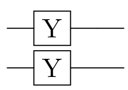
$\gamma_1\gamma_2 = \mathrm{i}$



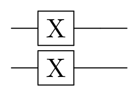
$\gamma_1\gamma_3 = -\mathrm{i}$



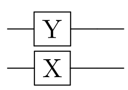
$\gamma_1\gamma_4 = -\mathrm{i}$



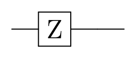
$\gamma_2\gamma_3 = -\mathrm{i}$



$\gamma_2\gamma_4 = \mathrm{i}$

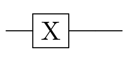


$\gamma_3\gamma_4 = \mathrm{i}$

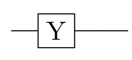


k = 3:

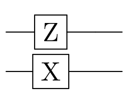
$\gamma_1\gamma_2\gamma_3 = \mathrm{i}$



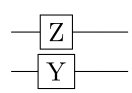
$\gamma_1\gamma_2\gamma_4 = \mathrm{i}$



$\gamma_1\gamma_3\gamma_4 = \mathrm{i}$

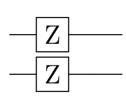


$\gamma_2\gamma_3\gamma_4 = \mathrm{i}$



k = 4:

$\gamma_1\gamma_2\gamma_3\gamma_4 =$



8 Representation of 2-fold twirling

Here, we aim to give a representation of $\int d\mu(Q)\mathcal{U}_Q^{\otimes 2}$. Due to Lemma. 3, we have

$$
\begin{aligned}
\int_{Q\sim M_n} d\mu(Q)\mathcal{U}_Q^{\otimes 2} =& |\gamma_\emptyset\rangle\rangle|\gamma_\emptyset\rangle\rangle\langle\langle\gamma_\emptyset|\langle\langle\gamma_\emptyset| + \frac{1}{4}\sum_{i,j}|\gamma_i\rangle\rangle|\gamma_i\rangle\rangle\langle\langle\gamma_j|\langle\langle\gamma_j| \\
&+ \frac{1}{6}\sum_{\substack{i_1\neq i_2\\ j_1\neq j_2}}|\gamma_{i_1}\gamma_{i_2}\rangle\rangle|\gamma_{i_1}\gamma_{i_2}\rangle\rangle\langle\langle\gamma_{j_1}\gamma_{j_2}|\langle\langle\gamma_{j_1}\gamma_{j_2}| \\
&+ \frac{1}{4}\sum_{\substack{i_1\neq i_2, j_1\neq j_2\\ i_1\neq i_3, j_1\neq j_3\\ i_2\neq i_3, j_2\neq j_3}}|\gamma_{i_1}\gamma_{i_2}\gamma_{i_3}\rangle\rangle|\gamma_{i_1}\gamma_{i_2}\gamma_{i_3}\rangle\rangle\langle\langle\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}|\langle\langle\gamma_{j_1}\gamma_{j_2}\gamma_{j_3}| \\
&+ |\gamma_1\gamma_2\gamma_3\gamma_4\rangle\rangle|\gamma_1\gamma_2\gamma_3\gamma_4\rangle\rangle\langle\langle\gamma_1\gamma_2\gamma_3\gamma_4|\langle\langle\gamma_1\gamma_2\gamma_3\gamma_4|
\end{aligned}
$$

Plugging the table in Sec. 7, we present the two-fold twirling to the Pauli basis. Denote the twirling as a 4-bond tensor as the following rule,

$$
T^{ij}_{\ \ ab} = \langle\langle P^{(a,b)}|\langle\langle P^{(a,b)}|\int_{Q\sim M_n} d\mu(Q)\mathcal{U}_Q^{\otimes 2}|P^{(i,j)}\rangle\rangle|P^{(i,j)}\rangle\rangle, \tag{23}
$$

where $(a,b),(i,j)\in\{(b_1,b_2)\mid b_1,b_2\in\{0,1\}\}$. We also write $T^{XY}_{\ \ \ YZ}$ instead of $T^{(1,1),(0,1)}_{\ \ (1,0),(1,1)}$ for simplification.

We write down every element of tensor T, which is shown in Table. 1. Here, we give an example of the calculation of $T^{IZ}_{\ \ XX}$.

$$
\begin{aligned}
T^{IZ}_{XX} =& \langle\langle XX|\langle\langle XX|\int_{Q\sim M_n} d\mu(Q)\mathcal{U}_Q^{\otimes 2}|IZ\rangle\rangle|IZ\rangle\rangle \\
=& \langle\langle\gamma_2\gamma_3|\langle\langle\gamma_2\gamma_3|(-\mathrm{i})^2\cdot\int_{Q\sim M_n} d\mu(Q)\mathcal{U}_Q^{\otimes 2}\cdot(-\mathrm{i})^2|\gamma_1\gamma_2\rangle\rangle|\gamma_1\gamma_2\rangle\rangle \\
=& \frac{1}{6}\langle\langle\gamma_2\gamma_3|\sum_{\substack{i_1\neq i_2\\ j_1\neq j_2}}|\gamma_{i_1}\gamma_{i_2}\rangle\rangle|\gamma_{i_1}\gamma_{i_2}\rangle\rangle\langle\langle\gamma_{j_1}\gamma_{j_2}|\langle\langle\gamma_{j_1}\gamma_{j_2}||\gamma_1\gamma_2\rangle\rangle|\gamma_1\gamma_2\rangle\rangle \\
=& \frac{1}{6}
\end{aligned}
$$

From the first line to the second line, we use the correspondence relation in Sec. 7, $|IZ\rangle\rangle = -\mathrm{i}|\gamma_1\gamma_2\rangle\rangle$, $|XX\rangle\rangle = \mathrm{i}|\gamma_2\gamma_3\rangle\rangle$. Then, terms $|\gamma_{S_1}\rangle\rangle\langle\langle\gamma_{S_1}|\langle\langle\gamma_{S_2}|\langle\langle\gamma_{S_2}|$ in $\int d\mu(Q)\mathcal{U}_Q^{\otimes 2}$ cancel if $|S_1|,|S_2|\neq 2$.

9 Calculate α

Lemma 7.

$$
\gamma_S^\dagger = (-1)^{\frac{|S|(|S|-1)}{2}}\gamma_S \tag{24}
$$

Proof.

$$
\begin{aligned}
\gamma_S^\dagger =& \left(\gamma_{l_1}\gamma_{l_2}\cdots\gamma_{l_{|S|}}\right)^\dagger \\
=& \gamma_{l_{|S|}}\gamma_{l_{|S|-1}}\cdots\gamma_{l_1}
\end{aligned}
$$

Because $\{\gamma_i,\gamma_j\} = 2\delta_{ij}$, it will generate a coefficient $(-1)^{|S|-1}$ when we swap the γ_{l_1} to the first place. Thus, we have

$$
\begin{aligned}
\gamma_S^\dagger =& (-1)^{|S|-1}\gamma_{l_1}\gamma_{l_{|S|}}\gamma_{l_{|S|-1}}\cdots\gamma_{l_2} \\
=& (-1)^{|S|-1+|S|-2}\gamma_{l_1}\gamma_{l_2}\gamma_{l_{|S|}}\gamma_{l_{|S|-1}}\cdots\gamma_{l_3} \\
=& (-1)^{\frac{|S|(|S|-1)}{2}}\gamma_S
\end{aligned}
$$

□

T	II	IX	IY	IZ	XI	XX	XY	XZ	YI	YX	YY	YZ	ZI	ZX	ZY	ZZ
II	1															
IX		1/4	1/4					1/4				1/4				
IY		1/4	1/4					1/4				1/4				
IZ				1/6		1/6	1/6			1/6	1/6		1/6			
XI					1/4				1/4					1/4	1/4	
XX				1/6		1/6	1/6			1/6	1/6		1/6			
XY				1/6		1/6	1/6			1/6	1/6		1/6			
XZ		1/4	1/4					1/4				1/4				
YI					1/4				1/4					1/4	1/4	
YX				1/6		1/6	1/6			1/6	1/6		1/6			
YY				1/6		1/6	1/6			1/6	1/6		1/6			
YZ		1/4	1/4					1/4				1/4				
ZI				1/6		1/6	1/6			1/6	1/6		1/6			
ZX					1/4				1/4					1/4	1/4	
ZY					1/4				1/4					1/4	1/4	
ZZ																1

Table 1: Values of tensor T. The head of columns represents the input of T while the head of rows represents the output of T. For example, the value in row 'XY' and column 'YZ' represent $T^{(1,1),(0,1)}_{(1,0),(1,1)}$. The blank space represents the value is 0.

Base on Lemma 7, the expression of α becomes

$$\begin{aligned}\alpha_{S,d} =&(-1)^{\frac{|S|(|S|-1)}{2}}\int_{Q\sim O_d} d\mu(Q)\, \langle 0|\, U_Q\gamma_S U_Q^\dagger\, |0\rangle^2 \\ =&(-1)^{\frac{|S|(|S|-1)}{2}}\int_{Q\sim O_d} d\mu(Q)\langle\langle 0,0|\mathcal{U}_Q\otimes\mathcal{U}_Q|\gamma_S\gamma_S\rangle\rangle \\ =&(-1)^{\frac{|S|(|S|-1)}{2}}\langle\langle 0,0|\left(\int_{Q\sim O_d} d\mu(Q)\mathcal{U}_Q^{\otimes 2}\right)|\gamma_S\gamma_S\rangle\rangle.\end{aligned}$$

Now, we focus on dealing with $\int d\mu(Q)\mathcal{U}_Q^{\otimes 2}$. In Sec. 8, we give a tensor representation of $\int_{M_2} d\mu(Q)\mathcal{U}_Q^{\otimes 2}$. Let the two-fold twirling of a layer of match gates be $T_l^{(2)}$, where

$$T_l^{(2)} := \begin{cases} T^{\otimes\lfloor\frac{n}{2}\rfloor} & l \mod 2 = 1 \\ \mathbb{1}_2\otimes T^{\otimes\lfloor\frac{n-1}{2}\rfloor}\otimes\mathbb{1}_0 & l \mod 2 = 0\end{cases}. \tag{25}$$

Then, we could bound the α to $1/6^d$

$$\begin{aligned}\alpha_{S,d} =&\langle\langle 0,0|\int_{Q\sim Q_d} d\mu(Q)\mathcal{U}_Q^{\otimes 2}|\gamma_S,\gamma_S\rangle\rangle = \langle\langle 0,0|\prod_{1\le l\le d} T_l^{(2)}|\gamma_S,\gamma_S\rangle\rangle \\ \ge&\inf\{\langle\langle\psi,\psi|T_l^{(2)}|\phi,\phi\rangle\rangle \mid |||\psi\rangle\rangle|| = |||\phi\rangle\rangle|| = 1\}^d \\ \ge&\frac{1}{6^d}\end{aligned}$$

References

[1] Christian Bertoni, Jonas Haferkamp, Marcel Hinsche, Marios Ioannou, Jens Eisert, and Hakop Pashayan. Shallow shadows: Expectation estimation using low-depth random clifford circuits. *arXiv preprint arXiv:2209.12924*, 2022.

[2] Kianna Wan, William J. Huggins, Joonho Lee, and Ryan Babbush. Matchgate shadows for fermionic quantum simulation, 2022.

[3] Yunchao Liu, Matthew Otten, Roozbeh Bassirianjahromi, Liang Jiang, and Bill Fefferman. Benchmarking near-term quantum computers via random circuit sampling. *arXiv preprint arXiv:2105.05232*, 2021.