Euler-Maclaurin formula

$$\sum_0^\infty e^{-\frac{k^2\pi^2 t}{2N^2}} \cos\left((i-j)\frac{\pi k}{N}\right)$$

let $f(n) = e^{-\frac{\beta^2 n^2 t}{2}} \cos(a\beta n)$, where $\beta = \frac{\pi}{N}$, $a = |i-j|$. Due to Euler-Maclaurin formula we have

$$\sum_0^\infty f(n) = \int_0^\infty f(x)dx + \frac{1}{2} + \int_0^\infty P_1(x)f'(x)dx$$

where $P_1(x) = B_1(x - \lfloor x \rfloor)$, B_1 is the first order Bernolli polynomial.

$$B_1(x) = x - \frac{1}{2}$$

$$\begin{aligned} f'(x) &= -e^{-\left(\frac{1}{2}\right)tx^2\beta^2} tx\beta^2 \cos[ax\beta] - ae^{-\left(\frac{1}{2}\right)tx^2\beta^2} \beta \sin[ax\beta] \\ &= -tx\beta^2 f(x) - a\beta \tan(ax\beta) f(x) \end{aligned}$$

Then

$$\sum_0^\infty f(n) = \int_0^\infty f(x)dx + \frac{1}{2} - \frac{1}{2}\int_0^\infty f'(x)dx + \int_0^\infty (x - \lfloor x \rfloor) f'(x)dx$$

$$\begin{aligned} \int_0^\infty (x-\lfloor x \rfloor)f'(x)dx &= \sum_{n=0}^\infty \int_n^{n+1} (x-n)f'(x)dx \\ &= \frac{1}{4\sqrt{t}\beta} e^{-\left(\frac{a^2}{2t}\right)} \sum_{n=0}^\infty \Bigg[2e^{\frac{(a-i(1+n)t\beta)^2}{2t}} \left(1 + e^{2ia(1+n)\beta}\right)\sqrt{t}\beta \\ &+ \sqrt{2\pi}\left(\operatorname{Erf}\left[\frac{ia + nt\beta}{\sqrt{2}\sqrt{t}}\right] - \operatorname{Erf}\left[\frac{ia + (1+n)t\beta}{\sqrt{2}\sqrt{t}}\right] - i\left(\operatorname{Erfi}\left[\frac{a + int\beta}{\sqrt{2}\sqrt{t}}\right] - \operatorname{Erfi}\left[\frac{a + i(1+n)t\beta}{\sqrt{2}\sqrt{t}}\right]\right)\right)\Bigg] \\ &= \frac{1}{4\sqrt{t}\beta} e^{-\left(\frac{a^2}{2t}\right)} \sum_{n=0}^\infty 2e^{\frac{(a-i(1+n)t\beta)^2}{2t}} \left(1 + e^{2ia(1+n)\beta}\right)\sqrt{t}\beta \\ &+ \frac{\sqrt{2\pi}}{4\sqrt{t}\beta} e^{-\left(\frac{a^2}{2t}\right)} \left[\left(\operatorname{Erf}\left[\frac{ia}{\sqrt{2}\sqrt{t}}\right] - \operatorname{Erf}\left[\frac{ia+\infty}{\sqrt{2}\sqrt{t}}\right]\right) - i\left(\operatorname{Erfi}\left[\frac{a}{\sqrt{2}\sqrt{t}}\right] - \operatorname{Erfi}\left[\frac{a+i\infty}{\sqrt{2}\sqrt{t}}\right]\right)\right] \\ &= \frac{1}{2} e^{-\left(\frac{a^2}{2t}\right)} \sum_{n=0}^\infty \left(e^{\frac{(a-i(1+n)t\beta)^2}{2t}} + e^{\frac{(a+i(1+n)t\beta)^2}{2t}}\right) - \frac{\sqrt{2\pi}}{2\sqrt{t}\beta} e^{-\left(\frac{a^2}{2t}\right)} \\ &= \frac{1}{2} e^{-\left(\frac{a^2}{2t}\right)} \frac{\sqrt{\frac{\pi}{2}}}{\sqrt{t}\beta} - \frac{\sqrt{2\pi}}{2\sqrt{t}\beta} e^{-\left(\frac{a^2}{2t}\right)} + \Delta_a \end{aligned}$$

where $\Delta_a \leq e^{-\left(\frac{a^2}{2t}\right)}$

Finally, we have

$$\frac{1}{2N} + \frac{1}{N}\sum_{n=1} f(n) = \frac{1}{2\sqrt{2\pi t}} e^{-\left(\frac{a^2}{2t}\right)} + \Delta_a$$

Compose results

$$
\begin{aligned}
\mathcal{P} &= \frac{1}{N} + \frac{1}{N}\sum_{k=1}^{\infty}\left(f_{a(k)} + f_{b(k)}\right) + O\left(t^{-\frac{3}{2}}\right) \\
&= \frac{1}{2\sqrt{2\pi t}}\left(e^{-\left(\frac{a^2}{2t}\right)} + e^{-\left(\frac{b^2}{2t}\right)}\right) + \Delta
\end{aligned}
$$

where $0 \leq \Delta \leq \frac{1}{N}\left(e^{-\left(\frac{a^2}{2t}\right)} + e^{-\left(\frac{b^2}{2t}\right)}\right)$