Notations

$$\alpha_k(t) := \frac{1}{2}\sum_{\mu=1}^{N-k}\left(P_i(\mu,t)P_j(\mu+k,t)+P_i(\mu+k,t)P_j(\mu,t)\right)$$

$$\beta_k(t) := \frac{1}{2}\sum_{\mu=1}^{N-k}\left(I_{i,j}(\mu,\mu+k,t)+I_{i,j}(\mu+k,\mu,t)\right)$$

$$a(t) := \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix},\quad b(t) := \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}$$

Get the recursive relation of β

Suppose $\beta_0 > 0, \beta_k < 0, k > 0$. When $t = \Omega(\log(N))$, suppose $P_i(1,t)P_j(1,t), P_i(1,t)P_j(1,t) = O\left(\frac{1}{N^2}\right)$

$$\begin{aligned}\beta_0(t+1) &= \frac{6}{16}\beta_0(t)+\frac{1}{4}\beta_1(t)-\frac{14}{144}(\beta_0(t)-\alpha_0(t))-\frac{1}{24}(\beta_1(t)-\alpha_1(t))+O\left(\frac{1}{N^2}\right)\\ &= \frac{5}{18}\beta_0(t)+\frac{5}{24}\beta_1(t)+\frac{7}{72}\alpha_0(t)+\frac{1}{12}\alpha_1(t)+O\left(\frac{1}{N^2}\right)\\ \beta_1(t+1) &\le \frac{5}{9}\beta_0(t)+\frac{5}{12}\beta_1(t)-\frac{1}{18}\alpha_0(t)-\frac{1}{24}\alpha_1(t)\end{aligned}$$

Thus, $\beta_k(t) \le \beta_k'(t)$,

$$b'(t+1) = C_b b'(t) + C_a a(t),$$

where $b'(t) = (\beta_0'(t), \beta_1'(t))^T$, $(\beta_0'(0), \beta_1'(0)) = (\beta_0(t), \beta_1(t))$,

$$C_b = \begin{pmatrix}\frac{5}{18} & \frac{5}{24}\\ \frac{5}{9} & \frac{5}{12}\end{pmatrix},\quad C_a = \begin{pmatrix}\frac{7}{72} & \frac{1}{12}\\ -\frac{1}{18} & -\frac{1}{24}\end{pmatrix}.$$

Then

$$b'(t) = C_b^t b'(0) + \sum_{k=0}^{t-1} C_b^k C_a a(t-k-1)$$

$$C_b = Q\Lambda Q^{-1}, \Lambda = \begin{pmatrix}\frac{25}{36} & \\ & 0\end{pmatrix}$$

$$C_b^k = \left(\frac{5}{6}\right)^{2k}\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}2 & \frac{3}{2}\end{pmatrix}$$

$$C_b^k C_a = \frac{1}{5}\left(\frac{5}{6}\right)^{2k}\begin{pmatrix}\frac{1}{9} & \frac{5}{48}\\ \frac{2}{9} & \frac{5}{24}\end{pmatrix}$$

Now, we consider the first element of $C_b^k C_a a(t-k)$, we could get

$$\lambda_k\alpha_0(t-k-1)+\eta_k\alpha_1(t-k-1)$$

where λ_k, η_k are constants.

$$\begin{aligned} \lambda_k &= \frac{1}{45}\left(\frac{5}{6}\right)^{2k}, & \eta_k &= \frac{1}{48}\left(\frac{5}{6}\right)^{2k} \\ \lambda_0 &= \frac{7}{72}, & \eta_0 &= \frac{1}{12} \end{aligned}$$

Then,

$$\beta_0'(t) = \sum_{k=0}^{t}(\lambda_k \alpha_0(t-k-1) + \eta_k \alpha_1(t-k-1))$$

And we have

$$\alpha_0(t+1) \geq \frac{3}{8}\alpha_0(t) + \frac{1}{2}\alpha_1(t)$$

Thus,

$$\beta_0'(t) \leq \sum_{k=0}^{t}\left(\lambda_k - \frac{3}{4}\eta_k + 2\eta_{k+1}\right)\alpha_0(t-k-1)$$

And

$$\begin{aligned} \lambda_k - \frac{3}{4}\eta_k + 2\eta_{k+1} &= \left(\frac{5}{6}\right)^{2k}\left(\frac{1}{45} - \frac{3}{4}\frac{1}{48} + 2\left(\frac{5}{6}\right)^{2}\frac{1}{48}\right) \\ &= \left(\frac{5}{6}\right)^{2k}\frac{307}{8640} \leq \left(\frac{5}{6}\right)^{2k}\frac{1}{28} \end{aligned}$$

Then

$$\begin{aligned} \beta_0'(t) &\leq \frac{1}{28}\sum\left(\frac{5}{6}\right)^{2k}\max_k\{\alpha_0(t-k)\} \\ &\leq \frac{1}{12}\max_{k>1}\{\alpha_0(t-k)\} + \frac{1}{16}\alpha_0(t-1) + \frac{1}{6}\alpha_0(t) \\ &\leq \frac{5}{16}\max_{k\geq 0}\{\alpha_0(t-k)\} \end{aligned}$$

As the same way, we have

$$\beta_1 \geq$$

Finally we have

$$\beta_0(t) \leq \beta_0'(t) \leq \frac{s+1}{28}\max_{k\in[0,s+1]}\left\{\left(\frac{5}{6}\right)^{2k}\alpha_0(t-k)\right\}$$