Calculate α

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1. Dealing PP

Theorem 1.1: Let $\mathcal{P}(t) = \sum_i P_1(i,t)P_N(i,t)$, and $M = \left\lfloor \frac{N-1}{2} \right\rfloor$, $c = N - 1 - 2M \in \{0, 1\}$. Then, $\mathcal{P}$ could be bonded by the following equation

$$\begin{aligned} L(t) &= \frac{1}{N} + \frac{1}{\pi} B_N^{(1)}(2t+1) + \frac{1}{\pi} B_N^{(2)}(2t+1) \\ &- \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{3}{2} + 2t\right)}{\Gamma(2+2t)} - \frac{2}{N} \sin^{4t+2}\left(\frac{\pi}{2N}\right) \\ U(t) &= \frac{1}{N} - \frac{1}{\pi} B_N^{(1)}(2t+1) - \frac{1}{\pi} B_N^{(2)}(2t+1) \\ &+ \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{3}{2} + 2t\right)}{\Gamma(2+2t)} \end{aligned}$$

Lemma 1.2 (✅Test Pass):

$$\begin{aligned} \mathcal{P}(t) &= \sum_i P_1(i,t)P_N(i,t) \\ &= \frac{1}{N} + \frac{2}{N} \sum_{m=1}^{M} \cos^{4t+2}\left(\frac{\pi m}{N}\right) - \frac{2}{N} \sum_{m=1}^{M} \cos^{4t+2}\left(\frac{\pi(2m-1)}{2N}\right) \\ &- \frac{2c}{N} \sin^{4t+2}\left(\frac{\pi}{2N}\right), \end{aligned}$$

where $M = \left\lfloor \frac{N-1}{2} \right\rfloor$, $c = N - 1 - 2M \in \{0, 1\}$.

Proof: Let

$$
\begin{aligned}
P_1(i,t) &= \frac{1}{N} + \frac{2}{N} A_1(i,t) \\
P_{N(i,t)} &= \frac{1}{N} + \frac{2}{N} A_N(i,t)
\end{aligned}
$$
,

where

$$
\begin{aligned}
A_1(i,t) &= \sum_k \cos\left(\frac{\pi\left(i-\frac{1}{2}\right)k}{N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right) \\
A_N(i,t) &= \sum_k \cos\left(\frac{\pi\left(i-\frac{1}{2}\right)k}{N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)
\end{aligned}
$$

Then we have

$$
\sum P_1(i,t) P_N(i,t) = \frac{1}{N} + \frac{2}{N^2} \sum_{i=1}^{N-1} (A_1(i,t) + A_N(i,t)) + \frac{4}{N^2} \sum_{i=1}^{N-1} A_1(i,t) A_N(i,t)
$$

Now, let's analysis the second term.

$$
\begin{aligned}
&\sum_{i=1}^{N-1} (A_1(i,t) + A_N(i,t)) \\
&= \sum \left(\cos\left(\frac{\pi k}{2N}\right) + \cos\left(\frac{\pi k\left(N-\frac{1}{2}\right)}{N}\right)\right) \cos\left(\frac{\pi\left(i-\frac{1}{2}\right)k}{N}\right) \cos^{2t}\left(\frac{\pi k}{2N}\right)
\end{aligned}
$$

The summation of the second factor is zero

$$
\begin{aligned}
&\sum_i \cos\left(\frac{\pi\left(i-\frac{1}{2}\right)k}{N}\right) \\
&= -\frac{1}{2} \cos\left(\frac{1}{2}\pi(2k+1)\right) \csc\left(\frac{\pi k}{2N}\right) \\
&= \sin(k\pi) \csc\left(\frac{\pi k}{2N}\right) \\
&= 0
\end{aligned}
$$

□

Thus, we have $\sum P_1(i,t) P_N(i,t) = \frac{1}{N} + \frac{4}{N^2} \sum_{i=1}^{N-1} A_1(i,t) A_N(i,t)$. Then we move to the term $\sum_{i=1}^{N-1} A_1(i,t) A_N(i,t)$.

$$\sum_{i=1}^{N-1} A_1(i,t) A_N(i,t)$$

$$= \sum_{i,j,k} \cos\left(\frac{\pi j\left(N-\frac{1}{2}\right)}{N}\right) \cos\left(\frac{\pi\left(i-\frac{1}{2}\right)j}{N}\right) \cos\left(\frac{\pi\left(i-\frac{1}{2}\right)k}{N}\right) \cos^{2t}\left(\frac{\pi j}{2N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

We also begin to deal with the factor

$$\begin{aligned}
&\sum_{i=1}^{N-1} \cos\left(\frac{\pi\left(i-\frac{1}{2}\right)j}{N}\right) \cos\left(\frac{\pi\left(i-\frac{1}{2}\right)k}{N}\right) \\
&= \frac{1}{2} \sum_{i=1}^{N-1} \cos\left(\frac{\pi\left(i-\frac{1}{2}\right)j}{N} - \frac{\pi\left(i-\frac{1}{2}\right)k}{N}\right) + \cos\left(\frac{\pi\left(i-\frac{1}{2}\right)j}{N} + \frac{\pi\left(i-\frac{1}{2}\right)k}{N}\right) \\
&= \frac{1}{2} \sum_{i=1}^{N-1} \cos\left(\frac{\pi(2i-1)(j-k)}{2N}\right) + \cos\left(\frac{\pi(2i-1)(j+k)}{2N}\right) \\
&= \frac{1}{4}\left(\sin(\pi(j+k)) \csc\left(\frac{\pi(j+k)}{2N}\right) - \sin(\pi(k-j)) \csc\left(\frac{\pi(j-k)}{2N}\right)\right)
\end{aligned}$$

We know that $\sin(\pi m) = 0$, but $\csc(\pi m) = \infty$. Thus, the above term is non-zero only when $\frac{\pi(j+k)}{2N} = a\pi$ or $\frac{\pi(j-k)}{2N} = b\pi$ for some integer a and b. This condition could be neatened to

$$\begin{cases} j+k = 2aN \quad \text{or } j-k = 2bN \\ a,\, b \in \mathbb{Z} \\ 1 < j,\, k < N-1 \end{cases} \implies j = k$$

In this case, we aim to get the value of $\sin(\pi(j-k)) \csc\left(\frac{\pi(j-k)}{2N}\right)$ when $j = k$. Thus, we evaluate the following term

$$\lim_{x\to 0} \sin(\pi x) \csc\left(\frac{\pi x}{2N}\right) = \lim_{x\to 0} \frac{\sin(\pi x)}{\sin\left(\frac{\pi x}{2N}\right)} = 2N$$

Then, we have

$$
\begin{aligned}
&\sum_{i=1}^{N-1} A_1(i,t)A_N(i,t) \\
&= \frac{N}{2}\sum_{k=1}^{N-1} \cos\left(\frac{\pi k\left(N-\frac{1}{2}\right)}{N}\right)\cos^{2t}\left(\frac{\pi k}{2N}\right)\cos^{2t+1}\left(\frac{\pi k}{2N}\right) \\
&= \frac{N}{2}\sum_{k=1}^{N-1} \cos\left(\frac{\pi k\left(N-\frac{1}{2}\right)}{N}\right)\cos^{4t+1}\left(\frac{\pi k}{2N}\right) \\
&= \frac{N}{4}\sum_{k=1}^{N-1} (\cos\left(\frac{\pi k(N-1)}{N}\right) + \cos(\pi k))\cos^{4t}\left(\frac{\pi k}{2N}\right)
\end{aligned}
$$

analysis the first factor

$$
\begin{aligned}
&\cos\left(\frac{\pi k(N-1)}{N}\right) + \cos(\pi k) \\
&= \cos(\pi k)\cos\left(\frac{\pi k}{N}\right) + \sin(\pi k)\sin\left(\frac{\pi k}{N}\right) + \cos(\pi k) \\
&= \cos(\pi k)\left(\cos\left(\frac{\pi k}{N}\right) + 1\right)
\end{aligned}
$$

Then, we sum these terms pairly. Let $M = \left\lfloor\frac{N-1}{2}\right\rfloor$, $c = N - 1 - 2M \in \{0, 1\}$.

$$
\begin{aligned}
&\sum_{i=1}^{N-1} A_1(i,t)A_N(i,t) \\
&= \frac{N}{4}\sum_{m=1}^{M}\left(\cos\left(\frac{2\pi m}{N}\right)+1\right)\cos^{4t}\left(\frac{\pi m}{N}\right) - \left(\cos\left(\frac{\pi(2m-1)}{N}\right)+1\right)\cos^{4t}\left(\frac{\pi(2m-1)}{2N}\right) \\
&-c\frac{N}{4}(\cos(\frac{\pi(N-1)}{N})+1)\cos^{4t}(\frac{\pi(N-1)}{2N}) \\
&= \frac{N}{2}\sum_{m=1}^{M}\left(\cos^{4t+2}\left(\frac{\pi m}{N}\right) - \cos^{4t+2}\left(\frac{\pi(2m-1)}{2N}\right)\right) \\
&-c\frac{N}{2}\cos^{4t+2}(\frac{\pi(N-1)}{2N})
\end{aligned}
$$

where $\cos\left(\frac{\pi(N-1)}{2N}\right) = \sin\left(\frac{\pi}{2N}\right)$.

Lemma 1.3 (✅ Test Pass):

$$\mathrm{B}_N^{(1)}(p) \le 2 \sum_{m=1}^{M} \frac{\pi}{N} \cos^{2p}\left(\frac{m\pi}{N}\right) \le \sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}+p\right)}{\Gamma(1+p)} - \mathrm{B}_N^{(2)}(p)$$

where Γ is gamma function, B is beta function, $\mathrm{B}_N^{(1)}(p) = \mathrm{B}\left(\cos^2\left(\frac{2\pi}{N}\right), \frac{1}{2}+p, \frac{1}{2}\right)$, $\mathrm{B}_N^{(2)}(p) = \mathrm{B}\left(\sin^2\left(\frac{\pi}{N}\right), \frac{1}{2}+p, \frac{1}{2}\right)$.

Proof: Because the $\cos^{2p}(x)$ is monotonic decreasing when $x \in \left[0, \frac{\pi}{2}\right]$, we have

$$\cos^{2p}\left(\frac{m\pi}{N}\right) \le \frac{1}{2}\left(\cos^{2p}\left(\frac{2m\pi}{2N}\right) + \cos^{2p}\left(\frac{(2m-1)\pi}{2N}\right)\right)$$

$$\text{and} \qquad \cos^{2p}\left(\frac{m\pi}{N}\right) \ge \frac{1}{2}\left(\cos^{2p}\left(\frac{2m\pi}{2N}\right) + \cos^{2p}\left(\frac{(2m+1)\pi}{2N}\right)\right)$$

for $p \in \mathbb{Z}_+$, $0 < m < \frac{N-1}{2}$. Similarly, we have

$$\cos^{2p}\left(\frac{m\pi}{N}\right) \le \frac{1}{B}\sum_{b=0}^{B-1} \cos^{2p}\left(\frac{(Bm-b)\pi}{BN}\right)$$

$$\text{and} \qquad \cos^{2p}\left(\frac{m\pi}{N}\right) \ge \frac{1}{B}\sum_{b=0}^{B-1} \cos^{2p}\left(\frac{(Bm+b)\pi}{BN}\right)$$

Thus, we have

$$\sum_{m=1}^{M} \frac{\pi}{N} \cos^{2p}\left(\frac{m\pi}{N}\right) \le \lim_{B\to\infty} \sum_{n=1}^{BM} \frac{\pi}{BN} \cos^{2p}\left(\frac{n\pi}{BN}\right)$$

$$\text{and} \qquad \sum_{m=1}^{M} \frac{\pi}{N} \cos^{2p}\left(\frac{m\pi}{N}\right) \ge \lim_{B\to\infty} \sum_{n=0}^{BM-1} \frac{\pi}{BN} \cos^{2p}\left(\frac{n\pi}{BN} + \frac{\pi}{N}\right)$$

Turn the summation to the integral, and using $M = \left\lfloor \frac{N-1}{2} \right\rfloor$

$$
\begin{aligned}
&\lim_{B\to\infty}\sum_{n=1}^{BM}\frac{\pi}{BN}\cos^{2p}\left(\frac{n\pi}{BN}\right) \\
&= \int_0^{\frac{M\pi}{N}}\cos^{2p}(\kappa)d\kappa \\
&\leq \int_0^{\frac{\pi}{2}-\frac{\pi}{2N}}\cos^{2p}(\kappa)d\kappa \\
&= \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1}{2}+p\right)}{\Gamma(1+p)}-\frac{1}{2}\mathrm{B}\left(\sin^2\left(\frac{\pi}{2N}\right),\frac{1}{2}+p,\frac{1}{2}\right)
\end{aligned}
$$

where Γ is gamma function, B is beta function. Similarly, we have

$$
\begin{aligned}
&\lim_{B\to\infty}\sum_{n=0}^{BM-1}\frac{\pi}{BN}\cos^{2p}\left(\frac{n\pi}{BN}+\frac{\pi}{N}\right) \\
&\geq \int_{\frac{2\pi}{N}}^{\frac{\pi}{2}}\cos^{2p}(\kappa)d\kappa \\
&= \frac{1}{2}\mathrm{B}\left(\cos^2\left(\frac{2\pi}{N}\right),\frac{1}{2}+p,\frac{1}{2}\right)
\end{aligned}
$$

Combine these results, we have

$$
\mathrm{B}_N^{(1)}(p) \leq 2\sum_{m=1}^{M}\frac{\pi}{N}\cos^{2p}\left(\frac{m\pi}{N}\right) \leq \sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}+p\right)}{\Gamma(1+p)} - \mathrm{B}_N^{(2)}(p)
$$

□

Corollary 1.3.1 *(☑ Test Pass)*:

$$
\mathrm{B}_N^{(1)}(p) \leq 2\sum_{m=1}^{M}\frac{\pi}{N}\cos^{2p}\left(\frac{(2m-1)\pi}{2}N\right) \leq \sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}+p\right)}{\Gamma(1+p)} - \mathrm{B}_N^{(2)}(p)
$$

Proof: The proof is similar to the Lemma 1.3. □

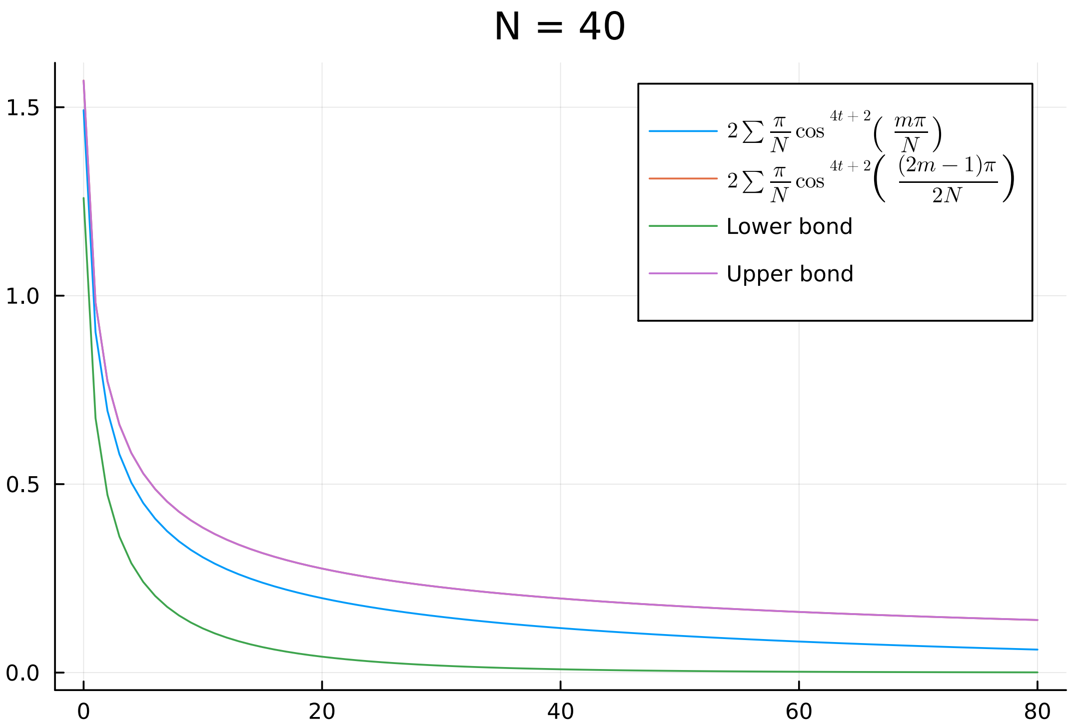

Figure 1: test Lemma 1.3. We could see that the upper bond is very tight.