Estimate $\cos^{4t}(x)$

A good approximation of $\cos^{4t}(x)$ is e^{-2tx^2}. The difference between the two terms is

$$\begin{aligned}
& e^{-2tx^2} - \cos^{4t}(x) \\
&= e^{-2tx^2} - e^{-2tx^2 + O(tx^4)} \\
&= e^{-2tx^2}\left(1 - e^{O(tx^4)}\right) \\
&\sim O\left(tx^4 e^{-2tx^2}\right)
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\mathcal{P} &= \frac{1}{N} + \frac{1}{N}\sum_{k=1}^{N-1}\left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right]\cos^{4t}\left(\frac{\pi k}{2N}\right) \\
&= -\frac{1}{N} + \frac{1}{N}\sum_{k=0}^{N-1}\left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right]\cos^{4t}\left(\frac{\pi k}{2N}\right) \\
&= -\frac{1}{N} + \frac{1}{N}\sum_{k} e^{-\frac{k^2\pi^2 t}{2N^2}}\left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right] + O\left(t^{-\frac{3}{2}}\right) \\
&= -\frac{1}{N} + \frac{1}{N}\Re\left[\sum_{k}\left(\exp\left(-\frac{k^2\pi^2 t}{2N^2} + ia\frac{\pi k}{2N}\right) + \exp\left(-\frac{k^2\pi^2 t}{2N^2} + ib\frac{\pi k}{2N}\right)\right)\right] + O\left(t^{-\frac{3}{2}}\right),
\end{aligned}$$

where $a = 2|i-j|$, $b = i+j-1$

Estimate $\sum_k \exp\left(-\frac{k^2\pi^2 t}{2N^2} + ia\frac{\pi k}{2N}\right)$

$$\begin{aligned}
& \Re\left[\frac{\pi}{2N}\sum_{k}\exp\left(-\frac{k^2\pi^2 t}{2N^2} + ia\frac{\pi k}{2N}\right)\right] \\
&< \Re\left[\int_0^\infty \exp(-2tx^2 + iax)\,dx\right] \\
&= \frac{1}{2}\Re\left[\int_{-\infty}^\infty \exp(-2tx^2 + iax)\,dx\right]
\end{aligned}$$

And we have

$$\int_{-\infty}^\infty \exp(-2tx^2 + iax)\,dx = \frac{e^{-\left(\frac{a^2}{8t}\right)}\sqrt{\frac{\pi}{2}}}{\sqrt{t}}$$

Now, we analysis how much we throw when we transfer the summation to integral.

$$\begin{aligned}
& \Re\left[\int_0^\infty \exp(-2tx^2 + iax)\,dx\right] - \Re\left[\frac{\pi}{2N}\sum_{k}\exp\left(-\frac{k^2\pi^2 t}{2N^2} + ia\frac{\pi k}{2N}\right)\right] \\
&= \sum_{i=0}^{N-1}\left(\int_{x_i}^{x_{i+1}} \exp(-2tx^2)\cos(ax)\,dx - \frac{\pi}{2N}\exp\left(-2t\frac{i^2\pi^2 t}{2N^2}\right)\cos\left(a\frac{\pi i}{2N}\right)\right) + \int_{\frac{\pi}{2}}^\infty \exp(-2tx^2 + iax)\,dx
\end{aligned}$$

The first term obey

$$\left|\int_{x_k}^{x_{k+1}} \exp(-2tx^2)\cos(ax)dx - \frac{\pi}{2N}\exp\left(-2t\frac{k^2\pi^2 t}{2N^2}\right)\cos\left(a\frac{\pi k}{2N}\right)\right|$$
$$\leq \frac{\pi}{2N}\left(\max_{x\in[x_i,x_{i+1}]}(\exp(-2tx^2)\cos(ax)) - \exp\left(-2t\frac{k^2\pi^2 t}{2N^2}\right)\cos\left(a\frac{\pi k}{2N}\right)\right)$$

The integral error

According to Trapezoidal Rule,

$$\Re\left[\int_0^{\frac{\pi}{2}} \exp(-2tx^2 + iax)dx\right] = \Re\left[\frac{\pi}{2N}\sum_{k=0}^{N}\exp\left(-\frac{k^2\pi^2 t}{2N^2} + ia\frac{\pi k}{2N}\right)\right] - \frac{\pi}{4N}\left(1 + e^{-\frac{\pi}{2}t}\cos\left(\frac{\pi a}{2}\right)\right) + R$$

where R is the error term

$$R = -\frac{\pi^3}{6N^2}f''(\eta),$$

where $f(x) = \exp(-2tx^2)\cos(ax)$, $\eta \in \left(0, \frac{\pi}{2}\right)$. Let $g(x) = f''(x)$,

$$g(x) = e^{-2tx^2}\left(-\left(a^2 + 4t\left(1 - 4tx^2\right)\right)\cos(ax) + 8atx\sin(ax)\right)$$

estimate again

$$\mathcal{P} = \frac{1}{N} + \frac{2}{\pi}\frac{\pi}{2N}\sum_{k=1}^{N}\left[\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right]\cos^{4t}\left(\frac{\pi k}{2N}\right)$$
$$= \frac{1}{N} + \frac{2}{\pi}\frac{\pi}{2N}\sum_{k=1}^{N}\Re\left[\exp\left(-\frac{k^2\pi^2 t}{2N^2} + ia\frac{\pi k}{2N}\right) + \exp\left(-\frac{k^2\pi^2 t}{2N^2} + ib\frac{\pi k}{2N}\right)\right] + O\left(t^{-\frac{3}{2}}\right)$$
$$= \frac{1}{N} + \frac{2}{\pi}\left(\Re\left[\int_1^{\frac{\pi}{2}} \exp(-2tx^2 + iax)dx\right] + \Re\left[\int_1^{\frac{\pi}{2}} \exp(-2tx^2 + ibx)dx\right] + \frac{\pi}{2N}e^{-\frac{\pi^2 t}{2N^2}}\cos\left(\frac{\pi a}{2N}\right) + \right)$$
$$= \frac{1}{\sqrt{2\pi t}}\left(e^{-\left(\frac{a^2}{8t}\right)} + e^{-\left(\frac{b^2}{8t}\right)}\right) + O\left(t^{-\frac{3}{2}}\right)$$