Analysis the order of PP

2024-06-03

Wu Bujiao

Bian Kaiming

1. The order of PP

Theorem 1.1: Let $\mathcal{A}_t := -\sum_{i=1}^{N-1} A_1(i,t)A_N(i,t)$. Suppose N is even, $\exists f_{t(N)} \in\in \mathcal{O}\left(\frac{1}{N}\right)$, s.t. $\mathcal{A}_T \leq f_t$ when $t \in \mathcal{O}(N)$.

Proof: Expand the $\mathcal{A}_t$ we have

$$\mathcal{A}_T = \frac{N}{4}\left[\sum_{k=1}^{N/2} \cos^{4t+2}\left(\frac{(2k-1)\pi}{2N}\right) - \sum_{k=1}^{N/2-1} \cos^{4t+2}\left(\frac{k\pi}{N}\right)\right]$$

Due to Lemma 1.2, we have

$$\begin{aligned}
&\sum_{k=1}^{N/2} \cos^{4t+2}\left(\frac{(2k-1)\pi}{2N}\right) - \sum_{k=1}^{N/2-1} \cos^{4t+2}\left(\frac{k\pi}{N}\right) \\
&\leq \frac{N}{\pi}\sum_{k=1}^{2t+1} \frac{(4t+1)!!}{(4t+2)!!}\frac{(2k-2)!!}{(2k-1)!!}\Bigg[\\
&\quad \sin\left(\frac{(2x-1)\pi}{2N}\right)\cos^{2k-1}\left(\frac{(2x-1)\pi}{2N}\right)\Bigg|_0^M \\
&\quad -\sin\left(\frac{x\pi}{N}\right)\cos^{2k-1}\left(\frac{x\pi}{N}\right)\Bigg|_1^{M+1}\Bigg]
\end{aligned}$$

The difference in the square brackets is equal to

$$\begin{aligned}
C = &\cos\left(\frac{\pi}{2M}\right)^{-1+2k}\sin\left(\frac{\pi}{2M}\right) \\
&+\cos\left(\frac{\pi}{M}\right)^{-1+2k}\sin\left(\frac{\pi}{M}\right) \\
&+\cos\left(\frac{3\pi}{2M}\right)\sin\left(\frac{3\pi}{2M}\right)^{-1+2k}
\end{aligned}$$

Now we try to estimate the order of C

$$C = \begin{cases} \frac{3\pi}{2N} + \mathcal{O}\left(\frac{1}{N^3}\right), & k > 1 \\ \frac{3\pi}{N} + \mathcal{O}\left(\frac{1}{N^3}\right), & k = 1 \end{cases}$$

$\square$

Lemma 1.2:

$$\mathcal{L}_T \leq F_T := \sum_{k=1}^{M} \cos^T\left(\frac{(2k+c)\pi}{2N}\right) \leq \mathcal{U}_T$$

where $1 < M < \lfloor \frac{N}{2} \rfloor$, $c \in \{0, 1\}$,

$$\mathcal{L}_T := \sum_{k=1}^{T/2} \frac{(T-1)!!}{T!!} \frac{(2k)!!}{(2k-1)!!} B_{2k}^{(1,M+1)} + \frac{(T-1)!!}{T!!} M$$

$$\mathcal{U}_T := \sum_{k=0}^{T/2} \frac{(T-1)!!}{T!!} \frac{(2k+1)!!}{(2k)!!} B_{2k}^{(0,M)} + \frac{(T-1)!!}{T!!} M$$

$$B_s^{(i,j)} := \frac{N}{\pi s} \sin\left(\frac{(2x+c)\pi}{2N}\right) \cos^{s-1}\left(\frac{(2x+c)\pi}{2N}\right) \Bigg|_i^j$$

Proof: We will begin with the lower bound.

Using the integral inequality, we have

$$F_T \geq \int_1^{M+1} \cos^T\left(\frac{(2x+c)\pi}{2N}\right) dx =: \mathcal{F}_T$$

Using integration by parts, we can transform the above equation into a recursive formula

$$\begin{aligned} & \frac{N}{\pi T} \int \sin\left(\frac{(2x+c)\pi}{2N}\right) d\cos^{T-1}\left(\frac{(2x+c)\pi}{2N}\right) \\ = & \frac{N}{\pi T} \sin\left(\frac{(2x+c)\pi}{2N}\right) \cos^{T-1}\left(\frac{(2x+c)\pi}{2N}\right) \Bigg|_1^{M+1} \\ & - \frac{1}{T} \int \cos^T\left(\frac{(2x+c)\pi}{2N}\right) dx \\ =: & B_T^{(1,M+1)} - \frac{1}{T} \mathcal{F}_T \end{aligned} \tag{1}$$

The left hand side of the Eqn. (1) can be written as

$$
\begin{aligned}
\text{l.h.s.} &= \frac{N}{\pi T}\int \sin\left(\frac{(2x+c)\pi}{2N}\right) d\cos^{T-1}\left(\frac{(2x+c)\pi}{2N}\right) \\
&= -\frac{T-1}{T}\int \sin^2\left(\frac{(2x+c)\pi}{2N}\right)\cos^{T-2}\left(\frac{(2x+c)\pi}{2N}\right)\mathtt{d}x \qquad (2) \\
&= -\frac{T-1}{T}\int \cos^{T-2}\left(\frac{(2x+c)\pi}{2N}\right)\mathtt{d}x + \frac{T-1}{T}\int \cos^{T}\left(\frac{(2x+c)\pi}{2N}\right)\mathtt{d}x
\end{aligned}
$$

Test Pass

Combine Eqn. (1) and (2), we have

$$
\begin{aligned}
\mathcal{F}_T &= B_T^{(1,M+1)} + \frac{T-1}{T}\mathcal{F}_{T-2} \\
&= \sum_{k=1}^{T/2}\frac{(T-1)!!}{T!!}\frac{(2k)!!}{(2k-1)!!}B_{2k}^{(1,M+1)} + \frac{(T-1)!!}{T!!}\mathcal{F}_0
\end{aligned}
$$

Similarly, we have

$$
F_T \le \mathcal{G}_T := \int_0^M \cos^T\left(\frac{(2x+c)\pi}{2N}\right)\mathtt{d}x
$$

$$
\begin{aligned}
\mathcal{G}_T &= B_T^{(0,M)} + \frac{T-1}{T}\mathcal{G}_{T-2} \\
&= \sum_{k=0}^{T/2}\frac{(T-1)!!}{T!!}\frac{(2k+1)!!}{(2k)!!}B_{2k+1}^{(0,M)} + \frac{(T-1)!!}{T!!}\mathcal{G}_0
\end{aligned}
$$

Notice that $\mathcal{F}_0 = \mathcal{G}_0 = M$.

□