Fermionic classical shallow shadows

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1. The variance of the fermionic classical shallow shadows

Let v be the estimation of $\mathrm{tr}(\rho\gamma_S)$ with shallow fermionic shadow $\mathcal{M}_d$ The variance of the fermionic classical shallow shadows is determined by the following equation

$$\mathrm{Var}[v] = \frac{1}{\alpha_{S,d}}.$$

The variance of the fermionic classical shallow shadows is inversely proportional to the α, where α could be calculated by the tensor network contraction.
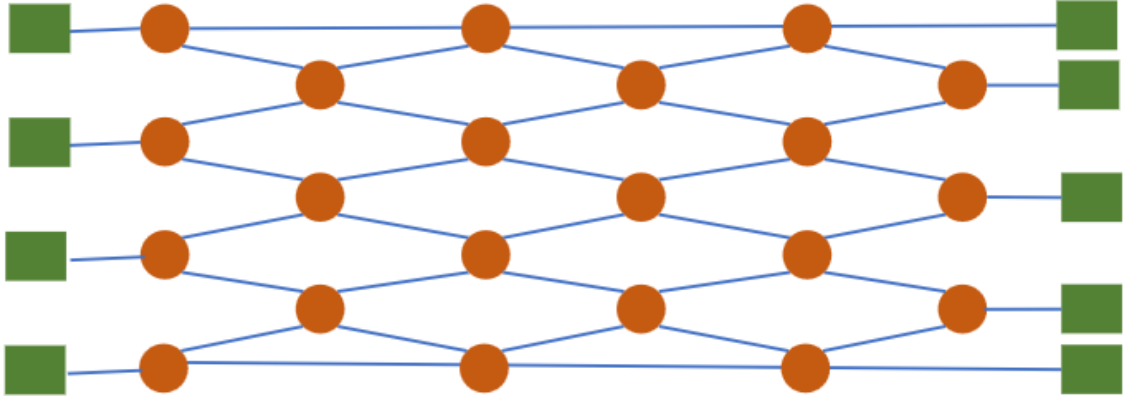

Figure 1: The green boxes are vectors with Pauli basis. The red circles are tensors, and the concrete form of these tensors are shown in <u>overleaf</u>.

After few layers of tensors acting on the initial vector, the result state could be writen as

$$\sum_{\{i \mid P_i \in \mathcal{P}_n\}} c_i P_i, \tag{1}$$

where $\mathcal{P}_n$ is the set of n bits Pauli group. Finally, the α could be given by the summation of the coefficients in Eqn. (1)

$$\alpha = \sum_{\{i \mid P_i \in \mathcal{Z}_n\}} c_i,$$

where $\mathcal{Z}_n$ is the set of n bits Pauli group with I or Z operator.

Thus, the key thing here is to calculate the distribution of the coefficients in Eqn. (1). And the calculation of the distribution could be mapped to a random walk problem, which is shown in `doc/deductions/CalculateAlp.pdf` (we will use `doc/deductions/` `*` as default path and use `*.pdf` as default format).

If we set the number of layers d as a odd number, and the number of qubits n as a even number, the random walk could be simplified to a neat form, which is a 2D-lattice.

Let $N = \frac{n}{2}$, $t = \frac{d-1}{2}$, $k := |S|$, and the vertice of the 2D-lattice is (μ, ν), $\mu, \nu = 1, 2, ..., N$. Denote the coefficient of (μ, ν) with d layers as $c^{i,j}(\mu, \nu, t)$, where (i, j) is the initial state. We will ignore the superscript (i, j) if it is not ambiguous.

Due to the note `CalculateAlp`, we could write down the transition equation of the random walk as

$$c^{i,j}(\mu, \nu, t) = P_i(\mu, t)P_j(\nu, t) + I^{i,j}(\mu, \nu, t) \tag{2}$$

Notice that the I is zero in most of place. It gets non-zero only when $\mu = \nu$ or $|\mu - \nu| = 1$. And it is a small term, too. Thus, the dominant term in Eqn. (2) is the first term. The α could be calculated by

$$\alpha_{2t+1} = \sum_\mu c(\mu, \mu, t)$$
$$= \sum_\mu P(\mu, t)P(\mu, t) + \sum v(\mu, \nu, t)$$

From numerical results, we should see that the summation $\mathcal{P} = \sum_\mu P(\mu, t)P(\mu, t)$ reflects the trend of the α. And the α could be approximated by the $\mathcal{P}$ with a constant coefficient.

2. Analysis $\mathcal{P}$

See `CalculatePP`.