Calculate α

2024-05-29

Bian Kaiming

1. for any S

Lemma 1.1: Let $S \subset [2n]$, $|S| = k$

$$\mathcal{P}_S(t) := \sum_{\mu} \prod_{i \in S} P_i(\mu, t)$$

Proof: Let $A_i(\mu) = \sum_{k=1}^{N-1} \cos\left(\left(\mu - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos\left(\left(i - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t}\left(\frac{\pi k}{2N}\right)$.

$$\begin{aligned}\mathcal{P}_S(t) &= \sum_{\mu} \prod_{i \in S} \frac{1}{N}(1 + 2A_i(\mu)) \\ &= \frac{1}{N^k} \sum_{\mu,\nu} 2^{\nu} \sum_{S_\nu \subset S} \prod_{i \in S_\nu} A_i(\mu),\end{aligned}$$

where $|S_\nu| = \nu$.

Now, we begin to deal with $\sum_{\mu} \prod_{i \in S_\nu} A_i(\mu)$ □

2. $P_i(\mu, t) P_j(\mu, t)$

Lemma 2.1: Let $S \subset [2n]$, $|S| = k$

$$\mathcal{P}_S(t) := \sum_{\mu} \prod_{i \in S} P_i(\mu, t)$$

Proof: As the same as the $\mathcal{P}_{1N}$, $\sum(A_1(\mu, t) + A_2(\mu, t)) = 0$. Thus,

$$\mathcal{P}_{ij}(t) = \frac{1}{N} + \frac{4}{N^2} \sum_{\mu} A_i(\mu, t) A_j(\mu, t)$$

where

$$A_i(\mu, t) = \sum \cos\left(\frac{\pi\left(i - \frac{1}{2}\right)k}{N}\right) \cos\left(\frac{\pi\left(\mu - \frac{1}{2}\right)k}{N}\right) \cos^{2t}\left(\frac{\pi k}{2N}\right)$$

As the same as the $\mathcal{P}_{1N}$,

$$\sum \cos\left(\frac{\pi\left(\mu - \frac{1}{2}\right)j}{N}\right) \cos\left(\frac{\pi\left(\mu - \frac{1}{2}\right)k}{N}\right) = \frac{N}{2}$$

Thus,

$$\begin{aligned}
&\sum A_i(\mu, t) A_j(\mu, t) \\
= &\frac{N}{2} \sum_{\mu} \cos\left(\frac{\pi\mu\left(i - \frac{1}{2}\right)}{N}\right) \cos\left(\frac{\pi\mu\left(j - \frac{1}{2}\right)}{N}\right) \cos^{4t}\left(\frac{\pi\mu}{2N}\right) \\
= &\frac{N}{4} \sum_{\mu} \cos\left(\frac{\pi\mu(i-j)}{N}\right) \cos^{4t}\left(\frac{\pi\mu}{2N}\right) - \frac{N}{4} \sum_{\mu} \cos\left(\frac{\pi\mu(i+j-1)}{N}\right) \cos^{4t}\left(\frac{\pi\mu}{2N}\right)
\end{aligned}$$

$\square$