Calculate α

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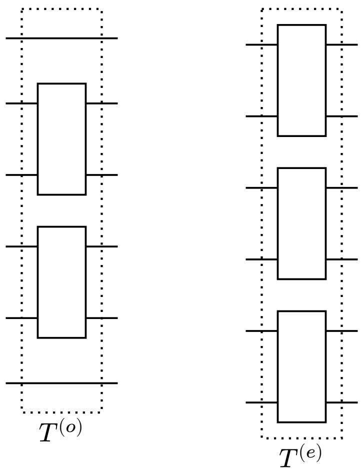
1. The tensor and the vector space

Let $T^{(o)}$ be the odd layer of T gates, and $T^{(e)}$ be the even layer of T gates. Then we have the following circuits:



Now, alternately apply the odd and even layers of T gates to the $|\gamma_1\gamma_{2n}\rangle\rangle$

$$T^{(\text{whole})}(b_1,t) = {T^{(o)}}^{b_2}\left(\prod_{i=0}^{t-b_2} T^{(e)}T^{(o)}\right){T^{(e)}}^{b_1},$$

where $b_1, b_2 \in \{0,1\}$, $t+b_1$ stands for the number of layers. The output states must in the vector space spaned by the following basis

$$|Z_i\rangle\rangle, \quad \left|X_i\left(\prod_{k=i+1}^{j-1} Z_k\right)Y_j\right\rangle\!\!\right\rangle$$

Let $M_i = \frac{1}{\sqrt{2}}(X_i + Y_i)$, the space

$$V := \operatorname{span}\left\{|Z_i\rangle\rangle,\ \left|M_i\left(\prod Z_k\right)M_j\right\rangle\!\!\right\rangle\right\}$$

is the image subspace for all $T^{(\text{whole})}(b_1,t)$ with $b_1 + t > 0$ if the input state is limit to Γ_2. Thus, the action of $T^{(\text{whole})}(b_1,t)$ could always be written in the following form

$$T^{(\text{whole})}(b_1,t)|\gamma_1\gamma_{2n}\rangle\rangle = \sum P(i,i,t)|Z_i\rangle\rangle + \sum_{i<j} P(i,j,t)\left|M_i\left(\prod Z_k\right)M_j\right\rangle\!\!\right\rangle.$$

The action of the tensor could be simplified by studying the coefficients $P(i,j,t)$.

2. Propagator

To simplify the discussion, we start with a special case: $b_1 = 1, b_2 = 0$, and the number of wires is an even number $2N$. In this case, the whole tensor $T^{(\text{whole})}$ could be written as

$$T^{(\text{whole})}(1,t) = \left(T^{(e)}T^{(o)}\right)^t T^{(e)}.$$

Then, we lift the V into the space of the second order 2N-dimensional polynomials

$$W := \operatorname{span}\left\{\sum_{i,j=1}^{2N} c_{i,j}x_i x_j\right\}$$

We can prove that the space V is isometric to the space W by ϕ,

$$\begin{aligned} \phi : V \to W \\ \left|M_i\left(\prod Z_k\right)M_j\right\rangle\!\!\rangle \to x_i x_j \\ |Z_i\rangle\!\rangle \to x_i^2. \end{aligned}$$

Let $T_p^{(2N)}(t)$ be the map $\phi T^{(\text{whole})}(1,t)\phi^{-1}$, the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "T^{(\text{whole})}(1,t)"] \arrow[d, leftrightarrow, "\phi"] & V \arrow[d, leftrightarrow, "\phi"] \\ W \arrow[r, "T_p^{(2N)}(t)"] & W \end{tikzcd}$$

Now, let's consider the action of the tensor $T_p^{(2N)}(t)$ on the space W. Similarly, we could write down the recursive relation of coefficients $a_{i,j}$.

When $t = 0$, the transforming state $T^{(e)}|\gamma_1\gamma_{4N}\rangle\!\rangle$ to W, and we got $\frac{1}{4}(x_1x_{2N} + x_1x_{2N-1} + x_2x_{2N} + x_1x_{2N-1})$. Suppose at t, the vector in W is $\sum c_i(t)c'_j(t)x_ix_j$,

$$\begin{aligned} & T_p^{(2N)}(t+1)\left(\frac{1}{4}(x_1x_{2N} + x_1x_{2N-1} + x_2x_{2N} + x_1x_{2N-1})\right) \\ &= T^{(e)}T^{(o)}T_p^{(2N)}(t)\left(\frac{1}{4}(x_1x_{2N} + x_1x_{2N-1} + x_2x_{2N} + x_1x_{2N-1})\right) \\ &= T^{(e)}T^{(o)}\sum c_i(t)c'_j(t)x_ix_j \end{aligned}$$

Then, the action of $\phi\, T^{(e)}T^{(o)}\phi^{-1}$ on this vector is

$c_i c'_j x_i x_j$	$\phi\, T^{(e)}T^{(o)}\phi^{-1}\big(c_i c'_j x_i x_j\big)$
$i = j = 1$	$\frac{1}{6}(c_1 x_1 + c_1 x_2)(c'_2 x_1 + c'_2 x_2)$
$i = 1, j = 2, 3$	$\frac{1}{8}(c_1 x_1 + c_1 x_2)\big(\frac{4}{3} c'_2 x_1 + \frac{4}{3} c'_2 x_2 + c'_2 x_3 + c'_2 x_4\big)$
$i = 1, j > 3$	$\frac{1}{8}(c_1 x_1 + c_1 x_2)\big(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j} + c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\big)$
$1 < i < 2N$, $i = j$	$\frac{1}{24}\big(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\big)$ $\big(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j} + c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\big)$ $-\frac{1}{72}\big(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i}\big)\big(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j}\big)$ $-\frac{1}{72}\big(c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\big)\big(c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\big)$
$1 < i < 2N$, i is even, $j = i + 1$	$\frac{1}{12} c_i c_{j'}\big(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\big)^2$ $-\frac{1}{36}\big(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i}\big)\big(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j}\big)$ $-\frac{1}{36}\big(c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\big)\big(c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\big)$
$1 < i < 2N$, $i + \eta_i \leq j \leq i + \eta_i + 2$	$\frac{1}{16}\big(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\big)$ $\big(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j} + c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\big)$ $+\frac{1}{48}\big(c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\big)\big(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j}\big)$
$1 < i < 2N$, $j > i + \eta_i + 2$	$\frac{1}{16}\big(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\big)$ $\big(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j} + c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\big)$
$i = j = 2N$	$\frac{1}{6}(c_{2N} x_{2N-1} + c_{2N} x_{2N})(c'_{2N} x_{2N-1} + c'_{2N} x_{2N})$

Table 1: The action of the tensor $\phi\, T^{(e)}T^{(o)}\phi^{-1}$ on the space of second order 2N-dimensional polynomials.

The subsript η_i is defined as $\eta_i := 1 - (i \bmod 2)$. We can see that, c_{2i-1} and c_{2i} are always the same. So do c'_{2j-1} and c'_{2j}. Let $b_i = 2c_{2i-1} = 2c_{2i}$ and $b'_j = 2c'_{2j-1} = 2c'_{2j}$, we could further simplify the action of the tensor in the space of second order N-dimensional polynomials W_N. For simplemess, we define a “free” recursive relation in W_N

$$P(i, t+1) = \begin{cases} \frac{1}{4}(P(i-1,t) + 2P(i,t) + P(i+1,t)) \text{ , } i \neq 1 \text{ or } N \\ \frac{1}{4}(P(i-1,t) + 2P(i,t) + P(i+1,t)) \end{cases}. \tag{1}$$

We call it “free” because $b_i(t) b'_j(t) = P(i,t)P(j,t)$ if i and j are not “collide” with each other (which means $|i-j| > 3$). And the solution of Eq. (1) is a propagating wave. Refs. [1] provide the solution of this equation,

$$P_{n_0}(n,t) = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N-1} \cos\left(\left(n - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos\left(\left(n_0 - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t} \frac{\pi k}{2N}, \tag{2}$$

where n_0 is the initial state. The term $P_{n_0}(n,t)$ also called the propagator.

For our case, there are 2 propagators, which are P_1 and P_N. And the initial state is $x_1 x_N$. Let $\phi()$

Now, let’s pluge $b_i = 2c_{2i-1} = 2c_{2i}$ and $b'_j = 2c'_{2j-1} = 2c'_{2j}$ into Table 1. More concretely, let ϕ' be the map

$$\begin{aligned} \phi' : W_N &\to W_{2N} \\ y_i y_j &\to \frac{1}{4}(x_{2i-1} + x_{2i})\big(x_{2j-1} + x_{2j}\big). \end{aligned}$$

If we consider the subspace W'_{2N} of W_{2N}, where $W'_{2N} := \operatorname{span}\big((x_{2i-1} + x_{2i})\big(x_{2j-1} + x_{2j}\big)\big)$ (it means $c_{2i-1} = c_{2i}$ and $c'_{2j-1} = c'_{2j}$), the ϕ' will be the isometric between W_N and W'_{2N}. Thus, the following diagram commutes.

$$
\begin{tikzcd}
V \arrow[r, "T^{(\mathrm{whole})}(1,t)"] \arrow[d, leftrightarrow, "\phi"] & V \arrow[d, leftrightarrow, "\phi"] \\
W'_{2N} \arrow[r, "T_p^{(2N)}(t)"] \arrow[d, leftrightarrow, "\phi'"] & W'_{2N} \arrow[d, leftrightarrow, "\phi'"] \\
W_N \arrow[r, "T_p^{(N)}(t)"] & W_N
\end{tikzcd}
$$

For simpliness, let

$$F(y_i) = \begin{cases} \frac{3}{4}y_1 + \frac{1}{4}y_2 \ , i = 1 \\ \frac{1}{4}y_{i-1} + \frac{1}{2}y_i + \frac{1}{4}y_{i+1} \ , 1 < i < N \\ \frac{3}{4}y_N + \frac{1}{4}y_{N-1} \ , i = N \end{cases}$$

Then we get

$y_i y_j$	$T^{(e)}T^{(o)}$
$i = j = 1$	$F(y_1)F(y_1) - \frac{5}{144}(y_1 + y_2)^2 + \frac{1}{36}y_1 y_2$
$i = 1, j = 2$	$F(y_1)F(y_2) + (\frac{5}{144})({y_1}^2) + (\frac{1}{24})y_1 y_2 + (\frac{1}{72})({y_2}^2)$
$i = 1, j = 3$	$F(y_1)F(y_3) + \frac{1}{48}{y_2}^2$
$1 < i < N, j = i$	$F(y_i)F(y_i) - \dfrac{5}{144}{y_{i-1}}^2 - \dfrac{5}{48}y_{i-1}y_i - \dfrac{1}{16}y_{i-1}y_{i+1} - \left(\dfrac{1}{9}\right)({y_i}^2) - \left(\dfrac{5}{48}\right)y_i y_{i+1} - \left(\dfrac{5}{144}\right)({y_{i+1}}^2)$
$1 < i < N, j = i + 1$	$F(y_i)F(y_{i+1}) + \dfrac{1}{72}{y_i}^2 + \dfrac{1}{24}y_i y_{i+1} + \dfrac{1}{72}{y_{i+1}}^2$

$y_i y_j$	$T^{(e)}T^{(o)}$
$1 < i < N, j = i+2$	$F(y_i)F(y_{i+2}) + \frac{1}{48} y_{i+1}^2$
$i = j = N$	$F(y_N)F(y_N) - \frac{5}{144}(y_N + y_{N-1})^2 + \frac{1}{36} y_{N-1} y_N$
other cases when $i \leq j$	$F(y_i)F(y_j)$

这个 table 的计算实在是太太太太太折磨人了。详细计算我放在了 Section 3 中。

3. Proofs

Proof of Table 2: 现在我们想要计算 $T_p^{(N)}(t)$ 在W_N 中的作用。因此我们需要计算

$$
\begin{aligned}
& \phi' \phi T^{(e)} T^{(o)} \phi^{-1} \phi'^{-1} (y_i y_j) \\
= & \phi T^{(e)} T^{(o)} \phi^{-1} \left(\frac{1}{4} (x_{2i-1} + x_{2i})(x_{2j-1} + x_{2j}) \right) \\
= & \phi T^{(e)} T^{(o)} \phi^{-1} \left(\frac{1}{4} (x_{2i-1} x_{2j-1} + x_{2i-1} x_{2j} + x_{2i} x_{2j-1} + x_{2i} x_{2j}) \right)
\end{aligned}
$$

相当于对于每一个 $x_{2i-1} x_{2j-1}$ 这样的项进行查表。我们现在来分析不同的 y 的映射。为了简化符号，这里记 $S_N = \phi' \phi T^{(e)} T^{(o)} \phi^{-1} \phi'^{-1}$, $S_{2N} = \phi T^{(e)} T^{(o)} \phi^{-1}$.

$$
\begin{aligned}
\phi'(y_1 y_1) &= \frac{1}{4}(x_1 x_1 + 2 x_1 x_2 + x_2 x_2) \\
S_N(y_1 y_1) &= \frac{1}{4}(S_{2N}(x_1 x_1) + S_{2N}(2 x_1 x_2) + S_{2N}(x_2 x_2))
\end{aligned}
$$

依据 Table 1, 我们有

$$S_N(y_1y_1) = \frac{1}{4}\Bigg($$

$$\frac{1}{6}(x_1 + x_2)(x_1 + x_2) +$$

$$\left(\frac{1}{4}\right)\left(\left(\frac{4}{3}\right)x_1 + \left(\frac{4}{3}\right)x_2 + x_3 + x_4\right)(x_1 + x_2) +$$

$$-\left(\frac{1}{72}\right)\left((x_1 + x_2)^2\right) + \left(\frac{1}{24}\right)\left((x_1 + x_2 + x_3 + x_4)^2\right) - \left(\frac{1}{72}\right)\left((x_3 + x_4)^2\right)$$

$$\Bigg)$$

$$\left(\frac{19}{144}\right)(x_1{}^2) + \left(\frac{19}{72}\right)x_1x_2 + \left(\frac{1}{12}\right)x_1x_3 + \left(\frac{1}{12}\right)x_1x_4$$

$$+\left(\frac{19}{144}\right)(x_2{}^2) + \left(\frac{1}{12}\right)x_2x_3 + \left(\frac{1}{12}\right)x_2x_4$$

$$+\left(\frac{1}{144}\right)(x_3{}^2) + \left(\frac{1}{72}\right)x_3x_4 + \left(\frac{1}{144}\right)(x_4{}^2)$$

我们想要和“自由演化”进行对比。而 y_1y_1 自由演化的结果是

$$\begin{aligned}
F(y_1y_1) &= \frac{1}{16}(3y_1 + y_2)(3y_1 + y_2) \\
&= \frac{1}{16}\left(\frac{3}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right)\left(\frac{3}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right) \\
&= \left(\frac{9}{64}\right)(x_1{}^2) + \left(\frac{9}{32}\right)x_1x_2 + \left(\frac{3}{32}\right)x_1x_3 + \left(\frac{3}{32}\right)x_1x_4 \\
&\quad + \left(\frac{9}{64}\right)(x_2{}^2) + \left(\frac{3}{32}\right)x_2x_3 + \left(\frac{3}{32}\right)x_2x_4 + \left(\frac{1}{64}\right)(x_3{}^2) \\
&\quad + \left(\frac{1}{32}\right)x_3x_4 + \left(\frac{1}{64}\right)(x_4{}^2)
\end{aligned}$$

二者之差为

$$
\begin{aligned}
S_N(y_1y_1)-F(y_1y_1)=&-\left(\frac{5}{576}\right)(x_1{}^2)-\left(\frac{5}{288}\right)x_1x_2-\left(\frac{1}{96}\right)x_1x_3-\left(\frac{1}{96}\right)x_1x_4\\
&-\left(\frac{5}{576}\right)(x_2{}^2)-\left(\frac{1}{96}\right)x_2x_3-\left(\frac{1}{96}\right)x_2x_4\\
&-\left(\frac{5}{576}\right)(x_3{}^2)-\left(\frac{5}{288}\right)x_3x_4-\left(\frac{5}{576}\right)(x_4{}^2)\\
=&-\frac{5}{576}(x_1+x_2+x_3+x_4)^2+\frac{1}{144}(x_1+x_2)(x_3+x_4)\\
=&-\frac{5}{144}(y_1+y_2)^2+\frac{1}{36}y_1y_2
\end{aligned}
$$

接下来，算 $S_N(y_1y_2)$

$$
\begin{aligned}
S_N(y_1y_2)=&\frac{1}{4}(S_{2N}(x_1x_3)+S_{2N}(x_1x_4)+S_{2N}(x_2x_3)+S_{2N}(x_2x_4))\\
=&\frac{1}{4}\Bigg(\\
&\frac{1}{8}\left(\frac{4}{3}x_1+\frac{4}{3}x_2+x_3+x_4\right)(x_1+x_2)\\
&+\frac{1}{8}(x_3+x_4+x_5+x_6)(x_1+x_2)\\
&-\left(\frac{1}{36}\right)\left((x_1+x_2)^2\right)+\left(\frac{1}{12}\right)\left((x_1+x_2+x_3+x_4)^2\right)-\left(\frac{1}{36}\right)\left((x_3+x_4)^2\right)\\
&+\frac{1}{16}(x_1+x_2+x_3+x_4)(x_3+x_4+x_5+x_6)+\frac{1}{48}(x_3+x_4)^2\\
\Bigg)=&\left(\frac{2}{9}\right)(y_1{}^2)+\left(\frac{23}{48}\right)y_1y_2+\left(\frac{3}{16}\right)y_1y_3+\left(\frac{5}{36}\right)(y_2{}^2)+\left(\frac{1}{16}\right)y_2y_3
\end{aligned}
$$

$F(y_1)F(y_2)=\left(\tfrac{3}{4}y_1+\tfrac{1}{4}y_2\right)\left(\tfrac{1}{4}y_1+\tfrac{1}{2}y_2+\tfrac{1}{4}y_3\right)$

$$
S_N(y_1y_2)=F(y_1)F(y_2)+\left(\frac{5}{144}\right)(y_1{}^2)+\left(\frac{1}{24}\right)y_1y_2+\left(\frac{1}{72}\right)(y_2{}^2)
$$

接下来，算 $S_N(y_1y_3)$

$$
S_N(y_1y_3)=F(y_1)F(y_3)+\frac{1}{48}y_2{}^2
$$

然后， $j>3$时

$$
S_N(y_1y_j)=F(y_1)F(y_j)
$$

然后， $i > 1, i = j$时,

$$S_N(y_i y_j) = \frac{1}{4}(S_N(x_{2i-1}x_{2j-1}) + S_N(x_{2i-1}x_{2j}) + S_N(x_{2i}x_{2j-1}) + S_N(x_{2i}x_{2j}))$$

$$= \frac{1}{4}(S_N(x_{2i-1}x_{2i-1}) + 2S_N(x_{2i-1}x_{2i}) + S_N(x_{2i}x_{2i}))$$

$$= \frac{1}{4}\Bigg($$

$$\frac{1}{24}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})^2$$

$$-\frac{1}{72}(x_{2i-3} + x_{2i-2})^2 - \frac{1}{72}(x_{2i-1} + x_{2i})^2$$

$$+\frac{1}{24}(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})^2$$

$$-\frac{1}{72}(x_{2i-1} + x_{2i})^2 - \frac{1}{72}(x_{2i+1} + x_{2i+2})^2$$

$$+\frac{1}{16}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})$$

$$+\frac{1}{48}(x_{2i-1} + x_{2i})^2$$

$$\Bigg)$$

$$= \frac{1}{24}(y_{i-1} + y_i)^2 - \frac{1}{72}(y_{i-1})^2 - \frac{1}{72}(y_i)^2$$

$$+\frac{1}{24}(y_i + y_{i+1})^2 - \frac{1}{72}(y_i)^2 - \frac{1}{72}(y_{i+1})^2$$

$$+\frac{1}{16}(y_{i-1} + y_i)(y_i + y_{i+1}) + \frac{1}{48}(y_i)^2$$

$$S_N(y_i y_j) = F(y_i)F(y_j) - \left(\frac{5}{144}\right)({y_{i-1}}^2) - \left(\frac{5}{48}\right)y_{i-1}y_i - \left(\frac{1}{16}\right)y_{i-1}y_{i+1}$$

$$-\left(\frac{1}{9}\right)({y_i}^2) - \left(\frac{5}{48}\right)y_i y_{i+1} - \left(\frac{5}{144}\right)({y_{i+1}}^2)$$

然后， $i > 1, j = i + 1$时,

$$
\begin{aligned}
S_N(y_i y_j) &= \frac{1}{4}\left(S_N(x_{2i-1}x_{2i+1}) + S_N(x_{2i-1}x_{2i+2}) + S_N(x_{2i}x_{2i+1}) + S_N(x_{2i}x_{2i+2})\right) \\
&= \frac{1}{4}\Bigg(\\
&\quad \frac{1}{16}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2}) \\
&\quad + \frac{1}{48}(x_{2i-1} + x_{2i})^2 \\
&\quad + \frac{1}{16}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})(x_{2i+1} + x_{2i+2} + x_{2i+3} + x_{2i+4}) \\
&\quad + \frac{1}{12}(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})^2 \\
&\quad - \frac{1}{36}(x_{2i-1} + x_{2i})^2 - \frac{1}{36}(x_{2i+1} + x_{2i+2})^2 \\
&\quad + \frac{1}{16}(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})(x_{2i+1} + x_{2i+2} + x_{2i+3} + x_{2i+4}) \\
&\quad + \frac{1}{48}(x_{2i+1} + x_{2i+2})^2 \\
&\Bigg) \\
&= \frac{1}{16}(y_{i-1} + y_i)(y_i + y_{i+1}) + \frac{1}{48}(y_i)^2 \\
&\quad + \frac{1}{16}(y_i + y_{i+1})(y_{i+1} + y_{i+2}) + \frac{1}{48}(y_{i+1})^2 \\
&\quad + \frac{1}{16}(y_{i-1} + y_i)(y_{i+1} + y_{i+2}) \\
&\quad + \frac{1}{12}(y_i + y_{i+1})^2 - \frac{1}{36}(y_i)^2 - \frac{1}{36}(y_{i+1})^2
\end{aligned}
$$

$$
S_N(y_i y_j) = F(y_i)F(y_j) + \left(\frac{1}{72}\right)({y_i}^2) + \left(\frac{1}{24}\right)y_i y_{i+1} + \left(\frac{1}{72}\right)({y_{i+1}}^2)
$$

$\square$

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