Calculate α

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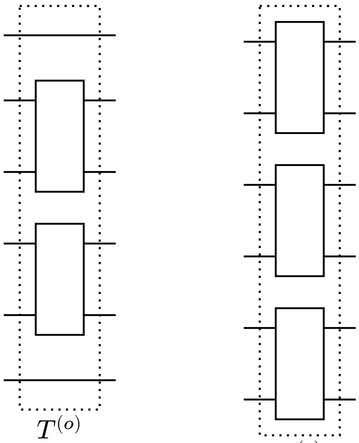
1. The tensor and the vector space

Let $T^{(o)}$ be the odd layer of T gates, and $T^{(e)}$ be the even layer of T gates. Then we have the following circuits:



Now, alternately apply the odd and even layers of T gates to the $|\gamma_1\gamma_{2n}\rangle\!\rangle$

$$T^{(\text{whole})}(b_1, t) = {T^{(o)}}^{b_2} \left(\prod_{i=0}^{t-b_2} T^{(e)} T^{(o)} \right) {T^{(e)}}^{b_1},$$

where $b_1, b_2 \in \{0, 1\}$, $t + b_1$ stands for the number of layers. The output states must in the vector space spaned by the following basis

$$|Z_i\rangle\!\rangle, \quad \left| X_i \left(\prod_{k=i+1}^{j-1} Z_k \right) Y_j \right\rangle\!\!\right\rangle$$

Let $M_i = \frac{1}{\sqrt{2}}(X_i + Y_i)$, the space

$$V := \operatorname{span}\left\{ |Z_i\rangle\!\rangle, \ \left| M_i \left(\prod Z_k \right) M_j \right\rangle\!\!\right\rangle \right\}$$

is the image subspace for all $T^{(\text{whole})}(b_1, t)$ with $b_1 + t > 0$ if the input state is limit to Γ_2. Thus, the action of $T^{(\text{whole})}(b_1, t)$ could always be written in the following form

$$T^{(\text{whole})}(b_1, t)|\gamma_1\gamma_{2n}\rangle\!\rangle = \sum P(i, i, t)|Z_i\rangle\!\rangle + \sum_{i<j} P(i, j, t) \left| M_i \left(\prod Z_k \right) M_j \right\rangle\!\!\right\rangle.$$

The action of the tensor could be simplified by studying the coefficients $P(i,j,t)$.

2. Propagator

To simplify the discussion, we start with a special case: $b_1 = 1, b_2 = 0$, and the number of wires is an even number $2N$. In this case, the whole tensor $T^{(\text{whole})}$ could be written as

$$T^{(\text{whole})}(1,t) = \left(T^{(e)}T^{(o)}\right)^t T^{(e)}.$$

Then, we lift the V into the space of the second order 2N-dimensional polynomials

$$W := \operatorname{span}\left\{\sum_{i,j=1}^{2N} c_{i,j} x_i x_j\right\}$$

We can prove that the space V is isometric to the space W by ϕ,

$$\begin{aligned} \phi : V &\to W \\ \left|M_i\left(\prod Z_k\right)M_j\right\rangle\!\rangle &\to x_i x_j \\ |Z_i\rangle\!\rangle &\to x_i^2. \end{aligned}$$

Let $T_p^{(2N)}(t)$ be the map $\phi T^{(\text{whole})}(1,t)\phi^{-1}$, the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "T^{(\text{whole})}(1,t)"] \arrow[d, "\phi"] & V \arrow[d, "\phi"] \\ W \arrow[r, "T_p^{(2N)}(t)"] & W \end{tikzcd}$$

Now, let's consider the action of the tensor $T_p^{(2N)}(t)$ on the space W. Similarly, we could write down the recursive relation of coefficients $a_{i,j}$.

When $t = 0$, the transforming state $T^{(e)}|\gamma_1\gamma_{4N}\rangle\!\rangle$ to W, and we got $\frac{1}{4}(x_1x_{2N} + x_1x_{2N-1} + x_2x_{2N} + x_1x_{2N-1})$. Suppose at t, the vector in W is $\sum c_i(t)c_j'(t)x_ix_j$,

$$\begin{aligned} & T_p^{(2N)}(t+1)\left(\frac{1}{4}(x_1x_{2N} + x_1x_{2N-1} + x_2x_{2N} + x_1x_{2N-1})\right) \\ &= T^{(e)}T^{(o)}T_p^{(2N)}(t)\left(\frac{1}{4}(x_1x_{2N} + x_1x_{2N-1} + x_2x_{2N} + x_1x_{2N-1})\right) \\ &= T^{(e)}T^{(o)}\sum c_i(t)c_j'(t)x_ix_j \end{aligned}$$

Then, the action of $\phi\, T^{(e)}T^{(o)}\phi^{-1}$ on this vector is

$c_i c'_j x_i x_j$	$\phi\, T^{(e)}T^{(o)}\phi^{-1}\left(c_i c'_j x_i x_j\right)$
$i=j=1$	$\frac{1}{6}(c_1 x_1 + c_1 x_2)(c'_2 x_1 + c'_2 x_2)$
$i=1, j=2,3$	$\frac{1}{8}(c_1 x_1 + c_1 x_2)\left(\frac{4}{3}c'_2 x_1 + \frac{4}{3}c'_2 x_2 + c'_2 x_3 + c'_2 x_4\right)$
$i=1, j>3$	$\frac{1}{8}(c_1 x_1 + c_1 x_2)\left(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j} + c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\right)$
$1<i<2N$, $i=j$	$\frac{1}{24}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)\left(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j} + c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\right) -\frac{1}{72}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i}\right)\left(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j}\right) -\frac{1}{72}\left(c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)\left(c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\right)$
$1<i<2N$, i is even, $j=i+1$	$\frac{1}{12}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)\left(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j} + c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\right) -\frac{1}{36}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i}\right)\left(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j}\right) -\frac{1}{36}\left(c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)\left(c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\right)$
$1<i<2N$, $i+\eta_i \le j \le i+\eta_i+2$	$\frac{1}{16}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)\left(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j} + c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\right) +\frac{1}{48}\left(c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)\left(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j}\right)$
$1<i<2N$, $j>i+\eta_i+2$	$\frac{1}{16}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)\left(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j} + c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j}\right)$
$i=j=2N$	$\frac{1}{6}(c_{2N} x_{2N-1} + c_{2N} x_{2N})(c'_{2N} x_{2N-1} + c'_{2N} x_{2N})$

Table 1: The action of the tensor $\phi\, T^{(e)}T^{(o)}\phi^{-1}$ on the space of second order 2N-dimensional polynomials.

We can see that, c_{2i-1} and c_{2i} are always the same. So do c'_{2j-1} and c'_{2j}. Let $b_i = 2c_{2i-1} = 2c_{2i}$ and $b'_j = 2c'_{2j-1} = 2c'_{2j}$, we could further simplify the action of the tensor in the space of second order N-dimensional polynomials W_N. For simplemess, we define a "free" recursive relation in W_N

$$P(i,t+1) = \begin{cases} \frac{1}{4}(P(i-1,t) + 2P(i,t) + P(i+1,t)) \ , i \neq 1 \text{ or } N \\ \frac{1}{4}(P(i-1,t) + 2P(i,t) + P(i+1,t)) \end{cases}. \tag{1}$$

We call it "free" because $b_i(t)b'_j(t) = P(i,t)P(j,t)$ if i and j are not "collide" with each other (which means $|i-j| > 3$). And the solution of Eq. (1) is a propagating wave. Refs. [1] provide the solution of this equation,

$$P_{n_0}(n,t) = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N-1} \cos\left(\left(n - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos\left(\left(n_0 - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t} \frac{\pi k}{2N}, \tag{2}$$

where n_0 is the initial state. The term $P_{n_0}(n,t)$ also called the propagator.

For our case, there are 2 propagators, which are P_1 and P_N. And the initial state is $x_1 x_N$. Let $\phi()$

Now, let's pluge $b_i = 2c_{2i-1} = 2c_{2i}$ and $b'_j = 2c'_{2j-1} = 2c'_{2j}$ into Table 1. More concretely, let ϕ' be the map

$$\begin{aligned} \phi' : W_N &\to W_{2N} \\ y_i y_j &\to \frac{1}{4}(x_{2i-1} + x_{2i})(x_{2j-1} + x_{2j}). \end{aligned}$$

If we consider the subspace W'_{2N} of W_{2N}, where $W'_{2N} := \operatorname{span}\big((x_{2i-1} + x_{2i})(x_{2j-1} + x_{2j})\big)$ (it means $c_{2i-1} = c_{2i}$ and $c'_{2j-1} = c'_{2j}$), the ϕ' will be the isometric between W_N and W'_{2N}. Thus, the following diagram commutes.

$$
\begin{tikzcd}
V \arrow[r, "T^{(\text{whole})}(1,t)"] \arrow[d, "\phi"] & V \arrow[d, "\phi"] \\
W'_{2N} \arrow[r, "T_p^{(2N)}(t)"] \arrow[d, "\phi'"] & W'_{2N} \arrow[d, "\phi'"] \\
W_N \arrow[r, "T_p^{(N)}(t)"] & W_N
\end{tikzcd}
$$

Then we get

y_i, y_j	$T^{(e)}T^{(o)}$
$i = j = 1$	

3. Proofs

Proof of : The latest layer of T gates is $T^{(e)}$, which behaves

$$
\begin{aligned}
&T^{(e)}\left|M_{2i+c}\left(\prod Z_k\right)M_{2j+c'}\right\rangle\!\!\!\rangle \\
&= \frac{1}{4}\left(\left|M_{2i+(-1)^{1-c}}\left(\prod Z_k\right)M_{2j+(-1)^{1-c'}}\right\rangle\!\!\!\rangle + \left|M_{2i+(-1)^{1-c}}\left(\prod Z_k\right)M_{2j+c'}\right\rangle\!\!\!\rangle\right. \\
&\quad \left. + \left|M_{2i+c}\left(\prod Z_k\right)M_{2j+(-1)^{1-c'}}\right\rangle\!\!\!\rangle + \left|M_{2i+c}\left(\prod Z_k\right)M_{2j+c'}\right\rangle\!\!\!\rangle\right),
\end{aligned}
$$

if $2j + c' > 2i + c + 1$. c and c' are the parity of the input state, which take the value 0 or 1. Similarly, we write down the output of $T^{(e)}$ in the other cases

$$
T^{(e)}|M_{2i+c}M_{2i+1+c}\rangle\!\!\rangle = \begin{cases} \frac{1}{4}\left(|M_{2i}M_{2i+1}\rangle\!\!\rangle + |M_{2i-1}Z_iM_{2i+1}\rangle\!\!\rangle \right. \\ \quad \left. |M_{2i}Z_{i+1}M_{2i+2}\rangle\!\!\rangle + |M_{2i-1}Z_iZ_{i+1}M_{2i+2}\rangle\!\!\rangle\right) \ , \quad c = 0 \\ \frac{1}{3}|Z_{2i+1}\rangle\!\!\rangle + \frac{1}{3}|Z_{2i+2}\rangle\!\!\rangle + \frac{1}{3}|M_{2i+1}M_{2i+2}\rangle\!\!\rangle \end{cases}
$$

□

Bibliography

[1] L. Giuggioli, "Exact spatiotemporal dynamics of confined lattice random walks in arbitrary dimensions: A century after Smoluchowski and Pólya," *Physical Review X*, vol. 10, no. 2, p. 21045–21046, 2020.