

Calculate α

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1. Dealing PP

Theorem 1.1: Let $\mathcal{P}(t) = \sum_i P_1(i, t)P_N(i, t)$, and $M = \lfloor \frac{N-1}{2} \rfloor$, $c = N - 1 - 2M \in \{0, 1\}$. Then, \mathcal{P} could be bonded by the following equation

$$\begin{aligned} L(t) &= \frac{1}{N} + \frac{1}{\pi} B_N^{(1)}(2t+1) + \frac{1}{\pi} B_N^{(2)}(2t+1) \\ &\quad - \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{3}{2} + 2t)}{\Gamma(2 + 2t)} - \frac{2}{N} \sin^{4t+2}\left(\frac{\pi}{2N}\right) \\ U(t) &= \frac{1}{N} - \frac{1}{\pi} B_N^{(1)}(2t+1) - \frac{1}{\pi} B_N^{(2)}(2t+1) \\ &\quad + \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{3}{2} + 2t)}{\Gamma(2 + 2t)} \end{aligned}$$

Lemma 1.2 (Test Pass):

$$\begin{aligned} \mathcal{P}(t) &= \sum_i P_1(i, t)P_N(i, t) \\ &= \frac{1}{N} + \frac{2}{N} \sum_{m=1}^M \cos^{4t+2}\left(\frac{\pi m}{N}\right) - \frac{2}{N} \sum_{m=1}^M \cos^{4t+2}\left(\frac{\pi(2m-1)}{2N}\right) \\ &\quad - \frac{2c}{N} \sin^{4t+2}\left(\frac{\pi}{2N}\right), \end{aligned}$$

where $M = \lfloor \frac{N-1}{2} \rfloor$, $c = N - 1 - 2M \in \{0, 1\}$.

Proof: Let

$$P_1(i, t) = \frac{1}{N} + \frac{2}{N} A_1(i, t)$$

$$P_{N(i,t)} = \frac{1}{N} + \frac{2}{N} A_N(i, t)$$

,

where

$$A_1(i, t) = \sum_k \cos\left(\frac{\pi(i - \frac{1}{2})k}{N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

$$A_N(i, t) = \sum_k \cos\left(\frac{\pi(i - \frac{1}{2})k}{N}\right) \cos\left(\frac{\pi(N - \frac{1}{2})k}{N}\right) \cos^{2t}\left(\frac{\pi k}{2N}\right)$$

Then we have

$$\sum P_1(i, t) P_N(i, t) = \frac{1}{N} + \frac{2}{N^2} \sum_{i=1}^{N-1} (A_1(i, t) + A_N(i, t)) + \frac{4}{N^2} \sum_{i=1}^{N-1} A_1(i, t) A_N(i, t)$$

Now, let's analysis the second term.

$$\begin{aligned} & \sum_{i=1}^{N-1} (A_1(i, t) + A_N(i, t)) \\ &= \sum \left(\cos\left(\frac{\pi k}{2N}\right) + \cos\left(\frac{\pi k(N - \frac{1}{2})}{N}\right) \right) \cos\left(\frac{\pi(i - \frac{1}{2})k}{N}\right) \cos^{2t}\left(\frac{\pi k}{2N}\right) \end{aligned}$$

The summation of the second factor is zero

$$\begin{aligned} & \sum_i \cos\left(\frac{\pi(i - \frac{1}{2})k}{N}\right) \\ &= -\frac{1}{2} \cos\left(\frac{1}{2}\pi(2k + 1)\right) \csc\left(\frac{\pi k}{2N}\right) \\ &= \sin(k\pi) \csc\left(\frac{\pi k}{2N}\right) \\ &= 0 \end{aligned}$$

□

Thus, we have $\sum P_1(i, t) P_N(i, t) = \frac{1}{N} + \frac{4}{N^2} \sum_{i=1}^{N-1} A_1(i, t) A_N(i, t)$. Then we move to the term $\sum_{i=1}^{N-1} A_1(i, t) A_N(i, t)$.

$$\begin{aligned}
& \sum_{i=1}^{N-1} A_1(i, t) A_N(i, t) \\
&= \sum_{i,j,k} \cos\left(\frac{\pi j(N - \frac{1}{2})}{N}\right) \cos\left(\frac{\pi(i - \frac{1}{2})j}{N}\right) \cos\left(\frac{\pi(i - \frac{1}{2})k}{N}\right) \cos^{2t}\left(\frac{\pi j}{2N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)
\end{aligned}$$

We also begin to deal with the factor

$$\begin{aligned}
& \sum_{i=1}^{N-1} \cos\left(\frac{\pi(i - \frac{1}{2})j}{N}\right) \cos\left(\frac{\pi(i - \frac{1}{2})k}{N}\right) \\
&= \frac{1}{2} \sum_{i=1}^{N-1} \cos\left(\frac{\pi(i - \frac{1}{2})j}{N} - \frac{\pi(i - \frac{1}{2})k}{N}\right) + \cos\left(\frac{\pi(i - \frac{1}{2})j}{N} + \frac{\pi(i - \frac{1}{2})k}{N}\right) \\
&= \frac{1}{2} \sum_{i=1}^{N-1} \cos\left(\frac{\pi(2i-1)(j-k)}{2N}\right) + \cos\left(\frac{\pi(2i-1)(j+k)}{2N}\right) \\
&= \frac{1}{4} \left(\sin(\pi(j+k)) \csc\left(\frac{\pi(j+k)}{2N}\right) - \sin(\pi(k-j)) \csc\left(\frac{\pi(j-k)}{2N}\right) \right)
\end{aligned}$$

We know that $\sin(\pi m) = 0$, but $\csc(\pi m) = \infty$. Thus, the above term is non-zero only when $\frac{\pi(j+k)}{2N} = a\pi$ or $\frac{\pi(j-k)}{2N} = b\pi$ for some integer a and b . This condition could be neatened to

$$\begin{cases} j+k = 2aN \text{ or } j-k = 2bN \\ a, b \in \mathbb{Z} \\ 1 < j, k < N-1 \end{cases} \implies j = k$$

In this case, we aim to get the value of $\sin(\pi(j-k)) \csc\left(\frac{\pi(j-k)}{2N}\right)$ when $j = k$. Thus, we evaluate the following term

$$\lim_{x \rightarrow 0} \sin(\pi x) \csc\left(\frac{\pi x}{2N}\right) = \lim_{x \rightarrow 0} \frac{\sin(\pi x)}{\sin\left(\frac{\pi x}{2N}\right)} = 2N$$

Then, we have

$$\begin{aligned}
& \sum_{i=1}^{N-1} A_1(i, t) A_N(i, t) \\
&= \frac{N}{2} \sum_{k=1}^{N-1} \cos\left(\frac{\pi k(N - \frac{1}{2})}{N}\right) \cos^{2t}\left(\frac{\pi k}{2N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right) \\
&= \frac{N}{2} \sum_{k=1}^{N-1} \cos\left(\frac{\pi k(N - \frac{1}{2})}{N}\right) \cos^{4t+1}\left(\frac{\pi k}{2N}\right) \\
&= \frac{N}{4} \sum_{k=1}^{N-1} (\cos\left(\frac{\pi k(N - 1)}{N}\right) + \cos(\pi k)) \cos^{4t}\left(\frac{\pi k}{2N}\right)
\end{aligned}$$

analysis the first factor

$$\begin{aligned}
& \cos\left(\frac{\pi k(N - 1)}{N}\right) + \cos(\pi k) \\
&= \cos(\pi k) \cos\left(\frac{\pi k}{N}\right) + \sin(\pi k) \sin\left(\frac{\pi k}{N}\right) + \cos(\pi k) \\
&= \cos(\pi k) \left(\cos\left(\frac{\pi k}{N}\right) + 1 \right)
\end{aligned}$$

Then, we sum these terms pairly. Let $M = \lfloor \frac{N-1}{2} \rfloor$, $c = N - 1 - 2M \in \{0, 1\}$.

$$\begin{aligned}
& \sum_{i=1}^{N-1} A_1(i, t) A_N(i, t) \\
&= \frac{N}{4} \sum_{m=1}^M \left(\cos\left(\frac{2\pi m}{N}\right) + 1 \right) \cos^{4t}\left(\frac{\pi m}{N}\right) - \left(\cos\left(\frac{\pi(2m-1)}{N}\right) + 1 \right) \cos^{4t}\left(\frac{\pi(2m-1)}{2N}\right) \\
&\quad - c \frac{N}{4} \left(\cos\left(\frac{\pi(N-1)}{N}\right) + 1 \right) \cos^{4t}\left(\frac{\pi(N-1)}{2N}\right) \\
&= \frac{N}{2} \sum_{m=1}^M \left(\cos^{4t+2}\left(\frac{\pi m}{N}\right) - \cos^{4t+2}\left(\frac{\pi(2m-1)}{2N}\right) \right) \\
&\quad - c \frac{N}{2} \cos^{4t+2}\left(\frac{\pi(N-1)}{2N}\right)
\end{aligned}$$

where $\cos\left(\frac{\pi(N-1)}{2N}\right) = \sin\left(\frac{\pi}{2N}\right)$.

Lemma 1.3 (Test Pass):

$$B_N^{(1)}(p) \leq 2 \sum_{m=1}^M \frac{\pi}{N} \cos^{2p} \left(\frac{m\pi}{N} \right) \leq \sqrt{\pi} \frac{\Gamma(\frac{1}{2} + p)}{\Gamma(1 + p)} - B_N^{(2)}(p)$$

where Γ is gamma function, B is beta function, $B_N^{(1)}(p) = B(\cos^2(\frac{2\pi}{N}), \frac{1}{2} + p, \frac{1}{2})$, $B_N^{(2)}(p) = B(\sin^2(\frac{\pi}{N}), \frac{1}{2} + p, \frac{1}{2})$.

Proof: Because the $\cos^{2p}(x)$ is monotonic decreasing when $x \in [0, \frac{\pi}{2}]$, we have

$$\cos^{2p} \left(\frac{m\pi}{N} \right) \leq \frac{1}{2} \left(\cos^{2p} \left(\frac{2m\pi}{2N} \right) + \cos^{2p} \left(\frac{(2m-1)\pi}{2N} \right) \right)$$

$$\text{and } \cos^{2p} \left(\frac{m\pi}{N} \right) \geq \frac{1}{2} \left(\cos^{2p} \left(\frac{2m\pi}{2N} \right) + \cos^{2p} \left(\frac{(2m+1)\pi}{2N} \right) \right)$$

for $p \in \mathbb{Z}_+, 0 < m < \frac{N-1}{2}$. Similarly, we have

$$\cos^{2p} \left(\frac{m\pi}{N} \right) \leq \frac{1}{B} \sum_{b=0}^{B-1} \cos^{2p} \left(\frac{(Bm-b)\pi}{BN} \right)$$

$$\text{and } \cos^{2p} \left(\frac{m\pi}{N} \right) \geq \frac{1}{B} \sum_{b=0}^{B-1} \cos^{2p} \left(\frac{(Bm+b)\pi}{BN} \right)$$

Thus, we have

$$\sum_{m=1}^M \frac{\pi}{N} \cos^{2p} \left(\frac{m\pi}{N} \right) \leq \lim_{B \rightarrow \infty} \sum_{n=1}^{BM} \frac{\pi}{BN} \cos^{2p} \left(\frac{n\pi}{BN} \right)$$

$$\text{and } \sum_{m=1}^M \frac{\pi}{N} \cos^{2p} \left(\frac{m\pi}{N} \right) \geq \lim_{B \rightarrow \infty} \sum_{n=0}^{BM-1} \frac{\pi}{BN} \cos^{2p} \left(\frac{n\pi}{BN} + \frac{\pi}{N} \right)$$

Turn the summation to the integral, and using $M = \lfloor \frac{N-1}{2} \rfloor$

$$\begin{aligned}
& \lim_{B \rightarrow \infty} \sum_{n=1}^{BM} \frac{\pi}{BN} \cos^{2p} \left(\frac{n\pi}{BN} \right) \\
&= \int_0^{\frac{M\pi}{N}} \cos^{2p}(\kappa) d\kappa \\
&\leq \int_0^{\frac{\pi}{2} - \frac{\pi}{2N}} \cos^{2p}(\kappa) d\kappa \\
&= \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{1}{2} + p)}{\Gamma(1 + p)} - \frac{1}{2} B \left(\sin^2 \left(\frac{\pi}{2N} \right), \frac{1}{2} + p, \frac{1}{2} \right)
\end{aligned}$$

where Γ is gamma function, B is beta function. Similarly, we have

$$\begin{aligned}
& \lim_{B \rightarrow \infty} \sum_{n=0}^{BM-1} \frac{\pi}{BN} \cos^{2p} \left(\frac{n\pi}{BN} + \frac{\pi}{N} \right) \\
&\geq \int_{\frac{2\pi}{N}}^{\frac{\pi}{2}} \cos^{2p}(\kappa) d\kappa \\
&= \frac{1}{2} B \left(\cos^2 \left(\frac{2\pi}{N} \right), \frac{1}{2} + p, \frac{1}{2} \right)
\end{aligned}$$

Combine these results, we have

$$B_N^{(1)}(p) \leq 2 \sum_{m=1}^M \frac{\pi}{N} \cos^{2p} \left(\frac{m\pi}{N} \right) \leq \sqrt{\pi} \frac{\Gamma(\frac{1}{2} + p)}{\Gamma(1 + p)} - B_N^{(2)}(p)$$

□

Corollary 1.3.1 (*Test Pass*):

$$B_N^{(1)}(p) \leq 2 \sum_{m=1}^M \frac{\pi}{N} \cos^{2p} \left(\frac{(2m-1)\pi}{2} N \right) \leq \sqrt{\pi} \frac{\Gamma(\frac{1}{2} + p)}{\Gamma(1 + p)} - B_N^{(2)}(p)$$

Proof: The proof is similar to the Lemma 1.3. □

$N = 40$

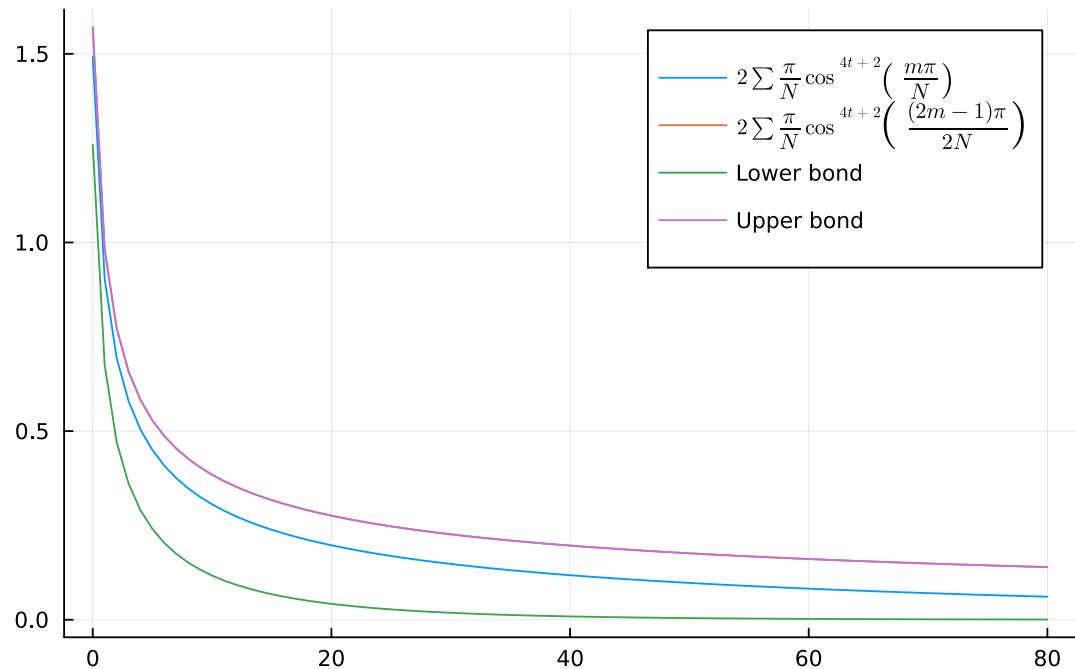


Figure 1: test Lemma 1.3. We could see that the upper bond is very tight.