In[56]:=
```
$Assumptions Element[x, PositiveIntegers]
```
默认假设 属于 正整数
```
$Assumptions Element[t, PositiveIntegers]
```
默认假设 属于 正整数
```
$Assumptions Element[N, PositiveIntegers]
```
默认假设 属于 ⋯ 正整数

```
P[i_, t_] := 1 / N + 2 / N * Sum[
    Cos[(i - 1 / 2) * Pi * k / N] * Cos[ Pi * k / N / 2] ^ (2 * t + 1),
    {k, 1, N - 1}
  ]
```
数值运算 ⋯ 求和
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数值运算

Out[56]=

True $(x \in \mathbb{Z} \&\& x > 0)$

Out[57]=

True $(t \in \mathbb{Z} \&\& t > 0)$

Out[58]=

True $(N \in \mathbb{Z} \&\& N > 0)$

Now we consider the difference of P(t) and P(t + 1) . We want to see the converge rate of P .

The difference of P $= \frac{2 \sum_{k=1}^{-1+N} \text{Cos}\left[\frac{k\pi}{2N}\right]^{1+2t} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right]}{N} - \frac{2 \sum_{k=1}^{-1+N} \text{Cos}\left[\frac{k\pi}{2N}\right]^{1+2(1+t)} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right]}{N}$,

while the middle term $\sum_{k=1}^{-1+N} \text{Cos}\left[\frac{k\pi}{2N}\right]^{1+2t} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right] - \text{Cos}\left[\frac{k\pi}{2N}\right]^{1+2(1+t)} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right]$ could be simplified.

In[5]:= $\text{Cos}\left[\frac{k\pi}{2N}\right]^{1+2t} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right] - \text{Cos}\left[\frac{k\pi}{2N}\right]^{1+2(1+t)} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right]$ // Simplify

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Out[5]= $\text{Cos}\left[\frac{k\pi}{2N}\right]^{1+2t} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right] \text{Sin}\left[\frac{k\pi}{2N}\right]^{2}$

Thus, we consider the Pdiff as

In[8]:= Pdiff[x_, t_] := $\frac{2}{N}\sum_{k=1}^{-1+N} \text{Cos}\left[\frac{k\pi}{2N}\right]^{1+2t} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right] \text{Sin}\left[\frac{k\pi}{2N}\right]^{2}$

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termInPdiff[k_, x_, t_] := $\text{Cos}\left[\frac{k\pi}{2N}\right]^{1+2t} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right] \text{Sin}\left[\frac{k\pi}{2N}\right]^{2}$

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$$\left(\sum_{k=1}^{-1+N} \text{Cos}\left[\frac{k\pi}{2N}\right]^{1+2t} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right] \text{Sin}\left[\frac{k\pi}{2N}\right]^{2}\right)^{2} \leq$$

$$\left(\sum_{k=1}^{-1+N} \text{Cos}\left[\frac{k\pi}{2N}\right]^{2+4t}\right) \left(\sum_{k=1}^{-1+N} \text{Cos}\left[\frac{k\pi\left(-\frac{1}{2}+x\right)}{N}\right]^{2} \text{Sin}\left[\frac{k\pi}{2N}\right]^{4}\right)$$

Thus, we want to bond the l.h.s by the r.h.s.

In[10]:=

$$\sum_{k=1}^{-1+N} \text{Cos}\left[\frac{k\,\pi}{2\,N}\right]^{2+4\,t}$$

$$\sum_{k=1}^{-1+N} \text{Cos}\left[\frac{k\,\pi\,\left(-\frac{1}{2}+x\right)}{N}\right]^{2} \text{Sin}\left[\frac{k\,\pi}{2\,N}\right]^{4}$$

Out[10]=

$$\sum_{k=1}^{-1+N} \text{Cos}\left[\frac{k\,\pi}{2\,N}\right]^{2+4\,t}$$

In[51]:= " The result of the second term is"

$$\text{sdTerm}[x\_] := \frac{1}{64}\left(-12+12\,N+8\,\text{Cos}\left[\frac{(-1+N)\,\pi}{2\,N}\right]\,\text{Csc}\left[\frac{\pi}{2\,N}\right]+\right.$$

$$8\,\text{Cos}\left[\frac{(1+N)\,\pi}{2\,N}\right]\,\text{Csc}\left[\frac{\pi}{2\,N}\right]-2\,\text{Cos}\left[\frac{(-2+N)\,\pi}{2\,N}\right]\,\text{Csc}\left[\frac{\pi}{N}\right]+2\,\text{Cos}\left[\frac{(2+N)\,\pi}{2\,N}\right]\,\text{Csc}\left[\frac{\pi}{N}\right]+$$

$$\left.4\,\text{Cos}\left[\frac{-N\,\pi+2\,\pi\,x}{2\,N}\right]\,\text{Csc}\left[\frac{\pi\,x}{N}\right]-4\,\text{Cos}\left[\frac{-N\,\pi-2\,\pi\,x+4\,N\,\pi\,x}{2\,N}\right]\,\text{Csc}\left[\frac{\pi\,x}{N}\right]\right)$$

sdTerm[1 / 2]

Out[51]=

The result of the second term is

Out[53]=

$$\frac{1}{64}\left(-12+12\,N+8\,\text{Cos}\left[\frac{(-1+N)\,\pi}{2\,N}\right]\,\text{Csc}\left[\frac{\pi}{2\,N}\right]+\right.$$

$$8\,\text{Cos}\left[\frac{(1+N)\,\pi}{2\,N}\right]\,\text{Csc}\left[\frac{\pi}{2\,N}\right]+4\,\text{Cos}\left[\frac{\pi-N\,\pi}{2\,N}\right]\,\text{Csc}\left[\frac{\pi}{2\,N}\right]-4\,\text{Cos}\left[\frac{-\pi+N\,\pi}{2\,N}\right]\,\text{Csc}\left[\frac{\pi}{2\,N}\right]-$$

$$\left.2\,\text{Cos}\left[\frac{(-2+N)\,\pi}{2\,N}\right]\,\text{Csc}\left[\frac{\pi}{N}\right]+2\,\text{Cos}\left[\frac{(2+N)\,\pi}{2\,N}\right]\,\text{Csc}\left[\frac{\pi}{N}\right]\right)$$

We want the upper bond of the second term. And we know $0 < \csc\left(\frac{\pi x}{2N}\right) \leq 1$ when $N \geq 1$, Thus, the upper term is less than

In[68]:=

```
D[sdTerm[x], x] * 64 N // Simplify
```

Out[68]=

$$4\,\pi\,\text{Csc}\left[\frac{\pi\,x}{N}\right]^{2}\left(-\left((-1+N)\,\text{Sin}[2\,\pi\,x]\right)+N\,\text{Sin}\left[\frac{2\,(-1+N)\,\pi\,x}{N}\right]\right)$$

In[79]:= $f[x\_] := (N-1)\ \mathrm{Sin}[2\ \pi\ x] + N\ \mathrm{Sin}\left[\frac{2\ (-1+N)\ \pi\ x}{N}\right]$
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$f[N/2]$
数值运算

Out[80]=

$N\ \mathrm{Sin}[(-1+N)\ \pi] + (-1+N)\ \mathrm{Sin}[N\ \pi]$

In[82]:= sdTerm[N / 2]
数值运算

Out[82]=

$$\frac{1}{64}\left(-8+12\ N-4\ \mathrm{Cos}\left[\frac{-2\ N\ \pi+2\ N^2\ \pi}{2\ N}\right]+8\ \mathrm{Cos}\left[\frac{(-1+N)\ \pi}{2\ N}\right]\ \mathrm{Csc}\left[\frac{\pi}{2\ N}\right]+8\ \mathrm{Cos}\left[\frac{(1+N)\ \pi}{2\ N}\right]\ \mathrm{Csc}\left[\frac{\pi}{2\ N}\right]-2\ \mathrm{Cos}\left[\frac{(-2+N)\ \pi}{2\ N}\right]\ \mathrm{Csc}\left[\frac{\pi}{N}\right]+2\ \mathrm{Cos}\left[\frac{(2+N)\ \pi}{2\ N}\right]\ \mathrm{Csc}\left[\frac{\pi}{N}\right]\right)$$

In[83]:=

$$\frac{1}{64}\left(-8+12\ N-4\ \mathrm{Cos}\left[\frac{-2\ N\ \pi+2\ N^2\ \pi}{2\ N}\right]+8\ \mathrm{Cos}\left[\frac{(-1+N)\ \pi}{2\ N}\right]\ \mathrm{Csc}\left[\frac{\pi}{2\ N}\right]+8\ \mathrm{Cos}\left[\frac{(1+N)\ \pi}{2\ N}\right]\ \mathrm{Csc}\left[\frac{\pi}{2\ N}\right]-2\ \mathrm{Cos}\left[\frac{(-2+N)\ \pi}{2\ N}\right]\ \mathrm{Csc}\left[\frac{\pi}{N}\right]+2\ \mathrm{Cos}\left[\frac{(2+N)\ \pi}{2\ N}\right]\ \mathrm{Csc}\left[\frac{\pi}{N}\right]\right)\ //\ \mathrm{Simplify}$$

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$$\frac{1}{16}\ (-3+3\ N+\mathrm{Cos}[N\ \pi])$$
… 余弦 数值运算

So the second term is less than $\frac{1}{16}\,(-2+3\,N)$

When it comes to the first term, we make the following approximation:

$\sum_{k=1}^{-1+N} \mathrm{Cos}\left[\frac{k\,\pi}{2\,N}\right]^{2+4\,t} \to \frac{N}{\pi}\sum_{k=1}^{-1+N}\frac{\pi}{N}\ \mathrm{Cos}\left[\frac{k\,\pi}{2\,N}\right]^{2+4\,t} \to \frac{N}{\pi}\int d\kappa\ \mathrm{Cos}\left[\frac{\kappa}{2}\right]^{2+4\,t}$

In[84]:= $\int_0^\pi \mathrm{Cos}\left[\frac{\kappa}{2}\right]^{2+4\,t} d\kappa$

Out[84]=

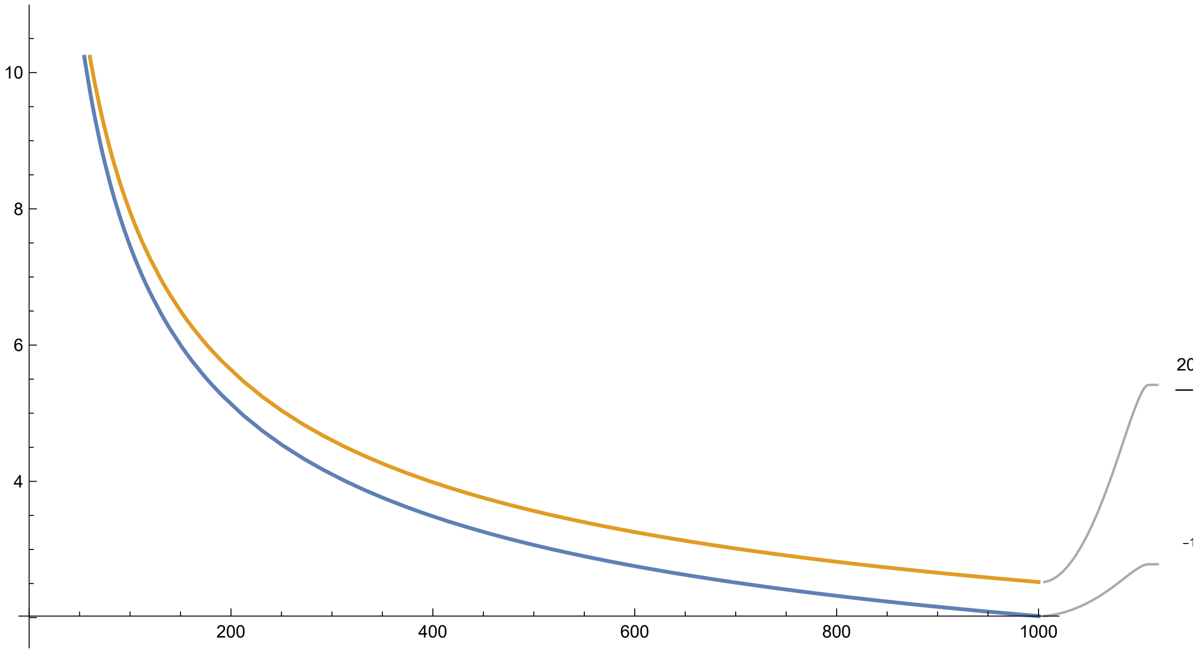
$$\frac{\sqrt{\pi}\ \mathrm{Gamma}\left[\frac{3}{2}+2\,t\right]}{\mathrm{Gamma}[2+2\,t]}\quad \text{if } \mathrm{Re}[t] > -\frac{3}{4}$$

Thus, the term $\sum_{k=1}^{-1+N} \mathrm{Cos}\left[\frac{k\,\pi}{2\,N}\right]^{2+4\,t}$ could be approximated by $\frac{N}{\sqrt{\pi}}\ \frac{\mathrm{Gamma}\left[\frac{3}{2}+2\,t\right]}{\mathrm{Gamma}[2+2\,t]}$

In[89]:= $$\text{Plot}\left[\left\{\sum_{k=1}^{-1+200}\text{Cos}\left[\frac{k\,\pi}{2*200}\right]^{2+4\,t},\ \frac{200}{\text{Pi}}*\frac{\sqrt{\pi}\ \text{Gamma}\left[\frac{3}{2}+2\,t\right]}{\text{Gamma}[2+2\,t]}\right\},\ \{t,10,1000\},\ \text{PlotLabels}\to\text{"Expressions"}\right]$$

绘图 数据绘制标签

Out[89]=



In[91]:= $$\text{Series}\left[\frac{\text{Gamma}\left[\frac{3}{2}+\frac{2}{w}\right]}{\text{Gamma}\left[2+\frac{2}{w}\right]},\ \{w,0,2\}\right]$$

Out[91]=

$$\text{RefLink}\left[\text{Series},\ \text{paclet}:\frac{\text{ref}}{\text{Series}}\right]\left[\frac{\text{Gamma}\left[\frac{3}{2}+\frac{2}{w}\right]}{\text{Gamma}\left[2+\frac{2}{w}\right]},\ \{w,0,2\}\right]$$

In[100]:=

$$\text{Series}\left[\frac{\text{Gamma}\left[\frac{3}{2}+\frac{2}{w}\right]}{\text{Gamma}\left[2+\frac{2}{w}\right]},\ \{w,0,2\},\ \text{Assumptions}\to w>0\right]$$

级数 假设

Out[100]=

$$\frac{\sqrt{w}}{\sqrt{2}}-\frac{5\,w^{3/2}}{16\,\sqrt{2}}+O[w]^{5/2}$$

In[97]:= `Series[Gamma[x], {x, 0, 2}]`
级数 伽玛函数

Out[97]=

$$\frac{1}{x} - \text{EulerGamma} + \frac{1}{12}\left(6\,\text{EulerGamma}^2 + \pi^2\right) x + \frac{1}{6}\left(-\text{EulerGamma}^3 - \frac{\text{EulerGamma}\,\pi^2}{2} + \text{PolyGamma}[2, 1]\right) x^2 + O[x]^3$$