Calculate α

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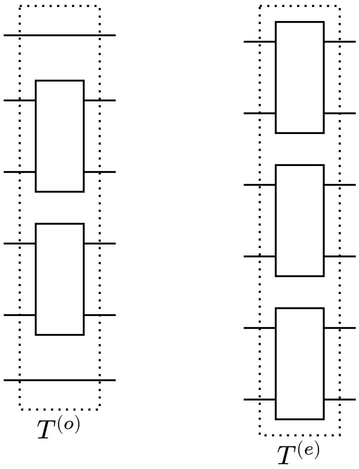
1. The tensor and the vector space

Let $T^{(o)}$ be the odd layer of T gates, and $T^{(e)}$ be the even layer of T gates. Then we have the following circuits:



Now, alternately apply the odd and even layers of T gates to the $|\gamma_1\gamma_{2n}\rangle\rangle$

$$T^{(\text{whole})}(b_1,t) = {T^{(o)}}^{b_2}\left(\prod_{i=0}^{t-b_2} T^{(e)}T^{(o)}\right){T^{(e)}}^{b_1},$$

where $b_1, b_2 \in \{0,1\}$, $t+b_1$ stands for the number of layers. The output states must in the vector space spaned by the following basis

$$|Z_i\rangle\rangle, \quad \left|X_i\left(\prod_{k=i+1}^{j-1} Z_k\right)Y_j\right\rangle\rangle$$

Let $M_i = \frac{1}{\sqrt{2}}(X_i + Y_i)$, the space

$$V := \operatorname{span}\left\{|Z_i\rangle\rangle, \ \left|M_i\left(\prod Z_k\right)M_j\right\rangle\rangle\right\}$$

is the image subspace for all $T^{(\text{whole})}(b_1,t)$ with $b_1+t>0$ if the input state is limit to Γ_2. Thus, the action of $T^{(\text{whole})}(b_1,t)$ could always be written in the following form

$$T^{(\text{whole})}(b_1,t)|\gamma_1\gamma_{2n}\rangle\rangle = \sum P(i,i,t)|Z_i\rangle\rangle + \sum_{i<j} P(i,j,t)\left|M_i\left(\prod Z_k\right)M_j\right\rangle\rangle.$$

The action of the tensor could be simplified by studying the coefficients $P(i,j,t)$.

2. Propagator

To simplify the discussion, we start with a special case: $b_1 = 1, b_2 = 0$, and the number of wires is an even number $2N$. In this case, the whole tensor $T^{(\text{whole})}$ could be written as

$$T^{(\text{whole})}(1,t) = \left(T^{(e)}T^{(o)}\right)^t T^{(e)}.$$

Then, we lift the V into the space of the second order 2N-dimensional polynomials

$$W := \operatorname{span}\left\{\sum_{i,j=1}^{2N} c_{i,j} x_i x_j\right\}$$

We can prove that the space V is isometric to the space W by ϕ,

$$\begin{aligned} \phi : V &\to W \\ \left|M_i\left(\prod Z_k\right)M_j\right\rangle\!\!\rangle &\to x_i x_j \\ |Z_i\rangle\!\!\rangle &\to x_i^2. \end{aligned}$$

Let $T_p^{(2N)}(t)$ be the map $\phi T^{(\text{whole})}(1,t)\phi^{-1}$, the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "T^{(\text{whole})}(1,t)"] \arrow[d, leftrightarrow, "\phi"] & V \arrow[d, leftrightarrow, "\phi"] \\ W \arrow[r, "T_p^{(2N)}(t)"] & W \end{tikzcd}$$

Now, let's consider the action of the tensor $T_p^{(2N)}(t)$ on the space W. Similarly, we could write down the recursive relation of coefficients $a_{i,j}$.

When $t = 0$, the transforming state $T^{(e)}|\gamma_1\gamma_{4N}\rangle\!\!\rangle$ to W, and we got $\frac{1}{4}(x_1 x_{2N} + x_1 x_{2N-1} + x_2 x_{2N} + x_1 x_{2N-1})$. Suppose at t, the vector in W is $\sum c_i(t) c'_j(t) x_i x_j$,

$$\begin{aligned} &T_p^{(2N)}(t+1)\left(\frac{1}{4}(x_1 x_{2N} + x_1 x_{2N-1} + x_2 x_{2N} + x_1 x_{2N-1})\right) \\ &= T^{(e)}T^{(o)}T_p^{(2N)}(t)\left(\frac{1}{4}(x_1 x_{2N} + x_1 x_{2N-1} + x_2 x_{2N} + x_1 x_{2N-1})\right) \\ &= T^{(e)}T^{(o)}\sum c_i(t) c'_j(t) x_i x_j \end{aligned}$$

Then, the action of $\phi\, T^{(e)}T^{(o)}\phi^{-1}$ on this vector is

$c_i c_j' x_i x_j$	$\phi\, T^{(e)}T^{(o)}\phi^{-1}(c_i c_j' x_i x_j)$
$i=j=1$	$\frac{1}{6}(c_1x_1+c_1x_2)(c_2'x_1+c_2'x_2)$
$i=1, j=2,3$	$\frac{1}{8}(c_1x_1+c_1x_2)\left(\frac{4}{3}c_2'x_1+\frac{4}{3}c_2'x_2+c_2'x_3+c_2'x_4\right)$
$i=1, j>3$	$\frac{1}{8}(c_1x_1+c_1x_2)\Big(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}+ c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\Big)$
$1<i<2N$, $i=j$	$\frac{1}{24}\Big(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}+c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\Big) \Big(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}+c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\Big) -\frac{1}{72}\Big(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}\Big)\Big(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}\Big) -\frac{1}{72}\Big(c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\Big)\Big(c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\Big)$
$1<i<2N$, i is even, $j=i+1$	$\frac{1}{12}\Big(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}+c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\Big) \Big(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}+c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\Big) -\frac{1}{36}\Big(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}\Big)\Big(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}\Big) -\frac{1}{36}\Big(c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\Big)\Big(c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\Big)$
$1<i<2N$, $i+\eta_i\le j\le i+\eta_i+2$	$\frac{1}{16}\Big(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}+c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\Big) \Big(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}+c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\Big) +\frac{1}{48}\Big(c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\Big)\Big(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}\Big)$
$1<i<2N$, $j>i+\eta_i+2$	$\frac{1}{16}\Big(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}+c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\Big) \Big(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}+c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\Big)$
$i=j=2N$	$\frac{1}{6}(c_{2N}x_{2N-1}+c_{2N}x_{2N})(c_{2N}'x_{2N-1}+c_{2N}'x_{2N})$

Table 1: The action of the tensor $\phi\, T^{(e)}T^{(o)}\phi^{-1}$ on the space of second order 2N-dimensional polynomials.

The subsript η_i is defined as $\eta_i := 1 - (i \bmod 2)$. We can see that, c_{2i-1} and c_{2i} are always the same. So do c'_{2j-1} and c'_{2j}. Let $b_i = 2c_{2i-1} = 2c_{2i}$ and $b'_j = 2c'_{2j-1} = 2c'_{2j}$, we could further simplify the action of the tensor in the space of second order N-dimensional polynomials W_N. For simplemess, we define a "free" recursive relation in W_N

$$P(i, t+1) = \begin{cases} \frac{1}{4}(P(i-1,t) + 2P(i,t) + P(i+1,t)) \ , i \neq 1 \text{ or } N \\ \frac{1}{4}(P(i-1,t) + 2P(i,t) + P(i+1,t)) \end{cases}. \tag{1}$$

We call it "free" because $b_i(t)b'_j(t) = P(i,t)P(j,t)$ if i and j are not "collide" with each other (which means $|i-j| > 3$). And the solution of Eq. (1) is a propagating wave. Refs. [1] provide the solution of this equation,

$$P_{n_0}(n,t) = \frac{1}{N} + \frac{2}{N}\sum_{k=1}^{N-1} \cos\left(\left(n - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos\left(\left(n_0 - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t}\frac{\pi k}{2N}, \tag{2}$$

where n_0 is the initial state. The term $P_{n_0}(n,t)$ also called the propagator.

For our case, there are 2 propagators, which are P_1 and P_N. And the initial state is $x_1 x_N$. Let $\phi()$

Now, let's pluge $b_i = 2c_{2i-1} = 2c_{2i}$ and $b'_j = 2c'_{2j-1} = 2c'_{2j}$ into Table 1. More concretely, let ϕ' be the map

$$\begin{gathered} \phi' : W_N \to W_{2N} \\ y_i y_j \to \frac{1}{4}(x_{2i-1} + x_{2i})\big(x_{2j-1} + x_{2j}\big). \end{gathered}$$

If we consider the subspace W'_{2N} of W_{2N}, where $W'_{2N} := \operatorname{span}\big((x_{2i-1} + x_{2i})\big(x_{2j-1} + x_{2j}\big)\big)$ (it means $c_{2i-1} = c_{2i}$ and $c'_{2j-1} = c'_{2j}$), the ϕ' will be the isometric between W_N and W'_{2N}. Thus, the following diagram commutes.

$$
\begin{tikzcd}
V \arrow[r, "T^{(\text{whole})}(1,t)"] & V \\
W'_{2N} \arrow[u, leftrightarrow, "\phi"] \arrow[r, "T_p^{(2N)}(t)"] & W'_{2N} \arrow[u, leftrightarrow, "\phi"] \\
W_N \arrow[u, leftrightarrow, "\phi'"] \arrow[r, "T_p^{(N)}(t)"] & W_N \arrow[u, leftrightarrow, "\phi'"]
\end{tikzcd}
$$

For simpliness, let

$$
F(y_i) = \begin{cases} \frac{3}{4}y_1 + \frac{1}{4}y_2 \ , i = 1 \\ \frac{1}{4}y_{i-1} + \frac{1}{2}y_i + \frac{1}{4}y_{i+1} \ , 1 < i < N \\ \frac{3}{4}y_N + \frac{1}{4}y_{N-1} \ , i = N \end{cases}
$$

Then we get

$y_i y_j$	$T^{(e)}T^{(o)}$
$i = j = 1$	$F(y_1)F(y_1) - \frac{5}{144}(y_1 + y_2)^2 + \frac{1}{36}y_1 y_2$

这个 table 的计算实在是太太太太太折磨人了。详细计算我放在了 Section 3 中。

3. Proofs

Proof of Table 2: 现在我们想要计算 $T_p^{(N)}(t)$ 在W_N 中的作用。因此我们需要计算

$$
\begin{aligned}
&\phi'\phi T^{(e)}T^{(o)}\phi^{-1}\phi'^{-1}(y_i y_j) \\
= &\phi T^{(e)}T^{(o)}\phi^{-1}\left(\frac{1}{4}(x_{2i-1} + x_{2i})(x_{2j-1} + x_{2j})\right) \\
= &\phi T^{(e)}T^{(o)}\phi^{-1}\left(\frac{1}{4}(x_{2i-1}x_{2j-1} + x_{2i-1}x_{2j} + x_{2i}x_{2j-1} + x_{2i}x_{2j})\right)
\end{aligned}
$$

相当于对于每一个 $x_{2i-1}x_{2j-1}$ 这样的项进行查表。我们现在来分析不同的 y 的映射。为了简化符号，这里记 $S_N = \phi'\phi T^{(e)}T^{(o)}\phi^{-1}\phi'^{-1}$，$S_{2N} = \phi T^{(e)}T^{(o)}\phi^{-1}$.

$$\phi'(y_1 y_1) = \frac{1}{4}(x_1 x_1 + 2x_1 x_2 + x_2 x_2)$$

$$S_N(y_1 y_1) = \frac{1}{4}(S_{2N}(x_1 x_1) + S_{2N}(2x_1 x_2) + S_{2N}(x_2 x_2))$$

依据 Table 1, 我们有

$$S_N(y_1 y_1) = \frac{1}{4}\Bigg($$

$$\frac{1}{6}(x_1 + x_2)(x_1 + x_2) +$$

$$\left(\frac{1}{4}\right)\left(\left(\frac{4}{3}\right)x_1 + \left(\frac{4}{3}\right)x_2 + x_3 + x_4\right)(x_1 + x_2) +$$

$$-\left(\frac{1}{72}\right)\left((x_1 + x_2)^2\right) + \left(\frac{1}{24}\right)\left((x_1 + x_2 + x_3 + x_4)^2\right) - \left(\frac{1}{72}\right)\left((x_3 + x_4)^2\right)$$

$$\Bigg)$$

$$\left(\frac{19}{144}\right)(x_1{}^2) + \left(\frac{19}{72}\right)x_1 x_2 + \left(\frac{1}{12}\right)x_1 x_3 + \left(\frac{1}{12}\right)x_1 x_4$$

$$+\left(\frac{19}{144}\right)(x_2{}^2) + \left(\frac{1}{12}\right)x_2 x_3 + \left(\frac{1}{12}\right)x_2 x_4$$

$$+\left(\frac{1}{144}\right)(x_3{}^2) + \left(\frac{1}{72}\right)x_3 x_4 + \left(\frac{1}{144}\right)(x_4{}^2)$$

我们想要和“自由演化”进行对比。而 $y_1 y_1$ 自由演化的结果是

$$\begin{aligned}
F(y_1 y_1) &= \frac{1}{16}(3y_1 + y_2)(3y_1 + y_2) \\
&= \frac{1}{16}\left(\frac{3}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right)\left(\frac{3}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right) \\
&= \left(\frac{9}{64}\right)(x_1{}^2) + \left(\frac{9}{32}\right)x_1 x_2 + \left(\frac{3}{32}\right)x_1 x_3 + \left(\frac{3}{32}\right)x_1 x_4 \\
&\quad + \left(\frac{9}{64}\right)(x_2{}^2) + \left(\frac{3}{32}\right)x_2 x_3 + \left(\frac{3}{32}\right)x_2 x_4 + \left(\frac{1}{64}\right)(x_3{}^2) \\
&\quad + \left(\frac{1}{32}\right)x_3 x_4 + \left(\frac{1}{64}\right)(x_4{}^2)
\end{aligned}$$

二者之差为

$$
\begin{aligned}
S_N(y_1y_1) - F(y_1y_1) = &-\left(\frac{5}{576}\right)({x_1}^2) - \left(\frac{5}{288}\right)x_1x_2 - \left(\frac{1}{96}\right)x_1x_3 - \left(\frac{1}{96}\right)x_1x_4 \\
&-\left(\frac{5}{576}\right)({x_2}^2) - \left(\frac{1}{96}\right)x_2x_3 - \left(\frac{1}{96}\right)x_2x_4 \\
&-\left(\frac{5}{576}\right)({x_3}^2) - \left(\frac{5}{288}\right)x_3x_4 - \left(\frac{5}{576}\right)({x_4}^2) \\
= &-\frac{5}{576}(x_1 + x_2 + x_3 + x_4)^2 + \frac{1}{144}(x_1 + x_2)(x_3 + x_4) \\
= &-\frac{5}{144}(y_1 + y_2)^2 + \frac{1}{36}y_1y_2
\end{aligned}
$$

接下来，算 $S_N(y_1y_2)$

$$
S_N(y_1y_2) = \frac{1}{4}(S_{2N}(x_1x_3) + S_{2N}(x_1x_4) + S_{2N}(x_2x_3) + S_{2N}(x_2x_4))
$$

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Bibliography

[1] L. Giuggioli, “Exact spatiotemporal dynamics of confined lattice random walks in arbitrary dimensions: A century after Smoluchowski and Pólya,” *Physical Review X*, vol. 10, no. 2, p. 21045–21046, 2020.