I. SAMPLING UNIFORMLY FROM GAUSSIAN CLIFFORDS

As described in Supplementary Appendix **??**, the matchgate group $\mathbb{M}_n$, modulo phases, is isomorphic to the continuous Lie group $\mathcal{O}(2n)$, i.e.

$$\mathbb{M}_n/\{e^{i\alpha}\}_{\alpha\in\mathbb{R}} \cong \mathcal{O}(2n). \tag{1}$$

The isomorphism is given by how a matchgate unitary acts on the Majorona operators. For practical purposes, we restrict ourselves to the discrete subgroup of Gaussian Cliffords, whose adjoint action maps Majorana operators to signed Majorana operators, $U\gamma_i U^\dagger = \pm\gamma_j$. In other words, the subgroup of Gaussian Cliffords is isomorphic to the subgroup of signed permutation matrices $\mathrm{Sym}(2,2n) \subset \mathcal{O}(2n)$. This restriction is justified, since the Gaussian Cliffords are a 3-design for the matchgate group, and they thus have matching third moments [**?**]. We now turn to the task of sampling Haar random Gaussian Cliffords. To this effect, we will identify the n-mode Fock space with n qubits, invoking the Jordan-Wigner isomorphism. Under the Jordan-Wigner isomorphism, the Majorana operators map to orthonormal Pauli strings, in particular, CITE HELSEN

$$\begin{aligned}\gamma_{2j-1} &= Z_1 \dots Z_{j-1} X_j I_{j+1} \dots I_n,\\ \gamma_{2j} &= Z_1 \dots Z_{j-1} Y_j I_{j+1} \dots I_n,\end{aligned}$$

for $j \in [n]$.

To accomplish our sampling task, we may draw a uniformly random permutation $\pi \in \mathrm{Sym}(2n)$, decomposed into a product of nearest-neighbor transpositions, which in turn can be easily translated into matchgate unitaries. This is permissible, as by the group homomorphism property, composition within $\mathrm{Sym}(2n)$ is compatible with the composition of the corresponding local matchgates. The so-called (modern) Fisher-Yates shuffle CITE KNUTH is an efficient algorithm that returns a uniformly random permutation of $[2n]$. Lastly, we must randomly apply reflections that map $\gamma_i \mapsto -\gamma_i$ $\forall i \in [2n]$, which lifts our permutation π to a signed permutation characterizing the adjoint action of the Gaussian Clifford.

Finally, we detail how to translate nearest-neighbour transpositions into the corresponding matchgate unitaries. Naturally, as is the case for the Majorana operators, there is a distinction between the odd and even cases:

For $j \in [n]$, transposing γ_{2j-1} and γ_{2j} (here, the lower index is odd) is accomplished by the adjoint action of the matchgate unitary

$$U = I_1 \dots I_{j-1} \left[Y_j \exp\left(-i\frac{\pi}{4} Z_j\right)\right] Z_{j+1} \dots Z_n, \tag{2}$$

whereas to transpose γ_{2j} and γ_{2j+1} (here, the lower index is even),

$$U = I_1 \dots I_{j-1} \left[Y_{j+1} \exp\left(-i\frac{\pi}{4} X_j X_{j+1}\right)\right] Z_{j+2} \dots Z_n \tag{3}$$

can be applied. For reflections $\gamma_{2j} \mapsto -\gamma_{2j}$, consider

$$U = I_1 \dots I_{j-1} X_j Z_{j+1} \dots Z_n, \tag{4}$$

and for reflections $\gamma_{2j-1} \mapsto -\gamma_{2j-1}$

$$U = I_1 \dots I_{j-1} Y_j Z_{j+1} \dots Z_n. \tag{5}$$

Note that in practice one can efficiently commute through all highly non-local Pauli Z-strings to the very left-hand side, giving us a decomposition in terms of 2-local unitaries – the final Pauli Z string does not affect the statistics of computational basis measurements, since the operator is diagonal in this basis.