Analysis the order of PP

2024-06-25

Wu Bujiao

Bian Kaiming

1. The order of PP

Theorem 1.1: Let $\mathcal{A}_t := -\sum_{i=1}^{N-1} A_1(i,t)A_N(i,t)$. Suppose N is even, $\exists f_t(N) \in \mathcal{O}(\frac{1}{N})$, s.t. $\mathcal{A}_T \leq f_t$ when $t \in \mathcal{O}(N)$.

Proof: Expand the $\mathcal{A}_t$ we have

$$\mathcal{A}_t = \frac{N}{2}\left[\sum_{k=1}^{N/2} \cos^{4t+2}\left(\frac{(2k-1)\pi}{2N}\right) - \sum_{k=1}^{N/2-1} \cos^{4t+2}\left(\frac{k\pi}{N}\right)\right]$$

Due to Lemma 1.2, we have

$$\begin{aligned}
&\sum_{k=1}^{N/2} \cos^{4t+2}\left(\frac{(2k-1)\pi}{2N}\right) - \sum_{k=1}^{N/2-1} \cos^{4t+2}\left(\frac{k\pi}{N}\right) \\
&\leq \frac{N}{\pi}\sum_{k=1}^{2t+1} \frac{(4t+1)!!}{(4t+2)!!}\frac{(2k-2)!!}{(2k-1)!!}\Bigg[\\
&\quad \sin\left(\frac{(2x-1)\pi}{2N}\right)\cos^{2k-1}\left(\frac{(2x-1)\pi}{2N}\right)\Bigg|_0^M \\
&\quad -\sin\left(\frac{x\pi}{N}\right)\cos^{2k-1}\left(\frac{x\pi}{N}\right)\Bigg|_1^{M+1}\Bigg]
\end{aligned} \tag{1}$$

The difference in the square brackets is equal to

$$\begin{aligned}
C = {} & \cos\left(\frac{\pi}{2N}\right)^{-1+2k}\sin\left(\frac{\pi}{2N}\right) \\
& +\cos\left(\frac{\pi}{N}\right)^{-1+2k}\sin\left(\frac{\pi}{N}\right) \\
& +\cos\left(\frac{3\pi}{2N}\right)\sin\left(\frac{3\pi}{2N}\right)^{-1+2k}
\end{aligned}$$

Now we try to estimate the order of C

$$C = \begin{cases} \frac{3\pi}{2N} + \frac{3}{8}(1-3k)\frac{\pi^3}{N^3} + \mathcal{O}\left(\frac{1}{N^4}\right), & k > 1 \\ \frac{3\pi}{N} - \frac{3\pi^3}{N^3} + \mathcal{O}\left(\frac{1}{N^4}\right), & k = 1 \end{cases}$$

Substituting the above result into Eq. (1), we have

$$\begin{aligned} &\sum_{k=1}^{N/2} \cos^{4t+2}\left(\frac{(2k-1)\pi}{2N}\right) - \sum_{k=1}^{N/2-1} \cos^{4t+2}\left(\frac{k\pi}{N}\right) \\ &\leq \frac{(4t+1)!!}{(4t+2)!!} \sum_{k=2}^{2t+1} \frac{(2k-2)!!}{(2k-1)!!}\left(\frac{3}{2} + \frac{3}{8}(1-3k)\frac{\pi^2}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)\right) + 3\frac{(4t+1)!!}{(4t+2)!!} \end{aligned}$$

Using Lemma 1.3, we have

$$\begin{aligned} \mathcal{A}_t &\leq \frac{N}{2\sqrt{2\pi t}} \sum_{k=2}^{2t+1} \left[\frac{\sqrt{\pi(k-1)}}{2k-1}\left(\frac{3}{2} + \frac{3}{8}(1-3k)\frac{\pi^2}{N^2} + \mathcal{O}\left(\frac{1}{N^3}\right)\right)\right] \\ &\quad + \frac{3N}{2\sqrt{2\pi t}} + \mathcal{O}(e^{-t}) \\ &\leq \frac{3N}{8\sqrt{2t}} \sum_{k=1}^{2t} \frac{1}{\sqrt{k}} - \frac{9\pi^2}{32N\sqrt{2t}} \sum_{k=2}^{2t+1} \frac{k}{\sqrt{k-1}} + \frac{3N}{2\sqrt{2\pi t}} + \mathcal{O}\left(\frac{1}{N}, e^{-t}\right) \\ &= \frac{3}{8\sqrt{2t}}\left(N - \frac{3\pi^2}{4N}\right) \sum_{k=1}^{2t} \frac{1}{\sqrt{k}} - \frac{9\pi^2}{32N\sqrt{2t}} \sum_{k=1}^{2t} \sqrt{k} + \frac{3N}{2\sqrt{2\pi t}} + \mathcal{O}\left(\frac{1}{N}, e^{-t}\right) \end{aligned} \tag{2}$$

And we will use the integral to estimate the order of the summation term in Eq. (2)

$$\sum_{k=1}^{2t} \frac{1}{\sqrt{k}} \leq \int_0^{2t} \frac{1}{\sqrt{k}} \,\mathtt{d}k + \mathcal{O}(1) = 2\sqrt{2t} + \mathcal{O}(1)$$

$$\sum_{k=1}^{2t} \sqrt{k} \leq \int_0^{2t} \sqrt{k} \,\mathtt{d}k + \mathcal{O}(1) = \frac{2}{3}(2t)^{\frac{3}{2}} + \mathcal{O}\left(\sqrt{t}\right)$$

And we know, $\sum_{k=1}^{2t} \frac{1}{\sqrt{k}} = 2\sqrt{2t} + \mathcal{O}(1)$. Thus, we have

$$\mathcal{A}_t \leq \frac{3N}{4} - \frac{3\pi^2}{8}\frac{t}{N} + \mathcal{O}\left(\frac{N}{\sqrt{t}}\right)$$

□

Lemma 1.2:

$$\mathcal{L}_T \leq F_T := \sum_{k=1}^{M} \cos^T\left(\frac{(2k+c)\pi}{2N}\right) \leq \mathcal{U}_T$$

where $1 < M < \lfloor \frac{N}{2} \rfloor$, $c \in \{0, 1\}$,

$$\mathcal{L}_T := \sum_{k=1}^{T/2} \frac{(T-1)!!}{T!!} \frac{(2k)!!}{(2k-1)!!} B_{2k}^{(1,M+1)} + \frac{(T-1)!!}{T!!} M$$

$$\mathcal{U}_T := \sum_{k=0}^{T/2} \frac{(T-1)!!}{T!!} \frac{(2k)!!}{(2k-1)!!} B_{2k}^{(0,M)} + \frac{(T-1)!!}{T!!} M$$

$$B_s^{(i,j)} := \frac{N}{\pi s} \sin\left(\frac{(2x+c)\pi}{2N}\right) \cos^{s-1}\left(\frac{(2x+c)\pi}{2N}\right) \Big|_i^j$$

Proof: We will begin with the lower bound.

Using the integral inequality, we have

$$F_T \geq \int_1^{M+1} \cos^T\left(\frac{(2x+c)\pi}{2N}\right) dx =: \mathcal{F}_T$$

Using integration by parts, we can transform the above equation into a recursive formula

$$\begin{aligned} &\frac{N}{\pi T} \int \sin\left(\frac{(2x+c)\pi}{2N}\right) d \cos^{T-1}\left(\frac{(2x+c)\pi}{2N}\right) \\ &= \frac{N}{\pi T} \sin\left(\frac{(2x+c)\pi}{2N}\right) \cos^{T-1}\left(\frac{(2x+c)\pi}{2N}\right) \Big|_1^{M+1} \\ &-\frac{1}{T} \int \cos^T\left(\frac{(2x+c)\pi}{2N}\right) dx \\ &=: B_T^{(1,M+1)} - \frac{1}{T} \mathcal{F}_T \end{aligned} \tag{3}$$

The left hand side of the Eqn. (3) can be written as

$$
\begin{aligned}
\text{l.h.s.} &= \frac{N}{\pi T} \int \sin\left(\frac{(2x+c)\pi}{2N}\right) d \cos^{T-1}\left(\frac{(2x+c)\pi}{2N}\right) \\
&= -\frac{T-1}{T} \int \sin^2\left(\frac{(2x+c)\pi}{2N}\right) \cos^{T-2}\left(\frac{(2x+c)\pi}{2N}\right) \,\mathtt{d}x \qquad (4) \\
&= -\frac{T-1}{T} \int \cos^{T-2}\left(\frac{(2x+c)\pi}{2N}\right) \,\mathtt{d}x + \frac{T-1}{T} \int \cos^{T}\left(\frac{(2x+c)\pi}{2N}\right) \,\mathtt{d}x
\end{aligned}
$$

✓ Test Pass

Combine Eq. (3) and (4), we have

$$
\begin{aligned}
\mathcal{F}_T &= B_T^{(1,M+1)} + \frac{T-1}{T} \mathcal{F}_{T-2} \\
&= \sum_{k=1}^{T/2} \frac{(T-1)!!}{T!!} \frac{(2k)!!}{(2k-1)!!} B_{2k}^{(1,M+1)} + \frac{(T-1)!!}{T!!} \mathcal{F}_0
\end{aligned}
$$

Similarly, we have

$$
F_T \leq \mathcal{G}_T := \int_0^M \cos^T\left(\frac{(2x+c)\pi}{2N}\right) \,\mathtt{d}x
$$

$$
\begin{aligned}
\mathcal{G}_T &= B_T^{(0,M)} + \frac{T-1}{T} \mathcal{G}_{T-2} \\
&= \sum_{k=0}^{T/2} \frac{(T-1)!!}{T!!} \frac{(2k)!!}{(2k-1)!!} B_{2k+1}^{(0,M)} + \frac{(T-1)!!}{T!!} \mathcal{G}_0
\end{aligned}
$$

Notice that $\mathcal{F}_0 = \mathcal{G}_0 = M$.

□

Lemma 1.3:

$$
\frac{(2k-1)!!}{(2k)!!} = \frac{1}{\sqrt{\pi k}} + \mathcal{O}\left(e^{-k}\right)
$$

Proof: Notice that

$$
(2k)!! = 2^k k!, \qquad (2k-1)!! = \frac{(2k)!}{2^k k!} = \frac{(2k-1)!}{2^{k-1}(k-1)!}
$$

Thus, we have

$$\frac{(2k-1)!!}{(2k)!!} = \frac{(2k)!}{2^k k!} \frac{1}{2^k k!}$$

using Stirling's formula $k! = \sqrt{2\pi k}\left(\frac{k}{e}\right)^k + \mathcal{O}(1)$, we have

$$\frac{(2k-1)!!}{(2k)!!} = \frac{1}{\sqrt{\pi k}} + \mathcal{O}\left(e^{-k}\right) \quad (5)$$

Thus, we get one of the results of the lemma. Now, we use Eq. (5) to prove the other part of the lemma.

$$\frac{(2k-2)!!}{(2k-1)!!} = \frac{\sqrt{\pi(k-1)}}{2k-1} + \mathcal{O}\left(e^{-k}\right)$$

□

2. The order of general PP

here we consider the order of general $P_i P_j$.

Theorem 2.1: when $t \sim \mathcal{O}(N^2)$, $\mathcal{P}(i,j,t) \sim \mathcal{O}\left(\frac{1}{N}\right)$

Let $t = cN^2$. from Lemma 2.4, we have $\cos^{4t}\left(\frac{k\pi}{2N}\right) \leq \frac{1}{N^3}$ if

$$k \geq \frac{2}{5}\sqrt{\frac{3\log(N)}{c}}$$

Let $T = \left\lceil \frac{2}{5}\sqrt{3\frac{\log(N)}{c}} \right\rceil$ Then the $\mathcal{P}(i,j,t)$ is

$$\begin{aligned}
\mathcal{P}(i,j,t) &= \frac{1}{N} + \frac{1}{N}\sum_k \left(\cos\left((i-j)\frac{k\pi}{N}\right) + \cos\left((i+j-1)\frac{k\pi}{N}\right)\right)\cos^{4t}\left(\frac{\pi k}{2N}\right) \\
&\geq \frac{1}{N} - \frac{T}{N} + \mathcal{O}\left(\frac{1}{N^3}\right) \\
&\sim \mathcal{O}\left(\frac{1}{N}\right)
\end{aligned}$$

We know, both $i-j$, $i+j-1$ are in $[0, N]$, thus, we have

$$\cos\left((i-j)\frac{k\pi}{N}\right) =$$

Lemma 2.2:

$$\mathcal{P}(i,j,t) = \sum_{\mu} P_i(\mu,t) P_j(\mu,t)$$
$$= \frac{1}{N} + \frac{1}{N} \sum_k \left(\cos\left((i-j)\frac{k\pi}{N} \right) + \cos\left((i+j-1)\frac{k\pi}{N} \right) \right) \cos^{4t}\left(\frac{\pi k}{2N} \right)$$

Proof:

$$P_i(\mu,t) = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N-1} \cos\left(\left(i - \frac{1}{2} \right) \frac{\pi k}{N} \right) \cos\left(\left(\mu - \frac{1}{2} \right) \frac{\pi k}{N} \right) \cos^{2t}\left(\frac{\pi k}{2N} \right)$$

then

$$\mathcal{P}(i,j,t) = \sum_{\mu} P_i(\mu,t) P_j(\mu,t).$$

Just like what we do on calculating PP, the above equation could be rewritten as

$$\mathcal{P}(i,j,t) = \frac{1}{N} + \frac{4}{N^2} \sum_{\mu,k,l} \cos\left(\left(i - \frac{1}{2} \right) \frac{\pi k}{N} \right) \cos\left(\left(j - \frac{1}{2} \right) \frac{\pi k}{N} \right)$$
$$\times \cos\left(\left(\mu - \frac{1}{2} \right) \frac{\pi k}{N} \right) \cos\left(\left(\mu - \frac{1}{2} \right) \frac{\pi l}{N} \right) \cos^{2t}\left(\frac{\pi k}{2N} \right) \cos^{2t}\left(\frac{\pi l}{2N} \right)$$
$$= \frac{1}{N} + \frac{2}{N} \sum_k \cos\left(\left(i - \frac{1}{2} \right) \frac{\pi k}{N} \right) \cos\left(\left(j - \frac{1}{2} \right) \frac{\pi k}{N} \right) \cos^{4t}\left(\frac{\pi k}{2N} \right)$$

substitute

$$\mathcal{P}(i,j,t) = \frac{1}{N} + \frac{1}{N} \sum_k \left(\cos\left((i-j)\frac{k\pi}{N} \right) + \cos\left((i+j-1)\frac{k\pi}{N} \right) \right) \cos^{4t}\left(\frac{\pi k}{2N} \right)$$

□

Lemma 2.3: $\cos(n \arccos(x))$ is a polynomial of x with degree n.

Proof: proof by induction.

when $n = 0$, $\cos(0) = 1$, $n = 1$, $\cos(\arccos(x)) = x$, satisfy the condition. Suppose for $n \leq k$, the statement is true, then we have

$$
\begin{aligned}
\cos((n+1)\arccos(x)) &= \cos(\arccos(x))\cos(n\arccos(x)) - \sin(\arccos(x))\sin(n\arccos(x)) \\
&= x\cos(n\arccos(x)) - \sqrt{1-x^2}\sin(n\arccos(x))
\end{aligned}
$$

and

$$
\begin{aligned}
\cos((n-1)\arccos(x)) &= \cos(\arccos(x))\cos(n\arccos(x)) + \sin(\arccos(x))\sin(n\arccos(x)) \\
&= x\cos(n\arccos(x)) - \sqrt{1-x^2}\sin(n\arccos(x))
\end{aligned}
$$

Thus, we have

$$
\cos((n+1)\arccos(x)) = 2x\cos(n\arccos(x)) - \cos((n-1)\arccos(x))
$$

is a polynomial of x with degree $n+1$. □

Lemma 2.4: $\cos^{4t}\left(\frac{\pi k}{2N}\right) \leq \frac{1}{N^d}$ if k satisfy

$$
k \geq \frac{2N}{5}\sqrt{\frac{d\log(N)}{t}}
$$

Proof: The condition $\cos^{4t}\left(\frac{\pi k}{2N}\right) \leq \frac{1}{N^d}$ is equivalent to the following inequality

$$
k \geq \frac{2N}{\pi}\arccos\left(N^{-\frac{d}{4t}}\right)
$$

Expanding the $\arccos\left(N^{-\frac{d}{4t}}\right)$ and we get

$$
\arccos\left(N^{-\frac{d}{4t}}\right) = \frac{\sqrt{1-N^{-\frac{d}{4t}}}\left(13-N^{-\frac{d}{4t}}\right)}{6\sqrt{2}} + \mathcal{O}\left(\left(1-N^{-\frac{d}{4t}}\right)^{\frac{5}{2}}\right).
$$

We further expand the $N^{-\frac{d}{4t}}$

$$
N^{-\frac{d}{4t}} = 1 - \frac{d\log(N)}{4t} + \frac{d^2\log(N)^2}{32t^2} + \mathcal{O}\left(\left(\frac{d\log(N)}{4t}\right)^3\right)
$$

Then, expanding the $\arccos\left(N^{-\frac{d}{4t}}\right)$ into

$$
\begin{aligned}
&\frac{2N}{\pi}\arccos\left(N^{-\frac{d}{4t}}\right) \\
&= \frac{N}{2\pi t}\sqrt{(d\log(N))(8t-d\log(N))}\left(1+\frac{d\log(N)}{48t} - \frac{d^2\log(N)^2}{384t^2}\right) + \mathcal{O}\left(\left(\frac{d\log(N)}{4t}\right)^2\right)
\end{aligned}
$$

Suppose that $4t > d\log(N)$, then

$$\frac{2N}{\pi} \arccos\left(N^{-\frac{d}{4t}}\right) \leq \frac{2N}{5} \sqrt{\frac{d \log(N)}{t}}$$

When $k \geq \frac{2N}{5} \sqrt{\frac{d \log(N)}{t}}$

$$\frac{\pi k}{N} \leq \frac{2\pi}{5} \sqrt{\frac{d \log(N)}{t}}$$

□

Lemma 2.5:

$$\frac{1}{N} \sum \cos^{4t}\left(\frac{\pi k}{2N}\right) = \int_0^{\frac{\pi}{2}} \cos^{4t}(x) \, \mathtt{d}x + \mathcal{O}(...)$$

Proof: Let x_i □