Calculate α

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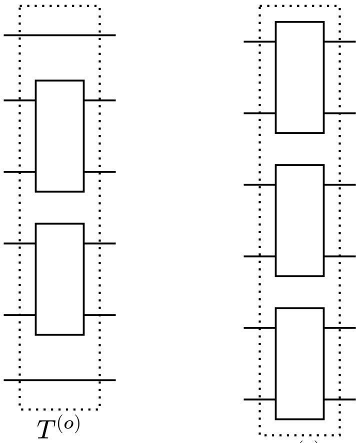
1. The tensor and the vector space

Let $T^{(o)}$ be the odd layer of T gates, and $T^{(e)}$ be the even layer of T gates. Then we have the following circuits:



Now, alternately apply the odd and even layers of T gates to the $|\gamma_1\gamma_{2n}\rangle\rangle$

$$T^{(\text{whole})}(b_1, t) = {T^{(o)}}^{b_2}\left(\prod_{i=0}^{t-b_2} T^{(e)}T^{(o)}\right){T^{(e)}}^{b_1},$$

where $b_1, b_2 \in \{0, 1\}$, $t + b_1$ stands for the number of layers. The output states must in the vector space spaned by the following basis

$$|Z_i\rangle\rangle, \quad \left|X_i\left(\prod_{k=i+1}^{j-1} Z_k\right)X_j\right\rangle\!\!\right\rangle, \quad \left|X_i\left(\prod_{k=i+1}^{j-1} Z_k\right)Y_j\right\rangle\!\!\right\rangle,$$
$$\left|Y_i\left(\prod_{k=i+1}^{j-1} Z_k\right)X_j\right\rangle\!\!\right\rangle, \quad \left|Y_i\left(\prod_{k=i+1}^{j-1} Z_k\right)Y_j\right\rangle\!\!\right\rangle$$

Let $M_i = \frac{1}{\sqrt{2}}(X_i + Y_i)$, the space

$$V := \operatorname{span}\left\{|Z_i\rangle\rangle, \ \left|M_i\left(\prod Z_k\right)M_j\right\rangle\!\!\right\rangle\right\}$$

is the image subspace for all $T^{(\text{whole})}(b_1, t)$ with $b_1 + t > 0$ if the input state is limit to Γ_2. Thus, the action of $T^{(\text{whole})}(b_1, t)$ could always be written in the following form

$$T^{(\text{whole})}(b_1,t)|\gamma_1\gamma_{2n}\rangle\!\rangle = \sum P(i,i,t)|Z_i\rangle\!\rangle + \sum_{i<j} P(i,j,t)\Big|M_i\Big(\prod Z_k\Big)M_j\Big\rangle\!\Big\rangle.$$

The action of the tensor could be simplified by studying the coefficients $P(i,j,t)$.

2. Propagation in polynomials space

To simplify the discussion, we start with a special case: $b_1 = 1, b_2 = 0$, and the number of wires is an even number $2N$. In this case, the whole tensor $T^{(\text{whole})}$ could be written as

$$T^{(\text{whole})}(1,t) = \left(T^{(e)}T^{(o)}\right)^t T^{(e)}.$$

Then, we lift the V into the space of the second order 2N-dimensional polynomials

$$W := \operatorname{span}\left\{\sum_{i,j=1}^{2N} c_{i,j}x_i x_j\right\}$$

We can prove that the space V is isometric to the space W by ϕ,

$$\begin{aligned} \phi : V &\to W \\ \Big|M_i\Big(\prod Z_k\Big)M_j\Big\rangle\!\Big\rangle &\to x_i x_j \\ |Z_i\rangle\!\rangle &\to x_i^2. \end{aligned}$$

Let $T_p^{(2N)}(t)$ be the map $\phi T^{(\text{whole})}(1,t)\phi^{-1}$, the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "T^{(\text{whole})}(1,t)"] \arrow[d, leftrightarrow, "\phi"] & V \arrow[d, leftrightarrow, "\phi"] \\ W \arrow[r, "T_p^{(2N)}(t)"] & W \end{tikzcd}$$

Now, let's consider the action of the tensor $T_p^{(2N)}(t)$ on the space W. Similarly, we could write down the recursive relation of coefficients $a_{i,j}$.

When $t = 0$, the transforming state $T^{(e)}|\gamma_1\gamma_{4N}\rangle\!\rangle$ to W, and we got $\frac{1}{4}(x_1x_{2N} + x_1x_{2N-1} + x_2x_{2N} + x_1x_{2N-1})$. Suppose at t, the vector in W is $\sum c_i(t)c_j'(t)x_ix_j$,

$$
\begin{aligned}
&T_p^{(2N)}(t+1)\left(\frac{1}{4}(x_1x_{2N}+x_1x_{2N-1}+x_2x_{2N}+x_1x_{2N-1})\right)\\
&=T^{(e)}T^{(o)}T_p^{(2N)}(t)\left(\frac{1}{4}(x_1x_{2N}+x_1x_{2N-1}+x_2x_{2N}+x_1x_{2N-1})\right)\\
&=T^{(e)}T^{(o)}\sum c_i(t)c_j'(t)x_ix_j
\end{aligned}
$$

Then, the action of $\phi\, T^{(e)}T^{(o)}\phi^{-1}$ on this vector is

x_ix_j	$\phi\, T^{(e)}T^{(o)}\phi^{-1}(x_ix_j)$
$i=j=1$	$\frac{1}{3}(x_1+x_2)^2-\frac{1}{6}(x_1^2+x_2^2)$
$i=1, j=2,3$	$\frac{1}{8}(x_1+x_2)(x_1+x_2+x_3+x_4)-\frac{1}{24}(x_1-x_2)^2$
$i=1, N>j>3$	$\frac{1}{8}(x_1+x_2)\left(x_{j-2+\eta_j}+x_{j-1+\eta_j}+x_{j+\eta_j}+x_{j+1+\eta_j}\right)$
$i=1, j=N$	$\frac{1}{4}(x_1+x_2)(x_{N-1}+x_N)$
$1<i<2N$, $i=j$	$\frac{1}{24}\left(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}+c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\right)$ $\left(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}+c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\right)$ $-\frac{1}{72}\left(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}\right)\left(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}\right)$ $-\frac{1}{72}\left(c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\right)\left(c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\right)$
$1<i<2N$, i is even, $j=i+1$	$\frac{1}{12}c_ic_{j'}\left(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}+c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\right)^2$ $-\frac{1}{36}\left(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}\right)\left(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}\right)$ $-\frac{1}{36}\left(c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\right)\left(c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\right)$
$1<i<2N$, $i+\eta_i\leq j\leq i+\eta_i+2$	$\frac{1}{16}\left(c_ix_{i-2+\eta_i}+c_ix_{i-1+\eta_i}+c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\right)$ $\left(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}+c_j'x_{j+\eta_j}+c_j'x_{j+1+\eta_j}\right)$ $+\frac{1}{48}\left(c_ix_{i+\eta_i}+c_ix_{i+1+\eta_i}\right)\left(c_j'x_{j-2+\eta_j}+c_j'x_{j-1+\eta_j}\right)$

$x_i x_j$	$\phi\, T^{(e)} T^{(o)} \phi^{-1} \left(x_i x_j \right)$
$1 < i < 2N,$ $j > i + \eta_i + 2$	$\frac{1}{16} \left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i} \right)$ $\left(c'_j x_{j-2+\eta_j} + c'_j x_{j-1+\eta_j} + c'_j x_{j+\eta_j} + c'_j x_{j+1+\eta_j} \right)$
$i = j = 2N$	$\frac{1}{6} (c_{2N} x_{2N-1} + c_{2N} x_{2N}) (c'_{2N} x_{2N-1} + c'_{2N} x_{2N})$

Table 1: The action of the tensor $\phi\, T^{(e)} T^{(o)} \phi^{-1}$ on the space of second order 2N-dimensional polynomials.

$$y_i y_j = \begin{cases} x_i^2 + 4 x_i x_{i+\eta_i} + x_{i+\eta_i}^2 \ , \ j = i \\ \left(x_i + x_{i+\eta_i} \right) \left(x_j + x_{j+\eta_j} \right) \ , \ \text{others} \end{cases}$$

$y_i y_j$	$\phi\, T^{(e)} T^{(o)} \phi^{-1} \left(x_i x_j \right)$
$i = j = 1$	$\frac{19}{36} y_1 y_1 + \frac{1}{36} y_2 y_2 + \frac{4}{9} y_1 y_2$
$i = 1, j = 2$	$\frac{1}{6} y_1 y_1 + \frac{23}{48} y_1 y_2 + \frac{3}{16} y_1 y_3 + \frac{5}{48} y_2 y_2 + \frac{1}{16} y_2 y_3$
$i = 1, N > j > 3$	$\frac{3}{16} y_1 y_{j-1} + \frac{3}{8} y_1 y_j + \frac{3}{16} y_1 y_{j+1} + \frac{1}{16} y_2 y_{j-1} +$ $\frac{1}{8} y_2 y_j + \frac{1}{16} y_2 y_{j+1}$
$i = 1, j = N$	$\frac{3}{16} y_1 y_{N-1} + \frac{9}{16} y_1 y_N + \frac{1}{16} y_2 y_{N-1} + \frac{3}{16} y_2 y_N$
$1 < i < N, j = i$	$\frac{1}{36} y_{i-1} y_{i-1} + \frac{5}{18} y_{i-1} y_i + \frac{1}{6} y_{i-1} y_{i+1} + \frac{2}{9} y_i y_i +$ $\frac{5}{18} y_i y_{i+1} + \frac{1}{36} y_{i+1} y_{i+1}$
$1 < i < N, j = i + 1$	$\frac{1}{16} y_{i-1} y_{i-1} + \frac{1}{8} y_{i-1} y_i + \frac{1}{16} y_{i-1} y_{i+1}$ $+ \frac{5}{48} y_i y_i + \frac{17}{48} y_i y_{i+1} + \frac{1}{8} y_{i+1} y_{i+2}$ $+ \frac{5}{48} y_{i+1} y_{i+1} + \frac{1}{16} y_{i+1} y_{i+2}$
$1 < i < N, N > j > i + 1$	$\frac{1}{16} y_{i-1} y_{j-1} + \frac{1}{8} y_{i-1} y_j + \frac{1}{16} y_{i-1} y_{j+1}$ $+ \frac{1}{8} y_i y_{j-1} + \frac{1}{4} y_i y_j + \frac{1}{8} y_{i+1} y_{j+1}$ $+ \frac{1}{16} y_{i+1} y_{j-1} + \frac{1}{8} y_{i+1} y_j + \frac{1}{16} y_{i+1} y_{j+1}$

$y_i y_j$	$\phi\, T^{(e)} T^{(o)} \phi^{-1}(x_i x_j)$
$1 < i < N, j = N$	$\frac{1}{16} y_{i-1} y_{N-1} + \frac{3}{16} y_{i-1} y_N + \frac{1}{8} y_i y_{N-1} + \frac{3}{8} y_i y_N$ $+ \frac{1}{16} y_{i+1} y_N + \frac{3}{16} y_i y_N$
$i = j = N$	$\frac{19}{36} y_N y_N + \frac{1}{36} y_{N-1} y_{N-1} + \frac{4}{9} y_{N-1} y_N$

Table 2: The action of the tensor $\phi\, T^{(e)} T^{(o)} \phi^{-1}$ on the space of second order 2N-dimensional polynomials.

$y_i y_j$	$\phi\, T^{(e)} T^{(o)} \phi^{-1}(x_i x_j)$
$i = j = 1$	$F(y_1, y_1) - \frac{5}{144} y_1 y_1 - \frac{5}{144} y_2 y_2 + \frac{5}{72} y_1 y_2$
$1 < i < N, j = i$	$F(y_i, y_i) - \frac{5}{144} y_{i-1} y_{i-1} + \frac{1}{36} y_{i-1} y_i + \frac{1}{24} y_{i-1} y_{i+1}$ $- \frac{1}{36} y_i y_i + \frac{1}{36} y_i y_{i+1} - \frac{5}{144} y_{i+1} y_{i+1}$
$1 \leq i < N, j = i + 1$	$F(y_i y_{i+1}) - \frac{1}{48} y_i y_i - \frac{1}{48} y_{i+1} y_{i+1} + \frac{1}{24} y_i y_{i+1}$
$i = j = N$	$F(y_N y_N) - \frac{5}{144} y_N y_N - \frac{5}{144} y_{N-1} y_{N-1} + \frac{5}{72} y_{N-1} y_N$
other case	$F(y_i y_j)$

Table 3: The action of the tensor $\phi\, T^{(e)} T^{(o)} \phi^{-1}$ on the space of second order 2N-dimensional polynomials.

The subsript η_i is defined as $\eta_i := 1 - (i \bmod 2)$. We can see that, c_{2i-1} and c_{2i} are always the same. So do c'_{2j-1} and c'_{2j}. Let $b_i = 2c_{2i-1} = 2c_{2i}$ and $b'_j = 2c'_{2j-1} = 2c'_{2j}$, we could further simplify the action of the tensor in the space of second order N-dimensional polynomials W_N. For simplemess, we define a "free" recursive relation in W_N

$$P(i, t+1) = \begin{cases} \frac{1}{4}(P(i-1,t) + 2P(i,t) + P(i+1,t)) \ , \ i \neq 1 \text{ or } N \\ \frac{1}{4}(P(i-1,t) + 2P(i,t) + P(i+1,t)), \ i = 1 \text{ or } N \end{cases}. \tag{1}$$

We call it "free" because $b_i(t) b'_j(t) = P(i,t) P(j,t)$ if i and j are not "collide" with each other (which means $|i - j| > 3$). And the solution of Eq. (1) is a propagating wave. Refs. [1] provide the solution of this equation,

$$P_{n_0}(n,t) = \frac{1}{N} + \frac{2}{N}\sum_{k=1}^{N-1} \cos\left(\left(n - \frac{1}{2}\right)\frac{\pi k}{N}\right)\cos\left(\left(n_0 - \frac{1}{2}\right)\frac{\pi k}{N}\right)\cos^{2t}\frac{\pi k}{2N}, \quad (2)$$

where n_0 is the initial state. The term $P_{n_0}(n,t)$ also called the propagator.

For our case, there are 2 propagators, which are P_1 and P_N. And the initial state is $x_1 x_N$. Let $\phi()$

Now, let's pluge $b_i = 2c_{2i-1} = 2c_{2i}$ and $b'_j = 2c'_{2j-1} = 2c'_{2j}$ into Table 1. More concretely, let ϕ' be the map

$$\phi' : W_N \to W_{2N}$$
$$y_i y_j \to \frac{1}{4}(x_{2i-1} + x_{2i})(x_{2j-1} + x_{2j}).$$

If we consider the subspace W'_{2N} of W_{2N}, where $W'_{2N} := \operatorname{span}\big((x_{2i-1} + x_{2i})(x_{2j-1} + x_{2j})\big)$ (it means $c_{2i-1} = c_{2i}$ and $c'_{2j-1} = c'_{2j}$), the ϕ' will be the isometric between W_N and W'_{2N}. Thus, the following diagram commutes.

$$\begin{tikzcd}
V \arrow[r, "T^{(\text{whole})}(1,t)"] & V \\
W'_{2N} \arrow[u, leftrightarrow, "\phi"] \arrow[r, "T_p^{(2N)}(t)"] & W'_{2N} \arrow[u, leftrightarrow, "\phi"] \\
W_N \arrow[u, leftrightarrow, "\phi'"] \arrow[r, "T_p^{(N)}(t)"] & W_N \arrow[u, leftrightarrow, "\phi'"]
\end{tikzcd}$$

For simpliness, let

$$F(y_i) = \begin{cases} \frac{3}{4}y_1 + \frac{1}{4}y_2 \ , \ i = 1 \\ \frac{1}{4}y_{i-1} + \frac{1}{2}y_i + \frac{1}{4}y_{i+1} \ , \ 1 < i < N \\ \frac{3}{4}y_N + \frac{1}{4}y_{N-1} \ , \ i = N \end{cases}$$

3. The interaction term

let $b(i,j,t) = P_1(i,t)P_N(j,t) + I(i,j,t)$. Then,

$$|\psi(t)\rangle = \sum_{i,j} P_1(i,t) P_N(j,t) y_i y_j - \sum_{i,j} I(i,j,t) y_i y_j. \tag{3}$$

From simpleness, let the terms in Table 3 be $F(y_i)F\left(y_j\right) + R(i,j)$. Then we get

$$\begin{aligned}|\psi(t+1)\rangle &= \sum_{i,j} P_1(i,t)P_N(j,t)\left(F(y_i)F\left(y_j\right)+R(i,j)\right) - \sum_{i,j} I(i,j,t)\left(F(y_i)F\left(y_j\right)+R(i,j)\right)\\ &= \sum_{i,j} P_1(i,t+1)P_N(j,t+1)y_iy_j + \sum_{i,j} P_1(i,t)P_N(i,t)R(i,j)\\ &\quad -\sum_{i,j} I(i,j,t)\left(F(y_i)F\left(y_j\right)+R(i,j)\right)\end{aligned}$$

这里又又又为了简洁性， 定义

$$F_f^{(1)} : \mathrm{Func}(i,j,t) \mapsto \begin{cases} \frac{1}{4}\ \mathrm{Func}(i-1,j,t) + \text{...} & \text{if ...} \\ \frac{3}{4}\ \mathrm{Func}(i,j,t) + \text{...} & \text{if i} = 1 \end{cases}$$

$$F_f^{(2)} : \mathrm{Func}(i,j,t) \mapsto \begin{cases} \frac{1}{4}\ \mathrm{Func}(i,j-1,t) + \text{...} & \text{if ...} \\ \frac{3}{4}\ \mathrm{Func}(i,j,t) + \text{...} & \text{if j} = 1 \end{cases}$$

By using Eq. (3) to expand $|\psi(t+1)\rangle$, we have

$$\begin{aligned}&I(i,j,t+1)\\ &= -\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\left(\sum_{i,j} P_1(i,t)P_N(i,t)R(i,j) - \sum_{i,j} I(i,j,t)\left(F(y_i)F\left(y_j\right)+R(i,j)\right)\right)\\ &= \frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\sum_{l,k} I(l,k,t)F(y_l)F(y_k) - \sum_{l,k}\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\left((P_1(l,t)P_N(k,t) - I(l,k,t))R(l,k)\right)\end{aligned}$$

Expand the first term as

$$\begin{aligned}&\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\sum_{l,k} I(l,k,t)F(y_l)F(y_k)\\ &= \sum_{l,k} I(l,k,t)\left(\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}F(y_l)F(y_k)\right)\end{aligned}$$

when $l, k \neq 1, N$, we have

$$
\begin{aligned}
F(y_l)F(y_k) = &\frac{1}{16}y_{l-1}y_{k-1} + \frac{1}{8}y_{l-1}y_k + \frac{1}{16}y_{l-1}y_{k+1} \\
&+\frac{1}{8}y_l y_{k-1} + \frac{1}{4}y_l y_k + \frac{1}{8}y_l y_{k+1} \\
&+\frac{1}{16}y_{l+1}y_{k-1} + \frac{1}{8}y_{l+1}y_k + \frac{1}{16}y_{l+1}y_{k+1}
\end{aligned}
$$

Thus, only when $|l-i| \leq 1, |k-j| \leq 1$, the term $\left(\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}F(y_l)F(y_k)\right)$ is not zero. And we have

$$
\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}F(y_l)F(y_k) = \begin{cases} \frac{1}{16}\ , \ |i-l| = |j-k| = 1 \\ \frac{1}{4}\ , \ |i-l| = |j-k| = 0 \\ \frac{1}{8}\ , \ \text{other cases} \end{cases}
$$

Thus,

$$
\begin{aligned}
&\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\sum_{l,k} I(l,k,t)F(y_l)F(y_k) \\
= &\frac{1}{16}I(l-1,k-1,t) + \frac{1}{8}I(l-1,k,t) + \frac{1}{16}I(l-1,k+1,t) \\
&+\frac{1}{8}I(l,k-1,t) + \frac{1}{4}I(l,k,t) + \frac{1}{8}I(l,k+1,t) \\
&+\frac{1}{16}I(l+1,k-1,t) + \frac{1}{8}I(l+1,k,t) + \frac{1}{16}I(l+1,k+1,t) \\
= &F_f^{(1)} \circ F_f^{(2)}(I)(i,j,t)
\end{aligned}
$$

combine the equation, we get

$$
\begin{aligned}
I(i,j,t+1) = &F_f^{(1)} \circ F_f^{(2)}(I)(i,j,t) \\
&-\sum_{l,k}\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\Big((P_1(l,t)P_N(k,t) - I(l,k,t))R(l,k)\Big)
\end{aligned} \tag{4}
$$

Let $D(i,j) = P_1(i,t)P_N(j,t) - I(i,j,t)$,

i, j	$I(i,j,t+1) - F_f^{(1)} \circ F_f^{(2)}(I)(i,j,t)$
$i = j = 1$	$\frac{5}{144}D(1,1) + \frac{5}{144}D(2,2) + \frac{1}{48}D(1,2) + \frac{1}{48}D(2,1)$

i,j	$I(i,j,t+1)-F_f^{(1)}\circ F_f^{(2)}(I)(i,j,t)$
$i=1,j=2$	$-\frac{5}{144}D(1,1)-\frac{1}{72}D(2,2)-\frac{1}{24}D(1,2)$
$1<i<N,j=i$	$\frac{5}{144}D(i-1,i-1)+\frac{1}{48}D(i-1,i)$ $\frac{1}{48}D(i,i-1)+\frac{1}{36}D(i,i)+\frac{1}{48}D(i,i+1)$ $+\frac{1}{48}D(i+1,i-1)+\frac{5}{144}D(i+1,i+1)$
$1\le i<N,j=i+1$	$-\frac{1}{24}D(i,j)-\frac{1}{72}D(i-1,j)-\frac{1}{72}D(i,j-1)$
$i=N-1,j=N$	$-\frac{5}{144}D(N,N)-\frac{1}{72}D(N-1,N-1)-\frac{1}{24}D(N,N-1)$
$i=j=N$	$\frac{5}{144}D(N,N)+\frac{5}{144}D(N-1,N-1)+\frac{1}{48}D(N,N-1)+\frac{1}{48}D(N-1,N)$
other case	0

Table 4: The action of the tensor $\phi\,T^{(e)}T^{(o)}\phi^{-1}$ on the space of second order 2N-dimensional polynomials.

4. Proofs

Proof of Table 3: 现在我们想要计算 $T_p^{(N)}(t)$ 在W_N 中的作用。因此我们需要计算

$$
\begin{aligned}
&\phi'\phi T^{(e)}T^{(o)}\phi^{-1}\phi'^{-1}(y_iy_j)\\
&=\phi T^{(e)}T^{(o)}\phi^{-1}\left(\frac{1}{4}(x_{2i-1}+x_{2i})(x_{2j-1}+x_{2j})\right)\\
&=\phi T^{(e)}T^{(o)}\phi^{-1}\left(\frac{1}{4}(x_{2i-1}x_{2j-1}+x_{2i-1}x_{2j}+x_{2i}x_{2j-1}+x_{2i}x_{2j})\right)
\end{aligned}
$$

相当于⊠于每一个 $x_{2i-1}x_{2j-1}$ 这样的项进行查表 。我们现在来分析不同的 y 的映射。⊠了简化符号，这里记 $S_N=\phi'\phi T^{(e)}T^{(o)}\phi^{-1}\phi'^{-1}$, $S_{2N}=\phi T^{(e)}T^{(o)}\phi^{-1}$.

$$\phi'(y_1y_1) = \frac{1}{4}(x_1x_1 + 2x_1x_2 + x_2x_2)$$

$$S_N(y_1y_1) = \frac{1}{4}(S_{2N}(x_1x_1) + S_{2N}(2x_1x_2) + S_{2N}(x_2x_2))$$

依据 Table 1, 我们有

$$S_N(y_1y_1) = \frac{1}{4}\Bigg($$

$$\frac{1}{6}(x_1 + x_2)(x_1 + x_2) +$$

$$\left(\frac{1}{4}\right)\left(\left(\frac{4}{3}\right)x_1 + \left(\frac{4}{3}\right)x_2 + x_3 + x_4\right)(x_1 + x_2) +$$

$$-\left(\frac{1}{72}\right)\left((x_1 + x_2)^2\right) + \left(\frac{1}{24}\right)\left((x_1 + x_2 + x_3 + x_4)^2\right) - \left(\frac{1}{72}\right)\left((x_3 + x_4)^2\right)$$

$$\Bigg)$$

$$= \left(\frac{19}{144}\right)({x_1}^2) + \left(\frac{19}{72}\right)x_1x_2 + \left(\frac{1}{12}\right)x_1x_3 + \left(\frac{1}{12}\right)x_1x_4$$

$$+\left(\frac{19}{144}\right)({x_2}^2) + \left(\frac{1}{12}\right)x_2x_3 + \left(\frac{1}{12}\right)x_2x_4$$

$$+\left(\frac{1}{144}\right)({x_3}^2) + \left(\frac{1}{72}\right)x_3x_4 + \left(\frac{1}{144}\right)({x_4}^2)$$

我们想要和“自由演化”进行对比。而 y_1y_1 自由演化的结果是

$$\begin{aligned}
F(y_1y_1) &= \frac{1}{16}(3y_1 + y_2)(3y_1 + y_2) \\
&= \frac{1}{16}\left(\frac{3}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right)\left(\frac{3}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right) \\
&= \left(\frac{9}{64}\right)({x_1}^2) + \left(\frac{9}{32}\right)x_1x_2 + \left(\frac{3}{32}\right)x_1x_3 + \left(\frac{3}{32}\right)x_1x_4 \\
&\quad + \left(\frac{9}{64}\right)({x_2}^2) + \left(\frac{3}{32}\right)x_2x_3 + \left(\frac{3}{32}\right)x_2x_4 + \left(\frac{1}{64}\right)({x_3}^2) \\
&\quad + \left(\frac{1}{32}\right)x_3x_4 + \left(\frac{1}{64}\right)({x_4}^2)
\end{aligned}$$

二者之差�

$$
\begin{aligned}
S_N(y_1y_1)-F(y_1y_1) &= -\left(\frac{5}{576}\right)({x_1}^2)-\left(\frac{5}{288}\right)x_1x_2-\left(\frac{1}{96}\right)x_1x_3-\left(\frac{1}{96}\right)x_1x_4 \\
&\quad -\left(\frac{5}{576}\right)({x_2}^2)-\left(\frac{1}{96}\right)x_2x_3-\left(\frac{1}{96}\right)x_2x_4 \\
&\quad -\left(\frac{5}{576}\right)({x_3}^2)-\left(\frac{5}{288}\right)x_3x_4-\left(\frac{5}{576}\right)({x_4}^2) \\
&= -\frac{5}{576}(x_1+x_2+x_3+x_4)^2+\frac{1}{144}(x_1+x_2)(x_3+x_4) \\
&= -\frac{5}{144}(y_1+y_2)^2+\frac{1}{36}y_1y_2
\end{aligned}
$$

接下来，算 $S_N(y_1y_2)$

$$
\begin{aligned}
S_N(y_1y_2) &= \frac{1}{4}(S_{2N}(x_1x_3)+S_{2N}(x_1x_4)+S_{2N}(x_2x_3)+S_{2N}(x_2x_4)) \\
&= \frac{1}{4}\Bigg(\\
&\quad \frac{1}{8}\left(\frac{4}{3}x_1+\frac{4}{3}x_2+x_3+x_4\right)(x_1+x_2) \\
&\quad +\frac{1}{8}(x_3+x_4+x_5+x_6)(x_1+x_2) \\
&\quad -\left(\frac{1}{36}\right)\left((x_1+x_2)^2\right)+\left(\frac{1}{12}\right)\left((x_1+x_2+x_3+x_4)^2\right)-\left(\frac{1}{36}\right)\left((x_3+x_4)^2\right) \\
&\quad +\frac{1}{16}(x_1+x_2+x_3+x_4)(x_3+x_4+x_5+x_6)+\frac{1}{48}(x_3+x_4)^2 \\
\Bigg) &= \left(\frac{2}{9}\right)({y_1}^2)+\left(\frac{23}{48}\right)y_1y_2+\left(\frac{3}{16}\right)y_1y_3+\left(\frac{5}{36}\right)({y_2}^2)+\left(\frac{1}{16}\right)y_2y_3
\end{aligned}
$$

$$F(y_1)F(y_2)=\left(\tfrac{3}{4}y_1+\tfrac{1}{4}y_2\right)\left(\tfrac{1}{4}y_1+\tfrac{1}{2}y_2+\tfrac{1}{4}y_3\right)$$

$$S_N(y_1y_2)=F(y_1)F(y_2)+\left(\frac{5}{144}\right)({y_1}^2)+\left(\frac{1}{24}\right)y_1y_2+\left(\frac{1}{72}\right)({y_2}^2)$$

接下来，算 $S_N(y_1y_3)$

$$S_N(y_1y_3)=F(y_1)F(y_3)+\frac{1}{48}{y_2}^2$$

然后，$j>3$时

$$S_N(y_1y_j)=F(y_1)F(y_j)$$

然后，$i > 1, i = j$时,

$$
\begin{aligned}
S_N(y_i y_j) &= \frac{1}{4}\left(S_N(x_{2i-1}x_{2j-1}) + S_N(x_{2i-1}x_{2j}) + S_N(x_{2i}x_{2j-1}) + S_N(x_{2i}x_{2j})\right) \\
&= \frac{1}{4}\left(S_N(x_{2i-1}x_{2i-1}) + 2S_N(x_{2i-1}x_{2i}) + S_N(x_{2i}x_{2i})\right) \\
&= \frac{1}{4}\Bigg(\\
&\quad \frac{1}{24}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})^2 \\
&\quad -\frac{1}{72}(x_{2i-3} + x_{2i-2})^2 - \frac{1}{72}(x_{2i-1} + x_{2i})^2 \\
&\quad +\frac{1}{24}(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})^2 \\
&\quad -\frac{1}{72}(x_{2i-1} + x_{2i})^2 - \frac{1}{72}(x_{2i+1} + x_{2i+2})^2 \\
&\quad +\frac{1}{16}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2}) \\
&\quad +\frac{1}{48}(x_{2i-1} + x_{2i})^2 \\
&\Bigg) \\
&= \frac{1}{24}(y_{i-1} + y_i)^2 - \frac{1}{72}(y_{i-1})^2 - \frac{1}{72}(y_i)^2 \\
&\quad +\frac{1}{24}(y_i + y_{i+1})^2 - \frac{1}{72}(y_i)^2 - \frac{1}{72}(y_{i+1})^2 \\
&\quad +\frac{1}{16}(y_{i-1} + y_i)(y_i + y_{i+1}) + \frac{1}{48}(y_i)^2
\end{aligned}
$$

$$
\begin{aligned}
S_N(y_i y_j) = F(y_i)F(y_j) &- \left(\frac{5}{144}\right)({y_{i-1}}^2) - \left(\frac{5}{48}\right)y_{i-1}y_i - \left(\frac{1}{16}\right)y_{i-1}y_{i+1} \\
&- \left(\frac{1}{9}\right)({y_i}^2) - \left(\frac{5}{48}\right)y_i y_{i+1} - \left(\frac{5}{144}\right)({y_{i+1}}^2)
\end{aligned}
$$

然后，$i > 1, j = i + 1$时,

$$
\begin{aligned}
S_N(y_i y_j) &= \frac{1}{4}(S_N(x_{2i-1}x_{2i+1}) + S_N(x_{2i-1}x_{2i+2}) + S_N(x_{2i}x_{2i+1}) + S_N(x_{2i}x_{2i+2})) \\
&= \frac{1}{4}\Bigg(\\
&\quad \frac{1}{16}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2}) \\
&\quad + \frac{1}{48}(x_{2i-1} + x_{2i})^2 \\
&\quad + \frac{1}{16}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})(x_{2i+1} + x_{2i+2} + x_{2i+3} + x_{2i+4}) \\
&\quad + \frac{1}{12}(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})^2 \\
&\quad - \frac{1}{36}(x_{2i-1} + x_{2i})^2 - \frac{1}{36}(x_{2i+1} + x_{2i+2})^2 \\
&\quad + \frac{1}{16}(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})(x_{2i+1} + x_{2i+2} + x_{2i+3} + x_{2i+4}) \\
&\quad + \frac{1}{48}(x_{2i+1} + x_{2i+2})^2 \\
&\Bigg) \\
&= \frac{1}{16}(y_{i-1} + y_i)(y_i + y_{i+1}) + \frac{1}{48}(y_i)^2 \\
&\quad + \frac{1}{16}(y_i + y_{i+1})(y_{i+1} + y_{i+2}) + \frac{1}{48}(y_{i+1})^2 \\
&\quad + \frac{1}{16}(y_{i-1} + y_i)(y_{i+1} + y_{i+2}) \\
&\quad + \frac{1}{12}(y_i + y_{i+1})^2 - \frac{1}{36}(y_i)^2 - \frac{1}{36}(y_{i+1})^2
\end{aligned}
$$

$$
S_N(y_i y_j) = F(y_i)F(y_j) + \left(\frac{1}{72}\right)({y_i}^2) + \left(\frac{1}{24}\right)y_i y_{i+1} + \left(\frac{1}{72}\right)({y_{i+1}}^2)
$$

□

Proof: 观察 Table 3，□有 $y_1y_1, y_1y_2, y_2y_1, y_2y_2$的项可以□ $I(1, 1, t + 1)$ □生贡献。因此，

$$
\begin{aligned}
&\frac{\partial^2}{\partial y_i y_j} P_1(l,t) P_N(k,t) R(l,k) \\
=&\frac{5}{144}(-P_1(1,t)P_N(1,t) - P_1(2,t)P_N(2,t) + P_1(1,t)P_N(2,t) + P_1(2,t)P_N(1,t)) \\
=&-\frac{5}{144}(P_1(1,t) - P_1(2,t))(P_N(1,t) - P_N(2,t))
\end{aligned}
$$

现在重要的就是确认这一项的符号。好在我们是知道 P 的具体形式的。由 Eq. (2) 可以得到

$$
\begin{aligned}
P_1(n,t) &= \frac{1}{N} + \frac{2}{N}\sum_{k=1}^{N-1} \cos\left(\left(n-\frac{1}{2}\right)\frac{\pi k}{N}\right)\cos^{2t+1}\left(\frac{\pi k}{2N}\right) \\
P_N(n,t) &= \frac{1}{N} + \frac{2}{N}\sum_{k=1}^{N-1} (-1)^k \cos\left(\left(n-\frac{1}{2}\right)\frac{\pi k}{N}\right)\cos^{2t+1}\left(\frac{\pi k}{2N}\right)
\end{aligned}
$$

由 Lemma 4.1 可以得到，$P_1(1,t) - P_1(2,t) < 0, P_N(1,t) - P_N(2,t) > 0$。因此，$I(1,1,t+1) > 0$。

当 $i=1, j=2$ 时

$$
\begin{aligned}
&\frac{\partial^2}{\partial y_i \partial y_j} P_1(l,t) P_N(k,t) R(l,k) \\
=&-\frac{1}{144}P_1(1)P_N(1) + \frac{1}{24}P_1(1)P_N(2) \\
&+\frac{1}{24}P_1(2)P_N(1) - \frac{5}{48}P_1(2)P_N(2)
\end{aligned}
$$

当 $i=j=2$ 时

$$
\begin{aligned}
&\frac{\partial^2}{\partial y_i \partial y_j} P_1(l,t) P_N(k,t) R(l,k) \\
&= -\frac{5}{144} P_1(1) P_N(1) + \frac{1}{72} P_1(1) P_N(2) + \frac{1}{48} P_1(1) P_N(3) \\
&\quad + \frac{1}{72} P_1(2) P_N(1) - \frac{1}{9} P_1(2) P_N(2) + \frac{1}{72} P_1(2) P_N(3) \\
&\quad + \frac{1}{48} P_1(3) P_N(1) + \frac{1}{72} P_1(3) P_N(2) - \frac{5}{144} P_1(3) P_N(3) \\
&= \begin{pmatrix} P_N(1) & P_N(2) & P_N(3) \end{pmatrix} \begin{pmatrix} -\frac{5}{144} & \frac{1}{72} & \frac{1}{48} \\ \frac{1}{72} & -\frac{1}{9} & \frac{1}{72} \\ \frac{1}{48} & \frac{1}{72} & -\frac{5}{144} \end{pmatrix} \begin{pmatrix} P_1(1) \\ P_1(2) \\ P_1(3) \end{pmatrix}
\end{aligned}
$$

可以看到，当 $1 < i = j < N$ 时 $\frac{\partial^2}{\partial y_i \partial y_j} P_1(l,t) P_N(k,t) R(l,k)$ 都符合上述形式。因此，我们可以得到

$$
\begin{aligned}
&\sum_i \frac{\partial^2}{\partial {y_i}^2} P_1(l,t) P_N(k,t) R(l,k) \\
&= \sum \begin{pmatrix} P_N(i-1) & P_N(i) & P_N(i+1) \end{pmatrix} \begin{pmatrix} -\frac{5}{144} & \frac{1}{72} & \frac{1}{48} \\ \frac{1}{72} & -\frac{1}{9} & \frac{1}{72} \\ \frac{1}{48} & \frac{1}{72} & -\frac{5}{144} \end{pmatrix} \begin{pmatrix} P_1(i-1) \\ P_1(i) \\ P_1(i+1) \end{pmatrix} \\
&= \sum \operatorname{tr} \left(\begin{pmatrix} -\frac{5}{144} & \frac{1}{72} & \frac{1}{48} \\ \frac{1}{72} & -\frac{1}{9} & \frac{1}{72} \\ \frac{1}{48} & \frac{1}{72} & -\frac{5}{144} \end{pmatrix} \begin{pmatrix} P_1(i-1) \\ P_1(i) \\ P_1(i+1) \end{pmatrix} \begin{pmatrix} P_N(i-1) & P_N(i) & P_N(i+1) \end{pmatrix} \right)
\end{aligned}
$$

现在来看看后面这个矩阵

$$
\mathcal{M} = \sum \begin{pmatrix} P_1(i-1) \\ P_1(i) \\ P_1(i+1) \end{pmatrix} \begin{pmatrix} P_N(i-1) & P_N(i) & P_N(i+1) \end{pmatrix}
$$

其矩阵元⊠

$$
\mathcal{M}_{l,m} = \sum_{k=2}^{N-1} P_1(k+\xi_l) P_N(k+\xi_m)
$$

其中，$\xi_l = -1, 0, 1$。我们知道 $\sum P_1(i,t) = \sum P_{N(i,t)} = 1$。 □

Lemma 4.1：$P_1(i,t)$ 随着i ⊠⊠，$P_N(i,t)$ 随着i ⊠增。

Proof:

先将 $P(i,t)$ 从整数域延拓到实数域上。这样就可以对 i 进行求导。此时，由 Eq.(2) 可以得到

$$P_1(x,t) = \frac{1}{N} + \frac{2}{N}\sum_{k=1}^{N-1}\cos\left(\left(x-\frac{1}{2}\right)\frac{\pi k}{N}\right)\cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

$$\frac{\partial}{\partial x}P_1(x,t) = -\frac{2\pi}{N}\sum\frac{k}{N}\sin\left(\left(x-\frac{1}{2}\right)\frac{\pi k}{N}\right)\cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

$$\frac{\partial}{\partial t}P_1(x,t) = \frac{2}{N}\sum\cos\left[\frac{k\pi}{2N}\right]^{1+2t}\cos\left[\frac{\left(-\left(\frac{1}{2}\right)+i\right)k\pi}{N}\right]\ln\left(\cos^2\left(\frac{k\pi}{2N}\right)\right)$$

$$P_N(x,t) = \frac{1}{N} + \frac{2}{N}\sum_{k=1}^{N-1}(-1)^k\cos\left(\left(x-\frac{1}{2}\right)\frac{\pi k}{N}\right)\cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

$$\frac{\partial}{\partial x}P_N(x,t) = -\frac{2\pi}{N}\sum(-1)^k\frac{k}{N}\sin\left(\left(x-\frac{1}{2}\right)\frac{\pi k}{N}\right)\cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

现在关注它们导数的符号。注意到 $\cos\left(\frac{\pi k}{2N}\right)$随着 k 增大⊠⊠。令 $\kappa = \frac{\pi k}{N} \in (0,\pi)$

$$\begin{aligned}&\sum k\frac{\pi}{N}\sin\left(\left(x-\frac{1}{2}\right)\frac{\pi k}{N}\right)\cos^{2t+1}\left(\frac{\pi k}{2N}\right)\\ &= \operatorname{Im}\left[\sum\kappa e^{i\left(x-\frac{1}{2}\right)\kappa}\cos^{2t+1}\left(\frac{\kappa}{2}\right)\right]\end{aligned}$$

假设 N 趋于无穷，那么

$$\frac{\partial}{\partial x}P_1(x,t) \to -2\int_0^\pi \kappa\sin\left(\left(x-\frac{1}{2}\right)\kappa\right)\cos^{2t+1}\left(\frac{\kappa}{2}\right)d\kappa$$

$$\begin{aligned}P_1(x,t) &= \frac{2}{\pi}\int_0^\pi\cos\left[\left(x-\frac{1}{2}\right)\kappa\right]*\cos\left[\frac{\kappa}{2}*(2t+1)\right]d\kappa\\ &= \frac{1}{\pi}\left[\frac{\sin[\pi(1+t-x)]}{1+t-x}+\frac{\sin[\pi(t+x)]}{t+x}\right]\end{aligned}$$

□

$$P(x,t+1)-P(x,t)=-\frac{2}{N}\sum\left(\cos\left[\frac{k\pi}{2N}\right]^{1+2t}\cos\left[\frac{k\pi(-(\frac{1}{2})+x)}{N}\right]\sin\left[\frac{k\pi}{2N}\right]^{2}\right)$$

Lemma 4.2: $I(i,j,i+1)$

Proof:

$$\begin{aligned}I(i,i,t+1)=&F_f^{(1)}(I)(i,j,t)F_f^{(2)}(I)(i,j,t)\\&-\sum_{l,k}\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\Big((P_1(l,t)P_N(k,t)-I(l,k,t))R(l,k)\Big)\end{aligned}$$

令 $\mathcal{M}=\begin{pmatrix}-\frac{5}{144} & \frac{1}{72} & \frac{1}{48}\\ \frac{1}{72} & -\frac{1}{9} & \frac{1}{72}\\ \frac{1}{48} & \frac{1}{72} & -\frac{5}{144}\end{pmatrix}$。 由 可以得到

□

Lemma 4.3: P(x, t+1)-P(x,t)

Proof:

x	$P(x,t+1)-P(x,t)$
$x=1$	$\frac{\Gamma\left[\frac{3}{2}+t\right]}{\sqrt{\pi}\Gamma[3+t]}$
$x=2$	$\frac{(-3+t)\Gamma\left[\frac{3}{2}+t\right]}{\sqrt{\pi}\Gamma[4+t]}$
$x=3$	$\frac{(-11+t)\Gamma\left[\frac{3}{2}+t\right]}{\sqrt{\pi}\Gamma[5+t]}$
$x=4$	$\frac{(-23+t)\Gamma\left[\frac{3}{2}+t\right]}{\sqrt{\pi}\Gamma[6+t]}$

从这里可以看出，分母就是简单的 $\Gamma(t+x+2)$。但分子有点诡异，在

□

Lemma 4.4: $|I(i,j,t)| \leq P_1(i,t)P_N(j,t)$

Proof: 用归纳法证明

当 $t=1$ 时，成立

假设 $t=n$ 时成立，那么 $t=n+1$ 时

$$I(t+1) \geq -P_1(i,t)P_N(j,t) - \sum_{l,k} \frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\Big((P_1(l,t)P_N(k,t) - I(l,k,t))R(l,k)\Big)$$

□

Bibliography

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