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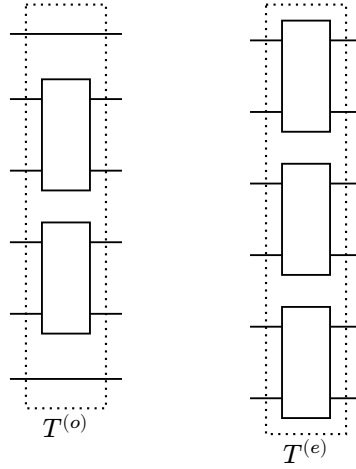
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1. The tensor and the vector space

Let $T^{(o)}$ be the odd layer of T gates, and $T^{(e)}$ be the even layer of T gates. Then we have the following circuits:

$$T^{(o)} \qquad T^{(e)}$$

Now, alternately apply the odd and even layers of T gates to the $|\gamma_1 \gamma_{2n}\rangle\!\rangle$

$$T^{(\text{whole})}(b_1, t) = T^{(o)^{b_2}} \left(\prod_{i=0}^{t-b_2} T^{(e)} T^{(o)} \right) T^{(e)^{b_1}},$$

where $b_1, b_2 \in \{0, 1\}$, $t + b_1$ stands for the number of layers. The output states must in the vector space spaned by the following basis

$$|Z_i\rangle\!\rangle, \quad \left| X_i \left(\prod_{k=i+1}^{j-1} Z_k \right) X_j \right\rangle\!\rangle, \quad \left| X_i \left(\prod_{k=i+1}^{j-1} Z_k \right) Y_j \right\rangle\!\rangle,$$

$$\left| Y_i \left(\prod_{k=i+1}^{j-1} Z_k \right) X_j \right\rangle\!\rangle, \quad \left| Y_i \left(\prod_{k=i+1}^{j-1} Z_k \right) Y_j \right\rangle\!\rangle$$

Let $M_i = \frac{1}{\sqrt{2}}(X_i + Y_i)$, the space

$$V := \text{span}\left\{ |Z_i\rangle\!\rangle, \left| M_i \left(\prod Z_k \right) M_j \right\rangle\!\rangle \right\}$$

is the image subspace for all $T^{(\text{whole})}(b_1, t)$ with $b_1 + t > 0$ if the input state is limit to Γ_2. Thus, the action of $T^{(\text{whole})}(b_1, t)$ could always be written in the following form

$$T^{(\mathrm{whole})}(b_1, t)|\gamma_1\gamma_{2n}\rangle\rangle = \sum P(i,i,t)|Z_i\rangle\rangle + \sum_{i<j} P(i,j,t)\Big|M_i\Big(\prod Z_k\Big)M_j\Big\rangle\rangle.$$

The action of the tensor could be simplified by studying the coefficients $P(i,j,t)$.

2. Propagation in polynomials space

To simplify the discussion, we start with a special case: $b_1 = 1, b_2 = 0$, and the number of wires is an even number $2N$. In this case, the whole tensor $T^{(\mathrm{whole})}$ could be written as

$$T^{(\mathrm{whole})}(1, t) = \big(T^{(e)}T^{(o)}\big)^t T^{(e)}.$$

Then, we lift the V into the space of the second order 2N-dimensional polynomials

$$W := \mathrm{span}\left\{\sum_{i,j=1}^{2N} c_{i,j}x_i x_j\right\}$$

We can prove that the space V is isometric to the space W by ϕ,

$$\phi : V \to W$$

$$\Big|M_i\Big(\prod Z_k\Big)M_j\Big\rangle\rangle \to x_i x_j$$

$$|Z_i\rangle\rangle \to x_i^2.$$

Let $T_p^{(2N)}(t)$ be the map $\phi T^{(\mathrm{whole})}(1,t)\phi^{-1}$, the following diagram commutes.

$$
\begin{array}{ccc}
V & \xrightarrow{\;T^{(\mathrm{whole})}(1,t)\;} & V \\[2pt]
\Big\downarrow{\phi} & & \Big\downarrow{\phi} \\[2pt]
W & \xrightarrow{\;T_p^{(2N)}(t)\;} & W
\end{array}
$$

Now, let's consider the action of the tensor $T_p^{(2N)}(t)$ on the space W. Similarly, we could write down the recursive relation of coefficients $a_{i,j}$.

When $t = 0$, the transforming state $T^{(e)}|\gamma_1\gamma_{4N}\rangle\rangle$ to W, and we got $\frac{1}{4}(x_1 x_{2N} + x_1 x_{2N-1} + x_2 x_{2N} + x_1 x_{2N-1})$. Suppose at t, the vector in W is $\sum c_i(t)c_j'(t)x_i x_j$,

$$T_p^{(2N)}(t+1)\left(\frac{1}{4}(x_1 x_{2N} + x_1 x_{2N-1} + x_2 x_{2N} + x_1 x_{2N-1})\right)$$

$$= T^{(e)}T^{(o)}T_p^{(2N)}(t)\left(\frac{1}{4}(x_1 x_{2N} + x_1 x_{2N-1} + x_2 x_{2N} + x_1 x_{2N-1})\right)$$

$$= T^{(e)}T^{(o)}\sum c_i(t)c_j'(t)x_i x_j$$

Then, the action of $\phi\, T^{(e)}T^{(o)}\phi^{-1}$ on this vector is

$c_i c_j' x_i x_j$	$\phi\, T^{(e)}T^{(o)}\phi^{-1}\left(c_i c_j' x_i x_j\right)$
$i = j = 1$	$\frac{1}{6}(c_1 x_1 + c_1 x_2)(c_2' x_1 + c_2' x_2)$
$i = 1, j = 2,3$	$\frac{1}{8}(c_1 x_1 + c_1 x_2)\left(\frac{4}{3}c_2' x_1 + \frac{4}{3}c_2' x_2 + c_2' x_3 + c_2' x_4\right)$
$i = 1, j > 3$	$\frac{1}{8}(c_1 x_1 + c_1 x_2)\left(c_j' x_{j-2+\eta_j} + c_j' x_{j-1+\eta_j} + c_j' x_{j+\eta_j} + c_j' x_{j+1+\eta_j}\right)$
$1 < i < 2N,$ $i = j$	$\frac{1}{24}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)$ $\left(c_j' x_{j-2+\eta_j} + c_j' x_{j-1+\eta_j} + c_j' x_{j+\eta_j} + c_j' x_{j+1+\eta_j}\right)$ $-\frac{1}{72}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i}\right)\left(c_j' x_{j-2+\eta_j} + c_j' x_{j-1+\eta_j}\right)$ $-\frac{1}{72}\left(c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)\left(c_j' x_{j+\eta_j} + c_j' x_{j+1+\eta_j}\right)$
$1 < i < 2N,$ i is even, $j = i+1$	$\frac{1}{12}c_i c_{j'}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)^2$ $-\frac{1}{36}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i}\right)\left(c_j' x_{j-2+\eta_j} + c_j' x_{j-1+\eta_j}\right)$ $-\frac{1}{36}\left(c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)\left(c_j' x_{j+\eta_j} + c_j' x_{j+1+\eta_j}\right)$
$1 < i < 2N,$ $i + \eta_i \le j \le i + \eta_i + 2$	$\frac{1}{16}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)$ $\left(c_j' x_{j-2+\eta_j} + c_j' x_{j-1+\eta_j} + c_j' x_{j+\eta_j} + c_j' x_{j+1+\eta_j}\right)$ $+\frac{1}{48}\left(c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)\left(c_j' x_{j-2+\eta_j} + c_j' x_{j-1+\eta_j}\right)$
$1 < i < 2N,$ $j > i + \eta_i + 2$	$\frac{1}{16}\left(c_i x_{i-2+\eta_i} + c_i x_{i-1+\eta_i} + c_i x_{i+\eta_i} + c_i x_{i+1+\eta_i}\right)$ $\left(c_j' x_{j-2+\eta_j} + c_j' x_{j-1+\eta_j} + c_j' x_{j+\eta_j} + c_j' x_{j+1+\eta_j}\right)$

$c_i c_j' x_i x_j$	$\phi\, T^{(e)} T^{(o)} \phi^{-1} \big(c_i c_j' x_i x_j \big)$
$i = j = 2N$	$\frac{1}{6}(c_{2N} x_{2N-1} + c_{2N} x_{2N})(c_{2N}' x_{2N-1} + c_{2N}' x_{2N})$

Table 1: The action of the tensor $\phi\, T^{(e)} T^{(o)} \phi^{-1}$ on the space of second order 2N-dimensional polynomials.

The subsript η_i is defined as $\eta_i := 1 - (i \bmod 2)$. We can see that, c_{2i-1} and c_{2i} are always the same. So do c_{2j-1}' and c_{2j}'. Let $b_i = 2c_{2i-1} = 2c_{2i}$ and $b_j' = 2c_{2j-1}' = 2c_{2j}'$, we could further simplify the action of the tensor in the space of second order N-dimensional polynomials W_N. For simplemess, we define a "free" recursive relation in W_N

$$P(i, t+1) = \begin{cases} \frac{1}{4}(P(i-1,t) + 2P(i,t) + P(i+1,t)) , & i \neq 1 \text{ or } N \\ \frac{1}{4}(P(i-1,t) + 2P(i,t) + P(i+1,t)), & i = 1 \text{ or } N \end{cases}. \quad (1)$$

We call it "free" because $b_i(t) b_j'(t) = P(i,t) P(j,t)$ if i and j are not "collide" with each other (which means $|i - j| > 3$). And the solution of Eq. (1) is a propagating wave. Refs. [1] provide the solution of this equation,

$$P_{n_0}(n, t) = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N-1} \cos\left(\left(n - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos\left(\left(n_0 - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t}\frac{\pi k}{2N}, \quad (2)$$

where n_0 is the initial state. The term $P_{n_0}(n, t)$ also called the propagator.

For our case, there are 2 propagators, which are P_1 and P_N. And the initial state is $x_1 x_N$. Let $\phi()$

Now, let's pluge $b_i = 2c_{2i-1} = 2c_{2i}$ and $b_j' = 2c_{2j-1}' = 2c_{2j}'$ into Table 1. More concretely, let ϕ' be the map

$$\phi' : W_N \to W_{2N}$$

$$y_i y_j \to \frac{1}{4}(x_{2i-1} + x_{2i})(x_{2j-1} + x_{2j}).$$

If we consider the subspace W_{2N}' of W_{2N}, where $W_{2N}' := \mathrm{span}\big((x_{2i-1} + x_{2i})(x_{2j-1} + x_{2j})\big)$ (it means $c_{2i-1} = c_{2i}$ and $c_{2j-1}' = c_{2j}'$), the ϕ' will be the isometric between W_N and W_{2N}'. Thus, the following diagram commutes.

$$
\begin{array}{ccc}
V & \xrightarrow{\ T^{(\mathrm{whole})}(1,t)\ } & V \\[4pt]
\phi\big\updownarrow & & \big\updownarrow\phi \\[4pt]
W'_{2N} & \xrightarrow{\ T_p^{(2N)}(t)\ } & W'_{2N} \\[4pt]
\phi'\big\updownarrow & & \big\updownarrow\phi' \\[4pt]
W_N & \xrightarrow{\ T_p^{(N)}(t)\ } & W_N
\end{array}
$$

For simpliness, let

$$
F(y_i) = \begin{cases}
\frac{3}{4}y_1 + \frac{1}{4}y_2 \ , \ i = 1 \\[6pt]
\frac{1}{4}y_{i-1} + \frac{1}{2}y_i + \frac{1}{4}y_{i+1} \ , \ 1 < i < N \\[6pt]
\frac{3}{4}y_N + \frac{1}{4}y_{N-1} \ , \ i = N
\end{cases}
$$

Then we get

$y_i y_j$	$T^{(e)}T^{(o)}$
$i = j = 1$	$F(y_1)F(y_1) - \frac{5}{144}(y_1 + y_2)^2 + \frac{1}{36}y_1 y_2$
$i = 1, j = 2$	$F(y_1)F(y_2) + \left(\frac{5}{144}\right)(y_1{}^2) + \left(\frac{1}{24}\right)y_1 y_2 + \left(\frac{1}{72}\right)(y_2{}^2)$
$i = 1, j = 3$	$F(y_1)F(y_3) + \frac{1}{48}y_2{}^2$
$1 < i < N, j = i$	$F(y_i)F(y_i) - \dfrac{5}{144}y_{i-1}{}^2 - \dfrac{5}{48}y_{i-1}y_i - \dfrac{1}{16}y_{i-1}y_{i+1}$ $-\dfrac{1}{9}y_i{}^2 - \dfrac{5}{48}y_i y_{i+1} - \dfrac{5}{144}y_{i+1}{}^2$
$1 < i < N, j = i + 1$	$F(y_i)F(y_{i+1}) + \dfrac{1}{72}y_i{}^2 + \dfrac{1}{24}y_i y_{i+1} + \dfrac{1}{72}y_{i+1}{}^2$
$1 < i < N, j = i + 2$	$F(y_i)F(y_{i+2}) + \dfrac{1}{48}y_{i+1}^2$

$y_i y_j$	$T^{(e)} T^{(o)}$
$i = j = N$	$F(y_N)F(y_N) - \frac{5}{144}(y_N + y_{N-1})^2 + \frac{1}{36} y_{N-1} y_N$
other cases when $i \leq j$	$F(y_i)F(y_j)$

Table 2: The action of the tensor $T^{(e)} T^{(o)}$ on the space of second order N-dimensional polynomials.

这个 table 的计算实在是太太太太太折磨人了。详细计算我放在了 Section 4 中。

3. The interaction term

let $b(i, j, t) = P_1(i, t) P_N(j, t) + I(i, j, t)$. Then,

$$|\psi(t)\rangle = \sum_{i,j} P_1(i, t) P_N(j, t) y_i y_j - \sum_{i,j} I(i, j, t) y_i y_j. \tag{3}$$

From simpleness, let the terms in Table 2 be $F(y_i)F(y_j) + R(i, j)$. Then we get

$$|\psi(t+1)\rangle = \sum_{i,j} P_1(i, t) P_N(j, t)\big(F(y_i)F(y_j) + R(i, j)\big) - \sum_{i,j} I(i, j, t)\big(F(y_i)F(y_j) + R(i, j)\big)$$

$$= \sum_{i,j} P_1(i, t+1) P_N(j, t+1) y_i y_j + \sum_{i,j} P_1(i, t) P_N(i, t) R(i, j)$$

$$- \sum_{i,j} I(i, j, t)\big(F(y_i)F(y_j) + R(i, j)\big)$$

这里又又又为了简洁性，定义

$$F_f^{(1)} : \text{Func}(i, j, t) \mapsto \begin{cases} \frac{1}{4} \text{Func}(i-1, j, t) + ... & \text{if } ... \\ \frac{3}{4} \text{Func}(i, j, t) + ... & \text{if i} = 1 \end{cases}$$

$$F_f^{(2)} : \text{Func}(i, j, t) \mapsto \begin{cases} \frac{1}{4} \text{Func}(i, j-1, t) + ... & \text{if } ... \\ \frac{3}{4} \text{Func}(i, j, t) + ... & \text{if j} = 1 \end{cases}$$

By using Eq. (3) to expand $|\psi(t+1)\rangle$, we have

$$I(i, j, t+1)$$

$$= -\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\left(\sum_{i,j} P_1(i,t)P_N(i,t)R(i,j) - \sum_{i,j} I(i,j,t)\big(F(y_i)F(y_j) + R(i,j)\big)\right)_{(4)}$$

$$= F_f^{(1)}(I)(i,j,t)F_f^{(2)}(I)(i,j,t) - \sum_{l,k}\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\Big((P_1(l,t)P_N(k,t) - I(l,k,t))R(l,k)\Big)$$

可以从表中看见，仅当 $l \in \{i-1, i, i+1\} \cap \mathbb{Z}_+$, $k \in \{j-1, j, j+1\} \cap \mathbb{Z}_+$, $|l-k| \le 2$ 时，Eq. (4) 中的第二项才不为 0。同时，我们知道，若其不为 0，则有

$$-\frac{1}{9} \le \frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}R(l,k) \le \frac{1}{24}.$$

由它们的系数 $b(i,j,t) > 0$，我们可以知道 $P_1(l,t)P_N(k,t) > I(l,k,t)$ 对所有时刻 t 恒成立。

由于初始状态下，$I(i,j,t) = 0 \forall i, j$。

现在我们来看一下第二项会是什么。这里主要就是看 R 是什么。这时候我们又开始快乐打表了。打表的一些细节仍然放在 Section 4 中。为了简单，在下面表格中用D_{ij}指代 $P_1(i,t)P_{j(i,t)}$ 或 $I(i,j,t)$。这里需要注意的是，$y_i y_i$ 产生的 $y_i y_{i+1}$ 会写成 $y_i y_{i+1} + y_{i+1} y_i$。因此，在生成下述表格时，$j = i+1$ 那一栏中，在从 Table 2 中写出D_{ii} 的系数时需要除一个 2。

$D(i, j, t+1)$	$\sum \frac{\partial^2}{\partial y_i \partial y_j} D_{lk} R(l, k)$
$i = j = 1$	$\frac{5}{144}(-D_{11} - D_{22} + D_{12} + D_{21})$
$i = 1, j = 2$	$-\frac{1}{48}D_{11} + \frac{1}{24}D_{12} - \frac{5}{96}D_{22}$
$1 < i = j < N$	$\begin{pmatrix} -\frac{5}{144} & \frac{1}{72} & \frac{1}{48} \\ \frac{1}{72} & -\frac{1}{9} & \frac{1}{72} \\ \frac{1}{48} & \frac{1}{72} & -\frac{5}{144} \end{pmatrix}$
$1 < i < N, j = i+1$	$-\frac{5}{96}D_{ii} + \frac{1}{24}D_{i,i+1} - \frac{5}{96}D_{i+1,i+1}$

$D(i,j,t+1)$	$\sum \frac{\partial^2}{\partial y_i \partial y_j} D_{lk} R(l,k)$
$i = N$	$\frac{5}{144}\left(-D_{N,N} - D_{N-1,N-1} + D_{N,N-1} + D_{N-1,N}\right)$

Table 3: The action of the operator $\frac{\partial^2}{\partial y_i \partial y_j}$ on the terms in Table 2.

令 $I_{i,j} = I(i,j,t+1) = I(i,j,t)$，则由 (4)

$$I_{i,j} = F_f^{(1)}(I)(i,j,t)F_f^{(2)}(I)(i,j,t) - \sum_{l,k} \frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\Big((P_1(l,t)P_N(k,t) - I(l,k,t))R(l,k)\Big)$$

再进行一个均匀的线性组合，有

$$0 = \frac{1}{N^2}\sum_{i,j}\Big(I_{i,j} - F_f^{(1)}(I)(i,j,t)F_f^{(2)}(I)(i,j,t)\Big)$$

$$= -\frac{1}{N^2}\sum_{i,j}\sum_{l,k}\frac{\partial}{\partial y_i}\frac{\partial}{\partial y_j}\Big((P_1(l,t)P_N(k,t) - I(l,k,t))R(l,k)\Big)$$

通过查表 Table 3，和一些代数游戏，可以得到

$$\frac{1}{48}\sum_{i=1}^{N-2}P_1(i,t)P_N(i+2,t) + \frac{1}{48}\sum_{i=3}^{N}P_1(i,t)P_N(i-2,t)$$

$$+\frac{5}{72}\sum_{i=1}^{N-1}P_1(i,t)P_N(i+1,t) + \frac{5}{72}\sum_{i=2}^{N}P_1(i,t)P_N(i-1,t)$$

$$-\frac{7}{18}\sum_{i=2}^{N-1}P_1(i,t)P_N(i,t) - \frac{1}{9}P_1(1,t)P_N(1,t) - \frac{1}{9}P_1(N,t)P_N(N,t)$$

i,j	$\sum \frac{\partial^2}{\partial y_i \partial y_j} D_{lk} R(l,k)$
$i = j = 1$	$\frac{5}{144}(-D_{11} - D_{22} + D_{12} + D_{21})$
$i = 1, j = 2$	$-\frac{1}{24}D_{11} + \frac{1}{24}D_{12} - \frac{5}{48}D_{22}$
$1 < i = j < N$	$\begin{pmatrix} -\frac{5}{144} & \frac{1}{72} & \frac{1}{48} \\ \frac{1}{72} & -\frac{1}{9} & \frac{1}{72} \\ \frac{1}{48} & \frac{1}{72} & -\frac{5}{144} \end{pmatrix}$

i, j	$\sum \frac{\partial^2}{\partial y_i \partial y_j} D_{lk} R(l, k)$
$1 < i < N, j = i+1$	$-\frac{5}{48} D_{ii} + \frac{1}{24} D_{i,i+1} - \frac{5}{48} D_{i+1,i+1}$
$i = N$	$\frac{5}{144}\left(-D_{N,N} - D_{N-1,N-1} + D_{N,N-1} + D_{N-1,N}\right)$

4. Proofs

Proof of Table 2: 现在我们想要计算 $T_p^{(N)}(t)$ 在W_N 中的作用。因此我们需要计算

$$\phi'\phi T^{(e)} T^{(o)} \phi^{-1} \phi'^{-1}(y_i y_j)$$

$$= \phi T^{(e)} T^{(o)} \phi^{-1}\left(\frac{1}{4}(x_{2i-1} + x_{2i})(x_{2j-1} + x_{2j})\right)$$

$$= \phi T^{(e)} T^{(o)} \phi^{-1}\left(\frac{1}{4}(x_{2i-1}x_{2j-1} + x_{2i-1}x_{2j} + x_{2i}x_{2j-1} + x_{2i}x_{2j})\right)$$

相当于对于每一个 $x_{2i-1}x_{2j-1}$ 这样的项进行查表。我们现在来分析不同的 y 的映射。为了简化符号，这里记 $S_N = \phi'\phi T^{(e)} T^{(o)} \phi^{-1} \phi'^{-1}$, $S_{2N} = \phi T^{(e)} T^{(o)} \phi^{-1}$.

$$\phi'(y_1 y_1) = \frac{1}{4}(x_1 x_1 + 2 x_1 x_2 + x_2 x_2)$$

$$S_N(y_1 y_1) = \frac{1}{4}(S_{2N}(x_1 x_1) + S_{2N}(2 x_1 x_2) + S_{2N}(x_2 x_2))$$

依据 Table 1, 我们有

$$S_N(y_1 y_1) = \frac{1}{4}\Bigg($$

$$\frac{1}{6}(x_1 + x_2)(x_1 + x_2) +$$

$$\left(\frac{1}{4}\right)\left(\left(\frac{4}{3}\right)x_1 + \left(\frac{4}{3}\right)x_2 + x_3 + x_4\right)(x_1 + x_2) +$$

$$-\left(\frac{1}{72}\right)\left((x_1 + x_2)^2\right) + \left(\frac{1}{24}\right)\left((x_1 + x_2 + x_3 + x_4)^2\right) - \left(\frac{1}{72}\right)\left((x_3 + x_4)^2\right)$$

$$\Bigg)$$

$$= \left(\frac{19}{144}\right)(x_1{}^2) + \left(\frac{19}{72}\right)x_1 x_2 + \left(\frac{1}{12}\right)x_1 x_3 + \left(\frac{1}{12}\right)x_1 x_4$$

$$+ \left(\frac{19}{144}\right)(x_2{}^2) + \left(\frac{1}{12}\right)x_2 x_3 + \left(\frac{1}{12}\right)x_2 x_4$$

$$+ \left(\frac{1}{144}\right)(x_3{}^2) + \left(\frac{1}{72}\right)x_3 x_4 + \left(\frac{1}{144}\right)(x_4{}^2)$$

我们想要和"自由演化"进行对比。而 $y_1 y_1$ 自由演化的结果是

$$F(y_1 y_1) = \frac{1}{16}(3y_1 + y_2)(3y_1 + y_2)$$

$$= \frac{1}{16}\left(\frac{3}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right)\left(\frac{3}{2}x_1 + \frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4\right)$$

$$= \left(\frac{9}{64}\right)(x_1{}^2) + \left(\frac{9}{32}\right)x_1 x_2 + \left(\frac{3}{32}\right)x_1 x_3 + \left(\frac{3}{32}\right)x_1 x_4$$

$$+ \left(\frac{9}{64}\right)(x_2{}^2) + \left(\frac{3}{32}\right)x_2 x_3 + \left(\frac{3}{32}\right)x_2 x_4 + \left(\frac{1}{64}\right)(x_3{}^2)$$

$$+ \left(\frac{1}{32}\right)x_3 x_4 + \left(\frac{1}{64}\right)(x_4{}^2)$$

二者之差为

$$S_N(y_1 y_1) - F(y_1 y_1) = -\left(\frac{5}{576}\right)(x_1{}^2) - \left(\frac{5}{288}\right)x_1 x_2 - \left(\frac{1}{96}\right)x_1 x_3 - \left(\frac{1}{96}\right)x_1 x_4$$

$$-\left(\frac{5}{576}\right)(x_2{}^2) - \left(\frac{1}{96}\right)x_2 x_3 - \left(\frac{1}{96}\right)x_2 x_4$$

$$-\left(\frac{5}{576}\right)(x_3{}^2) - \left(\frac{5}{288}\right)x_3 x_4 - \left(\frac{5}{576}\right)(x_4{}^2)$$

$$= -\frac{5}{576}(x_1 + x_2 + x_3 + x_4)^2 + \frac{1}{144}(x_1 + x_2)(x_3 + x_4)$$

$$= -\frac{5}{144}(y_1 + y_2)^2 + \frac{1}{36}y_1 y_2$$

接下来，算 $S_N(y_1 y_2)$

$$S_N(y_1 y_2) = \frac{1}{4}(S_{2N}(x_1 x_3) + S_{2N}(x_1 x_4) + S_{2N}(x_2 x_3) + S_{2N}(x_2 x_4))$$

$$= \frac{1}{4}\bigg($$

$$\frac{1}{8}\left(\frac{4}{3}x_1 + \frac{4}{3}x_2 + x_3 + x_4\right)(x_1 + x_2)$$

$$+\frac{1}{8}(x_3 + x_4 + x_5 + x_6)(x_1 + x_2)$$

$$-\left(\frac{1}{36}\right)\left((x_1 + x_2)^2\right) + \left(\frac{1}{12}\right)\left((x_1 + x_2 + x_3 + x_4)^2\right) - \left(\frac{1}{36}\right)\left((x_3 + x_4)^2\right)$$

$$+\frac{1}{16}(x_1 + x_2 + x_3 + x_4)(x_3 + x_4 + x_5 + x_6) + \frac{1}{48}(x_3 + x_4)^2$$

$$\bigg) = \left(\frac{2}{9}\right)(y_1{}^2) + \left(\frac{23}{48}\right)y_1 y_2 + \left(\frac{3}{16}\right)y_1 y_3 + \left(\frac{5}{36}\right)(y_2{}^2) + \left(\frac{1}{16}\right)y_2 y_3$$

$$F(y_1)F(y_2) = \left(\tfrac{3}{4}y_1 + \tfrac{1}{4}y_2\right)\left(\tfrac{1}{4}y_1 + \tfrac{1}{2}y_2 + \tfrac{1}{4}y_3\right)$$

$$S_N(y_1 y_2) = F(y_1)F(y_2) + \left(\frac{5}{144}\right)(y_1{}^2) + \left(\frac{1}{24}\right)y_1 y_2 + \left(\frac{1}{72}\right)(y_2{}^2)$$

接下来，算 $S_N(y_1 y_3)$

$$S_N(y_1 y_3) = F(y_1)F(y_3) + \frac{1}{48}y_2{}^2$$

然后，$j > 3$时

$$S_N(y_1 y_j) = F(y_1)F(y_j)$$

然后，$i > 1, i = j$时，

$$S_N(y_i y_j) = \frac{1}{4}\big(S_N(x_{2i-1}x_{2j-1}) + S_N(x_{2i-1}x_{2j}) + S_N(x_{2i}x_{2j-1}) + S_N(x_{2i}x_{2j})\big)$$

$$= \frac{1}{4}\big(S_N(x_{2i-1}x_{2i-1}) + 2S_N(x_{2i-1}x_{2i}) + S_N(x_{2i}x_{2i})\big)$$

$$= \frac{1}{4}\bigg($$

$$\frac{1}{24}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})^2$$

$$-\frac{1}{72}(x_{2i-3} + x_{2i-2})^2 - \frac{1}{72}(x_{2i-1} + x_{2i})^2$$

$$+\frac{1}{24}(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})^2$$

$$-\frac{1}{72}(x_{2i-1} + x_{2i})^2 - \frac{1}{72}(x_{2i+1} + x_{2i+2})^2$$

$$+\frac{1}{16}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})$$

$$+\frac{1}{48}(x_{2i-1} + x_{2i})^2$$

$$\bigg)$$

$$= \frac{1}{24}(y_{i-1} + y_i)^2 - \frac{1}{72}(y_{i-1})^2 - \frac{1}{72}(y_i)^2$$

$$+\frac{1}{24}(y_i + y_{i+1})^2 - \frac{1}{72}(y_i)^2 - \frac{1}{72}(y_{i+1})^2$$

$$+\frac{1}{16}(y_{i-1} + y_i)(y_i + y_{i+1}) + \frac{1}{48}(y_i)^2$$

$$S_N(y_i y_j) = F(y_i)F(y_j) - \left(\frac{5}{144}\right)(y_{i-1}{}^2) - \left(\frac{5}{48}\right)y_{i-1}y_i - \left(\frac{1}{16}\right)y_{i-1}y_{i+1}$$

$$-\left(\frac{1}{9}\right)(y_i{}^2) - \left(\frac{5}{48}\right)y_i y_{i+1} - \left(\frac{5}{144}\right)(y_{i+1}{}^2)$$

然后，$i > 1, j = i + 1$时，

$$S_N(y_i y_j) = \frac{1}{4}\left(S_N(x_{2i-1}x_{2i+1}) + S_N(x_{2i-1}x_{2i+2}) + S_N(x_{2i}x_{2i+1}) + S_N(x_{2i}x_{2i+2})\right)$$

$$= \frac{1}{4}\bigg($$

$$\frac{1}{16}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})$$

$$+\frac{1}{48}(x_{2i-1} + x_{2i})^2$$

$$+\frac{1}{16}(x_{2i-3} + x_{2i-2} + x_{2i-1} + x_{2i})(x_{2i+1} + x_{2i+2} + x_{2i+3} + x_{2i+4})$$

$$+\frac{1}{12}(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})^2$$

$$-\frac{1}{36}(x_{2i-1} + x_{2i})^2 - \frac{1}{36}(x_{2i+1} + x_{2i+2})^2$$

$$+\frac{1}{16}(x_{2i-1} + x_{2i} + x_{2i+1} + x_{2i+2})(x_{2i+1} + x_{2i+2} + x_{2i+3} + x_{2i+4})$$

$$+\frac{1}{48}(x_{2i+1} + x_{2i+2})^2$$

$$\bigg)$$

$$= \frac{1}{16}(y_{i-1} + y_i)(y_i + y_{i+1}) + \frac{1}{48}(y_i)^2$$

$$+\frac{1}{16}(y_i + y_{i+1})(y_{i+1} + y_{i+2}) + \frac{1}{48}(y_{i+1})^2$$

$$+\frac{1}{16}(y_{i-1} + y_i)(y_{i+1} + y_{i+2})$$

$$+\frac{1}{12}(y_i + y_{i+1})^2 - \frac{1}{36}(y_i)^2 - \frac{1}{36}(y_{i+1})^2$$

$$S_N(y_i y_j) = F(y_i)F(y_j) + \left(\frac{1}{72}\right)(y_i{}^2) + \left(\frac{1}{24}\right)y_i y_{i+1} + \left(\frac{1}{72}\right)(y_{i+1}{}^2)$$

$$\square$$

Proof of Table 3: 观察 Table 2，仅有 $y_1 y_1, y_1 y_2, y_2 y_1, y_2 y_2$的项可以对 $I(1, 1, t + 1)$ 产生贡献。因此，

$$\frac{\partial^2}{\partial y_i y_j} P_1(l,t) P_N(k,t) R(l,k)$$

$$= \frac{5}{144}(-P_1(1,t)P_N(1,t) - P_1(2,t)P_N(2,t) + P_1(1,t)P_N(2,t) + P_1(2,t)P_N(1,t))$$

$$= -\frac{5}{144}(P_1(1,t) - P_1(2,t))(P_N(1,t) - P_N(2,t))$$

现在重要的就是确认这一项的符号。好在我们是知道 P 的具体形式的。由 Eq. (2) 可以得到

$$P_1(n,t) = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N-1} \cos\left(\left(n - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

$$P_N(n,t) = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N-1} (-1)^k \cos\left(\left(n - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

由 Lemma 4.1 可以得到，$P_1(1,t) - P_1(2,t) < 0, P_N(1,t) - P_N(2,t) > 0$。因此，$I(1,1,t+1) > 0$。

当 $i = 1, j = 2$ 时

$$\frac{\partial^2}{\partial y_i \partial y_j} P_1(l,t) P_N(k,t) R(l,k)$$

$$= -\frac{1}{144} P_1(1) P_N(1) + \frac{1}{24} P_1(1) P_N(2)$$

$$+ \frac{1}{24} P_1(2) P_N(1) - \frac{5}{48} P_1(2) P_N(2)$$

当 $i = j = 2$ 时

$$\frac{\partial^2}{\partial y_i \partial y_j} P_1(l,t) P_N(k,t) R(l,k)$$

$$= -\frac{5}{144} P_1(1) P_N(1) + \frac{1}{72} P_1(1) P_N(2) + \frac{1}{48} P_1(1) P_N(3)$$

$$+ \frac{1}{72} P_1(2) P_N(1) - \frac{1}{9} P_1(2) P_N(2) + \frac{1}{72} P_1(2) P_N(3)$$

$$+ \frac{1}{48} P_1(3) P_N(1) + \frac{1}{72} P_1(3) P_N(2) - \frac{5}{144} P_1(3) P_N(3)$$

$$= (P_N(1) \ P_N(2) \ P_N(3)) \begin{pmatrix} -\frac{5}{144} & \frac{1}{72} & \frac{1}{48} \\ \frac{1}{72} & -\frac{1}{9} & \frac{1}{72} \\ \frac{1}{48} & \frac{1}{72} & -\frac{5}{144} \end{pmatrix} \begin{pmatrix} P_1(1) \\ P_1(2) \\ P_1(3) \end{pmatrix}$$

可以看到, 当 $1 < i = j < N$ 时 $\frac{\partial^2}{\partial y_i \partial y_j} P_1(l,t) P_N(k,t) R(l,k)$ 都符合上述形式。因此, 我们可以得到

$$\sum_i \frac{\partial^2}{\partial {y_i}^2} P_1(l,t) P_N(k,t) R(l,k)$$

$$= \sum (P_N(i-1) \ P_N(i) \ P_N(i+1)) \begin{pmatrix} -\frac{5}{144} & \frac{1}{72} & \frac{1}{48} \\ \frac{1}{72} & -\frac{1}{9} & \frac{1}{72} \\ \frac{1}{48} & \frac{1}{72} & -\frac{5}{144} \end{pmatrix} \begin{pmatrix} P_1(i-1) \\ P_1(i) \\ P_1(i+1) \end{pmatrix}$$

$$= \sum \mathrm{tr} \left(\begin{pmatrix} -\frac{5}{144} & \frac{1}{72} & \frac{1}{48} \\ \frac{1}{72} & -\frac{1}{9} & \frac{1}{72} \\ \frac{1}{48} & \frac{1}{72} & -\frac{5}{144} \end{pmatrix} \begin{pmatrix} P_1(i-1) \\ P_1(i) \\ P_1(i+1) \end{pmatrix} (P_N(i-1) \ P_N(i) \ P_N(i+1)) \right)$$

现在来看看后面这个矩阵

$$\mathcal{M} = \sum \begin{pmatrix} P_1(i-1) \\ P_1(i) \\ P_1(i+1) \end{pmatrix} (P_N(i-1) \ P_N(i) \ P_N(i+1))$$

其矩阵元为

$$\mathcal{M}_{l,m} = \sum_{k=2}^{N-1} P_1(k + \xi_l) P_N(k + \xi_m)$$

其中, $\xi_l = -1, 0, 1$。我们知道 $\sum P_1(i,t) = \sum P_{N(i,t)} = 1$。 $\qquad \square$

Lemma 4.1：$P_1(i, t)$ 随着 i 单减，$P_N(i, t)$ 随着 i 单增。

Proof:

先将 $P(i, t)$ 从整数域延拓到实数域上。这样就可以对 i 进行求导。此时，由 Eq.(2) 可以得到

$$P_1(x, t) = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N-1} \cos\left(\left(x - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

$$\frac{\partial}{\partial x} P_1(x, t) = -\frac{2\pi}{N} \sum \frac{k}{N} \sin\left(\left(x - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

$$\frac{\partial}{\partial t} P_1(x, t) = \frac{2}{N} \sum \cos\left[\frac{k\pi}{2N}\right]^{1+2t} \cos\left[\frac{\left(-\left(\frac{1}{2}\right) + i\right)k\pi}{N}\right] \ln\left(\cos^2\left(\frac{k\pi}{2N}\right)\right)$$

$$P_N(x, t) = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{N-1} (-1)^k \cos\left(\left(x - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

$$\frac{\partial}{\partial x} P_N(x, t) = -\frac{2\pi}{N} \sum (-1)^k \frac{k}{N} \sin\left(\left(x - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

现在关注它们导数的符号。注意到 $\cos\left(\frac{\pi k}{2N}\right)$ 随着 k 增大单减。令 $\kappa = \frac{\pi k}{N} \in (0, \pi)$

$$\sum k \frac{\pi}{N} \sin\left(\left(x - \frac{1}{2}\right)\frac{\pi k}{N}\right) \cos^{2t+1}\left(\frac{\pi k}{2N}\right)$$

$$= \text{Im}\left[\sum \kappa e^{i\left(x - \frac{1}{2}\right)\kappa} \cos^{2t+1}\left(\frac{\kappa}{2}\right)\right]$$

假设 N 趋于无穷，那么

$$\frac{\partial}{\partial x} P_1(x, t) \to -2 \int_0^\pi \kappa \sin\left(\left(x - \frac{1}{2}\right)\kappa\right) \cos^{2t+1}\left(\frac{\kappa}{2}\right) d\kappa$$

$$P_1(x, t) = \frac{2}{\pi} \int_0^\pi \cos\left[\left(x - \frac{1}{2}\right)\kappa\right] * \cos\left[\frac{\kappa}{2} * (2t + 1)\right] d\kappa$$

$$= \frac{1}{\pi}\left[\frac{\sin[\pi(1 + t - x)]}{1 + t - x} + \frac{\sin[\pi(t + x)]}{t + x}\right]$$

$\square$

$$P(x, t+1) - P(x, t) = -\frac{2}{N} \sum \left(\cos\left[\frac{k\pi}{2N}\right]^{1+2t} \cos\left[\frac{k\pi(-(\frac{1}{2}) + x)}{N}\right] \sin\left[\frac{k\pi}{2N}\right]^2 \right)$$

Lemma 4.2: $I(i, j, i+1)$

Proof:

$$I(i, i, t+1) = F_f^{(1)}(I)(i, j, t) F_f^{(2)}(I)(i, j, t)$$

$$- \sum_{l,k} \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \Big((P_1(l, t) P_N(k, t) - I(l, k, t)) R(l, k) \Big)$$

令 $\mathcal{M} = \begin{pmatrix} -\frac{5}{144} & \frac{1}{72} & \frac{1}{48} \\ \frac{1}{72} & -\frac{1}{9} & \frac{1}{72} \\ \frac{1}{48} & \frac{1}{72} & -\frac{5}{144} \end{pmatrix}$。 由 Table 3 可以得到

$\square$

Lemma 4.3: P(x, t+1)-P(x,t)

Proof:

x	$P(x, t+1) - P(x, t)$
$x = 1$	$\dfrac{\Gamma\left[\frac{3}{2} + t\right]}{\sqrt{\pi}\Gamma[3 + t]}$
$x = 2$	$\dfrac{(-3 + t)\Gamma\left[\frac{3}{2} + t\right]}{\sqrt{\pi}\Gamma[4 + t]}$
$x = 3$	$\dfrac{(-11 + t)\Gamma\left[\frac{3}{2} + t\right]}{\sqrt{\pi}\Gamma[5 + t]}$
$x = 4$	$\dfrac{(-23 + t)\Gamma\left[\frac{3}{2} + t\right]}{\sqrt{\pi}\Gamma[6 + t]}$

从这里可以看出，分母就是简单的 $\Gamma(t+x+2)$。但分子有点诡异

□

Lemma 4.4: $|I(i,j,t)| \le P_1(i,t)P_N(j,t)$

Proof: 用归纳法证明

当 $t=1$ 时，成立

假设 $t=n$ 时成立，那么 $t=n+1$ 时

$$I(t+1) \ge -P_1(i,t)P_N(j,t) - \sum_{l,k} \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \Big((P_1(l,t)P_N(k,t) - I(l,k,t))R(l,k) \Big)$$

□

Bibliography

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