Introduction to Financial Mathematics

Lecture 9: Bonds: Duration, Convexity and immunization.

December 14, 2024

1 Yield Curve and Bonds

A bond entitles its holder specified payments at future dates. As an example consider a treasury note of two years with a coupon rate of 0.38%. It was auctioned on April 27, 2020 and issued on April, 30, 2020. For a face value of one million dollars, the future payments were the followings:

- On 10/30/2020, $1,900;
- On 04/30/2021, $1,900;
- On 10/30/2021, $1,900;
- On 04/30/2022, $1,900 + $1,000,000.

The total *un-discounted total* payment is $1,007,600. Its price is determined in an auction and the coupon rate determined so as to make this price close to its face value.

The prices are quoted for a face value of $100. Say the price is $100.31. Then, for the above bond payments, one needs to pay $1,003,100. In four years, the total *un-discounted* gains is $4,500.

1.1 Internal Rate of Return

A *cash flow* is a sequence of future payments. Prices of them depend on the times and sizes of the payments, making it is hard to compare. The following yield definition provides a comparison mechanism.

Definition 1.1.
Let $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}$ *be a sequence of future payments that will be made at times* $0 < t_1 < t_2 < \ldots < t_n$. *Suppose that its price is* $p(\mathbf{c})$. *The* internal rate of return *(IRR) of the cash flow* $\mathbf{c}$ *is the yield value* r *satisfying the following equation,*

$$\boxed{p(\mathbf{c}) = \sum_{k=1}^{n} c_k\, e^{-r\, t_k}.} \tag{1.1}$$

In the case of a bond, the internal rate of return is called *yield-to-maturity*. Form the above definition, it is not clear if the internal rate of return is a well-defined quantity. The following simple result resolves this issue.

Proposition 1.2. *Suppose that $c_k > 0$ for every k and $p(\mathbf{c}) > 0$. Then, the internal rate of return r of $\mathbf{c}$ is uniquely defined. Moreover, it is inversely related to the price and it is strictly positive whenever $p(\mathbf{c}) < \sum_{k=1}^n c_k$.*

Proof. Define a function,

$$r \in \mathbb{R} \ \mapsto \ pv(r) := \sum_{k=1}^n c_k \, e^{-r\, t_k}.$$

Then,

$$pv'(r) = -\sum_{k=1}^n t_k \, c_k \, e^{-r\, t_k} < 0 \qquad \Rightarrow \qquad pv \text{ is strictly decreasing.}$$

Moreover,

$$\lim_{r \to \infty} pv(r) = 0, \quad \text{and} \qquad \lim_{r \to 0} pv(r) = pv(0) = \sum_k c_k.$$

Hence, the function pv attains any positive value at a unique point. This proves that the internal rate of return is uniquely defined. The positivity of it follows from the monotonicity and the fact that $pv(0) = \sum_{k=1}^n c_k$. □

Example 1.3. *Consider the cash flow discussed at the beginning of the chapter. In this example, the equation* (**??**) *is given by,*

$$p(\mathbf{c}) = 1,003,100 = pv(r) \ = \ 1,900[e^{-r/2} + e^{-r} + e^{-3r/2} + e^{-2r}] + 1,000,000 e^{-2r}.$$

As the price is greater than the face value, we immediately observe that the internal rate of return r is less than the coupon rate. However, they should be close to each other as the difference between the price and the face value is not much.

The solution obtained by a root solver is given by $r = 0.224\%$. Hence, this bonds yield-to-maturity is 0.224%.

Exercise. This exercise connects the coupon of a bond to its internal rate of return.

1. Consider a bound with face value of one, and annual coupon of c. Suppose that the price of this bond is equal to the face value. Show that the internal rate ϱ is given by,

$$\varrho = \ln(1+c), \quad \Leftrightarrow \quad c = e^{\varrho} - 1.$$

2. Consider the same question if the coupons are paid semi-annually, i.e., $c/2$ is paid every half year. Show that

$$\varrho = 2\ln(1+\tfrac{c}{2}), \quad \Leftrightarrow \quad \frac{c}{2} = e^{\frac{1}{2}\varrho} - 1.$$

Note that in both cases, if c is small then $\varrho \approx c$.

1.2 Yield Curve

The internal rate of returns of zero-coupon bonds, also called *strips*, as a function of their maturity provide an effective overview of how the markets view the future interest rates. The relationship between the yield-to-maturity for zero-coupon tresuary bonds and the time to maturity is called the *term structure of interest rates*. This is a rich area for research and applications.

We start with a definition.

Definition 1.4.

The yield curve *is the function that maps time t to the internal rate of return $r(t)$ of the treasury bond with zero coupon and maturity t, i.e.,*

$$\boxed{r(t) = -\frac{\ln(B(t))}{t}, \qquad \Leftrightarrow \qquad B(t) = e^{-tr(t)},} \tag{1.2}$$

where $B(t)$ is the price of the treasury bond with zero coupon and maturity t.

The yield curve allows us to price all cash flows. Indeed, if a cash flow does not cary any default risk and other frictions, then its theoretical price must be given by,

$$\boxed{\text{Present Value of } \mathbf{c} = PV(\mathbf{c}) := \sum_{k=1}^{n} c_k \, e^{-t_k r(t_k)}.}$$

In the case of coupon bearing treasury bonds, above price is the actual price. For other cash flows, due to default risk $PV(\mathbf{c})$ is in fact an upper bound that one expects in practice and its distance to the actual price provides valuable information.

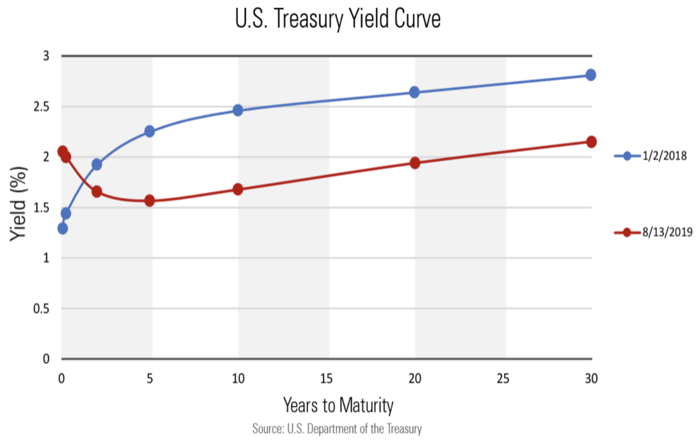
Followings are some yield curves from 2018 and 2020. It is important to note that the yield curve theoretically can be of any shape of the yield and historically almost all have been realized. The below example shows two days in 2018 with very different term structures.

Constructing the Yield Curve.

The definition of the yield curve $r(t)$ requires to have a zero coupon treasury bond or a strip with that exact maturity t. In practice, zero-coupon bonds are issued only for very short maturities. However, each coupon bearing bond is a linear combination of many strips and thus carry information about not one but many points on the yield curve.

Suppose that there are M liquidly traded treasury bonds with prices B_m and annual coupon of c_m paid at time $t_1^m < \ldots < t_{n_m}^m$ which are half a year apart, and t_{n_m} is its maturity. We assume that face value is one. Suppose $t \mapsto r(t)$ is the yield curve. To simplify the notation, set $B(t) := e^{-r(t)\,t}$ be the price of the strip with maturity t. Then,

$$B_m = \sum_{k=1}^{n_m} \frac{c_m}{2} B(t_k^m) \; + \; B(t_k^{n_m}), \qquad\qquad m = 1, \ldots, M.$$



This provides M equations and typically M is a large number. The unknowns are all strip prices

$$\{B(t_k^m)\ :\ k=1,\ldots,n_m,\ m=1,\ldots M\}.$$

There could be at most $\sum_{m=1}^{M} n_m$ of them and n_m is much larger than one. So the number of unknowns is generally much larger than the number of equations. The main observation to get an 'almost' unique yield curve is to use the fact that strip value must be a continuous function of their maturities. So $B(t_k^m)$ values with close maturities can be equated, reducing the number of unknowns.

2 Duration and Convexity

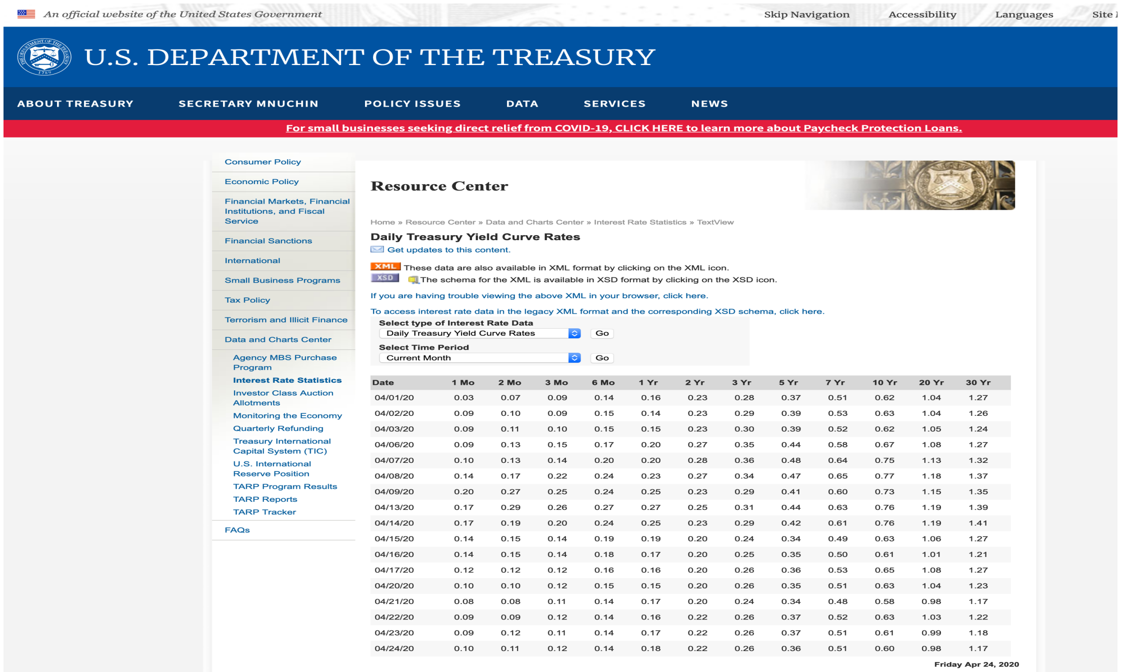
This notion tries to find a way to summarize a complex future cash payments with one lump-payment at some future date called the maturity. Since in addition to the date, we can also change the amount of the payment, this is not a well-posed question.

A better posed question is this. Consider a cash flow of $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}$ at times $t_1, t_2, \ldots, t_n$. Suppose that the yields $r(t_1), r(t_2), \ldots, r(t_n)$ are known. Then, the sensitivity of this cash flow with respect to the interest rate is defined by a considering small change ρ in all yields. Such a change moves the present value of this cash flow to

$$PV(\mathbf{c};\rho) := \sum_{k=1}^{n} c_k\, e^{-[r(t_k)+\rho]t_k}.$$

Note that $PV(\mathbf{c}) = PV(\mathbf{c};0)$. Then, the derivative of the above function with respect to ρ evaluated at $\rho=0$ gives us its sensitivity to the interest rate changes:

$$\frac{d}{d\rho}PV(\mathbf{c};0) = -\sum_{k=1}^{n} t_k\ PV_k, \quad \text{where} \quad PV_k := c_k\, e^{-r(t_k)t_k}.$$



An official website of the United States Government — Skip Navigation — Accessibility — Languages

U.S. DEPARTMENT OF THE TREASURY

ABOUT TREASURY — SECRETARY MNUCHIN — POLICY ISSUES — DATA — SERVICES — NEWS

For small businesses seeking direct relief from COVID-19, CLICK HERE to learn more about Paycheck Protection Loans.

Consumer Policy
Economic Policy
Financial Markets, Financial Institutions, and Fiscal Service
Financial Sanctions
International
Small Business Programs
Tax Policy
Terrorism and Illicit Finance
Data and Charts Center
Agency MBS Purchase Program
Interest Rate Statistics
Investor Class Auction Allotments
Monitoring the Economy
Quarterly Refunding
Treasury International Capital System (TIC)
U.S. International Reserve Position
TARP Program Results
TARP Reports
TARP Tracker
FAQs

Resource Center

Home » Resource Center » Data and Charts Center » Interest Rate Statistics » TextView

Daily Treasury Yield Curve Rates

Get updates to this content.

XML These data are also available in XML format by clicking on the XML icon.

XSD The schema for the XML is available in XSD format by clicking on the XSD icon.

If you are having trouble viewing the above XML in your browser, click here.

To access interest rate data in the legacy XML format and the corresponding XSD schema, click here.

Select type of Interest Rate Data: Daily Treasury Yield Curve Rates — Go

Select Time Period: Current Month — Go

Date	1 Mo	2 Mo	3 Mo	6 Mo	1 Yr	2 Yr	3 Yr	5 Yr	7 Yr	10 Yr	20 Yr	30 Yr
04/01/20	0.03	0.07	0.09	0.14	0.16	0.23	0.28	0.37	0.51	0.62	1.04	1.27
04/02/20	0.09	0.10	0.09	0.15	0.14	0.23	0.29	0.39	0.53	0.63	1.04	1.26
04/03/20	0.09	0.11	0.10	0.15	0.15	0.23	0.30	0.39	0.52	0.62	1.05	1.24
04/06/20	0.09	0.13	0.15	0.17	0.20	0.27	0.35	0.44	0.58	0.67	1.08	1.27
04/07/20	0.10	0.13	0.14	0.20	0.20	0.28	0.36	0.48	0.64	0.75	1.13	1.32
04/08/20	0.14	0.17	0.22	0.24	0.23	0.27	0.34	0.47	0.65	0.77	1.18	1.37
04/09/20	0.20	0.27	0.25	0.24	0.25	0.23	0.29	0.41	0.60	0.73	1.15	1.35
04/13/20	0.17	0.29	0.26	0.27	0.27	0.25	0.31	0.44	0.63	0.76	1.19	1.39
04/14/20	0.17	0.19	0.20	0.24	0.25	0.23	0.29	0.42	0.61	0.76	1.19	1.41
04/15/20	0.14	0.15	0.14	0.19	0.19	0.20	0.24	0.34	0.49	0.63	1.06	1.27
04/16/20	0.14	0.15	0.14	0.18	0.17	0.20	0.25	0.35	0.50	0.61	1.01	1.21
04/17/20	0.12	0.12	0.12	0.16	0.16	0.20	0.26	0.36	0.53	0.65	1.08	1.27
04/20/20	0.10	0.10	0.12	0.15	0.15	0.20	0.26	0.35	0.51	0.63	1.04	1.23
04/21/20	0.08	0.08	0.11	0.14	0.17	0.20	0.24	0.34	0.48	0.58	0.98	1.17
04/22/20	0.09	0.09	0.12	0.14	0.16	0.22	0.26	0.37	0.52	0.63	1.03	1.22
04/23/20	0.09	0.12	0.11	0.14	0.17	0.22	0.26	0.37	0.51	0.61	0.99	1.18
04/24/20	0.10	0.11	0.12	0.14	0.18	0.22	0.26	0.36	0.51	0.60	0.98	1.17

Friday Apr 24, 2020

Definition 2.1. *The* duration *of the cash flow* $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}$ *paid at times* $t_1, t_2, \ldots, t_n$ *is defined to be*

$$\boxed{D(\mathbf{c}) := -\frac{\frac{d}{d\rho} PV(\mathbf{c}; 0)}{PV(\mathbf{c})} = \sum_{k=1}^{n} t_k \, \frac{PV_k}{PV(\mathbf{c})}, \quad \textit{where} \quad PV_k := c_k \, e^{-r(t_k) t_k}.}$$

Note that we may consider the original cash flow $\mathbf{c} = \{c_1, \ldots, c_n\}$ as a sum of n separate cash flows each paying c_k at time t_k. These cash flows are simply zero-coupon bonds with face value c_k, duration t_k, and their value is PV_k. Also,

$$PV(\mathbf{c}) = PV(\mathbf{c}; 0) = \sum_{k=1}^{n} PV_k \quad \Leftrightarrow \quad \sum_{k=1}^{n} \frac{PV_k}{PV(\mathbf{c})} = 1.$$

Hence, the duration of $\mathbf{c}$ is simply the weighted average of the durations of its components, and their weights are proportional to their present value.

By considering the second derivatives we also have the following definition.

Definition 2.2. *The* convexity *of the cash flow* $\mathbf{c} = \{c_1, c_2, \ldots, c_n\}$ *paid at times* $t_1, t_2, \ldots, t_n$ *is defined to be*

$$\boxed{C(\mathbf{c}) := \frac{\frac{d^2}{d\rho^2} PV(\mathbf{c}; 0)}{PV(\mathbf{c})} = \sum_{k=1}^{n} (t_k)^2 \, \frac{PV_k}{PV(\mathbf{c})}.}$$

Since $\sum_k PV_k = PV$, $\sum_k (PV_k/PV) = 1$ and consequently, the duration of a cash flow is the convex combination of the future payment times and the weights are given by the present value of the payment that will be made at that time.

The duration of the cash flow consists of a one lump-payment payment at time T is equal to T and its convexity is T^2. Although the duration and convexity are not linear, the duration and the convexity of a cash flow obtained by combining two cash flows can be easily calculated.

Consider M cash flows $\mathbf{c}_1, \ldots, \mathbf{c}_M$ where for $i = 1, \ldots, M$, $\mathbf{c}_i = (c_1^i, \ldots c_{n_i}^i)$ paid at times $t_1^i < \ldots < t_{n_i}^i$ and let (P_i, D_i, C_i) be their present value, duration and convexities.

Theorem 2.3. *Consider above* M *cash flows. Let* $\mathbf{c}$ *be the cash flow obtained by combining them. Then,*

$$PV(\mathbf{c}) =: P = \sum_{i=1}^{M} P_i, \qquad D(\mathbf{c}) = \sum_{i=1}^{M} \frac{P_i}{P} D_i, \qquad C(\mathbf{c}) = \sum_{i=1}^{M} \frac{P_i}{P} C_i.$$

Proof. First statement follows directly from the definition.

$$D(\mathbf{c}) = \sum_{i=1}^{M} \sum_{k=1}^{n_i} t_k^i \, \frac{PV(c_k^i)}{P} = \sum_{i=1}^{M} \frac{P_i}{P} \sum_{k=1}^{n_i} t_k^i \, \frac{PV(c_k^i)}{P_i} = \sum_{i=1}^{M} \frac{P_i}{P} D_i.$$

The statement for convexity is proved exactly the same way. $\square$

3 Risk Reduction

Portfolios with fixed income securities are subject to interest rate risk. In practice one can reduce this risk by design the portfolio in an appropriate way. Duration matching and more generally immunization is the main approach that we now outline.

3.1 Duration Matching

Suppose that we have sold a financial contract that will pay $\mathbf{c} = \{c_1, c_2, \dots, c_n\}$ at times $t_1, \dots, t_n$. The price of the cash flow now is equal to its present value $PV(\mathbf{c})$ and we receive this amount. The promised future cash flow $\mathbf{c}$ induces a liability on us that will move randomly in the future creating a risk. We may hedge this risk, by buying one zero-coupon bond that pays x dollars at time T. The question is how to choose x and T. Firstly, the value of the zero-coupon bond must be equal to the present value of the cash flow:

$$xe^{-r(T)T} = PV(\mathbf{c}), \quad \Rightarrow \quad x = e^{r(T)T} PV(\mathbf{c}). \tag{3.1}$$

To determine T, we match the durations of the cash flow $\mathbf{c}$ and the zero-coupon bond. Since the duration of any zero-coupon bond is equal to its maturity, we have

$$T = D(\mathbf{c}) = \sum_{k=1}^{n} t_k \, \frac{PV_k}{PV(\mathbf{c})}.$$

After we sell the cash flow $\mathbf{c}$ for $PV(\mathbf{c})$ and buy the zero-coupon bond (the strip) as above, our position is as follows. As the prices of the cash flow and the strip are equal to each other, we initially paid zero dollars. In the future, we have to pay

$$\hat{\mathbf{c}} = \{c_1, \dots, -\, x, \dots, c_n\},$$

dollars at times $t_1, \dots, T, \dots, t_n$. Notice that the minus sign in $-\, x$ indicates that we will receive x dollars at time T. The present value of $\hat{\mathbf{c}}$ with the current yield curve is zero. But if the yields of all maturities move by an amount of ρ, its value will become

$$PV(\hat{\mathbf{c}}; \rho) = \sum_{k=1}^{n} c_k \, e^{-[r(t_k)+\rho]t_k} - x \, e^{-[r(T)+\rho]T}.$$

The sensitivity of this value to the interest rate is equal to its derivative with respect to ρ at $\rho = 0$. It is given by,

$$-\frac{d}{d\rho} PV(\hat{\mathbf{c}}; 0) = \sum_{k=1}^{n} t_k \, PV_k \; - \; T \, x \, e^{-r(T)T}.$$

where as before $PV_k := c_k \, e^{-r(t_k)t_k}$. Hence, by (**??**),

$$-\frac{d}{d\rho} PV(\hat{\mathbf{c}}; 0) = \sum_{k=1}^{n} t_k \, PV_k - T \, x \, e^{-r(T)T} = D(\mathbf{c}) PV(\mathbf{c}) - T \, PV(\mathbf{c}) = 0.$$

In summary, *by matching the durations, we obtain a portfolio whose first order sensitivity with respect to interest rate is zero.* This is exactly the same philosophy that is behind delta hedging in the options market.

In practice, we can not find bonds with all maturities. Then, we use two bonds with maturities $T_1 \leq D(\mathbf{c}) \leq T_2$. The yields of each are $r_i = r(T_i)$. Then, the present values with face value x_i are

$$p_i = x_i e^{-r_i T_i}, \quad i = 1, 2.$$

Clearly their durations are equal to T_i. The flexibility we now have are the values x_i. Matching the present values values and the durations, we obtain the following two equations by using Theorem **??**,

$$\begin{aligned} PV(\mathbf{c}) &= x_1\, e^{-r_1 T_1} + x_2\, e^{-r_2 T_2}, \\ D(\mathbf{c}) &= x_1\, \frac{e^{-r_1 T_1}}{PV(\mathbf{c})}\, T_1 + x_2\, \frac{e^{-r_2 T_2}}{PV(\mathbf{c})}\, T_2. \end{aligned}$$

The above is a system of two linear equations with two unknowns x_1, x_2.

3.2 Immunization

If there is enough flexibility, we would like to match the convexity as well, which is called immunization. We give the following example to illustrate the ideas:

Consider a 30 year bond with a coupon rate of 1.8%. Using the yield curve, its price (or present value), duration and convexity are given by,

$$\text{price} = 99.8642, \quad \text{duration} = 16.8813, \quad \text{convexity} = 317.4994.$$

To immunize this bond we can use 3 bonds with maturities 5, 10 and 20 respectively. Then, the amounts to be bought x_1, x_2, x_3 must solve the equations

$$\begin{aligned} 99.8642 &= x_1 p_1 + x_2 p_2 + x_3 p_3 =: p^* \\ 16.8813 &= x_1 d_1\, \frac{p_1}{p^*} + x_2 d_2\, \frac{p_2}{p^*} + x_3 d_3\, \frac{p_3}{p^*} \\ 317.4994 &= x_1 c_1\, \frac{p_1}{p^*} + x_2 c_2\, \frac{p_2}{p^*} + x_3 c_3\, \frac{p_3}{p^*} \end{aligned}$$

where p_i's are prices, d_i's are durations and c_i's are convexities of the bonds.

Exercise: Suppose that the yield curve and two cash flows are given in the table below:

Year	0.5	1	1.5	2	2.5	3	3.5	4
Yield	1%	1%	3%	4%	4%	3%	5%	5%
Cash Flow 1	0	100	50	0	0	50	0	100
Cash Flow 2	2	2	2	2	2	2	2	102

a. Compute the present value of the cash flow 1.

b. Compute the present value of the cash flow 2.

c. Compute the duration of the cash flow 1.

d. Compute the duration of the cash flow 2.

e. What would be the cash flow of a 3 year bond with a 4% semi-annual coupon and face value 100?

f. To which bond cash flow 2 corresponds?

g. Compute the present value of the bond from part[e].

h. Compute the duration of the bond from part[e].