Introduction to Financial Mathematics

Lecture 3: One and Two step Binomial model.

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1 Calls and Puts

These are the most important financial instruments after the stocks, futures and bonds. They are very similar to forward contracts, but they give to the holder of the contract the *option* either to buy or to sell as opposed to the forward contracts which require the sale is made. Therefore, the owner of the options are sure not to lose money at maturity. On the other hand, they have to pay an initial fee to obtain these contracts. Hence, their net position after maturity can be both positive or negative. Options are written on a particular stock which is called the *underlying*.

We start with defining these instruments. Let S_T is the future random value of the stock and $(a)^+ := \max\{a, 0\}$ for a real number a. Note that the pay-off of a forward contact is $S_T - K$. The owners of a Call option, however, are not required to exercise the option and they will exercise their option of no-action when the future stock price S_T is less than the agreed strike price of K. Thus, the the payoff of a call is the positive part of $S_T - K$.

Definition 1.1.
The European Call option *with maturity T and strike K gives its holder the* option *but* not the obligation *to* buy *the stock at time T for a price K (regardless the price of the stock at time T). The future random pay-off of this option is*

$$(S_T - K)^+ ,$$

where S_T is the future random value of the stock and $(a)^+ := \max\{a, 0\}$ for a real number a.

The product dual to the Call is the Put and is defined as follows.

Definition 1.2.
The European Put option *with maturity T and strike K gives its holder the option (not the obligation) to* sell *the stock at time T for a price K (regardless the price of the stock at time T). The future random pay-off of this option is*

$$(K - S_T)^+ .$$

Since the pay-offs of either the Call or the Put options are positive, one needs to pay an initial fee in order to obtain them. A nontrivial question is *how to compute this initial price*.

1.1 Put-Call Parity

We start with the simple identity which holds for any values of S_T:

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K.$$

The right side of the above identity is the pay-off of forward contract with strike K. This simple relation has a pricing implication called the *Put-Call parity* which states that the prices of these quantities should satisfy the same relationship.

Let $C(K,T)$ be the price of the European Call option and respectively, $P(K,T)$ be the price of the European Put option with strike K maturity T. Then, the above identity implies that $C(K,T) - P(K,T)$ must be equal to the price of S_T minus the price of K that will be paid at time T. It is clear that the price of of S_T is the initial stock value S_0, and the price of K that will be paid at time T is equal to K times the price of a zero-coupon bond with face value 1. Let $B(T)$ be the price of this zero-coupon bond with maturity T. With these notations, *Put-Call parity* states that

$$\boxed{C(K,T) - P(K,T) = S_0 - KB(T).}$$

In particular, if r_T is the interest rate, then $B(T) = e^{-r_T T}$.

This simple identity highlights the fact that pricing is a nontrivial task and must be done in a careful way taking into account all existing liquid instruments.

2 Contingent Claims

One can also construct general financial contracts whose future random pay-off is a general deterministic function of the random stock price process of a an underlying asset. While some of them, as calls and puts, depend only on the final value of the stock, others may depend on the whole history of the prices. As their payments are derived from or contingent upon another financial instrument, they are called *contingent claims*, *derivatives* or simply *options*.

Definition 2.1.

The European contingent claim *with maturity T and the specified pay-off*

$$\Phi : \textit{Stock paths} \to \mathbb{R},$$

is a financial contract that pays $\Phi((S_t)_{t\in[0,T]})$, where $(S_t)_{t\in[0,T]}$ is the future random values of the stock. The function Φ is described in the financial contract.
If the contract pay-off depends only on the stock value at maturity, we use the notation $\varphi(S_T)$.

We next give important examples.

2.1 Butterflies

These options are designed to combine bull and bear spreads with a fixed risk and capped profit, and they are market-neutral strategies. Namely, the pay off of a butterfly option is the largest if the underlying asset does not move much prior to option expiration in either direction. When the butterfly spread is centered around the spot (which we always do), then the holder of this option will be "short" in volatility. Namely, if the stock price does not oscillate much, or equivalently, if the volatility is small, then the holder makes money.

The call butterfly is created by selling two shares of at-the-money call options (i.e., strike is $K = S_0$), buying a one share of in-the-money call option with a low strike price, say $K_1 = K - \Delta$, and buying one share out-of-the-money call option with a higher strike price, say $K_2 = K + \Delta$. Then, the payoff is given by $(S_T - K + \Delta)^+ - 2 \times (S_T - K)^+ + (S_T - K - \Delta)+$, i.e.

$$\text{pay-off} = \begin{cases} 0 & \text{if} \quad S_T \geq K + \Delta, \text{ or } S_T \leq K - \Delta, \\ S_T - K + \Delta & \text{if} \quad S_T \in [K - \Delta, K], \\ K + \Delta - S_T & \text{if} \quad S_T \in [K, K + \Delta]. \end{cases}$$

2.2 Straddles

A *straddle* is formally known as *going long volatility*. Its construction involves in buying both a call option and a put option on the same underlying. These options are bought at the same strike price and expire at the same time. The owner of a straddle makes a profit if the underlying price moves a long way from the strike price, either above or below. Thus, investors may take a long straddle position if they think the market is more volatile than option prices suggest, but does not know in which direction it is going to move. This position is a limited risk, since the most a purchaser may lose is the cost of both options. At the same time, there is unlimited profit potential. Its payoff is given by $(S_T - K)^+ + (K - S_T)^+$, i.e.

$$\text{pay-off} = \begin{cases} K - S_T & \text{if} \quad S_T \leq K, \\ S_T - K & \text{if} \quad S_T \geq K. \end{cases} = |S_T - K|.$$

2.3 Different types of Options

As butterflies and straddles, there are many other combinations including Back spread, Box spread, Calendar spread, Collar, Condor, Covered option, Credit spread, Debit spread, Diagonal spread, Fence, Intermarket spread, Iron butterfly, Iron condor, Jelly roll, Ladder, Naked option, Strangle, Protective option, ... All these options are complex constructions for a certain particular investment strategy or a risk management reason.

Other flexibility is over time and one uses the stock price evolution between the initiation of the option and its maturity. European options are settled at maturity and they use the stock price process in a direct manner. Other versions are:

- *American* options can be settled at any time before the maturity. This time is chosen by the holder of the option. For instance an American put option with maturity T and strike K would pay
$$(K - S_\tau)^+ ,$$
where $\tau \in [0, T]$ is a random stopping time chosen by the holder of the option. An American call pays $(S_\tau - K)^+$ with a chosen τ. These options may pay more than their European counterparts. So, they cost more.

- *Bermudan* options are like American options, but stopping dates are pre-restricted to a finite set, like every Friday.

- *Asian* options use the average of the stock price. There are European and American versions. For example, the European Asian call would pay
$$\left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ .$$
The American version of the above is called *Amerasian*.

- An important class of *path-dependent options* use the maximum or the minimum of the whole path process in addition to the value at maturity. An example is the *up-and-out Call option*. With a given strike K and a *barrier* $B > K$, its payoff is given by,
$$\Phi((S_t)_{t\in[0,T]}) = \begin{cases} (S_T - K)^+, & \text{if } \max_{t\in[0,T]} \ S_t \ \leq B, \\ 0, & \text{if } \max_{t\in[0,T]} \ S_t \ > B. \end{cases}$$

3 One-Step Binomial Model

This is a very simple model in order to price options. We will make it sufficiently more complex in the subsequent lectures. This simple structure on the other hand is rich enough to demonstrate several fundamental concepts in pricing theory.

3.1 Model

In this simple market we can trade only at time zero and all financial contracts are closed at maturity denoted by 1. We make the following modeling assumptions:

One-Step Binomial Model:

- The stock price at time zero is S_0 and is observed.
- There are two possible values of S_1: dS_1 and uS_1 for factors $d < u$.
  These factors are known to everybody.
- There is a zero-coupon bond maturing at $T = 1$. The price of this bond at time zero is B_0 and its price or value B_1 at maturity is always equal to 1.

  Equivalently, one may assume that the investors may lend or borrow money for a fixed one-step rate of r (compounded once). Then, $B_0 = 1/(1+r)$.

We consider a general option which pays $V_1^{up} =: v_u$ or $V_1^{down} =: v_d$ depending on S_1. Let V_0 be the price of this option at time zero. Then, there are two states of the world: "down" or "up". In the down case, the holder of this option will receive v_d and in the up case will receive v_u. Schematically,

	up: $S_1 = uS_0, \quad V_1 = v_u,$
S_0 given, $V_0 =?$	
	down: $S_1 = dS_0, \quad V_1 = v_d.$

In the above diagram, known quantities are: $S_0, d < 1 + r < u, v_u, v_d$. We want to calculate V_0.

We first consider a Call option to introduce the idea of a replicating portfolio. Then, we apply this idea to the general problem.

3.2 Replicating Portfolio (an example).

We continue with the specific example of a Call option with

$$S_0 = 4, \;\; K = 5, \;\; d = 1/2, \;\; u = 2, \;\; r = 0.$$

We assume that the interest rate r is zero for simplicity. Then,

	up:	$S_1 = 8,$	$B_1 = 1,$	$v_u = 3,$
$S_0 = 4, B_0 = 1,$ $V_0 =?$				
	down:	$S_1 = 2,$	$B_1 = 1,$	$v_d = 0.$

Suppose the investors buy $1/2$ shares of the stock for \$2 dollars. They finance this purchase by \$1 dollars from their sources and borrow \$1 dollars with no interest. Then, this portfolio consists of half of a stock and \$1 dollar debt and its value at time zero is $V_0 = 1$ (the amount provided by the investor). Let V_1 be the value of this portfolio at maturity. At that time the debt is still $\$1$ and the value of this portfolio is

$$V_1 = \frac{1}{2}S_1 - 1.$$

Hence we have the following diagram:

	up:	$S_1 = 8,$	$V_1 = 3,$
$S_0 = 4, \ V_0 = 1$			
	down:	$S_1 = 2,$	$V_1 = 0.$

Note that in all cases of the world, the value of this portfolio is same of the pay-off of the Call option. By the law of one price, the price of this option must be equal to $V_0 = \$1$.

4 Hedging and Pricing

We apply the same methodology of the above example of a call option to the general option problem. Suppose that there is a replicating portfolio and it contains θ shares of the stock and the price of the option is V_0. At this point, we do not know the values of V_0 and θ and our goal is to calculate them as a function of the known quantities.

Towards the goal of computing θ and V_0, assume that the investors have V_0 dollars and instead of buying the option, they buy θ shares of the stock for θS_0 dollars. If $V_0 \geq \theta S_0$ they buy θ shares and invest the remainder $V_0 - \theta S_0$. On the other hand if $V_0 > \theta S_0$ the investors borrow $(\theta S_0 - V_0)$ dollars to buy θ shares. More compactly, we say that they invest $(V_0 - \theta S_0)$ dollars in the bond: this is an investment if it is positive and a loan if it is negative. At maturity this investment or debt will become $(V_0 - \theta S_0)(1 + r)$. So the value of their portfolio at maturity is

$$V_1 = \theta S_1 + (V_0 - \theta S_0)(1 + r).$$

Since this portfolio is replicating, at time T its value V_1 should be equal to the option value in both cases of the world. Hence, we have the following two equations:

$$\begin{aligned} \text{if up:} \qquad & \theta u S_0 - (\theta S_0 - V_0)(1 + r) = v_u, \\ \text{if down:} \qquad & \theta d S_0 - (\theta S_0 - V_0)(1 + r) = v_d. \end{aligned}$$

We need to solve two equations and there are two unknowns: V_0 and θ. Simple algebra yields,

$$V_0 = \frac{1}{1 + r}\left[\frac{(1 + r) - d}{u - d}v_u + \frac{u - (1 + r)}{u - d}v_d\right]$$

and

$$\theta = \frac{v_u - v_d}{(u - d)S_0}.$$

So we have proved the following result:

Theorem 4.1. *In the one-step Binomial model,*

$$\boxed{V_0 = \frac{1}{1+r}\left[\frac{(1+r)-d}{u-d}v_u + \frac{u-(1+r)}{u-d}v_d\right].} \tag{4.1}$$

Moreover, the portfolio which holds

$$\boxed{\theta = \frac{v_u - v_d}{(u-d)S_0}} \tag{4.2}$$

many shares of the stock and invests $(V_0 - \theta S_0)$ dollars in a zero-coupon bond (it is a debt if negative) replicates the option.

Consider the Call option discussed in the previous subsection:

$$S_0 = 4, \;\; K = 5, \;\; d = 1/2, \;\; d = 2, \;\; r = 0.$$

Then, $v_u = 3, v_d = 0$ and

$$\theta = \frac{v_u - v_d}{(u-d)S_0} = \frac{3-0}{(2-0.5)4} = \frac{3}{6} = \frac{1}{2},$$

$$V_0 = \frac{1}{1+r}\left[\frac{(1+r)-d}{u-d}v_u + \frac{u-(1+r)}{u-d}v_d\right] = \frac{1-0.5}{2-0.5}\,3 = \frac{0.5}{1.5}\,3 = 1.$$

These are exactly the results obtained earlier.

Consider now a Put option with same parameters. Then, in the up case, the option value is $v_u = (5-8)^+ = 0$ and the in the down case, the option value is $v_d = (5-2)^+ = 3$. So the value of the option is equal to

$$\frac{1}{1+r}\left[\frac{(1+r)-d}{u-d}v_u + \frac{u-(1+r)}{u-d}v_d\right] = \frac{2-1}{2-0.5}\,3 = \frac{1}{1.5}\,3 = 2.$$

The number of shares in the replication portfolio is

$$\frac{v_u - v_d}{(u-d)S_0} = \frac{0-3}{(2-0.5)4} = -\frac{1}{2}.$$

Hence, to replicate this Put option we must have *minus* $1/2$ shares of the stock. This means that we short sell the stock. We assume that this allowed.

4.1 Risk Neutral Measure

We assume that

$$d \leq 1 + r \leq u. \tag{4.3}$$

Set

$$\boxed{p^* := \frac{(1+r) - d}{u - d}.} \tag{4.4}$$

Then, under our assumption, $p^* \in [0, 1]$ and one may think of it as the probability of the stock price going up. Indeed, consider a probability measure $\mathbb{Q}$ on the probability set of $\Omega = \{\text{“}up\text{”}, \text{“}down\text{”}\}$ given by,

$$\mathbb{Q}(\text{“}up\text{”}) = p^* = \frac{(1+r) - d}{u - d}, \quad \mathbb{Q}(\text{“}down\text{”}) = 1 - p^* = \frac{u - (1+r)}{u - d}.$$

Then, the option price formula (4.1) can be re-written as

$$\boxed{V_0 = \frac{1}{1+r} \mathbf{E}_{\mathbb{Q}}[V_1].} \tag{4.5}$$

where $\mathbf{E}_{\mathbb{Q}}$ is expected value with respect to the probability measure $\mathbb{Q}$ and in case of above

$$\mathbf{E}_{\mathbb{Q}}[V_1] = \mathbb{Q}(\text{“}up\text{”}) v_u + \mathbb{Q}(\text{“}down\text{”}) v_d.$$

Moroever,

$$\mathbf{E}_{\mathbb{Q}}[S_1] = p^* u S_0 + (1 - p^*) d S_0 = \frac{(1+r) - d}{u - d} u S_0 + \frac{u - (1+r)}{u - d}) d S_0 = (1+r) S_0.$$

This is a very important identity which we restate it for future reference.

$$\boxed{S_0 = \frac{1}{1+r} \mathbf{E}_{\mathbb{Q}}[S_1].} \tag{4.6}$$

In fact, (4.6) is a special case of the more general pricing formula (4.5). Indeed, consider the option that pays one share of the stock at maturity. This is exactly the same as the cash flow of the original stock and by the law of one price, they must have the same price. Then, the left hand side of (4.6) is the price of the stock which is S_0, the left hand side is the option formula (4.5) with pay-off S_1.

The notion of risk-neutral measure will be further discussed in coming lectures.

4.2 Arbitrage in one-step model

The assumption (4.3) is important to ensure that the risk neutral up-probability p^* is indeed a probability, i.e., $p^* \in [0, 1]$. This assumption also is needed to ensure that the one-step Binomial model is economically viable. In fact, when this assumption does not hold, then there is a portfolio which results in positive future value starting with zero investment (an *arbitrage*). Such a possibility would destabilize the market and any viable model should not allow arbitrage possibilities. Here we show how the condition (4.3) is equivalent to the lack of arbitrages.

Definition 4.2.

In a one-dimensional Binomial model,

- *A portfolio is a pair of θ shares of stocks and x_0 dollars. (Both θ and x_0 are real-valued. In particular they can take negative values).*
- *The initial value of the portfolio is $V_0 = \theta S_0 + x_0$.*
- *The future value of the portfolio is $V_1 = \theta S_1 + (1 + r)x_0$. (As S_1 is random, so is V_1).*
- *A portfolio is an arbitrage if $V_0 \leq 0$ and if $V_1 \geq 0$ in both cases (up or down) and is strictly positive in at least one of the cases.*

Remark 4.3. *Sometimes one defines arbitrage by requiring $V_0 = 0$. However, it is easy to show that they are equivalent definitions.*

Theorem 4.4.

In a one-dimensional Binomial model, there are no arbitrages if and only if (4.3) *holds.*

Proof. Suppose (4.3) holds. We can then define the risk-neutral probability measure $\mathbb{Q}$. Now, consider any portfolio (θ, x_0) with $V_0 = x_0 + \theta S_0 \leq 0$ and take the expected value of its future value. The result is the following,

$$\mathbf{E}_{\mathbb{Q}}[V_1] = \mathbf{E}_{\mathbb{Q}}[\theta S_1 + (1 + r)x_0] = \theta \mathbf{E}_{\mathbb{Q}}[S_1] + (1 + r)x_0 = (1 + r)[\theta S_0 + x_0] =\leq 0.$$

If $V_1 \geq 0$ in both cases, then it has to be identically equal to zero. Hence, it cannot be an arbitrage. This shows that when (4.3) holds, there are no arbitrages.

For the converse, suppose (4.3) does not hold. Then,

1. either $d < u < (1 + r)$,
2. or $(1 + r) < d < u$.

In case 1, consider the portfolio $(\theta, x_0) = (-1, S_0)$ (i.e., short-sell one stock and invest the proceeds in the bond market). This portfolio has zero initioal value:

$$V_0 = \theta S_0 + x_0 = -1 \cdot S_0 + S_0 = 0.$$

Since in both cases, $S_1 \leq u S_0 < (1+r) S_0$,

$$V_1 = \theta S_1 + (1+r) x_0 > -(1+r)[S_0 + x_0] = 0.$$

Hence, $(-1, S_0)$ is an arbitrage when the parameters are as in case 1. Similarly one can show that the portfolio $(1, -S_0)$ is an arbitrage when the parameters are as in case 2.

The above argument shows that whenever (4.3) does not hold, there is an arbitrage. This completes the proof of equivalence between (4.3) and no-arbitrage. $\square$

5 Two Step Model

This is exactly as the one-step model but we have two times at which the investors can trade. For a given initial stock price s_0 and up factor u, and a down factor d, the Binomial tree is as follows:

$$\boxed{\begin{array}{lll} & & s_{2,2} = u^2 s_0,\ v_{2,2} = ? \\ & s_{1,1} = u s_0,\ v_{1,1} = ?,\ \theta_{1,1} = ?, & \\ s_0,\ v_0 = ?,\ \theta_0 = ?, & & s_{2,1} = u d s_0,\ v_{2,1} = ? \\ & s_{1,0} = d s_0,\ v_{1,0} = ?,\ \theta_{1,0} = ?, & \\ & & s_{2,0} = d^2 s_0,\ v_{2,0} = ? \end{array}}$$

5.1 An example

Start with a two-step model with

$$s_0 = 4,\ d = \frac{1}{2},\ 1 + r = \frac{3}{2},\ u = 2, \quad \Rightarrow \quad p^* = \frac{2}{3}.$$

The Binomial tree is as follows:

$$\boxed{\begin{array}{lll} & & s_{2,2} = 16 \\ & s_{1,1} = 8 & \\ s_0 = 4 & & s_{2,1} = 4 \\ & s_{1,0} = 2 & \\ & & s_{2,0} = 1 \end{array}}$$

We consider a Put option with $K = 7$. Then, the pay-off at the final nodes is given by

$$v_{2,i} = (7 - s_{2,i})^+, \qquad i = 0, 1, 2.$$

So we have

$$\boxed{\begin{array}{lll} & & s_{2,2} = 16,\ v_{2,2} = 0, \\ & s_{1,1} = 8, & \\ s_0 = 4, & & s_{2,1} = 4,\ v_{2,1} = 3, \\ & s_{1,0} = 2, & \\ & & s_{2,0} = 1,\ v_{2,0} = 6. \end{array}}$$

We continue by considering 3 subtrees:

Subtree 1 = node (1,1):

$$\boxed{\begin{array}{ll} & s_{2,2} = 16,\ v_{2,2} = 0, \\ s_{1,1} = 8,\ v_{1,1} = ? & \\ & s_{2,1} = 4,\ v_{2,1} = 3. \end{array}}$$

This is a one-step Binomial tree that we know how to solve:

$$v_{1,1} = \frac{1}{1+r}\ [p^* v_{2,2} + (1-p^*) v_{2,1}] = \frac{2}{3}, \quad \theta_{1,1} = \frac{v_{2,2} - v_{2,1}}{s_{2,2} - s_{2,1}} = -\ \frac{1}{4}.$$

Subtree 2 = node (1,0):

$$\boxed{\begin{array}{ll} & s_{2,1} = 4,\ v_{2,2} = 3, \\ s_{1,0} = 2,\ v_{1,0} = ? & \\ & s_{2,0} = 1,\ v_{2,0} = 6. \end{array}}$$

Again we use the one-step formula to calculate:

$$v_{1,0} = \frac{1}{1+r}\ [p^* v_{2,1} + (1-p^*) v_{2,0}] = \frac{8}{3}, \quad \theta_{1,0} = \frac{v_{2,1} - v_{2,0}}{s_{2,1} - s_{2,0}} = -\ 1.$$

Subtree 3 = node (0,0):

$$\boxed{\begin{array}{ll} & s_{1,1} = 8,\ v_{1,1} = \frac{2}{3}, \\ s_0 = 4,\ v_0 = ? & \\ & s_{1,0} = 2,\ v_{1,0} = \frac{8}{3}. \end{array}}$$

Then,

$$v_0 = \frac{1}{1+r}\ [p^* v_{1,1} + (1-p^*) v_{1,0}] = \frac{8}{9}, \quad \theta_0 = \frac{v_{1,1} - v_{1,0}}{s_{1,1} - s_{1,0}} = -\frac{1}{3}.$$

We now have the following solution:

$$\boxed{\begin{array}{ccc} & & s_{2,2} = 16,\ v_{2,2} = 0, \\ & s_{1,1} = 8,\ v_{1,1} = \frac{2}{3},\ \theta_{1,1} = -\frac{1}{4}, & \\ s_0 = 4,\ v_0 = \frac{8}{9},\ \ \theta_0 = -\frac{1}{3} & & s_{2,1} = 4,\ v_{2,1} = 3, \\ & s_{1,0} = 2,\ v_{1,0} = \frac{8}{3},\ \ \theta_{1,0} = -1 & \\ & & S_{2,0} = 1,\ v_{2,0} = 6. \end{array}}$$

5.2 Hedging - example

We show in the example above that by holding appropriate amount of then stock, we can perfectly replicate the option. We start with time zero:

Time 0**:**
According to the above calculations $v_0 = \frac{8}{9}$ and $\theta_0 = -\frac{1}{3}$. So we sell one option for the price of $v_0 = \frac{8}{9}$ and buy $\theta_0 = -\frac{1}{3}$ shares of the stock. Since θ_0 is negative this means short-selling the stock. Then, the amount of money put in the bank is

$$x_0 = v_0 - \theta_0 s_0 = \frac{8}{9} + \frac{1}{3}\, 4 = \frac{20}{9}.$$

Summarizing, our portfolio is the following:

$$\boxed{\text{short } \tfrac{2}{7} \text{ shares of stock, } \tfrac{20}{9} \text{ dollars in the bank.}}$$

There are two possibilities:

1. Time 1 **up:** $S_1 = s_{1,1} = 8$. Then, the value of the portfolio is

$$\theta_0 s_{1,1} + (1+r) x_0 = -\frac{1}{3}\, 8 + \frac{3}{2}\, \frac{20}{9} = \frac{2}{3} = v_{1,1}.$$

So our portfolio is worth $\frac{2}{3}$ dollars. We first convert our portfolio to money giving us $v_{1,1} = \frac{2}{3}$ dollars. According to the solution, we need to have $\theta_{1,1} = -\frac{1}{4}$ many stocks. This means that the amount of money in the bank is

$$x_{1,1} = v_{1,1} - \theta_{1,1}\, s_{1,1} = \frac{2}{3} + \frac{1}{4}\, 8 = \frac{8}{3}.$$

Summarizing, our portfolio has become:

$$\boxed{\text{short } \tfrac{1}{4} \text{ shares of stock, } \tfrac{8}{3} \text{ dollars in the bank.}}$$

Again there are two possibilities:

1a. Time 2 **up again:** $S_2 = s_{2,2} = 16$. Then, the value of the portfolio is

$$\theta_{1,1}s_{2,2} + (1+r)x_{1,1} = -\frac{1}{4}\,16 + \frac{3}{2}\frac{8}{3} = 0 = v_{2,2}.$$

Thus we have exactly the amount to cover the option, which is worthless in this case.

1b. Time 2 **is down:** $S_2 = s_{2,1} = 4$. Then, the value of the portfolio is

$$\theta_{1,1}s_{2,1} + (1+r)x_{1,1} = -\frac{1}{4}\,4 + \frac{3}{2}\frac{8}{3} = 3 = v_{2,1}.$$

We again have exactly the amount to cover the option.

2. Time 1 **down:** $S_1 = s_{1,0} = 2$. Then, the value of the portfolio is

$$\theta_0 s_{1,0} + (1+r)x_0 = -\frac{1}{3}\,2 + \frac{3}{2}\frac{20}{9} = \frac{8}{3} = v_{1,0}.$$

So our portfolio is worth $\frac{8}{3}$ dollars. We first convert our portfolio to money giving us $v_{1,0} = \frac{8}{3}$ dollars. According to the solution, we need to have $\theta_{1,0} = -1$ many stocks. This means that the amount of money in the bank is

$$x_{1,0} = v_{1,0} - \theta_{1,0}\, s_{1,0} = \frac{8}{3} + 2 = \frac{14}{3}.$$

Summarizing, our portfolio has become:

$$\boxed{\text{short } 1 \text{ share of stock, } \tfrac{14}{3} \text{ dollars in the bank.}}$$

Again there are two possibilities:

2a. Time 2 **up:** $S_2 = s_{2,1} = 4$. Then, the value of the portfolio is

$$\theta_{1,1}s_{2,1} + (1+r)x_{1,1} = -\frac{1}{4}\,4 + \frac{3}{2}\frac{8}{3} = 3 = v_{2,1}.$$

We have exactly the amount to cover the option.

2b. Time 2 is down again: $S_2 = s_{2,0} = 1$. Then, the value of the portfolio is

$$\theta_{1,0} s_{2,0} + (1+r) x_{1,0} = -1 + \frac{3}{2}\frac{14}{3} = 6 = v_{2,0}.$$

Once again, we have exactly the amount to cover the option.

We analyzed all cases of the possible stock movements. By dynamically changing our portfolio, we are able to have exactly the same amount of money needed to cover the option liability. This is *perfect replication* of the option by starting with an initial amount equal to the option price.

We note for future reference that at each node the composition. of the portfolio is given according to the following formula.

Hedging Portfolio at node (k,i) consists of

- $\theta_{k,i}$ many shares of the stock, and
- $x_{k,i} := v_{k,i} - \theta_{k,i}\, s_{k,i}$ dolars invested in the bank.

The total value of this portfolio is $v_{k,i}$.

5.3 Equation

We have argued in the above example that the pricing equation for a European option with a general pay-off of $\varphi(S_2)$ in the Binomial model is

$$\boxed{v_{k,i} = \frac{1}{1+r}\left[p^* v_{k+1,i+1} + (1-p^*) v_{k+1,i}\right], \quad k = 0, 1, \ i = 0, \ldots k}$$

with the final condition

$$\boxed{v_{2,i} = \varphi(s_{2,i}), \quad i = 0, 1, 2.}$$

6 Homework

1. A stock price is currently $\$50$. It is known that at the end of three months it will be either $\$60$ or $\$40$. The three-month interest rate is $r = 0.05$ (compounded once).

   a. What is the value of a three-month European call option with a strike price of $\$55$?

   b. What is the value of a three-month European Put option with a strike price of $\$55$?

   c. Does the put-call parity hold?

2. Consider a two-step Binomial model with

$$s_0 = \$108, \qquad d = \frac{2}{3}, \qquad u = \frac{3}{2}, \qquad r = 0.$$

a. *Compute the risk-neutral up probability*, p^*.

b. Compute the price of an option with pay-off

$$\text{pay-off} = \begin{cases} \$50, & \text{if } S_2 \geq 100, \\ 00, & \text{if } S_2 < 100. \end{cases}$$

c. Compute the price of an option with pay-off

$$\text{pay-off} = \begin{cases} \$50, & \text{if } S_2 \leq 100, \\ 00, & \text{if } S_2 > 100. \end{cases}$$

d. Compute the price of an option with pay-off

$$\text{pay-off} = \begin{cases} \$50, & \text{if } S_2 = 108, \\ 00, & \text{otherwise.} \end{cases}$$

3. Consider a call option with strike $K = 8$ in a two step Binomial model with

$$S_0 = 4, \quad d = \frac{1}{2}, \quad u = 2, \quad r = \frac{1}{4}.$$

a. *Compute the risk-neutral up probability*, p^*.

b. As in the lectures, let $v_{k,i}$ be the price of the call option, and $\theta_{k,i}$ be the shares to be held in the replicating portfolio at node (k, i), where k is the time step and i is the number of up movements up to step k. We compute that

$$v_0 = \frac{32}{25}, \ v_{1,0} = 0, \ \theta_{1,0} = 0.$$

Compute $v_{1,1}$, $\theta_{1,1}$, and θ_0.