Introduction to Financial Mathematics

Lecture 4: Multi step Binomial models.

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1 One Step Binomial Model

We apply the same methodology of the above example of a call option to the general option problem. Suppose that there is a replicating portfolio and it contains θ shares of the stock and the price of the option is V_0. At this point, we do not know the values of V_0 and θ and our goal is to calculate them as a function of the known quantities.

Towards the goal of computing θ and V_0, assume that the investors have V_0 dollars and instead of buying the option, they buy θ shares of the stock for θS_0 dollars. If $V_0 \geq \theta S_0$ they buy θ shares and invest the remainder $V_0 - \theta S_0$. On the other hand if $V_0 > \theta S_0$ the investors borrow $(\theta S_0 - V_0)$ dollars to buy θ shares. More compactly, we say that they invest $(V_0 - \theta S_0)$ dollars in the bond: this is an investment if it is positive and a loan if it is negative. At maturity this investment or debt will become $(V_0 - \theta S_0)(1 + r)$. So the value of their portfolio at maturity is

$$V_1 = \theta S_1 + (V_0 - \theta S_0)(1 + r).$$

Since this portfolio is replicating, at time T its value V_1 should be equal to the option value in both cases of the world. Hence, we have the following two equations:

$$\begin{aligned} \text{if up:} \qquad & \theta u S_0 - (\theta S_0 - V_0)(1 + r) = v_u, \\ \text{if down:} \qquad & \theta d S_0 - (\theta S_0 - V_0)(1 + r) = v_d. \end{aligned}$$

We need to solve two equations and there are two unknowns: V_0 and θ. Simple algebra yields,

$$V_0 = \frac{1}{1+r}\left[\frac{(1+r)-d}{u-d} v_u + \frac{u-(1+r)}{u-d} v_d\right]$$

and

$$\theta = \frac{v_u - v_d}{(u-d)S_0}.$$

So we have proved the following result:

Theorem 1.1. *In the one-step Binomial model,*

$$\boxed{V_0 = \frac{1}{1+r}\left[\frac{(1+r)-d}{u-d} v_u + \frac{u-(1+r)}{u-d} v_d\right].} \tag{1.1}$$

Moreover, the portfolio which holds

$$\boxed{\theta = \frac{v_u - v_d}{(u-d)S_0}} \tag{1.2}$$

many shares of the stock and invests $(V_0 - \theta S_0)$ dollars in a zero-coupon bond (it is a debt if negative) replicates the option.

Consider the Call option discussed in the previous subsection:

$$S_0 = 4, \;\; K = 5, \;\; d = 1/2, \;\; d = 2, \;\; r = 0.$$

Then, $v_u = 3, v_d = 0$ and

$$\theta = \frac{v_u - v_d}{(u-d)S_0} = \frac{3-0}{(2-0.5)4} = \frac{3}{6} = \frac{1}{2},$$

$$V_0 = \frac{1}{1+r}\left[\frac{(1+r)-d}{u-d}v_u + \frac{u-(1+r)}{u-d}v_d\right] = \frac{1-0.5}{2-0.5}\,3 = \frac{0.5}{1.5}\,3 = 1.$$

These are exactly the results obtained earlier.

Consider now a Put option with same parameters. Then, in the up case, the option value is $v_u = (5-8)^+ = 0$ and the in the down case, the option value is $v_d = (5-2)^+ = 3$. So the value of the option is equal to

$$\frac{1}{1+r}\left[\frac{(1+r)-d}{u-d}v_u + \frac{u-(1+r)}{u-d}v_d\right] = \frac{2-1}{2-0.5}\,3 = \frac{1}{1.5}\,3 = 2.$$

The number of shares in the replication portfolio is

$$\frac{v_u - v_d}{(u-d)S_0} = \frac{0-3}{(2-0.5)4} = -\frac{1}{2}.$$

Hence, to replicate this Put option we must have *minus* $1/2$ shares of the stock. This means that we short sell the stock. We assume that this allowed.

1.1 Risk Neutral Measure

We assume that

$$d \le 1 + r \le u. \tag{1.3}$$

Set

$$\boxed{p^* := \frac{(1+r)-d}{u-d}.} \tag{1.4}$$

Then, under our assumption, $p^* \in [0,1]$ and one may think of it as the probability of the stock price going up. Indeed, consider a probability measure $\mathbb{Q}$ on the probability set of $\Omega = \{\text{"}up\text{"}, \text{"}down\text{"}\}$ given by,

$$\mathbb{Q}(\text{"}up\text{"}) = p^* = \frac{(1+r)-d}{u-d}, \quad \mathbb{Q}(\text{"}down\text{"}) = 1-p^* = \frac{u-(1+r)}{u-d}.$$

Then, the option price formula (1.1) can be re-written as

$$\boxed{V_0 = \frac{1}{1+r}\mathbf{E}_{\mathbb{Q}}[V_1].} \tag{1.5}$$

where $\mathbf{E}_{\mathbb{Q}}$ is expected value with respect to the probability measure $\mathbb{Q}$ and in case of above

$$\mathbf{E}_{\mathbb{Q}}[V_1] = \mathbb{Q}(\text{"}up\text{"})v_u + \mathbb{Q}(\text{"}down\text{"})v_d.$$

Moroever,

$$\mathbf{E}_{\mathbb{Q}}[S_1] = p^* u S_0 + (1-p^*) d S_0 = \frac{(1+r)-d}{u-d} u S_0 + \frac{u-(1+r)}{u-d}) d S_0 = (1+r) S_0.$$

This is a very important identity which we restate it for future reference.

$$\boxed{S_0 = \frac{1}{1+r}\mathbf{E}_{\mathbb{Q}}[S_1].} \tag{1.6}$$

In fact, (1.6) is a special case of the more general pricing formula (1.5). Indeed, consider the option that pays one share of the stock at maturity. This is exactly the same as the cash flow of the original stock and by the law of one price, they must have the same price. Then, the left hand side of (1.6) is the price of the stock which is S_0, the left hand side is the option formula (1.5) with pay-off S_1.

The notion of risk-neutral measure will be further discussed in coming lectures.

1\. Example: 2-step binomial model with

$$S_0 = 4, \quad u = 2, \quad d = \tfrac{1}{2}, \quad r = \tfrac{1}{4} \quad \text{and call}$$

option with strike $K = 4$.

$(0,0)$: $\boxed{S_0 = 4}$

$(1,1)$: $\boxed{S_1 = 8}$

$(1,0)$: $\boxed{S_1 = 2}$

$(2,2)$: $\boxed{S_2 = 16}$ payoff $= (S_2 - 4)^+ = 12$

$(2,1)$: $\boxed{S_2 = 4}$ payoff $= (S_2 - 4)^+ = 0$

$(2,0)$: $\boxed{S_2 = 1}$ payoff $= (S_2 - 4)^+ = 0$

Notation: $v_{k,i}$ = price of the option at node (k,i) in a binomial tree. node (k,i) is the point that we reach at step k after i many ups.

$S_{k,i}$ = stock price at node (k,i)

We use 0 instead of $(0,0)$

$\theta_{1,1}$

$S_2 = 16 = S_{2,2}$ $\quad v_{2,2} = 12$

$S_1 = 8 = S_{1,1}$ $\quad v_{1,1}$

$S_2 = 4 = S_{2,1}$ $\quad v_{2,1} = 0$

$S_0 = 4$ $\quad v_0 = ?$

θ_0

$S_1 = 2 = S_{1,0}$ $\quad v_{1,0}$

$\theta_{1,0}$

$S_2 = 1 = S_{2,0}$ $\quad v_{2,0} = 0$

step 0 $(k=0)$ | step 1 $(k=1)$ | Step 2 $(k=2)$

We continue at compute $v_{1,1}$ $v_{1,0}$ v_0 and also the hedges $\theta_{1,1}$ $\theta_{1,0}$ θ_0.

Recall 1-step binomial formula

$$\text{price} = \frac{1}{1+r}\left[p^* \cdot v_{up} + (1-p^*)\, v_{down}\right]$$

where $p^* = \dfrac{1+r-d}{u-d} = \dfrac{1+\frac{1}{4}-\frac{1}{2}}{2-\frac{1}{2}} = \dfrac{1}{2}$

and $\text{hedge} = \dfrac{v_{up} - v_{down}}{S_{up} - S_{down}}$

node (1,1):

$$v_{1,1} = \frac{1}{1+\frac{1}{4}}\left[\frac{1}{2}\cdot \underbrace{v_{2,2}}_{=12} + \frac{1}{2}\, \underbrace{v_{2,1}}_{=0}\right] = \frac{48}{10}$$

$$\theta_{1,1} = \frac{v_{2,2} - v_{2,1}}{S_{2,2} - S_{2,1}} = \frac{12-0}{16-4} = 1.$$

node (1,0):

$$v_{1,0} = \frac{1}{1+1/4}\left[\frac{1}{2}\, \underbrace{v_{2,1}}_{=0} + \frac{1}{2}\, \underbrace{v_{2,0}}_{=0}\right] = 0$$

$$\theta_{1,0} = \frac{v_{2,1} - v_{2,0}}{S_{2,1} - S_{2,0}} = \frac{0-0}{4-1} = 0.$$

node 0

$$v_0 = \frac{1}{1+1/4}\left[p^*\, \underbrace{v_{1,1}}_{=48/10} + (1-p^*)\, \underbrace{v_{1,0}}_{=0}\right] = \frac{48}{25}$$

$$\theta_0 = \frac{v_{1,1} - v_{1,0}}{S_{1,1} - S_{1,0}} = \frac{48/10 - 0}{8-2} = \frac{4}{5}$$

2 Multi-Step Model

Consider the multi-step Binomial model already introduced yesterday with a constant down factor d, up factor u and a unit interest rate r. We assume there are N trading dates and we refer to the k-th trading date as step k. Let S_k be the random value of the stock at step k. Then, S_k only takes the values

$$\boxed{s_{i,k} = u^k\, d^{i-k}\, S_0 \qquad\qquad i = 0, 1, \ldots, N, \;\; k = 0, 1 \ldots, i.} \tag{2.1}$$

2.1 Option

In this market, we consider a *European* option with a general pay-off $\varphi(S_N)$. Let V_k be the random value of this option at step k. As the stock values, option values can take finitely many values denoted by $v_{i,k}$ for $i = 0, \ldots, N$, and $k = 0, \ldots, i$. Initial value V_0 takes only one value $v_{0,0}$ (hence, is deterministic) and we write v_0. The goal is to compute $V_0 = v_0$.

We achieve this by computing all $v_{i,k}$ values by dynamic programming. Indeed, at maturity the value of the option is equal to its exercise value. Hence,

$$v_{N,k} = \varphi(s_{N,k}), \qquad k = 0, 1, \ldots, N.$$

Starting with this information at maturity, we calculate the value of this option at earlier times by backwards recursion or by dynamic programming.

2.2 Equation

To develop this recursion, we start with the node $(T-1, k)$. From this node the stock price can make only one up or down movement. After these movements we know the value of the option. Hence, we have the following diagram:

$$\boxed{\begin{array}{ll} & S_T = s_{T,k+1} = u s_{T-1,k}, \quad v_{T,k+1} = \varphi(s_{T,k+1}) \\ S_{T-1} = s_{T-1,k},\; v_{T-1,k} = ? & \\ & S_T = s_{T,k} = d s_{T-1,k}, \quad v_{T,k} = \varphi(s_{T,k}). \end{array}}$$

This is exactly a one-step Binomial model with initial stock price $s_{T-1,k}$ and

$$v_u = \varphi(s_{T,k+1}) = \varphi(u s_{T-1,k}), \quad v_d = \varphi(s_{T,k}) = \varphi(d s_{T-1,k}).$$

Since d, r, u are time independent, p^* is given by,

$$p^* = \frac{(1+r) - d}{u - d}. \tag{2.2}$$

Hence,

$$v_{T-1,k} = \frac{1}{1+r}\left[p^* \varphi(s_{T,k+1}) + (1-p^*)\varphi(s_{T,k})\right]. \tag{2.3}$$

Now suppose (by backwards induction) that all values $v_{t+1,j}$ for $j = 0, 1 \ldots, t+1$ are already computed and we would like to compute $v_{t,k}$. Then, we have the following one-step Binomial model:

$$\boxed{\begin{array}{ll} & S_{t+1} = s_{t+1,k+1} = u s_{t,k},\ v_{t+1,k+1}(\text{known}) \\ S_t = s_{t,k} \text{ observed},\ v_{t,k} = ? & \\ & S_{t+1} = s_{t+1,k} = d s_{t,k}, \quad v_{t+1,k}\ (\text{known}). \end{array}}$$

Using the one-step pricing formula derived in Lecture 6, we arrive at the pricing equation for the Binomial model. We summarize this equation in the following theorem.

Theorem 2.1. *Consider a Binomial model with N steps, up factor u, down factor d, interest rate r satisfying*

$$d \ < \ 1 + r \ < \ u. \tag{2.4}$$

Let $v_{i,k}$ be the value of a European option with pay-off $\varphi(S_N)$ at step k after i many up movements. Then, $v_{i,k}$ is the unique solution of

$$\boxed{v_{i,k} = \frac{1}{1+r}\left[p^* v_{i+1,k+1} + (1-p^*) v_{i+1,k}\right], \qquad i = 0, 1, \ldots N-1,\ k = 0, 1 \ldots i,} \tag{2.5}$$

together with the final condition

$$\boxed{v_{N,k} = \varphi(s_{N,k}), \qquad k = 0, 1 \ldots N,} \tag{2.6}$$

where $s_{i,k}$ are as in (2.1).

The above recursive equation can be easily solved numerically. Several formulae are also discussed later in these notes as well.

2.3 Hedge

In the one step Binomial model, the hedging portfolio consists of holding

$$\theta = \frac{V_u - V_d}{(u-d)S_0} = \frac{V^{up} - V^{down}}{S^{up} - S^{down}}$$

many stocks and a cash amount of

$$c = v_0 - \theta\, S_0.$$

Indeed, suppose that we hold c dollars and own θ shares of the stock in a one-step Binomial model. Then, the value of the portfolio at the next step is given by,

$$V_1 = (1+r)c \, + \, \theta \, S_1.$$

The formulae for v_0, θ, c are such that V_1 is equal to v_u if the stock moves up and is equal to v_d if the stock movement is down. Therefore, this portfolio of θ many shares of the stocks and c dollars in the bank, *perfectly replicated* the option. As the multi-dimensional Binomial model consists of many one-step Binomial models, the same perfect hedging holds.

Consider European option with a pay-off φ in a multi-step Binomial model with nodes (i, k), $k = 0, \dots, N$, $k = 0, \dots, i$. Let $v_{i,k}$ be the value of the option given by Theorem 2.1. Set

$$\boxed{\theta_{i,k} := \frac{v_{i+1,k+1} - v_{i+1,k}}{s_{i,k}\,(u-d)}, \qquad i = 0, 1, \dots N-1, \;\; k = 0, 1 \dots i.} \tag{2.7}$$

$$\boxed{c_{i,k} := v_{i,k} \, - \, \theta_{i,k}\, s_{i,k}, \qquad i = 0, 1, \dots N-1, \;\; k = 0, 1 \dots i.} \tag{2.8}$$

Theorem 2.2. *The hedging strategy that starts with v_0 dollars and at each node investing $\theta_{i,k}$ shares in the stock and holding $c_{i,k}$ dollars in the bank (borrowing if negative), perfectly replicates the option pay-off $\varphi(S_N)$ without any additional funds. Additionally, the value of this portfolio at each node is exactly equal to $v_{i,k}$.*

Proof. We prove this by forward induction. The induction statement at step $i < N$ is this: suppose that at any $k = 0, \dots, i$ the value of the portfolio is $v_{i,k}$. Suppose we rebalance the portfolio to have $\theta_{i,k}$ many shares of the stock. Then, the value of the portfolio at the next step $i+1$ is equal to $v_{i,k+1}$ if stock moves up and is equal to $v_{i,k}$ if stock moves down.

Initial step. At $(i, k) = (0, 0)$, we have v_0 dollars. We buy θ_0 shares of the stock for $\theta_0 \, s_0$ dollars. As we have v_0 dollars, the difference $v_0 - \theta_0 \, s_0 = c_0$ is kept in the bank (borrowed if negative). At the next step this portfolio has the value

$$V_1 = (1+r)c_0 \, + \, \theta_0 \, S_1.$$

As this is a one-step Binomial model with $v_u = v_{1,1}$ and $v_d = v_{1,0}$, by the discussion before the theorem, if we move to node $(1, 1)$ we would have $V_1 = v_u = v_{1,1}$ and if we move to node $(1, 0)$ we would have $V_1 = v_d = v_{1,0}$. Therefore the induction step is proved at $k = 0$.

Induction step. At node (i, k) the value of the portfolio is $v_{i,k}$. Independent of the number of shares that was held prior, we can rebalance our portfolio by buying $\theta_{i,k}$ many shares of the stock and investing the rest, namely $v_{i,k} - \theta_{i,k} s_{i,k} = c_{i,k}$ dollars in the bank. Then, the value of this portfolio at the next step is

$$V_{\text{next}} = (1+r)c_{i,k} \, - \, \theta_{i,k}\, S_{\text{next}}.$$

Since we are at node (i,k), S_{next} is either equal to $s_{i+1,k}$ or $s_{i+1,k+1}$. We first consider the case $S_{\text{next}} = s_{i+1,k} = d\, s_{i,k}$. Then,

$$\begin{aligned}
V_{\text{next}} &= (1+r)\,[v_{i,k} \;-\; \theta_{i,k}\, s_{i,k}]\,\theta_{i,k}\, d\, s_{i,k} \\
&= (1+r)\, v_{i,k} + \theta_{i,k}\, s_{i,k}\,[d-(1+r)] \\
&= \frac{(1+r)-d}{u-d}\, v_{i+1,k+1} + \frac{u-(1+r)}{u-d}\, v_{i+1,k} + \frac{v_{i+1,k+1}-v_{i+1,k}}{u-d}\,(d-(1+r)) \\
&= v_{i+1,k}.
\end{aligned}$$

A similar calculation yields that $V_{\text{next}} = v_{i+1,k+1}$ when $S_{\text{next}} = s_{i+1,k+1}$.

This completes the induction step and the proof. □

3 European Pricing Formula

In this section, we use the European pricing equation to obtain compact formulae for the price.

Using the risk-neutral probability measure $\mathbb{Q}$, we re-write the pricing equation obtained ion the previous lecture as follows:

$$v_{i,k} = \frac{1}{1+r}\,\mathbf{E}_{\mathbb{Q}}\left[V_{i+1} \mid S_i = s_{i,k}\right], \qquad 0 \le k \le i < N.$$

This is equivalent to the following more compact expression.

$$\boxed{V_i = \frac{1}{1+r}\,\mathbf{E}_{\mathbb{Q}}\left[V_{i+1} \mid S_i\right], \qquad 0 \le i < N.} \tag{3.1}$$

As the tree is recombining and the option pay-off depends only on the final stock price, the price of the option is determined by the current stock value. In other words,

$$V_i = \frac{1}{1+r}\,\mathbf{E}_{\mathbb{Q}}\left[V_{i+1} \mid S_i\right] = \frac{1}{1+r}\mathbf{E}_{\mathbb{Q}}\left[V_{i+1} \mid S_0, S_1, \dots S_i\right].$$

This means that the only relevant information for pricing the current stock value. Equivalently, even tough all stock price information is available at step i, one uses only the current stock value. Such processes are called *Markov*.

By iterating the above formula towards $i = T$, we obtain the formula compact formula.

Theorem 3.1. *Let $V = (V_0, \dots, V_N)$ be the price (or value) process of a European option with pay-off $V_N = \varphi(S_N)$ for some given function φ. Then,*

$$\boxed{V_i = \frac{1}{(1+r)^{N-i}}\,\mathbf{E}_{\mathbb{Q}}\left[\varphi(S_N) \mid S_i\right] = \frac{1}{(1+r)^{N-i}}\,\mathbf{E}_{\mathbb{Q}}\left[\varphi(S_N) \mid S_0, S_1, \dots S_i\right].} \tag{3.2}$$

In particular, initial price is given as,

$$\boxed{V_0 = \frac{1}{(1+r)^N}\,\mathbf{E}_{\mathbb{Q}}\left[\varphi(S_N)\right].} \tag{3.3}$$

Proof. We prove (3.2) by backwards induction on $i = N, N-1, \ldots, 0$. The case $i = N$ is trivial and the case $i = N-1$ follows from (3.1). Suppose that (3.2) holds for $i+1$:

$$V_{i+1} = \frac{1}{(1+r)^{N-i-1}}\,\mathbf{E}_{\mathbb{Q}}\left[\varphi(S_N) \mid S_0, S_1, \ldots S_i, S_{i+1}\right].$$

Then,

$$\begin{aligned}
V_i &= \frac{1}{1+r}\mathbf{E}_{\mathbb{Q}}\left[V_{i+1} \mid S_0, S_1, \ldots S_i\right] \\
&= \frac{1}{1+r}\mathbf{E}_{\mathbb{Q}}\left[\frac{1}{(1+r)^{N-i-1}}\,\mathbf{E}_{\mathbb{Q}}\left[\varphi(S_N) \mid S_0, S_1, \ldots S_i, S_{i+1}\right]\ \mid S_0, S_1, \ldots S_i\right] \\
&= \frac{1}{(1+r)^{N-i}}\,\mathbf{E}_{\mathbb{Q}}\left[\,\mathbf{E}_{\mathbb{Q}}\left[\varphi(S_N) \mid S_0, S_1, \ldots S_i, S_{i+1}\right]\ \mid S_0, S_1, \ldots S_i\right].
\end{aligned}$$

We now use the "tower property" of the conditional expectations, to conclude that (3.2) holds for i as well.

$\square$

We can also take $\varphi(S_N) = S_N$ is the above formula. This contract simply pays the spot price of the stock at the time of maturity. So its value by the law-of-one-price is the spot price of the stock now, S_0. We state this observation as it has important consequences.

Theorem 3.2. *For any* $0 \le k \le n \le N$,

$$\boxed{S_i = \frac{1}{(1+r)^{n-i}}\,\mathbf{E}_{\mathbb{Q}}\left[\,S_n \mid S_i\right].} \tag{3.4}$$

4 Up and Out Call Option

This option is like a Call option unless the stock price goes above a certain barrier during the life of the option. Mathematically, let $S_0, S_1, \ldots, S_N$ be the random stock price evolution. The running maximum of this process is define by,

$$M_k := \max\left\{\,S_0, \ldots, S_k\,\right\}.$$

Then, the pay-off of an *up and out call option* with strike K and barrier B is given by,

$$\varphi(S_0, S_1, \ldots, S_N) = \begin{cases} (S_N - K)^+, & \text{if } M_N < B, \\ 0, & \text{if } M_N \geq B. \end{cases}$$

This option is *path-dependent* as its pay-off depends not only on the price at maturity but also the maximum value. In general, pricing of path-dependent options is computationally a hard problem. However, in the case of option depending on the path only through the maximum or the minimum or both, one can use the dynamic programmimng technique. Amin observation is this. As before let $s_{i,k}$ be the stock price at node (i, k), i.e., it is the stock value at step k after i many up movements. Further let $v_{i,k}$ be the value of the up and out call option at that node *provided that the stock path has not gine above the barier*. Then,

$$s_{i,k} \geq B \qquad \Rightarrow \qquad v_{i,k} = 0.$$

Then, the pricing computation goes like this:

Maturity:

$$v_{N,k} = \begin{cases} (s_{N,k} - K)^+, & \text{if } s_{N,k} < B, \\ 0, & \text{if } s_{N,k} \geq B. \end{cases} \qquad \forall\, k = 0, \ldots, N.$$

At any $i = 0, 1 \ldots, N-1$:

$$v_{i,k} = \begin{cases} \frac{1}{1+r} \left[p^* v_{i+1,k+1} + (1-p^*) v_{i+1,k} \right], & \text{if } s_{i,k} < B, \\ \\ 0, & \text{if } s_{i,k} \geq B. \end{cases} \qquad \forall\, k = 0, \ldots, i.$$

We solve the above equations, by backward recursion.

Example 4.1. Consider the simple example with

$$S_0 = K = 1, \; B = 3, \; d = 1, \; u = 2, \; r = 0.25.$$

Then, $p^* = 0.25$ and the calculations are shown in the diagram below.

5 How to choose the parameters?

In an n-step binomial model with large n, it becomes important to choose the parameters d, r, u appropriately so that the resulting model is relevant. One way to choose them depending on n is as follows:

$$\boxed{r_n = e^{rh} - 1, \quad u_n = e^{\sigma\sqrt{h}}, \quad d_n = e^{-\sigma\sqrt{h}}, \qquad \text{with} \qquad h = \frac{T}{n},}$$

where $r > 0$ is the *continuously compounded annual interest rate*, T is maturity measured in years and $\sigma > 0$ is the annual *volatility*. A discussion of the above choice will be discussed later.

The above procedure has the advantage of reducing the number of parameters. Indeed, the only parameter to be calibrated to the real financial markets is the volatility σ.

6 Exercises

1. Consider a 3-step Binomial model with
$$S_0 = 8, \quad d = \frac{1}{2}, \quad u = 2, \quad r = 0.$$

   a. Compute the *initial price* of an *up-and-out digital* option with strike $K = 2$ and a barrier $b = 20$, i.e., the pay-off of this option at maturity $N = 3$ is given by,
$$\text{pay-off} = \begin{cases} 1, & \text{if} \quad \max\{S_0, S_1, S_2, S_3\} < 20, \text{ and } S_3 \geq 2, \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

   b. Compute the *initial price* of a *forward-start Put* option that pays $(S_1 - S_3)^+$ at maturity $N = 3$. Notice that this is a path-dependent pay-off.

   c. Compute the *initial price* of a *forward-start Call* option that pays $(S_3 - S_1)^+$ at maturity $N = 3$. Notice that this is a path-dependent pay-off.

2. Consider a 3-step Binomial model with
$$S_0 = 8, \quad d = \frac{1}{2}, \quad u = \frac{3}{2}, \quad r = 0.$$

   a. Compute the *initial price* of a down-and-out *Put* option with strike 11 and the down barrier $b = 2.5$, i.e., the pay-off of this option at maturity $N = 3$ is given by,
$$\text{pay-off} = \begin{cases} (11 - S_3)^+, & \text{if} \quad min\{S_0, S_1, S_2, S_3\} \geq 2.5, \\ 0, & \text{if} \quad min\{S_0, S_1, S_2, S_3\} < 2.5. \end{cases}$$

   b. Compute the *initial price* of a *chooser option* with strike $K = 10$ allows the owner of the option to choose whether the payoff is a call or a put with strike $K = 10$ at step one, i.e., at step one ($k = 1$) the owner decides whether the pay-off will be $(S_3 - 10)^+$ or $(10 - S_3)^+$. This choice is made after observing the first stock movement, and therefore can depend on the first stock movement being up or down. After the choice is made at step one, it cannot be changed later.

3. Consider the same 3-step Binomial model in Problem 2.

   A digital option with a parameter b has the following pay-off:
$$\text{pay-off} = \begin{cases} 1, & \text{if} \quad S_3 \geq b, \\ 0, & \text{if} \quad S_3 < b. \end{cases}$$

   Let $p(b)$ be the price of this option as a function of the parameter b.

   a. Compute $p(30)$.

   b. Compute $p(20)$.

   c. Compute $p(10)$.

   d. Compute $p(5)$.