Lecture 5

American Options

Presented by: Yan
09-10/11/2024

Introduction

1. **American Options**
2. **Mid-term Review(other slides)**

01 American Options

American Options

- **Definition:**
  - ✓ American Option is a type of financial derivative that allows the holder to exercise the option at any time prior to the expiration date, unlike the European Option which can only be exercised on the expiration date. The American Option offers greater flexibility as investors can choose the optimal time to exercise based on market conditions.
- **Characteristics:**
  - ✓ Early Exercise Rights
  - ✓ Diverse Exercise Strategies
  - ✓ Hedging Requirements
  - ✓ Incentive to Exercise Early

01 American Options

The payoff of American Options

- **European Options:**

  The European Opitons pays: $\varphi(S_T)$

  where T is the pre-determined maturity.

  ✓ Call option: $\varphi(S_T) = (S_T - K)^+$

  ✓ Put option: $\varphi(S_T) = (K - S_T)^+$

- **American Options:**

  The American Opitons pays: $\varphi(S_\tau)$, the holder can choose to stop anytime τ between initiation and the maturity,$0 \le \tau \le T$.

  ✓ Call option: $\varphi(S_\tau) = (S_\tau - K)^+$

  ✓ Put option: $\varphi(S_\tau) = (K - S_\tau)^+$

01 American Options

American Option Binomial Model Pricing:

➢ **Example:**

$$S_0 = 4,\ d = \frac{1}{2},\ 1 + r = \frac{3}{2},\ u = 2$$

We consider a Put option with K=7. Then, the potential pay-off at any node is given by $\varphi_{k,i} = \left(7 - s_{k,i}\right)^+$, $k = 0,1,2,\ \ i = 0,\ \ldots,\ k$

✓ Compute the risk-neutral up probability, $p^* = \frac{(1+r)-d}{u-d} = \frac{2}{3}$

✓ At maturity 2, the option has to be exercised and therefore,

$$v_{2,i} = \varphi_{2,i}, \quad i = 0, 1, 2.$$

✓ Compute the stock values ($s_{k,i}$) and potential pay-offs ($\varphi_{k,i}$):

$$s_{2,2} = 16,\ \varphi_{2,2} = v_{2,2} = 0,$$

$$s_{1,1} = 8,\ \varphi_{1,1} = 0,$$

$$s_0 = 4,\ \varphi_0 = 3,$$

$$s_{2,1} = 4,\ \varphi_{2,1} = v_{2,1} = 3,$$

$$s_{1,0} = 2,\ \varphi_{1,0} = 5,$$

$$S_{2,0} = 1,\ \varphi_{2,0} = v_{2,0} = 6.$$

01 American Options

American Option Binomial Model Pricing:

Subtree 1 = node (1,1):

$$s_{1,1} = 8,\ \varphi_{1,1} = 0,\ v_{1,1} = ? \qquad s_{2,2} = 16,\ v_{2,2} = 0, \qquad s_{2,1} = 4,\ v_{2,1} = 3.$$

$$stop\ price = \varphi_{1,1} = (7-8)^{+} = 0$$

$$continuation\ \ price = \frac{1}{1+r}\left[p^{*}v_{2,2} + (1-p^{*})v_{2,1}\right] = \frac{2}{3}$$

$$v_{1,1} = max\left\{stop\ price; continuation\ price\right\}$$

$$= max\left\{\varphi_{1,1}; \frac{1}{1+r}\left[p^{*}v_{2,2} + \left(1+p^{*}\right)v_{2,1}\right]\right\}$$

$$\theta_{1,1} = \frac{v_{2,2}-v_{2,1}}{s_{2,2}-s_{2,1}} = -\frac{1}{4}$$

Continue

$$\boldsymbol{v_{1,1} = \frac{2}{3}}$$

01 American Options

American Option Binomial Model Pricing:

Subtree 2 = node (1,0):

$$s_{1,0} = 2,\ \varphi_{1,0} = 5,\ v_{1,0} = ? \qquad \begin{array}{l} s_{2,1} = 4,\ v_{2,1} = 3, \\ s_{2,0} = 1,\ v_{2,0} = 6. \end{array}$$

$$stop\ price = \varphi_{1,0} = (7-2)^+ = 5$$

$$continuation\ \ price = \frac{1}{1+r}\left[p^* v_{2,1} + (1-p^*)v_{2,0}\right] = \frac{8}{3}$$

$$v_{1,0} = max\{stop\ price; continuation\ price\}$$

$$= max\left\{\varphi_{1,0}; \frac{1}{1+r}\left[p^* v_{2,1} + \left(1+p^*\right)v_{2,0}\right]\right\}$$

Stop

$\boldsymbol{v_{1,1} = 5}$

$$\theta_{1,0} = \frac{v_{2,1} - v_{2,0}}{s_{2,1} - s_{2,0}} = -1$$

01 American Options

American Option Binomial Model Pricing:

Subtree 3 = node (0,0):

$$s_{1,1} = 8,\ v_{1,1} = \frac{2}{3},$$

$$s_0 = 4,\ \varphi_0 = 3,\ v_0 = ?$$

$$s_{1,0} = 1,\ v_{1,0} = 5.$$

$$stop\ price = \varphi_0 = (7-4)^+ = 3$$

$$continuation\ price = \frac{1}{1+r}\left[p^* v_{1,1} + (1-p^*)v_{1,0}\right] = \frac{38}{27}$$

$$v_{1,0} = max\{stop\ price; continuation\ price\}$$

$$= max\left\{\varphi_0; \frac{1}{1+r}\left[p^* v_{1,1} + \left(1+p^*\right)v_{1,0}\right]\right\}$$

Stop

$v_0 = 3$

$$\theta_0 = \frac{v_{1,1} - v_{1,0}}{s_{1,1} - s_{1,0}} = -\frac{13}{18}$$

01 American Options

Hedging (super-replicate the American option):

- **Put Option Seller:** $v_0 = 3$
  - ✓ If it is immediately exercised, then the seller gives back this amount and the hedging is done.
  - ✓ But if the investors decide to continue (despite the fact that it is not optimal), then the seller will hedge.
- **Step 0**
  - ✓ The Put Option seller of the option shorts $\theta_0 = -\frac{13}{18}$ shares of the stock. Together with the initial proceed of 3 from the sale, the seller now has

$$3 + 4 \times \frac{13}{18} = \frac{53}{9}$$

- **Step 1**

If up: $-\frac{13}{18} \times 8 + \frac{53}{6} > v_{1,1} = \frac{2}{3}.$

If down: $-\frac{13}{18} \times 2 + \frac{53}{6} = \frac{133}{18} > v_{1,0} = 5.$

01 American Options

Hedging (super-replicate the American option):

- **Step 1**
  - ✓ If the investors decide to continue at node (1,0), the investos bring their short position to $\theta_{1,0} = -1$ by shorting an additional $-\theta_{1,0} - (-\theta_0) = 1 - \frac{13}{18} = \frac{5}{18}$ shares.
  - ✓ The cash amout (in bond): $\frac{53}{6} + \frac{5}{18} \times 2 = \frac{169}{18}$
- **Step 2**

$$\textit{If up:} \qquad -1 \times 4 + \frac{169}{12} = \frac{121}{12} > v_{2,1} = 3.$$

$$\textit{If down:} \qquad -1 \times 1 + \frac{169}{12} = \frac{157}{12} > v_{2,0} = 6.$$

The above hedging strategy super-replicates the Put option with initial price of 3.

01 American Options

The Equation of Amercian Options Pricing:

We have argued in the above example that the pricing equation for the American option with a general pay-off of $\varphi(S_\tau)$ in the Binomial model is

$$\boxed{v_{k,i} = \max\left\{\varphi(s_{k,i})\ ,\ \frac{1}{1+r}\left[p^* v_{k+1,i+1} + (1-p^*)v_{k+1,i}\right]\right\},\ \ k = 0, 1, \ldots N-1,\ i = 0, 1 \ldots, k}$$

with the final condition

$$\boxed{v_{N,i} = \varphi(s_{N,i}), \quad i = 0, 1 \ldots N.}$$

One may rewrite the above equation as

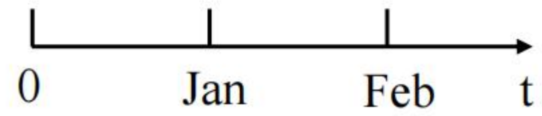
$$V_k = \max\left\{\varphi(S_k)\ ,\ \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left[V_{k+1} \mid S_k\right]\right\}$$

Important to note that, as opposed to European pricing formula, we cannot express $v_{k,i}$ simply in terms of $v_{N,k}$'s.

01 American Options

Two-step Binomial :

- $S_0 = 50,\ u = 1.05,\ d = 0.95,\ r = 0.05,\ K = 50, T = 1/6.$



$$\boldsymbol{S_0 = 50}$$

$$S_{1,1} = uS_0 = 52.5$$

$$S_{1,0} = dS_0 = 47.5$$

$$S_{2,2} = uuS_0 = 55.125$$

$$S_{2,1} = duS_0 / udS_0 = 49.875$$

$$S_{2,0} = dS_1 = ddS_0 = 45.125$$

T_0 T_1 T_2

01 American Options

Two-step Binomial Model (American Call Option)

$S_0 = 50$

- $S_{1,1}=52.5$
  - $S_{2,2}=55.125$
  - $S_{2,1}=49.875$
- $S_{1,0}=47.5$
  - $S_{2,1}=49.875$
  - $S_{2,0}=45.125$

$$p^* = \frac{e^{r\Delta t}\text{-}d}{u\text{-}d} = \frac{e^{0.05\times\frac{1}{12}}\text{-}0.95}{1.05\text{-}0.95} \approx 0.54$$

$$v_{1,1}=e^{-r\Delta t}\left[p^* v_{2,2}+(1-p^*)v_{2,1}\right]\approx 2.765$$

$$v_{1,0}=e^{-r\Delta t}\left[p^* v_{2,1}+(1-p^*)v_{2,0}\right]=0$$

$$v_0=e^{-r\Delta t}\left[p^* v_{1,1}+(1-p^*)v_{1,0}\right]\approx 1.492$$

$\varphi_0 = 0$, $v_0=1.492$

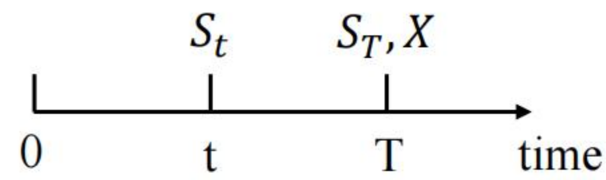
- $\varphi_{1,1}=2.5$, $v_{1,1}=2.765$
  - $\varphi_{2,2}=5.125$, $v_{2,2}=5.125$
  - $\varphi_{2,1}=0$, $v_{2,1}=0$
- $\varphi_{1,0}=0$, $v_{1,0}=0$
  - $\varphi_{2,1}=0$, $v_{2,1}=0$
  - $\varphi_{2,0}=0$, $v_{2,0}=0$

> The best option for American call options on non-dividend paying stocks is not to exercise them early.

Why?

01 American Options

American Call Option



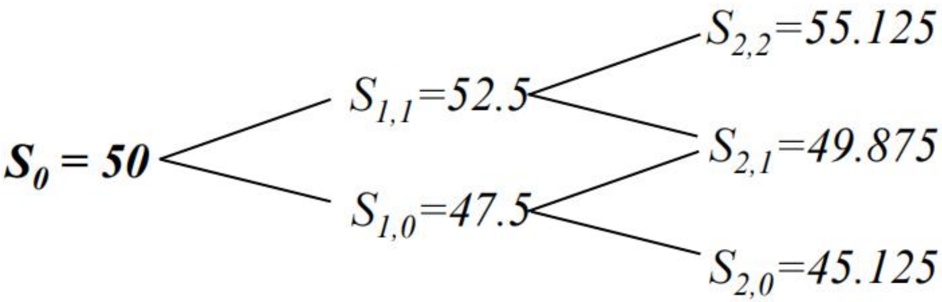
- **Four Staretigies:**
  A. Exercise the call option, and hold the stock
  B. Exercise the call option, and sell the stock
  C. Not exercise the call option
  D. Not exercise the call option, but sell it.

- **At maturity T:**
  A. $S_T - Xe^{r(T-t)}$
  B. $(S_t - X)e^{r(T-t)}$
  C. $max\{S_T - X, 0\}$
  D. $\geq \left(S_t - Xe^{-r(T-t)}\right)e^{r(T-t)} = S_t\, e^{r(T-t)} - X$

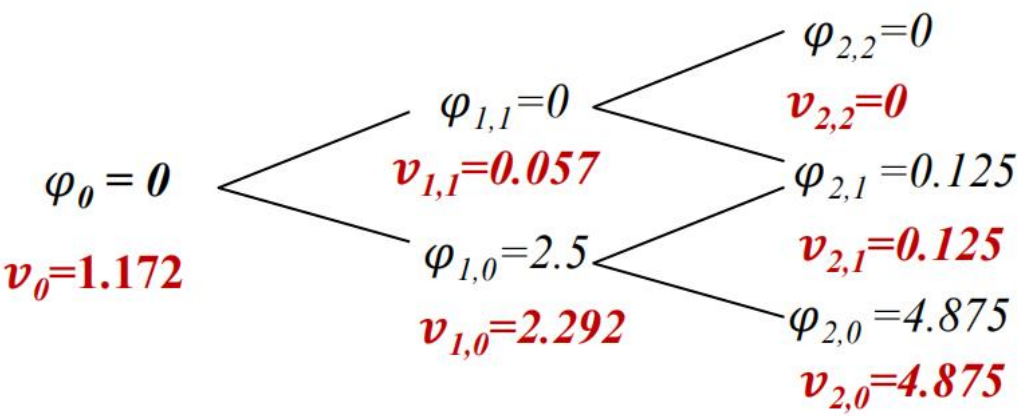
- **Comparisons:**
  - ✓ Holding staretiges (Expected the stock price will increase): C > A
  - ✓ Selling staretiges (Expected the stock price will decrease): D > B

01 American Options

Two-step Binomial Model (American Put Option)



$$p^* = \frac{e^{r\Delta t}\text{-}d}{u\text{-}d} = \frac{e^{0.05\times\frac{1}{12}}\text{-}0.95}{1.05\text{-}0.95} \approx 0.54$$



No early exercise

$$v_{1,1}=e^{-r\Delta t}\left[p^* v_{2,2}+(1-p^*)v_{2,1}\right]\approx 0.057$$

$$v_{1,0}=e^{-r\Delta t}\left[p^* v_{2,1}+(1-p^*)v_{2,0}\right]\approx 2.292$$

$$v_0=e^{-r\Delta t}\left[p^* v_{1,1}+(1-p^*)v_{1,0}\right]\approx 1.077$$

Early exercise at T=1

$$P_0=e^{-0.05\times\frac{1}{12}}\left[0.057p^*+2.5(1-p^*)\right]\approx 1.172$$

Thanks!

Presented by: Yan
09-10/11/2024