Lecture 9

Bonds：Duration, Convexity and immunization

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14-15/12/2024

Introduction

1. **Homework 3 explanation**

2. **Lecture 9 review**

① Homework 3 explanation

问题 1: $E[Y_{100}]$

公式:

随机游走 $Y_k = \sum_{i=1}^{k} x_i$，其中 x_i 是 i.i.d. 变量，且 $P(x_i = 1) = P(x_i = -1) = 0.5$。

因此，$E[x_i] = 0$，而 Y_k 是 x_i 的和，因此期望 $E[Y_k] = k \cdot E[x_i] = 0$。

答案: $a.0$。

问题 2: $P(Y_5 = 10)$

随机游走中，Y_k 不可能超出 $\pm k$ 的范围。对于 $Y_5 = 10$，它超出了可能的范围，因此概率为 0。

答案: $a.0$。

① Homework 3 explanation

问题 3: $P(Y_5 = 5)$

要使 $Y_5 = 5$，意味着 5 次抛掷中全是 $x_i = 1$。

$P(Y_5 = 5) = (0.5)^5 = \frac{1}{32}$

答案: $b.1/32$。 P(Y5 = 5) = P(um= 5)
= 5!/(5!0!)*1/2^5 = 1/32

$$\mathbb{P}(Y_m = 2k - m) = \mathbb{P}(U_m = k)$$

for $m = 1, \ldots$
$k = 0, \ldots, k$.

$$= \frac{k \text{ choose from } m}{2^m}$$

$$= \frac{m!}{k!\,(m-k)!} \cdot \frac{1}{2^m}$$

Note that $\ell = 2k - m \Rightarrow k = \frac{\ell + m}{2}$

问题 4: $P(Y_6 = 0)$

随机游走的 $Y_6 = 0$ 表示 6 次抛掷中，正负次数相同，即正负各为 3 次。可能的组合数为 $\binom{6}{3} = 20$ 。总样本空间为 $2^6 = 64$，因此：

$$P(Y_6 = 0) = \frac{\binom{6}{3}}{2^6} = \frac{20}{64} = \frac{5}{16}$$

答案: $d.5/16$。

① Homework 3 explanation

问题 5: $E[(Y_2)^2]$

$E[Y_k^2] = k$

随机游走 $Y_2 = x_1 + x_2$，其中 $x_1, x_2 \in \{1, -1\}$。可能的取值为:

$Y_2 = 2, 0, -2$

其平方为 $Y_2^2 = 4, 0, 4$。概率分布为 $P(Y_2^2 = 4) = \frac{1}{2}$，$P(Y_2^2 = 0) = \frac{1}{2}$。期望为:

$E[(Y_2)^2] = 4 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2} = 2$

答案: $c.2$。

问题 6: $E[(Y_8)^2]$

类似于问题 5，但 Y_8 是 8 次随机游走的和。由于 x_i 是独立的:

$\mathrm{Var}(Y_8) = \sum_{i=1}^{8} \mathrm{Var}(x_i) = 8 \cdot 1 = 8$

$E[(Y_8)^2] = \mathrm{Var}(Y_8) + [E(Y_8)]^2 = 8 + 0 = 8$

答案: $b.8$。

① Homework 3 explanation

问题 7: $E[Y_1Y_2]$

随机游走中:

$Y_1 = x_1, \quad Y_2 = x_1 + x_2$

因此:

$E[Y_1Y_2] = E[x_1(x_1 + x_2)] = E[x_1^2] + E[x_1x_2]$

由于 $x_1^2 = 1$，且 x_1 和 x_2 独立，$E[x_1x_2] = 0$:

$E[Y_1Y_2] = 1 + 0 = 1$

答案: $b.1$。

问题 8: $E[Y_4Y_7]$

$Y_4 = \sum_{i=1}^{4} x_i, \quad Y_7 = \sum_{i=1}^{7} x_i$

$E[Y_4Y_7] = E\left[\left(\sum_{i=1}^{4} x_i\right) \cdot \left(\sum_{j=1}^{7} x_j\right)\right]$

展开并利用独立性，只有 $i = j$ 时有贡献:

$E[Y_4Y_7] = \sum_{i=1}^{4} E[x_i^2] = 4$

答案: $b.4$。

① Homework 3 explanation

问题 9: $P(M_4 \geq 6)$

最大值 $M_k = \max(0, Y_1, Y_2, \ldots, Y_k)$。对 4 次抛掷，最大值不可能超过 4，因此 $P(M_4 \geq 6) = 0$。

Theorem. For any $0 \leq B \leq m$,

$$\mathbb{P}(M_m \geq B) = 2\,\mathbb{P}(Y_m > B) + \mathbb{P}(Y_m = B)$$

where

$$M_m = \max\{Y_0 = 0, Y_1, \ldots, Y_m\}$$

= "running maximum"

答案: $a.0$。

问题 10: $P(M_4 \geq 2)$

我们计算在 4 次抛掷中，最大值达到或超过 2 的概率。通过枚举：

- $M_4 \geq 2$ 的情况有 Y_k 曾达到 2，共 $\binom{4}{2} = 6$ 种方式（正 2 次，负 2 次）。总样本数为 $2^4 = 16$，概率为：$P(M_4 \geq 2) = \frac{6}{16} = \frac{3}{8}$

Y4 = {4,-4,2,-2,0}

= 2P(Y4>2)+ P(Y4= 2)
= 2P(Y4 = 4)+ P(Y4=2)
=2P(u4=4)+P(u4=3)
=2*4!/(4!0!)*1/2^4 + 4!/(3!1!)*1/2^4 = 3/8

答案: $c.3/8$。

① Homework 3 explanation

11. Suppose $A = \{2, 20, 4\}$. What is the best initial move of the first player ?

a. 2, $\boxed{\text{b. } 4,}$ c. 20, d. 24.

12. Compute $v^* = v_0(0)$ for $A = \{2, 20, 4\}$:

a. 4, $\boxed{\text{b. } 6,}$ c. 20, d. 24.

① if the first player chooses 4,
then the second player chooses 20, then the first player chooses 2.
so, the first player has 6 golds, the second player has 20 golds

② if the first player chooses 2,
then the second player chooses 20, then the first player chooses 4.
so, the first player has 6 golds, the second player has 20 golds

$$v_k(i) = \max\{ x_{i+1} + w_{k+1}(i+1) \,;\; x_{n-k+i} + w_{k+1}(i) \}$$

$$w_k(i) = \begin{cases} v_{k+1}(i) & : \text{if right optimal} \\ v_{k+1}(i+1) & : \text{if left optimal} \end{cases}$$

$$k = 0, 1, \ldots, n-1$$
$$i = 0, 1, \ldots, k.$$

① Homework 3 explanation

13. Compute $v^* = v_0(0)$ for $A = \{2, 7, 20, 16\}$:

a. 16, b. 22, c. 23, d. 27.

14. Suppose $A = \{2, 7, 20, 16\}$. What is the best initial move of the first player?

a. pick 2, b. pick 7, c. pick 20, d. pick 16.

① if the first player chooses 16,
then the second player chooses 20, then the first player chooses 7,
then the second player chooses 2
so, the first player has 23 golds, the second player has 22 golds

② if the first player chooses 2,
then the second player chooses 16, then the first player chooses 20,
then the second player chooses 7
so, the first player has 22 golds, the second player has 23 golds

① Homework 3 explanation

15. Consider $A = \{5, 9, 3, 7, 3, 3, 4, 11, 20, 7, 2\}$. We know that $w_1(0) = 35$ and $w_1(1) = 30$. What is the initial optimal move of the first player?

$\boxed{\text{a. pick 2,}}$ b. pick 5, c. pick 20, d. pick 35.

16. As in Prob. 15, $A = \{5, 9, 3, 7, 3, 3, 4, 11, 20, 7, 2\}$, $w_1(0) = 35$, $w_1(1) = 30$. Compute v^*:

a. 20, b. 30, c. 35, $\boxed{\text{d. 37.}}$

$$v_k(i) = \max\{ x_{i+1} + w_{k+1}(i+1);\ x_{n-k+i} + w_{k+1}(i) \}$$

$$w_k(i) = \begin{cases} v_{k+1}(i) & : \text{if right optimal} \\ v_{k+1}(i+1) & : \text{if left optimal} \end{cases}$$

$$k = 0, 1, \ldots, n-1$$
$$i = 0, 1, \ldots, k.$$

v* = v0(0)
= Max{5+w1(1), 2+w1(0)}
= max{5+30, 2+35}
= 37

so, initial optimal move of the first player should be 2 and v* = 37

① Homework 3 explanation

17. First player always get at least as much as the second player.

a. True $\boxed{\text{b. False.}}$

18. Suppose $A = \{2, x_2, x_3, 20\}$ with some unknown x_2 and x_3. Then, it is always optimal to choose 20 .

a. True $\boxed{\text{b. False.}}$

T17 false.
从T11 可知，无论 first player 如何选择，始终有6 <20，the first player lose.
从T13可知，first player 有概率胜过second player, 23>22

T18 the first player will have:
20+max(min(x3,2), x2) or 2+max(min(x2,20), x3).
if x3 = 100, x2 = 0,
20+max(min(x3,2), x2) = 22，2+max(min(x2,20), x3) = 102,
it’s optimal to choose 2.
So, we could **not** say it’s always optimal to choose 20

① Homework 3 explanation

19. Suppose $A = \{2, x_2, 20\}$ with some unknown x_2. Then, it is always optimal to choose 20.

$\boxed{\text{a. True}}$ b. False.

① if the first player chooses 20,
then the second player chooses x2(x2>2), then the first player chooses 2.
so, the first player has 22 golds, the second player has x2(x2>2) golds
② if the first player chooses 20,
then the second player chooses 2(x2<2), then the first player chooses x2.
so, the first player has 20+x2 golds, the second player has 2 golds
' 20+min(2, x2)'

③ if the first player chooses 2,
then the second player chooses 20(x2<20), then the first player chooses x2.
so, the first player has 2+x2 golds, the second player has 20 golds
④ if the first player chooses 2,
then the second player chooses x2(x2>20), then the first player chooses 20.
so, the first player has 22 golds, the second player has x2(x2>20) golds
'2+min(x2, 20)'
∵ 20+min(2, x2) always larger than or equal to 2+min(x2, 20)
∴ it's optimal to choose 20

① Homework 3 explanation

20. Consider $A = \{5, 9, 3, 7, 3, 3, 4\}$. Then, $v^* + w^* = 34$.

$\boxed{\text{a. True}}$ b. False.

两个玩家轮流选数字，最终选完所有的数字，
所以v* + w* = total sum number = 34

Thank you!

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14-15/12/2024