Introduction to Financial Mathematics

Lecture 7: Random Walk: properties and simulations

November 30, 2024

1 Random Walk

The *random walk* is a simple discrete analogue of the Brownian motion. Its increments $\{x_k\}_{k=1,2,\ldots}$ is a sequence of independent, identically distributed (i.i.d.) random variables with values in $\{-1,+1\}$ with equal probabilities, i.e.,

$$\mathrm{P}(x_i = 1) = \mathrm{P}(x_i = -1) = \frac{1}{2}, \qquad i = 1, 2, \ldots.$$

For any positive integer m, set

$$Y_m = \sum_{i=1}^{m} x_i.$$

The process $(Y_m)_{m=0,1,\ldots}$ is called the *random walk*. For each m, Y_m has the Binomial distribution with success probability a half, i.e.,

$$\mathrm{P}(Y_m = 2k - m) = \binom{m}{k}\frac{1}{2^m} = \frac{m!}{k!(m-k)!}\,\frac{1}{2^m}, \qquad k = 0, 1, \ldots, m.$$

Example 1.1. Suppose we get the random sequence of $+1, +1, -1, +1, +1, -1, -1, -1, -1, -1, -1, +1$. Then,

$$\boxed{\begin{array}{rcl}
\text{always} & & Y_0 = \ 0, \\
x_1 = +1, & \Rightarrow & Y_1 = +1, \\
x_2 = +1, & \Rightarrow & Y_2 = +2, \\
x_3 = -1, & \Rightarrow & Y_3 = +1, \\
x_4 = +1, & \Rightarrow & Y_4 = +2, \\
x_5 = +1, & \Rightarrow & Y_5 = +3, \\
x_6 = -1, & \Rightarrow & Y_6 = +2, \\
x_7 = -1, & \Rightarrow & Y_7 = +1, \\
x_8 = -1, & \Rightarrow & Y_8 = \ 0, \\
x_9 = -1, & \Rightarrow & Y_9 = -1, \\
x_{10} = -1, & \Rightarrow & Y_{10} = -2, \\
x_{11} = -1, & \Rightarrow & Y_{11} = -3, \\
x_{12} = +1, & \Rightarrow & Y_{12} = -2.
\end{array}}$$

This is a 'typical' path. But its probability is $(\frac{1}{2})^{12} = 0.00024414 = 0.024\%$. That is 24 times in $100,000$.

Example 1.2. Suppose that $m = 12$ as in the above example. Then, $Y_{12} = -2$ means that -1's are two more than $+1$s, or equivalently, the number of -1s is 7, and the number of $+1$s is 5. But the order in which $+1$s or -1s appear is not important. Therefore,

$$\mathrm{P}(Y_{12} = -2) = \mathrm{P}(\text{there are exactly five +1s in 12 steps}) =?$$

Combinatorial calculations give the result. Each case has probability $(\frac{1}{2})^{12}$. Hence,

$$\begin{aligned}&\mathrm{P}(\text{there are exactly five +1s in 12 steps})\\ &\qquad = \text{number of different ways of placing five +1s in 12 stops)} \left(\frac{1}{2}\right)^{12}.\end{aligned}$$

Also,

$$\text{number of different ways of placing five +1s in 12 stops)} = \binom{12}{5} = \frac{12!}{5!\,7!} = \frac{95,040}{120} = 792.$$

Hence,

$$\mathrm{P}(Y_{12} = -2) = \frac{792}{2^{12}} = \frac{792}{4,096} = 0.19336 = 19.336\%.$$

Homework. Compute the following probabilities

a. $\mathrm{P}(Y_{10} = 5)$.

b. $\mathrm{P}(Y_{11} = 5)$.

c. $\mathrm{P}(Y_{11} > 5)$.

1.1 Definition and Properties

The random walk has several immediate properties:

Theorem 1.3 (Properties of Random Walk)**.** *For all integers $0 < m < k$,*

1. *$-Y$ is also a random walk. In particular, $-Y_m$ has the same distribution as Y_m.*
2. *Y_m is independent of $Y_k - Y_m$;*
3. *$Y_k - Y_m$; has the same distribution of Y_{k-m}.*

Proof. Note that

$$-Y_m = -\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} (-x_i),$$

and the sequence $\{-x_i\}_{i=1,2,...}$ is also an i.i.d. sequence of random variables taking values ± 1 with equal probability. Hence $-Y$ is another random walk.

It is clear that

$$Y_k - Y_m = \sum_{i=m+1}^{k} x_i = \sum_{j=1}^{k-m} x_{j+(k-m)}.$$

Since $x_i's$ identical, $x_{j+(k-m)}$ has the same distribution as x_i. Hence, $Y_k - Y_m$ also has the same distribution as

$$\sum_{i=1}^{k-m} x_i = Y_{k-m}.$$

Similarly, since all x_i's with $i \geq m+1$ are independent of x_j's with $j \leq m$,

$$Y_k - Y_m = \sum_{i=m+1}^{k} x_i$$

is independent of

$$Y_m = \sum_{j=1}^{m} x_j.$$

$\square$

Example 1.4. Consider the probability $\mathrm{P}(Y_5 = 1 \text{ and } Y_{17} = 3)$. First note that

$$\mathrm{P}(Y_5 = 1 \text{ and } Y_{17} = 3) = \mathrm{P}(Y_5 = 1 \text{and } Y_{17} - Y_5 = 2).$$

As $Y_{12} - Y_5$ is independent of Y_M,

$$\mathrm{P}(Y_5 = 1 \text{ and } Y_{17} = 3) = \mathrm{P}(Y_5 = 1)\, \mathrm{P}(Y_{17} - Y_5 = 2).$$

Also, $Y_{12} - Y_5$ has the same distribution as $Y_{17-5} = Y_{12}$, and Y_{12} has the same distribution as $-Y_{12}$:

$$\mathrm{P}(Y_{17} - Y_5 = 2) = \mathrm{P}(Y_{12} = 2) = \mathrm{P}(-Y_{12} = -2) = \mathrm{P}(Y_{12} = -2).$$

We have calculated the probabilty above and it is equal to $\frac{792}{4{,}096}$ Moreover,

$$\mathrm{P}(Y_5 = 1) = \mathrm{P}(\text{three +1s in five steps}) = \frac{\binom{5}{3}}{2^5} = \frac{10}{32}.$$

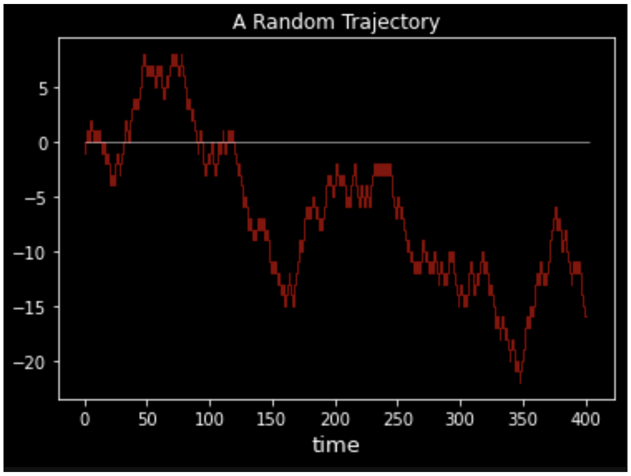
Hence,

$$\mathrm{P}(Y_5 = 1 \text{ and } Y_{17} = 3) = \frac{10}{32} \frac{792}{4,096} = 0.060425 = 6.0425\%.$$

Homework. Compute $\mathrm{P}(Y_5 > 1 \text{ and } Y_{17} = 3)$.

1.2 Simulations

We may use the python program to generate paths alike above. Below are two graphs and the python is attached at the end of the notes. A typical path from that code with $N = 400$ is given below:



2 Reflection Principle

The following property of random walk is useful. Fix a level $B > 0$ and define the random hitting time T_B by

$$T_B = \min \{ \, j = 1, 2, \ldots \; : \; Y_j \geq B \, \} \, .$$

This is first time the random walk Y has a value larger or equal to B. We now define the process reflected around B by,

$$R_m := \begin{cases} Y_m, & m \leq T_B, \\ 2B - Y_m, & m > T_B. \end{cases}$$

This process has the following representation also,

$$R_m = \sum_{j=1}^{m} z_j,$$

where

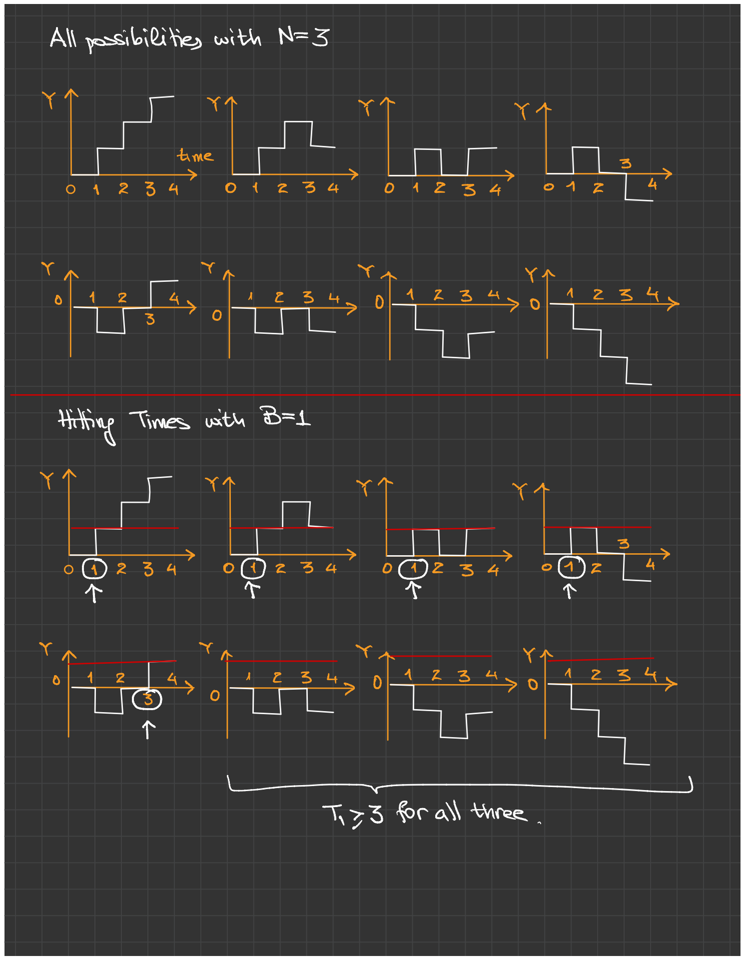
$$z_j := \begin{cases} x_j, & j \leq T_B, \\ -x_j, & j > T_B. \end{cases}$$

It is immediate that $\{z_j\}_{j=1,2,\ldots}$ is also an i.i.d. sequence of random variables with values in $\{-1, +1\}$ and equal probability each. Therefore, R is also a random walk.

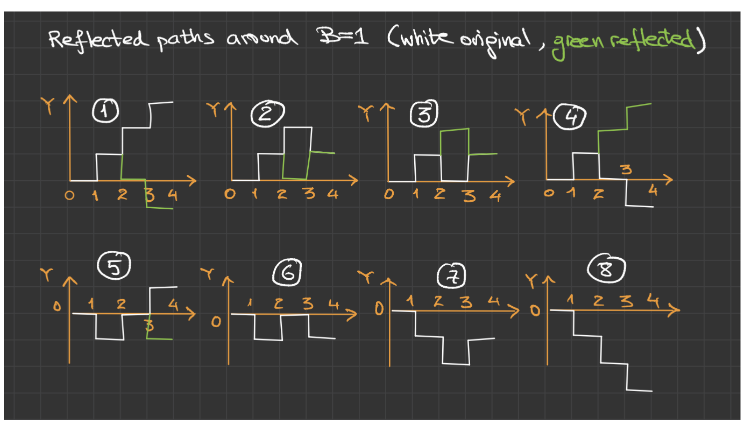
This construction is useful in calculating probabilities related to maximum of the random walk. In fact, it can be also used to price barrier type options. Define the *running maximum* of Y by,

$$M_m := \max \{0, Y_1, \ldots, Y_m\} \, , \qquad m = 1, 2, \ldots$$

Example 2.1. Let's take $N = 3$ and $B = 1$. all possibilities are given below



It is clear that the maximum value M_3 is greater or equal than 1 if and only id the hitting time T_1 is less than or equal to 4. From above picture we calculate that $\mathrm{P}(M_3 \geq 1) = \mathrm{P}(T_1 \geq 3) = \frac{5}{8}$.



The reflected and the original walks are equal to each until until the hitting time. Therefore, we do not see the reflected walk in the last three cases.

Let R the random walk reflected at 1 and M^R be its running maximum. Whenever $Y_3 > 1$, we have $M_3 \geq Y_3 > 1$. So,

$$\mathcal{O}_1 := \{M_3 \geq 1 \text{ and } Y_3 > 1\} = \{Y_3 \geq 1\}.$$

This happens only in case 1 above. The set $\{M_3 \geq 1 \text{ and } Y_3 < 1\}$ is the case 4. Note that this is also the only case $\left\{M_3^R \geq 1 \text{ and } R_3 > 1\right\}$. We also see that the set

$$\mathrm{P}(M_3 \text{ and } Y_3 > 1) = \mathrm{P}(M_3^R \text{ and } R_3 > 1) = \mathrm{P}(M_3 \text{ and } Y_3 < 1) = \frac{1}{8}.$$

Also $\{M_3 \text{ and } Y_3 = 1\}$ are the cases 2, 3, 5. So

$$\mathrm{P}(M_3 \text{ and } Y_3 = 1) = \frac{3}{8}.$$

Now,

$$\begin{aligned}
\mathrm{P}(M_3 \leq 1) &= \mathrm{P}(M_3 \text{ and } Y_3 > 1) + \mathrm{P}(M_3 \text{ and } Y_3 < 1) + \mathrm{P}(M_3 \text{ and } Y_3 = 1) \\
&= \mathrm{P}(M_3 \text{ and } Y_3 > 1) + \mathrm{P}(M_3^R \text{ and } R_3 > 1) + \mathrm{P}(M_3 \text{ and } Y_3 = 1) \\
&= 2\mathrm{P}(M_3 \text{ and } Y_3 > 1) + \mathrm{P}(M_3 \text{ and } Y_3 = 1) \\
&= \frac{2}{8} + \frac{3}{8} = \frac{5}{8}.
\end{aligned}$$

The calculation given in the above example generalizes and we have the following result.

Theorem 2.2. *For any positive integers* $0 < B \leq m$*,*

$$\mathrm{P}(M_m \geq B) = 2\mathrm{P}(Y_m > B) + \mathrm{P}(Y_m = B).$$

Note that the right hand side of the above identity can be calculated by binomial coefficients.

Proof. Note that if $Y_m > B$, then $M_m \geq Y_m > B$. Hence,

$$\mathcal{O}_B := \{M_m \geq B \text{ and } Y_m > B\} = \{Y_m > B\}.$$

Let R the random walk reflected at B and M^R be its running maximum. As R is a random walk itself, we have

$$\mathrm{P}\left(M_m^R \geq B \text{ and } R_m > B\right) = \mathrm{P}\left(M_m \geq B \text{ and } Y_m > B\right).$$

Also, for each $k \leq T_B$, $Y_k = R_k$ and consequently, $M_k = M_k^R$. Hence, on $\{T_B < N\}$,

$$M_m^R \geq M_{\tau_B}^R = B, \quad \text{and} \quad B - R_m = Y_m - B,$$

and,

$$\{M_m \geq B \text{ and } Y_m < B\} = \left\{M_m^R \geq B \text{ and } R_m > B\right\}.$$

Hence,

$$\mathrm{P}\left(M_m \geq B \text{ and } Y_m < B\right) = \mathrm{P}\left(M_m^R \geq B \text{ and } R_m > B\right) = \mathrm{P}\left(M_m \geq B \text{ and } Y_m > B\right).$$

Then,

$$\begin{aligned}\mathrm{P}(M_m \geq B) =& \mathrm{P}\left(M_m \geq B \text{ and } Y_m > B\right) + \mathrm{P}\left(M_m \geq B \text{ and } Y_m < B\right)\\ &+ \mathrm{P}\left(M_m \geq B \text{ and } Y_m = B\right)\\ =& 2\mathrm{P}\left(M_m \geq B \text{ and } Y_m > B\right) + \mathrm{P}\left(M_m \geq B \text{ and } Y_m = B\right)\\ =& 2\mathrm{P}\left(Y_m > B\right) + \mathrm{P}\left(Y_m = B\right).\end{aligned}$$

□

3 Applications to option pricing

As an application of the reflection principle in option pricing, consider the Binomial model with Consider a 5-step Binomial model with

$$S_0 = 1, \;\; u = \frac{3}{2}, \;\; d = \frac{2}{3}, \;\; r = \frac{1}{12}, \;\; n = 5.$$

The risk-neutral up probability is given by,

$$p^* = \frac{1 + r - d}{u - d} = \frac{\frac{13}{12} - \frac{2}{3}}{\frac{3}{2} - \frac{2}{3}} = \frac{1}{2}.$$

We know that

$$S_k = s_0\,(\frac{u}{d})^{\text{number of up movements in } k \text{ steps}}\, d^k.$$

Let U_k be the number of up movements in k steps. Then,

$$S_k = 1\,(\frac{9}{4})^{U_k}\,(\frac{2}{3})^k = (\frac{3}{2})^{2U_k}\,(\frac{2}{3})^k.$$

Take the logarithm to obatin,

$$\ln(S_k) = 2\ln(3/2)\,U_k - k\ln(3/2).$$

Let $x_i = +1$ is the stock movement in the i-th step is up and let $x_i = -1$ is the stock movement in the i-th step is down. Set

$$Y_k = (x_1 + \ldots + x_k) = \text{number of ups} - \text{number of downs} = U_k - (k - U_k) = 2U_5 - k.$$

Hence,

$$\ln(S_k) = \ln(3/2)\,(Y_k + k) - k\ln(3/2) = \ln(3/2)\,Y_k.$$

3.1 Up-and-in Digital option

In this market, we consider the up-and-in option with pay-off

$$\varphi(S_0, S_1, \ldots S_5) = \begin{cases} 1, & \text{if } \max\{S_0, S_1, \ldots S_5\} \geq \binom{3}{2} \\ 0, & \text{if } \max\{S_0, S_1, \ldots S_5\} < \binom{3}{2}. \end{cases}$$

Then,

$$\text{price of this option} = \frac{1}{1+r^5}\, \mathbf{E}_*[\text{pay-off}].$$

But the pay-off is either one or zero, so

$$\mathbf{E}_*[\text{pay-off}] = \mathrm{P}(\text{pay-off} = 1) = \mathrm{P}(\max\{S_0, S_1, \ldots S_5\} \geq \binom{3}{2}).$$

Observe that

$$\begin{aligned} \max\{S_0, S_1, \ldots S_5\} \geq \binom{3}{2} &\Leftrightarrow \ln(\max\{S_0, S_1, \ldots S_5\}) \geq \ln(3/2) \\ &\Leftrightarrow \max\{\ln(S_0), \ln(S_1), \ldots \ln(S_5)\} \geq \ln(3/2) \\ &\Leftrightarrow \max\{0, \ln(3/2)Y_1, \ldots \ln(3/2)Y_5\} \geq \ln(3/2) \\ &\Leftrightarrow \max\{0, Y_1, \ldots Y_5\} \geq 1. \end{aligned}$$

Note that Y is a random walk and M is running maximum. So we have shown that

$$\max\{S_0, S_1, \ldots S_5\} \geq \binom{3}{2} \quad \Leftrightarrow \quad M_5 \geq 1.$$

Therefore,

$$\mathrm{P}(\max\{S_0, S_1, \ldots S_5\} \geq \binom{3}{2}) = \mathrm{P}(M_5 \geq 1).$$

By reflection principle,

$$\mathrm{P}(M_5 \geq 1) = 2\mathrm{P}(Y_5 > 1) + \mathrm{P}(Y_5 = 1).$$

We now calculate above probabilities: Y_5 can take values $-5, -3, -1, 1, 3, 5$. So,

$$\begin{aligned} \mathrm{P}(Y_5 > 1) &= \mathrm{P}(Y_5 = 3) + \mathrm{P}(Y_5 = 5) \\ &= [\, \binom{5}{4} + \binom{5}{5} \,]\, (\frac{1}{2})^5 \\ &= \frac{5+1}{2^5} = \frac{6}{32}. \end{aligned}$$

Also,

$$\mathrm{P}(Y_5 = 1) = \binom{5}{3}\, (\frac{1}{2})^5 = \frac{10}{32}$$

Hence,

$$P(M_5 \geq 1) = 2P(Y_5 > 1) + P(Y_5 = 1) = \frac{12}{32} + \frac{10}{32} = \frac{22}{32} = \frac{11}{16} = 0.6875 = 68.75\%.$$

We combine all these to compute the price of the option:

$$\begin{aligned} \text{price of this option} &= \frac{1}{1+r^5}\mathbf{E}_*[\text{pay-off}] \\ &= \left(\frac{12}{13}\right)^5 \frac{11}{16} = 0.46074. \end{aligned}$$

Homework. In the above market, we consider the up-and-in option with pay-off

$$\varphi(S_0, S_1, \dots S_5) = \begin{cases} 1, & \text{if } \max\{S_0, S_1, \dots S_5\} \geq \left(\frac{3}{2}\right)^3 \text{ and } S_5 \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Compute its price.

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt; plt.style.use('dark_background')
```

Example

In [2]:
```
N = 10
xx=np.random.choice(2,N)
x=xx*2-1
Y=[sum(x[:k]) for k in range(1,N+2)]
Y.insert(0,0)
print(x)
print(Y)
```

```
[-1 -1  1  1  1  1  1  1 -1  1]
[0, -1, -2, -1, 0, 1, 2, 3, 4, 3, 4, 4]
```

Simulating a Random Path

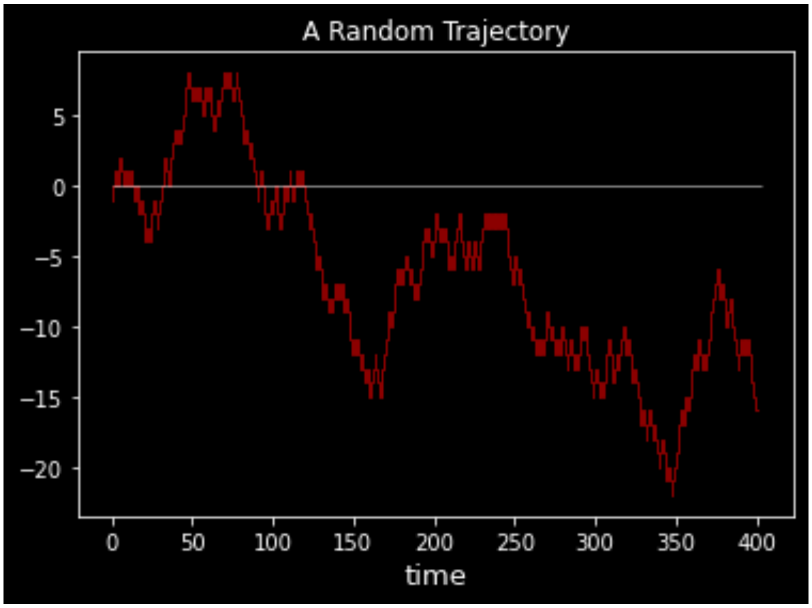
In [3]:
```
def randomwalk(N):
    xx=np.random.choice(2,N)
    x=xx*2-1
    Y=[sum(x[:k]) for k in range(1,N+2)]
    Y.insert(0,0)
    return Y
```

Ploting

In [6]:
```
def plotrw(Y):
    N = len(Y)
    ts = np.arange(N)
    #fig,ax = plt.subplots(figsize = (7,5))
    # Use plt.step to plot piecewise constant functions
    plt.step(ts,Y,color="darkred",lw=1)
    plt.hlines(0,0,N+1,color="white",lw=1,alpha=.5)
    #plt.ylim([*K,1.1*K])
    plt.xlabel(r"time",fontsize = 13)
    #plt.ylabel(r"Random Walk",rotation =0,fontsize = 13,labelpad = 20)
    plt.title("A Random Trajectory")
    plt.draw()
```

Example

In [9]:
```
N=400
Y= randomwalk(N)
plotrw(Y)
```


A Random Trajectory
5
0
−5
−10
−15
−20
0
50
100
150
200
250
300
350
400
time

In [ ]: