Introduction to Financial Mathematics

Lecture 1: Time Value of Money

October 12, 2024

This parts covers

- Bonds and cash flows.
- Different types of interest rates.
- Present value of a cash flow.
- Bond characteristics.
- Yield Curve.

1 Cash Flows

A *bond* entitles its holder specified payments at future dates. As an example consider a treasury note of two years with a coupon rate of 0.38%. It was auctioned on April 27, 2020 and issued on April, 30, 2020. For a face value of one million dollars, the future payments will be the followings:

- On 10/30/2020, \$1,900;
- On 04/30/2021, \$1,900;
- On 10/30/2021, \$1,900;
- On 04/30/2022, \$1,900 + \$1,000,000.

The total *un-discounted total* payment is \$1,007,600. Its price was determined in an auction and the coupon rate determined so as to make this price close to its face value.

The prices are quoted for a face value of \$100. Say the price is \$100.31. Then, for the above bond payments, one needs to pay \$1,003,100. In four years, the total *un-discounted* gains is \$4,500.

A *cash flow* is a sequence of future payments. Prices of them depend on the times and sizes of the payments, making it is hard to compare.

Example 1.1. Consider the bond discussed above and the following cash flows:

Cash Flow 1:
10/30/2020, \$1,900; 10/30/2021, \$1,900; 10/30/2022, \$1,900; 10/30/2023, \$1,001,900.

Cash Flow 2:
10/30/2020, \$2,200; 10/30/2021, \$2,200; 10/30/2022, \$2,200; 10/30/2023, \$1,002,200.

Which one do you prefer:

a. Cash Flow 1 or the Bond ?

b. Cash Flow 1 or Cash Flow 2?

c. Cash Flow 2 or the Bond ?

At the fist analysis we consider the total payment and the payment dates.

a. The *un-discounted total* payment of cash flow 1 is \$1,007,600. It is exactly equal to the total *un-discounted* payments of the bond. But the bond pays these amounts *earlier* than the cash flow 1. So the bond is preferred to the cash flow 1.

b. Cash flow 2 pays *more* than cash flow 1 exactly at the *same days*. So cash flow 2 is clearly better than cash flow 1.

c. The *un-discounted total* payment of cash flow 2 is \$1,008,800 which is more than the *un-discounted total* payment of the bond. But cash flow 2 pays later than the bond. So we comparison between the bond and the cash flow is *not immediate*.

To have a more refined comparison we need to discount the future payments.

□

Mathematically, we represent a cash flow by the amount of money $c(t_i)$ to be paid in the future and t_i is the date at which it will be paid:

$$\text{We write } \mathbf{c} := \{c(t_i)\}_{i=1,\ldots,n} \text{ for the cash flow } c(t_1), c(t_2), \cdots, c(t_n).$$

Consider two cash flows $\mathbf{c} := \{c(t_i)\}_{i=1,\ldots,n}$ and $\widehat{\mathbf{c}} := \{\widehat{c}(t_i)\}_{i=1,\ldots,n}$. Then,

$$c(t_i) \geq \widehat{c}(t_i) \; \forall \, i = 1, \cdots, n \quad \Rightarrow \quad \text{we prefer } \mathbf{c} \text{ to } \widehat{\mathbf{c}}.$$

If we do not have the above simple inequality between the cash flows, then it is hard to compare them. The following sections are related to this question but they are also independently importnat concepts.

1.1 Interest Rate

Interest is the amount charged when one takes a loan or the amount earned when one makes a cash investment in a bank or in a bond. For instance, suppose that we take a loan of $\$200,000$ and pay it back 25 days later. In addition to the loan amount, we also pay an interest of $\$685$. Clearly, the amount of interest depends on *length of the loan* called the maturity and the *amount of the loan* called the face value. In almost all cases, interest amount depends *linearly* on the loan amount. For example, if have taken a loan of $\$100,000$ we would have paid an interest of $\$342.5$, and for a loan $\$800,000$ interest would be $\$2,740$. In other words, the ratio of the interest to the loan amount is constant and in this example it is 0.34246% percent. This ratio depends on the duration of the loan.

In fact suppose instead we pay the same loan back in 35 days, then the bank would charge a larger interest. Similarly when a cash investment is made, one receives the interest as a payment.

Suppose we take a loan of *one dollars* and pay it back in n days. Then, the face value $F = 1$, and the maturity is $T = \frac{n}{365}$ many years. There are several conventions for reporting the interest rate. In fact, the rate is always annualized but the compounding of the interest may differ.

1. Quote an *annual* interest rate $r \in (0, 1)$, and charge an interest proportional to the duration of the loan. In this case, the interest amount is

$$r \times \frac{n}{365} = rT = r_d\, n, \qquad \text{where } \; r_d := \frac{r}{365}.$$

2. Again quote an *annual* interest rate r, but every day charge r_d interest to the original loan amount *and to the accrued interest* up to that day. The *accrued interest* is the amount of interest coming from previous days. To make the computations, let us define

$$F_k := \text{loan and the accrued interest at the end of day } k.$$

Note that $F_0 = 1$ is the original loan amount.

- At the end of day one, the accrued interest is r_d and therefore, $F_1 = (1 + r_d)$.
- At the end of day two,

$$\begin{aligned} F_2 &= \text{loan} + \text{interest from day one} + \text{interest for the second day} \\ &= F_1 + r_d\, F_1 = (1 + r_d)\, F_1 = (1 + r_d)^2. \end{aligned}$$

- Similarly, for any $k \geq 0$,

$$\begin{aligned} F_{k+1} &= \text{loan and the accrued interest at the end of day } k + \text{interest for day } (k+1) \\ &= F_k + r_d\, F_k = (1 + r_d)\, F_k = F_{k-1}\, (1 + r_d)^2 = \ldots = (1 + r_d)^{k+1}. \end{aligned}$$

So in this case, at the end of day n, interest payment is equal to

$$(1 + r_d)^n - 1 = (1 + \frac{r}{365})^{365\,T} - 1.$$

The above amount is greater than the amount $r_d\, n$ paid in the first case. This happens because we pay interest on the interest.

3. Quote an *annual* interest rate, and charge interest continuously. This is done to make calculations simpler. To understand the mechanism, imagine that bank collects interest more frequently than every day; say, every hour or even every second and so on.

Mathematically, we divide a year into M-equal pieces. For instance, in the case of daily interests $M = 365$. Now we assume that M is very, very large. Then, interest rate for each unit time interval is $r_M := \frac{r}{M}$. The loan is paid after T years which corresponds to $M\,T$

time intervals. Then, arguing as in the second case we obtain that the interest that is paid after T-years is

$$(1+\frac{r}{M})^{MT}-1.$$

Continuous interest compounding means that the number M is taken to infinity. Therefore,

$$\text{interest paid after } T \text{ years} = \lim_{M\to\infty}(1+\frac{r}{M})^{MT}-1 = e^{rT}-1.$$

We summarize this in the following.

Consider a loan or an investment of F dollar. Suppose that the annual interest rate is r and the maturity is T years. Then, the total interest after T years is

- $rT\,F$ if the *interest is compounded once*;
- $[(1+\frac{rT}{n})^n-1]\,F$ if the *interest is compounded n times*;
- $[e^{rT}-1]\,F$ if the *interest is compounded continuously*.

We continue with an example.

Example 1.2. Suppose we take a loan of $\$200,000$ and the loan provider charges 5% *annual interest*. Then, $T=25/365$ and the interest would be

- $\$684.93$ if the interest is compounded once;
- $\$686.06$ if the interest is compounded daily (i.e., 25 times);
- $\$686.10$ if the interest is compounded continuously.

Notice that the differences are very small. Now suppose that $T=2$ years. Then, the interest after two years would be

- $\$20,000$ if the interest is compounded once;
- $\$21,032.67$ if the interest is compounded daily (i.e., 730 times);
- $\$21,034.19$ if the interest is compounded continuously.

The difference between daily and continuous is still very small. But compounding it only once is substantially smaller. $\square$

1.2 Bonds

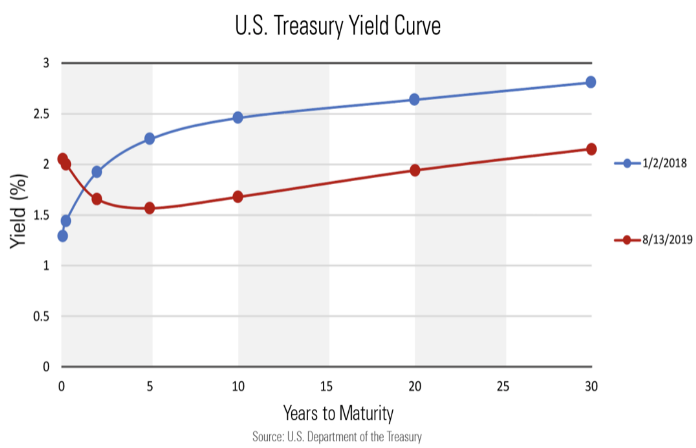
A government bond or sovereign bond is a contract issued by a national government, that promises to pay periodic payments called coupon payments and to repay the face value on the maturity date. The aim of a government bond is to support government spending. Government bonds are usually denominated in the country's own currency.

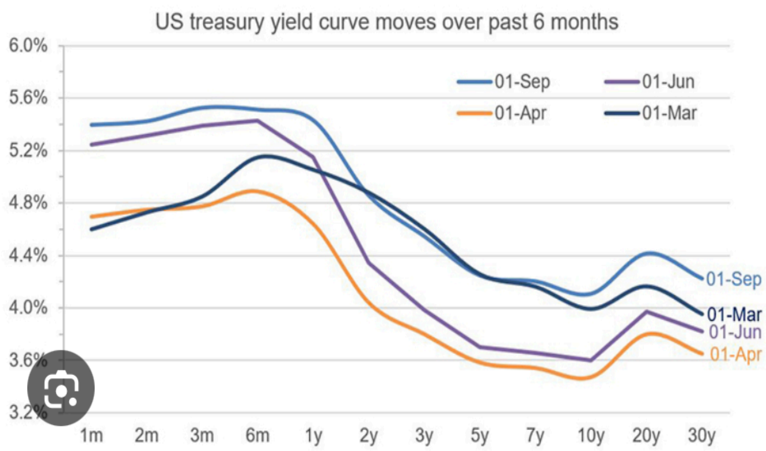
Government bonds have three important characteristics:

- *Maturity* T is the length of the bond. It can vary from few weeks to 30/40 years.
- *Face Value* F is the amount of the debt that will paid in full at maturity. We assume any F value is possible. The convention is to quote prices for $F = 100$.
- *Coupon rate* C is percentage of the face value that will paid annually or semi-annually. Typical values of C are zero for short maturities and close to the inflation rate for longer maturities.

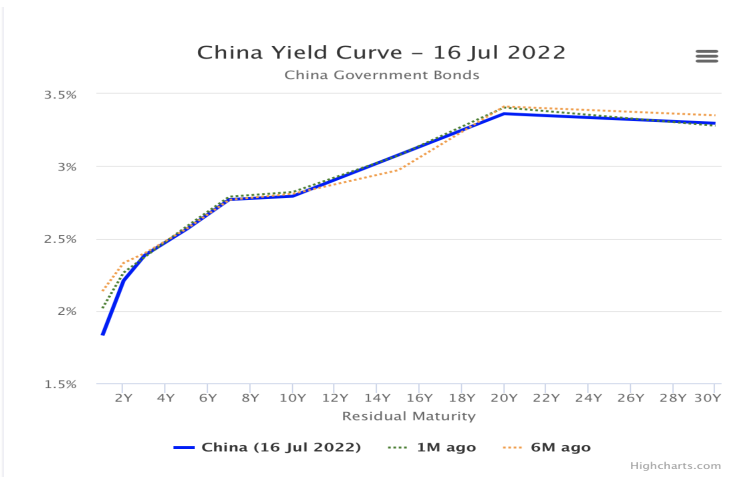
1.3 Yield Curve

The interest rates are always quoted annually and mostly compounded continuously. Moreover, the interest rate for loans or investments of different maturities are different. The following diagram shows the interest paid by the US government in February 2018 (blue) and August 2018 (red). Note that the interest rate or equivalently the yields are annualized. Still they are *not constant* over maturities and also they change in time in a random and an unpredictable manner.





The current Chinese yield curve given below also shows the same features.



Mathematically, we make the following definition.

Definition 1.3. The *yield curve* is a function $r : [0, \infty) \mapsto (0, \infty)$. The value $r(t)$ represents the annualized continuously compounded interest rate for maturity t.

Let $p(t)$ be *the price of the zero coupon* bond with maturity t and face value F. Then,

$$p(t) = e^{-tr(t)}\, F \qquad \Leftrightarrow \qquad r(t) = -\frac{\ln(p(t)/F)}{t}.$$

Thus, there exists a one-to-one correspondence between the yields and the prices of zero-coupon bonds.

In these lectures, the value $r(t)$ is not the percentage but the real number representing that percentage. For example $r(0.75) = 0.056$ means that the annualized rate for maturity of 9 months is 5.6%. In the Chinese yield curve graph above, twenty year yield curve is 3.362%. Hence, $r(20) = 0.03362$. Also in that yield curve $r(2) = 0.02208$.

1.4 Discounting

Consider the Chinese yield curve given above. Suppose that we will receive $¥15,000$ in two years. What would be its value today? Observe that getting this payment is equivalent to a zero coupon Chinese bond with maturity two years. Therefore, its value is exactly equal to the price of this zero coupon bond with face value $¥15,000$. By the above formulae, its price is equal to

$$e^{-2r(2)}15,000 = e^{-2\times 0.02208}15,000 = 14,352.$$

Similarly, the current value of a future cash payment of c dollars at time t, is equal to $ce^{-tr(t)}$ today. This is called *discounting*, and by discounting we can find the present value of a cash flow.

Definition 1.4. Suppose that we are given a yield curve $r(\cdot)$.

a. The price $e^{-tr(t)}$ of the zero-coupon bond with face value zero is called the *discount factor.*

b The *present value the cash flow* $\mathbf{c}\{c(t_i)\}_{i=1,\ldots,n}$ is given by,

$$PV(\mathbf{c}) := \sum_{i=1}^{n} c(t_i)\, e^{-t_i r(t_i)}.$$

Example 1.5. Consider the cash flows and the bond introduced in Example **??**. Suppose that the yield curve is constant 5%, i.e., $r(t) = 0.05$ for every t. Then, we can calculate the present values of each cash flows and the bond:

$$\begin{aligned}
PV(\text{bond}) &= 1,900[e^{-0.025} + e^{-0.05} + e^{-0.075}] + 1,001,000\, e^{-0.1} &&= \$911,979.75\\
PV(\text{cf } 1) &= 1,900[e^{-0.05} + e^{-0.1} + e^{-0.15}] + 1,001,000\, e^{-0.2} &&= \$825,448.21\\
PV(\text{cf } 2) &= 1,900[e^{-0.05} + e^{-0.1} + e^{-0.15}] + 1,001,000\, e^{-0.2} &&= \$826,508.87.
\end{aligned}$$

□

If we know the yield curve, then we can compute the present value of a cash flow or equivalently the prices or values of them. This gives us a way to compare different cash flows.

1.5 Internal Rate of Return

Suppose we now *know the prices of the cash flows* and not the yield curve. The following yield definition provides a comparison mechanism.

Definition 1.6.

Let $\mathbf{c} = \{c(t_i)\}_{i=1,\ldots,n}$ be a cash flow with a price of $p(\mathbf{c})$. The *internal rate of return* (IRR) of this cash flow $\mathbf{c}$ is the yield value r satisfying the following equation,

$$\boxed{p(\mathbf{c}) = \sum_{i=1}^{n} c(t_i)\, e^{-r\, t_i}.} \tag{1.1}$$

In the case of a bond, the internal rate of return is called *yield-to-maturity*. Form the above definition, it is not clear if the internal rate of return is a well-defined quantity. The following simple result resolves this issue.

Theorem 1.7. *Suppose that $c(t_i) > 0$ for every i and $p(\mathbf{c}) > 0$. Then, the internal rate of return r of $\mathbf{c}$ is uniquely defined. Moreover, it is inversely related to the price and it is strictly positive whenever $p(\mathbf{c}) < \sum_{i=1}^n c(t_i)$.*

Proof. Define a function,

$$r \in \mathbb{R} \;\mapsto\; pv(r) \;:=\; \sum_{i=1}^n c(t_i)\, e^{-r\, t_i}.$$

Then,

$$pv'(r) = -\sum_{i=1}^n t_i\, c(t_i)\, e^{-r\, t_i} < 0 \qquad \Rightarrow \qquad pv \text{ is strictly decreasing.}$$

Moreover,

$$\lim_{r\to\infty} pv(r) = 0, \quad \text{and} \qquad \lim_{r\to 0} pv(r) = pv(0) = \sum_{i=1}^n c(t_i).$$

Hence, the function pv attains any positive value at a unique point. This proves that the internal rate of return is uniquely defined. The positivity of it follows from the monotonicity and the fact that $pv(0) = \sum_{i=1}^n c(t_i)$. $\square$

Example 1.8. Consider the bond discussed at the beginning of the chapter and in examples **??** and **??**. In this example, the equation (**??**) is given by,

$$p(\mathbf{c}) = 1,003,100 = pv(r) \;=\; 1,900[e^{-r/2} + e^{-r} + e^{-3r/2} + e^{-2r}] + 1,000,000e^{-2r}.$$

As the price is greater than the face value, we immediately observe that the internal rate of return r is less than the coupon rate. However, they should be close to each other as the difference between the price and the face value is not much.

The solution obtained by a root solver is given by $r = 0.00224$. Hence, this bonds yield-to-maturity is 0.224%. $\square$

2 Exercises

1. Consider a bond with face value ¥$100,000$, semi-annual coupon of 3.5% and maturity 5 years.

   a. If the price is ¥$100,000$, find the internal rate of return.

   b. If the price is ¥$101,586.07$, find the internal rate of return.

c. If the price is ¥$98,939.72$, find the internal rate of return.

2. Consider a bond with face value F, semi-annual coupon of $c\%$ and maturity T years. Decide on the folloiwng statement

   a. If the price is exactly equal to F, then internal rate of return is approximately equal to c. Correct? Not correct?

   b. If the price is strictly more than F, then internal rate of return is approximately less than c. Correct? Not correct?

   c. If the price is strictly more than F, then internal rate of return is approximately greater than c. Correct? Not correct?

   d. I If the price is strictly less than F, then internal rate of return is approximately greater than c. Correct? Not correct?

   e. I If the price is strictly less than F, then internal rate of return is approximately less than c. Correct? Not correct?