Lecture 4

Multi step Binomial model

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02-03/11/2024

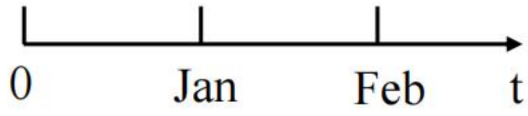
Introduction

1. Two-step Binomial Model
2. Binomial Pricing
3. Hedge

01 Two-step Binomial Model

Two-step Binomial Model (European Call Option):

- $S_0= 50,\ u= 1.05,\ d= 0.95,\ r=0.05,\ K=50, T=1/6.$



$S_2=uS_1 = uuS_0 = 55.125$

$S_1=uS_0 = 52.5$

$\boldsymbol{S_0 = 50}$

$S_2=dS_1 / uS_1 = duS_0 / udS_0 = 49.875$

$S_1=dS_0 = 47.5$

$S_2=dS_1 = ddS_0 = 45.125$

T_0 T_1 T_2

$C_2=5.125$

$C_1=2.5$

$\boldsymbol{C_0 = ?}$

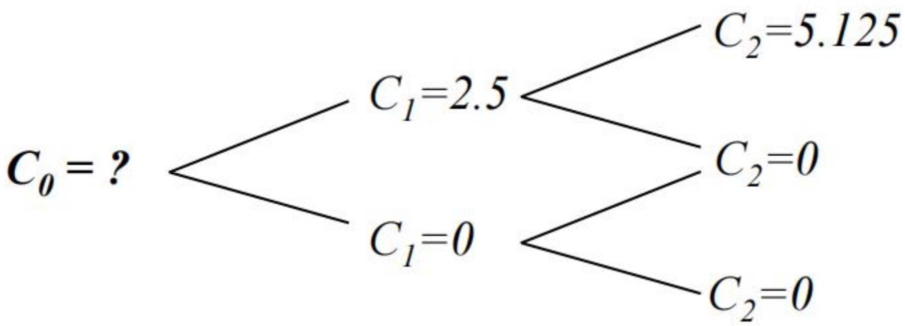
$C_2=0$

$C_1=0$

$C_2=0$

01 Two-step Binomial Model

Two-step Binomial Model (European Call Option):

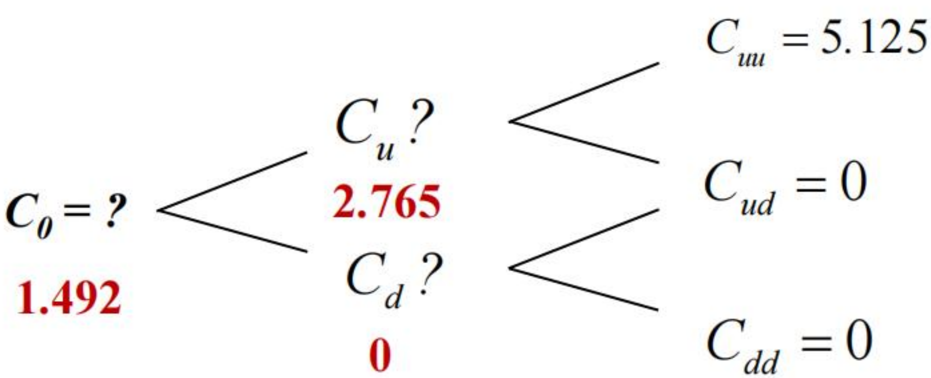


$$p^* = \frac{e^{r\Delta t}\text{-}d}{u\text{-}d} = \frac{e^{0.05\times\frac{1}{12}}\text{-}0.95}{1.05\text{-}0.95} \approx 0.54$$

$$C_u = e^{-r\Delta t}\left[p^* C_{uu} + (1-p^*) C_{ud}\right] \approx 2.765$$

$$C_d = e^{-r\Delta t}\left[p^* C_{ud} + (1-p^*) C_{dd}\right] = 0$$

$$C_0 = e^{-r\Delta t}\left[p^* C_u + (1-p^*) C_d\right] \approx 1.492$$



01 Two-step Binomial Model

Two-step Binomial Model (European Put Option):

$S_0 = 50$

$S_1 = 52.5$ $\quad$ $S_1 = 47.5$

$S_2 = 55.125$ $\quad$ $S_2 = 49.875$ $\quad$ $S_2 = 45.125$

$$p^* = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05 \times \frac{1}{12}} - 0.95}{1.05 - 0.95} \approx 0.54$$

$P_0 = ?$

1.077

$P_u\,?$

0.057

$P_d\,?$

2.292

$P_{uu} = 0$

$P_{ud} = 0.125$

$P_{dd} = 4.875$

$$P_u = e^{-r\Delta t}\left[p^* P_{uu} + (1 - p^*) P_{ud}\right] \approx 0.057$$

$$P_d = e^{-r\Delta t}\left[p^* P_{ud} + (1 - p^*) P_{dd}\right] \approx 2.292$$

$$P_0 = e^{-r\Delta t}\left[p^* P_u + (1 - p^*) P_d\right] \approx 1.077$$

$$f = e^{-2r\Delta t}\left[p^{*2} f_{uu} + 2p^*(1 - p^*) f_{ud} + (1 - p^*)^2 f_{dd}\right]$$

02 Binomial Pricing

2.1 Stock

Consider the multi-step binomial model already introduced in Section 4.4 with a constant down factor d, up factor u and a unit interest rate r. We assume there are N trading dates and we refer to the k-th trading date as step k. As in Section 4.4, let S_k be the random value of the stock at step k. Then, S_k only takes the values

$$\boxed{s_{k,i} = u^i d^{k-i} S_0 \qquad k = 0,1,\ldots,N, \;\; i = 0,1\ldots,k.} \tag{1.1}$$

02 Binomial Pricing

2.2 Option

In this market, we consider a *European* option with a general pay-off $\varphi(S_N)$. Let V_k be the random value of this option at step k. As the stock values, option values can take finitely many values denoted by $v_{k,i}$ for $k = 0,\ldots,N$, and $i = 0,\ldots,k$. Initial value V_0 takes only one value $v_{0,0}$ (hence, is deterministic) and we write v_0. The *goal is to compute* v_0.

We achieve this by computing all $v_{k,i}$ values by dynamic programming. Indeed, at maturity the value of the option is equal to its exercise value. Hence,

$$\boxed{v_{N,i} = \varphi(s_{N,i}), \qquad i = 0,1,\ldots,N.}$$

Starting with this information at maturity, we calculate the value of this option at earlier times by backwards recursion or by dynamic programming.

2.3 Pricing Equation

To develop this recursion, we start with the node $(T-1, i)$. From this node the stock price can make only one up or down movement. After these movements, we know the value of the option. Hence, we have the following diagram:

$$\boxed{\begin{array}{ll} & S_T = s_{T,i+1} = us_{T-1,i}, \quad v_{T,i+1} = \varphi(s_{T,i+1}) \\ S_{T-1} = s_{T-1,i},\ v_{T-1,i} = ? & \\ & S_T = s_{T,i} = ds_{T-1,i}, \quad v_{T,i} = \varphi(s_{T,i}). \end{array}}$$

This is exactly a one-step binomial model with initial stock price $s_{T-1,i}$ and

$$v_u = \varphi(s_{T,i+1}) = v_{T,i+1}, \qquad v_d = \varphi(s_{T,i}) = v_{T,i}.$$

Since d, r, u are time independent, p^* is given by,

$$p^* = \frac{(1+r) - d}{u - d}.$$

Hence,

$$v_{T-1,i} = \frac{1}{1+r}\left[p^*\ v_{T,i+1} + (1-p^*)\ v_{T,i}\right].$$

02 Binomial Pricing

2.3 Pricing Equation

Theorem 1.1.

Consider a Binomial model with N steps, up factor u, down factor d, interest rate r satisfying

$$d<1+r<u \tag{1.2}$$

Let $v_{k,i}$ *be the value of this option at step k after i many up movements. Then,* $v_{k,i}$ *is computed recursively by the equation*

$$\boxed{v_{k,i}=\frac{1}{1+r}\left[p^{*}v_{k+1,i+1}+\left(1-p^{*}\right)v_{k+1,i}\right],k=0,1,\cdots,N-1,i=0,1,\cdots,k,} \tag{1.3}$$

Starting form the final condition

$$\boxed{v_{N,i}=\varphi\left(s_{N,i}\right),i=0,1,\cdots,N,} \tag{1.4}$$

where $s_{k,i}$ *are as in* (1.1).

03 Hedge

3.1 Stock

In the one step Binomial model, the hedging portfolio consists of holding

$$\boxed{\theta = \frac{V_u - V_d}{(u-d)S_0} = \frac{V^{up} - V^{down}}{S^{up} - S^{down}}}$$

many stocks and a cash amount of

$$\boxed{c = v_0 - \theta\, S_0.}$$

Indeed, suppose that we hold c dollars and own θ shares of the stock in a one-step Binomial model. Then, the value of the portfolio at the next step is given by,

$$V_1 = (1+r)c + \theta\, S_1.$$

The formulae for v_0, θ, c are such that V_1 is equal to v_u if the stock moves up and is equal to v_d if the stock movement is down. Therefore, this portfolio of θ many shares of the stocks and c dollars in the bank, *perfectly replicated* the option. As the multi-dimensional Binomial model consists of many one-step Binomial models, the same perfect hedging holds.

03 Hedge

3.1 Stock

Consider European option with a pay-off φ in a multi-step binomial model with nodes (k,i), $k=0,\dots,N$, $i=0,\dots,k$. Let $v_{k,i}$ be the value of the option computed in Theorem 1.1 . Set

$$\boxed{\theta_{k,i} := \frac{v_{k+1,i+1}-v_{k+1,i}}{s_{k,i}(u-d)}, \qquad k=0,1,\dots N-1,\ i=0,1\dots k.} \tag{2.1}$$

$$\boxed{c_{k,i} := v_{k,i}-\theta_{k,i}s_{k,i}, \qquad k=0,1,\dots N-1,\ i=0,1\dots k.} \tag{2.2}$$

03 Hedge

Theorem 2.1.

The hedging strategy that starts with v0 dollars, and at each node

a. *investing* $\theta_{k,i}$ *shares in the stock and*

b. *holding* $c_{k,i}$ *dollars in the bank (borrowing if negative)*

- ✓ *Perfectly replicates the option pay-off* $\varphi(S_N)$ *without any additional funds.*
- ✓ *Additionally, the value of this portfolio at each node is exactly equal to* .

04 How to choose the parameters?

In an n-step binomial model with large n, it becomes important to choose the parameters d, r, u appropriately so that the resulting model is relevant. One way to choose them is as follows:

$$\boxed{r_n = e^{rh} - 1, \quad u_n = e^{\sigma\sqrt{h}}, \quad d_n = e^{-\sigma\sqrt{h}}, \qquad \text{with} \qquad h = \frac{T}{n},}$$

where $r > 0$ is the *continuously compounded annual interest rate*, T is maturity measured in years and $\sigma > 0$ is the annual *volatility*. The above procedure has the advantage of reducing the number of parameters. Indeed, the only parameter to be calibrated to the real financial markets is the volatility σ.

Thanks!

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02-03/11/2024