Lecture 3

One step Binomial model

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Introduction

1. **Option Review**
2. **One-Step Binomial Model**
3. **Risk Neutral Probability**

01 Option Review

Options

- **Definition:** Options are derivative financial instruments that provide the holder with the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a predetermined price within a specific time frame or at expiration.

- **Classification:**
  - ✓ **Call Option:** Grants the holder the right to buy the underlying asset.
  - ✓ **Put Option:** Grants the holder the right to sell the underlying asset.

  - ✓ **European Option:** Can only be exercised on the expiry date.
  - ✓ **American Option:** Can be exercised at any time.

  - ✓ **Physical Option:** Options where the underlying asset is a physical commodity.
  - ✓ **Financial Option:** Options where the underlying asset is a financial instrument.

01 Option Review

Call Options

- **Definition:** Options are derivative financial instruments that provide the holder with the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a predetermined price within a specific time frame or at expiration.

- **Call Options:** The European Call option with maturity T and strike K gives its holder the option but not the obligation to buy the stock at time T for a price K, regardless the price of the stock at maturity. The future random pay-off of this option is $(S_T - K)+$, where S_T is the future random value of the stock and $(a)+ := \max\{a,0\}$ for a real number a.

- **Put Options:** The European Put option with maturity T and strike K gives its holder the option (not the obligation) to sell the stock at time T for a price K, regardless the price of the stock at maturity. The future random pay-off of this option is $(K - S_T)+$.

01 Option Review

Moneyness: In-the-money, at-the-money, and out-of-the-money

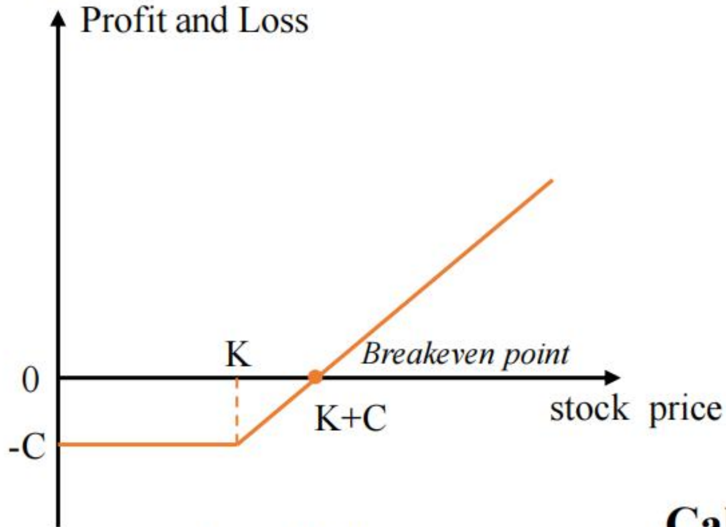
- **Call Options:**

$$\begin{cases} \boldsymbol{S_T} > \text{K, in-the-money option} \\ \boldsymbol{S_T} = \text{K, at-the-money option} \\ \boldsymbol{S_T} < \text{K, out-of-the-money option} \end{cases}$$

- **Put Options:**

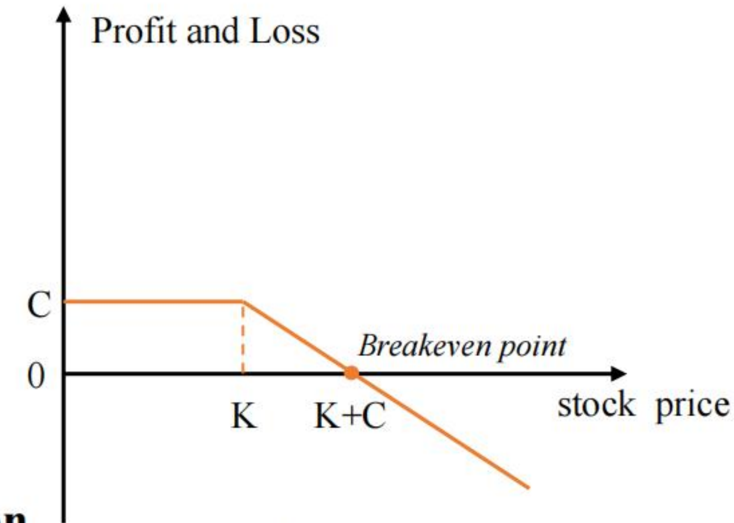
$$\begin{cases} \text{K} > \boldsymbol{S_T}\text{, in-the-money option} \\ \text{K} = \boldsymbol{S_T}\text{, at-the-money option} \\ \text{K} < \boldsymbol{S_T}\text{, out-of-the-money option} \end{cases}$$

01 Option Review

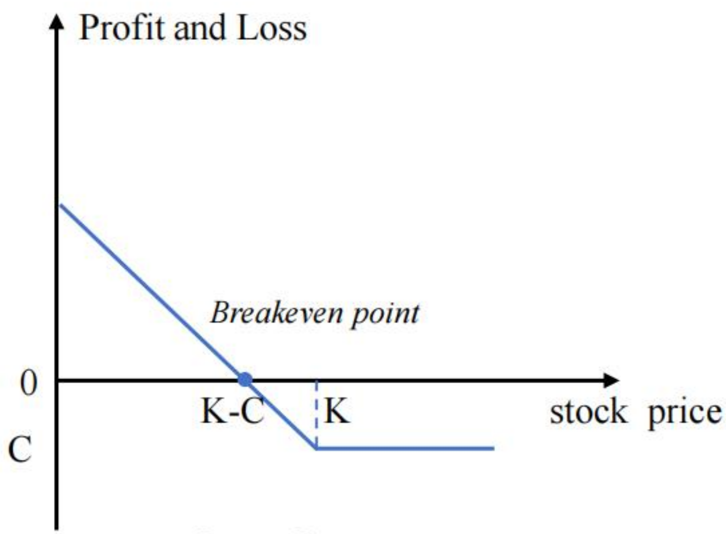


Long Call

Call option

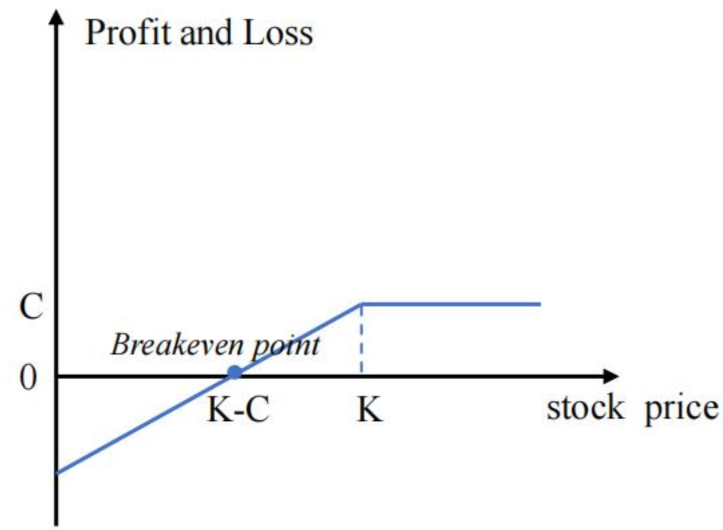


Short Call



Long Put

Put option



Short Put

01 Option Review

Put-Call Parity (PCP)

- Constructing two portfolios
  - ✓ Portfolio A: Buy one European Call option and K many shares of the zero-coupon bonds with face value 1 and maturity T.
  - ✓ Portfolio B： Buy one European Put option and one share of the stock.
- **At time T,** the value of these financial portfolio will be as follows:

		$S_T > K$	$S_T < K$
	Bond	K	K
Portfolio A	**Call option**	$(S_T - K)^+$	0
	Total	$(S_T - K)^+ + K = S_T$	K
	Stock	S_T	S_T
Portfolio B	**Put option**	0	$(K\text{-}S_T)^+$
	Total	S_T	$(K\text{-}S_T)^+ + S_T = K$

01 Option Review

Put-Call Parity (PCP)

- **Law of One Price:** In the absence of trade frictions, manipulations and tariffs, contracts possibly different locations must have the same price, if they produce identical cash-flows when expressed in a common currency.
- At time T, when the option expires, both portfolios have equal value.

$$max(S_T\ ,K)$$

- At present ($T=0$), the price of these financial portfolio are as follows:
  - ✓ **Portfolio A:**
    - Bonds: $KB(T)= Ke^{-rT}$
    - Call option: $C(K,T)$
    - Portfolio A: $C(K,T)+ KB(T)= C(K,T)+ Ke^{-rT}$
  - ✓ **Portfolio B:**
    - Stock: S_0
    - Put option: $P(K,T)$
    - Portfolio B: $P(K,T)+S_0$

$$C(K,T)+KB(T)=P(K,T)+S_0$$

$$C(K,T)-P(K,T)=S_0-KB(T)$$

01 Option Review

Put-Call Parity (PCP)

- Suppose the stock price is \$31, the risk-free interest rate is 10%, and both the put and call options have a strike price of \$30, with a term of 3 months each.

Arbitrage opportunity when the Put-Call Parity does not hold

	P(K, T) =P (30,0.25) =\$2.25	P (K, T) =P (30,0.25) =\$1.00
Current Trade	Buy a call option, pay \$3	Borrow \$29 for 3 months
	Sell a put option, receive \$2.25	Sell a call option, receive \$3
	Short sell the stock, receive \$31	Buy a put option at a price of \$1
	Invest \$30.25 at the risk-free interest rate for 3 months	Buy the stock at a price of \$31
Trade after 3 months when $S_T>30$	Receive \$31.02 from investment	Call option is exercised, sell the stock at a price of \$30
	Exercise the call option, buy the stock at a price of \$30	Repay the loan of \$29.73
	Net profit of \$1.02	Net profit of \$0.27
Trade after 3 months when $S_T<30$	Receive \$31.02 from investment	Exercise the put option, sell the stock at a price of \$30
	Put option is exercised, buy the stock at a price of \$30	Repay the loan of \$29.73
	Net profit of \$1.02	Net profit of \$0.27

01 Option Review

Thinking:

- Why does the discussion of the European option put-call parity relational equation not apply to American options?

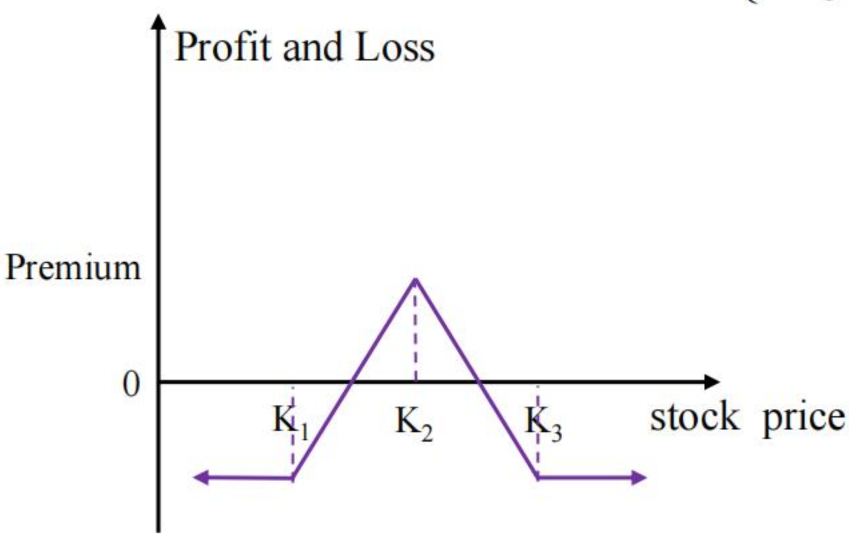
$$S_0\text{-}K \le C - P \le S_0\text{-}KB\left(T\right)$$

01 Option Review

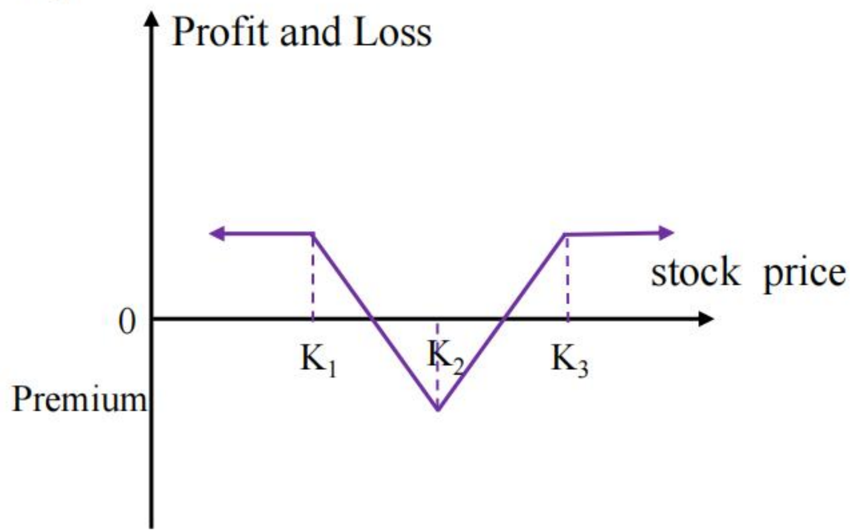
Butterflies:

- Buy one share of European call (put) option with a lower strike K_1 (in-the-moeny)
- Buy one share of European call (put) option with a higher strike K_3 (out-of-thoeny)
- Sell two shares of European call (put) option with a middle strike K_2 (at-the-me-moeny), $K_2 = (K_1 + K_3)/2$, $K_2 = S_0$.

$$pay-off = \begin{cases} 0, S_T \le K_1 \\ S_T - K_1, K_1 < S_T \le K_2 \\ K_3 - S_T, K_2 < S_T \le K_3 \\ 0, S_T \ge K_3 \end{cases}$$



Long Butterfly



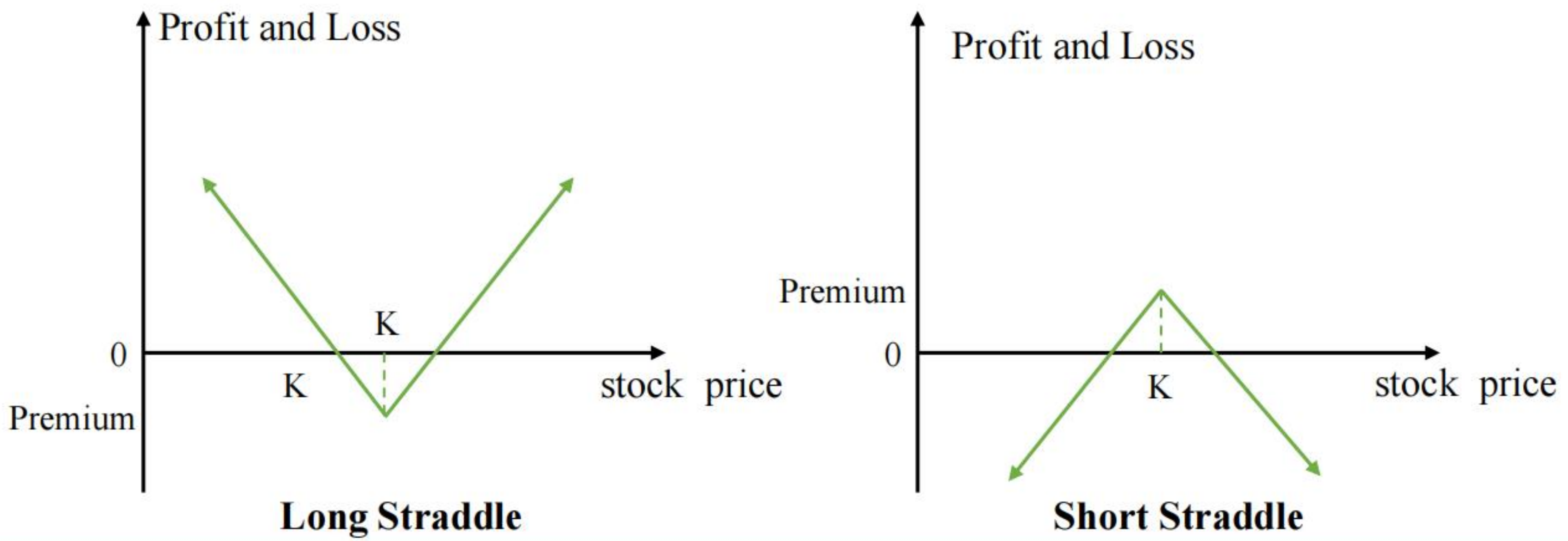
Short Butterfly

01 Option Review

Straddles:

- Buy one share of European call option with strike K and maturity T.
- Sell one share of European call option with strike K (out-of-thoeny) and maturity T.

$$pay-off=\begin{cases}K-S_T,S_T\le K\\ S_T-K,S_T>K\end{cases}$$



02 One-Step Binomial Model

Definition:

- The model simulates the future paths of asset prices by dividing them into a series of binary choices (either up or down at each step). At each time step, the asset price can move according to a specific up-factor (u) or down-factor (d).

Core:

- Construct a no-arbitrage or risk-free replicating portfolio

Characteristics:

- Advantages: intuitiveness and flexibility.
- Disadvantages: The significant increase in computational effort required when asset prices are highly volatile or when the option has a long time to maturity.

Applicaiton:

- Options pricing, risk management, and derivative structure design

02 One-Step Binomial Model

Example.

A stock is currently priced at \$100. It is anticipated that after one year, the stock price may either rise to \$150 or fall to \$50. Assuming an annual simple interest risk-free rate of 25%, consider a European call option with a term of one year, which gives the holder the right to buy one share of the stock at a strike price of \$100 upon maturity.

Calculate the price of this European call option.

03 One-Step Binomial Model

One-Step Binomial Model

Stock

S_0 — up: p → $S_1 = uS_0$

S_0 — down: $1-p$ → $S_1 = dS_0$

$d < 1 + r < u$

European option

$V_0?$ — up: p → v_u

$V_0?$ — down: $1-p$ → v_d

Structured portfolio (option): *buy a shares of the stock and b shares of the zero-coupon bond*

portfolio $aS_0 + bB(T)$ → $auS_0 + b$

portfolio $aS_0 + bB(T)$ → $adS_0 + b$

$$auS_0 + b = v_u$$
$$adS_0 + b = v_d$$

$$\Rightarrow \quad a = \frac{v_u - v_d}{(u-d)S_0} \qquad b = \frac{uv_d - dv_u}{u-d}$$

$$aS_0 + bB(T) = V_0,\; B(T) = \frac{1}{1+r}$$

$$\Rightarrow \quad V_0 = \frac{1}{1+r}\left[\frac{(1+r)-d}{u-d}v_u + \frac{u-(1+r)}{u-d}v_d\right]$$

$$p = \frac{(1+r)\text{-}d}{u\text{-}d}$$

$$V_0 = \frac{1}{1+r}\left[pv_u + (1-p)v_d\right]$$

e^{-rT}

03 Risk-Neutral Probability

The definition of risk-neutral probability:

- In financial mathematics, the risk-neutral probability is a hypothetical probability that assumes market participants do not account for risk when evaluating and pricing financial assets.

The characteristics of risk-neutral world:

- A risk-neutral world is a theoretical financial environment where all investors have a neutral attitude towards risk.
- The current price of an asset is equal to the present value of its expected future value, discounted at the risk-free interest rate.
- The expected return on all securities is equal to the risk-free rate, regardless of the actual risk preferences of the market.

03 Risk Neutral Probability

- **Risk Neutral Probability:** $p^* = \dfrac{(1+r)-d}{u-d}$

p*** is the probability of a stock price increase in a risk-neutral world.

- **The price of an European option:** $V_0 = \dfrac{1}{1+r}\left[p^* v_u + (1-p^*) v_d\right] = \dfrac{1}{1+r} E(V_1)$

- **The expected return of the stock at time T=1 is when the probability of rise is *p****:**

$$E(S_1) = p^* u S_0 + (1-p^*) d S_0$$

$$E(S_1) = p^*(u-d) S_0 + d S_0$$

$$E(S_1) = (1+r) S_0 \quad \Rightarrow \quad S_0 = \frac{1}{(1+r)} E(S_1)$$

The above formula illustrates that stock prices increase at the risk-free rate when the probability of stock price increase is the risk-neutral probability ***p****.

Thanks!

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