**Introduction to Financial Mathematics**

# Lecture 8: Brownian Motion and Black & Scholes model

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## 1 Scaled Random Walk and Brownian Motion

As in the Binomial model, we want to let $n$ tend to infinity. However, without scaling the process $Y$ explodes. So we rescale and define a new processes by,

$$X_k^{(n)} := \frac{1}{\sqrt{n}} Y_k, \quad k = 0, 1, 2, \ldots,$$

and

$$\boxed{W_t^{(n)} := X_{\lfloor nt \rfloor}^{(n)}, \quad t \geq 0,}$$

where $\lfloor a \rfloor$ is the largest integer less than or equal to $a$.

Then, for any $t \geq 0$,

$$W_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} x_i.$$

As $\mathbf{E}[x_i] = 0$ and $\mathbf{E}[(x_i)^2] = 1$ for each $i$,

$$\mathbf{E}[W_t^{(n)}] = \mathbf{E}[X_{\lfloor nt \rfloor}^{(n)}] = 0,$$

and

$$\begin{aligned}
\mathbf{E}\left[\left(W_t^{(n)}\right)^2\right] &= \frac{1}{n}\,\mathbf{E}[\left(\sum_{i=1}^{\lfloor nt \rfloor} x_i\right)^2 ] \\
&= \frac{1}{n}\,\mathbf{E}\left[\left(\sum_{i=1}^{\lfloor nt \rfloor} x_i \sum_{j=1}^{\lfloor nt \rfloor} x_j\right)\right] \\
&= \frac{1}{n}\,\mathbf{E}[\sum_{i=1}^{\lfloor nt \rfloor}\sum_{j=1}^{\lfloor nt \rfloor} x_i\, x_j ] \\
&= \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor}\sum_{j=1}^{\lfloor nt \rfloor} \mathbf{E}\left[\, x_i\, x_j \,\right].
\end{aligned}$$

When $i \neq j$, $x_i$ is independent of $x_j$. Hence, $\mathbf{E}\,[\,x_i\,x_j\,] = \mathbf{E}\,[\,x_i\,]\;\mathbf{E}\,[x_j\,] = 0$. On the other hand, when $i = j$, $x_i x_j = x_i^2 = 1$. Hence,

$$\begin{aligned}\mathbf{E}\left[\left(W_t^{(n)}\right)^2\right] &= \frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{\lfloor nt\rfloor}\mathbf{E}\,[\,x_i\,x_j\,]\\ &= \frac{1}{n}\sum_{i=1}^{\lfloor nt\rfloor} 1\\ &= \frac{\lfloor nt\rfloor}{n} \approx t, \qquad \text{for large } n.\end{aligned}$$

## 1.1 Central Limit Theorem

For a large positive integer $N$, let

$$Z_N := \frac{1}{\sqrt{N}}\sum_{i=1}^{N} x_i.$$

Then, we can calculate as above to obtain,

$$\mathbf{E}[Z_N] = 0, \qquad \mathbf{E}[Z_N^2] = 1.$$

**Theorem 1.1** (Central Limit Theorem)**.** *As $n$ tends to infinity $Z_N$ converges to a Gaussian random variable with mean zero and variance $1$, which we denotes by $\mathcal{N}$ and called the standard Gaussian random variable.*

This convergence implies the following: for any nice function $g$,

$$\lim_{N\to\infty}\mathbf{E}[g(Z_N)] = \mathbf{E}[g(\mathcal{N})].$$

## 1.2 Gaussian Random Variables

The *distribution function* of the called the standard Gaussian random variable is given by,

$$f_{\mathcal{N}}(x) = \frac{1}{\sqrt{2\pi}}\,\exp(-\frac{1}{2}\,x^2), \qquad x \in \mathbb{R}.$$

Then, for any $a < b$,

$$\mathrm{P}(a \leq \mathcal{N} \leq b) = \int_a^b f_{\mathcal{N}}(x)\,dx.$$

Also, for any nice function $g$,

$$\mathbf{E}[f(\mathcal{N})] = \int_{-\infty}^{\infty} g(x)\,f_{\mathcal{N}}(x)\,dx.$$

By the Central Limit Theorem, we have

$$\mathbf{E}[f(\mathcal{N})] = \int_{-\infty}^{\infty} g(x)\, f_{\mathcal{N}}(x)\, dx = \lim_{N\to\infty} \mathbf{E}[g(Z_N)].$$

# 2 Brownian Motion

Recall that

$$W_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} x_i.$$

We rewrite it as

$$W_t^{(n)} = \frac{\sqrt{\lfloor nt \rfloor}}{\sqrt{n}} \frac{1}{\sqrt{\lfloor nt \rfloor}} \sum_{i=1}^{\lfloor nt \rfloor} x_i.$$

Now,

$$\lim_{n\to\infty} \frac{\sqrt{\lfloor nt \rfloor}}{\sqrt{n}} = \sqrt{t}.$$

Since as $n$ goes to infinity, $N := \sqrt{\lfloor nt \rfloor}$ also goes to infinity, we have,

$$\lim_{n\to\infty} \frac{1}{\sqrt{\lfloor nt \rfloor}} \sum_{i=1}^{\lfloor nt \rfloor} x_i = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i = \mathcal{N}.$$

Putting them together, we conclude that:

$$\lim_{n\to\infty} W_t^{(n)} = \sqrt{t}\,\mathcal{N}.$$

In words, we have show that $W_t^{(n)}$ converges to a Gaussian random variable with mean zero and variance $t$. But in fact one can prove that the whole process $W^{(n)}$ converges to a continuous process $W$ called *Brownian motion*. The precise statement is the following:

**Theorem 2.1** (Donsker Invariance Principle (1951))**.** *As $n$ tends to infinity the rescaled random walk $W^{(n)}$ converges to a process $W$ called Brownian motion.*

The meaning of the above convergence is the following. For any real-valued function $\Phi$ of stock price paths, we have

$$\lim_{n\to\infty} \mathbf{E}[\,\Phi((W_t^{(n)})_{t\in[0,T]})] \;=\; \mathbf{E}[\,\Phi((W_t)_{t\in[0,T]})]. \tag{2.1}$$

Note that in the above we can take functions of the average value of the scaled random walk or its running maximum, or functions its values at more than one time points.

## 2.1 Definition

The *Brownian motion* is the stochastic process given by,

$$W_t = \lim_{n\to\infty} W_t^{(n)},$$

where

$$W_t^{(n)} := X_{\lfloor nt \rfloor}^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} x_i, \quad t \geq 0,$$

and the i.i.d. random sequence $\{x_i\}$ takes values in $\{-1, +1\}$ with probability.

Mathematically, there are several other equivalent definitions. The followings are essentially defining properties of the Brownian motion.

Properties of Brownian Motion:

1. For each $t \geq 0$, $W_t$ has the same distribution as $\sqrt{t}\mathcal{N}$, where $\mathcal{N}$ is a Gaussian random variable with mean zero and variance 1. Then, for any nice function $g$,

$$\begin{aligned}
\mathbf{E}[g(W_t)] = \mathbf{E}[g(\sqrt{t}\,\mathcal{N}] &= \int_{-\infty}^{\infty} g(\sqrt{t}\,x)\, f_{\mathcal{N}}(x)\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\sqrt{t}\,x)\, e^{-\frac{x^2}{2}}\, dx \\
&= \frac{1}{\sqrt{2t\pi}} \int_{-\infty}^{\infty} g(y)\, e^{-\frac{y^2}{2t}}\, dy.
\end{aligned}$$

2. For each $0 < s < t$, $W_s$ is independent of $W_t - W_s$;

3. For each $0 < s < t$, $W_t - W_s$ has the same distribution of $W_{t-s}$;

4. $t \to W_t$ is continuous.

First three of these properties follow immediately from the properties of the random walk. The final statement that the process $t \to W_t$ is continuous, is however a harder statement to prove. For future reference, we record an important consequence of the first property.

**Exercise:**

**a.** Compute $\mathbf{E}[\, e^{W_2}\, ]$.

**b.** Compute $\mathbf{E}[\, e^{W_t}\, ]$ for any $t > 0$.

**c.** Compute $\mathbf{E}[W_1\, W_2]$.

# 3 Black & Scholes model

This famous model considered by Fisher Black and Myron Scholes postulates that the stock price process is given by

$$\boxed{S_t = S_0 e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}, \quad t \geq 0,}$$

where the *mean-return rate* $\mu$ and the *volatility* $\sigma$ are two model parameters. They will be discussed thoroughly in the subsequent chapters.

In the following chapter, we will derive the option pricing formula directly in the continuous time starting from the above model. In order to achieve that we need to develop the *Ito calculus* that allows us to work with functions of the Brownian like the stock process. This computational tool also helps us to efficiently derive hedging strategies.

## 3.1 Black & Scholes as the limit of Binomial

Consider a Binomial model with $n$-steps. We would like to let $n$ go to infinity and obtain a continuous-time model. Towards that goal, we fix a maturity $T$ (measured in years), partition the time interval $[0, T]$ uniformly by $n+1$ points,

$$\{t_0 = 0, t_1 = h, \ldots, t_k = kh, \ldots, t_n = T\}, \quad \text{where} \quad h := h_n = \frac{T}{n}.$$

We assume that the trading is restricted to these time points and the stock price is modeled by a recombining $n$-step Binomial tree. We let $(S_k)_{k=0,1,\ldots,n}$ be the random stock values in this multi-step Binomial model. The random variable $S_k$ is a model for the stock value $S_{t_k}$ at time $t_k$. Equivalently, the $k$th step of the Binomial model corresponds to the actual time $t_k = T\,k/n$. So we define a process for all continuous time by,

$$S_t^{(n)} := S_{\lfloor t/h \rfloor} = S_{\lfloor nt/T \rfloor}, \qquad t \in [0, T].$$

We choose the down factors as well as the interest rate depending on $n$ as follows:

$$\boxed{r_n = e^{rh} - 1, \quad u_n = e^{\sigma\sqrt{h}}, \quad d_n = e^{-\sigma\sqrt{h}}, \quad \text{where} \quad h = \frac{T}{n},} \tag{3.1}$$

$r > 0$ is the *continuously compounded annual interest rate* and $\sigma > 0$ is the annual *volatility* that is discussed later. A discussion of this choice is given in the Appendix at the end of the Chapter.

Recall that in the Binomial model, we have the following representation of the stock process:

$$\ln(S_t) = \ln(S_0) + \sigma\,\sqrt{t}\,\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i,$$

where $x_i$ is an i.i.d. sequence of random variables with values in $\{-1, +1\}$. The value $+1$ corresponds to an up movement and its probability is equal to $p_n^* = ((1 + r_n) - d_n)/(u_n - d_n)$ which also depends on $n$. The following theorem is proved in the Appendix of the Lecture Notes.

**Theorem 3.1.** *For any $t \in [0,T]$, under the risk neutral measure, $S_t^{(n)}$ converges to*

$$S_t = S_0\, e^{\sigma W_t + (r-\sigma^2/2)t},$$

*where $W_t$ is the standard Brownian motion. In particular, for any function $h$,*

$$\mathbf{E}_{\mathbb{Q}}[\, h(S_t)\,] = \lim_{n\to\infty} \mathbf{E}_{\mathbb{Q}_n}[h(S_t^{(n)})\,], \qquad \forall\, t \in [0,T],$$

*where $\mathbb{Q}$ is the risk-neutral measure and $\mathbb{Q}_n$ is the risk neutral measure in the n-step Binomial model, i.e., the up probability is equal to $p_n^*$.*

The following is an important computational statement.

**Corollary 3.2** (Numerical approximation of Black & Scholes)**.**

*The multi-step Binomial model with parameters given by* (**??**) *and sufficiently large $n$, approximates the option values and the hedges of the continuous-time Black & Scholes model with volatility $\sigma$ and annual interest-rate $r$.*

## 3.2 Option prices

Let $v_n$ be the price of an option in the $n$-step Binomial model, and let $v$ be its price in the Black & Scholes model. Suppose the pay-off of the option is $h$. Then,

$$v_n = \frac{1}{(1+r_n)^n}\, \mathbf{E}_{\mathbb{Q}_n}[h(S_T^{(n)}\,].$$

As $r_n = e^{rh} - 1$ and $h = T/n$,

$$(1+r_n)^n = (e^{rh})^n = e^{rnh} = e^{rT}.$$

By the above theorem,

$$\boxed{v = e^{rT}\, \mathbf{E}_{\mathbb{Q}}[h(S_T) \;= \lim_{n\to\infty} \frac{1}{(1+r_n)^n}\, \mathbf{E}_{\mathbb{Q}_n}[h(S_T^{(n)})\,] = \lim_{n\to\infty} v_n.}$$

Since $S_T = \sqrt{T}\,\mathcal{N}$,

$$\boxed{\text{option price} = v = e^{rT}\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(\sqrt{T}x)\, \exp(-\frac{x^2}{2})\, dx.} \qquad (3.2)$$

# 4 Call Option

In the example of a Call option, option price can be computed explicitly. Indeed, for a Call option with a general strike $K$,

$$\varphi(s) = (s-K)^+.$$

## 4.1 Black & Scholes Formula

Let $C(K,T)$ be the price of this option. In view of the equation (**??**),

$$\begin{aligned}
C(K,T) &= \exp(-rT)\mathbf{E}\left[\left(S_0\exp(\sigma\sqrt{T}\mathcal{N}+(r-\sigma^2/2)T)-K\right)^+\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-rT}\left(S_0\, e^{\sigma\sqrt{T}x+(r-\sigma^2/2)T}-K\right)^+ e^{-x^2/2}\boldsymbol{d}x, \\
&= \frac{1}{\sqrt{2\pi}}\int_{d^*}^{\infty} e^{-rT}\left(S_0\, e^{\sigma\sqrt{T}x+(r-\sigma^2/2)T}-K\right) e^{-x^2/2}\boldsymbol{d}x, \\
&=: I_1 - I_2,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &:= S_0\frac{1}{\sqrt{2\pi}}\int_{d^*}^{\infty}\exp(-rT+\sigma\sqrt{T}x+(r-\sigma^2/2)T-x^2/2)\boldsymbol{d}x \\
I_2 &:= Ke^{-rT}\frac{1}{\sqrt{2\pi}}\int_{d^*}^{\infty} e^{-x^2/2}\boldsymbol{d}x,
\end{aligned}$$

and $d^*$ is given by

$$S_0\, e^{\sigma\sqrt{T}d^*+(r-\sigma^2/2)T}-K=0 \quad \Rightarrow \quad d^* = \frac{(\sigma^2/2-r)T+\ln(K/S_0)}{\sigma\sqrt{T}}.$$

We continue by calculating the above integrals in term of the *c.d.f. of standard Gaussian* defined by,

$$\boxed{N(d) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-x^2/2}dx.} \tag{4.1}$$

We know that $N(-\infty)=0, N(0)=1/2, N(\infty)=1$ and by symmetry,

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-d}^{\infty} e^{-x^2/2}dx = 1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d} e^{-x^2/2}dx = 1-N(-d).$$

Moreover, $N$ is closely related to a function sometimes called the *error function*. Indeed,

$$N(x) = \frac{1}{2}\left[erf\left(\frac{x}{\sqrt{2}}\right)+1\right], \quad \text{where} \quad erf(\xi) := \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-y^2}\,\boldsymbol{d}y.$$

In particular,

$$I_2 = N(d_2), \quad \text{where} \quad d_2 = -d^* = \frac{(r - \sigma^2/2)T + \ln(S_0/K)}{\sigma\sqrt{T}}.$$

To express $I_1$ in terms of $N$, we observe that

$$\begin{aligned} -rT + \sigma\sqrt{T}x + (r - \sigma^2/2)T - x^2/2 &= -\frac{1}{2}\left(x^2 + \sigma^2 T - 2\sigma\sqrt{T}\right) \\ &= -\frac{1}{2}\left(x - \sigma\sqrt{T}\right)^2. \end{aligned}$$

Hence,

$$\begin{aligned} I_1 := S_0 \frac{1}{\sqrt{2\pi}} &\int_{d^*}^{\infty} \exp(-\frac{1}{2}\left(x - \sigma\sqrt{T}\right)^2) dx \\ = S_0 \frac{1}{\sqrt{2\pi}} &\int_{(d^* - \sigma\sqrt{T})}^{\infty} \exp(-y^2/2) \boldsymbol{d}y \\ = S_0 N(\sigma\sqrt{T} - d^*). \end{aligned}$$

Therefore,

$$C(K, T) = S_0 N(d_1) - Ke^{-rT} N(d_2),$$

where

$$d_1 = \frac{(r + \sigma^2/2)T + \ln(S_0/K)}{\sigma\sqrt{T}}, \quad d_2 = \frac{(r - \sigma^2/2)T + \ln(S_0/K)}{\sigma\sqrt{T}}.$$

We summarize this classical formula in the following theorem.

**Theorem 4.1** (Black & Scholes call price)**.**
*Let $v(t, s)$ be the price of the Call option at time $t$ with $S_t = s$. Then,*

$$\boxed{v(t, s) = sN(d_1(t, s)) - e^{-r(T-t)} KN(d_2(t, s)),} \tag{4.2}$$

*where*

$$d_1 = \frac{(r + \sigma^2/2)(T - t) + \ln(s/K)}{\sigma\sqrt{T - t}}, \quad d_2 = \frac{(r - \sigma^2/2)(T - t) + \ln(s/K)}{\sigma\sqrt{T - t}}.$$

**Exercise:**

Let $S_0 = 100$, $\sigma = 20\% = 0.2$, $r = 2\% = 0.02$.

We are given that

$$N(-0.1) = 0.460, \quad N(-0.2) = 0.421, \quad N(-0.3) = 0.382, \quad N(-0.4) = 0.345.$$

**a.** What is $N(0)$?

**b.** Compute $N(0.1), N(0.2), N(0.3), N(0.4)$.

**c.** . Compute the price of Call option with strike $K = 100$ and maturity $1$ year.

**d.** Compute the price of Put option with strike $K = 100$ and maturity $1$ year.

**Theoretical Exercise:**

**a.** Show that with $h(s) = s$, the expression in (**??**) is equal to $S_0$, i.e.,

$$e^{-rT}\, \mathbf{E}\left[S_0 \exp(\sigma\sqrt{T}\mathcal{N} + (r - \sigma^2/2)T)\right] = S_0.$$

**b.** Derive the Put formula.

**c.** Let $v(t, s)$ be as in the above Theorem. Show that

$$\frac{\partial v}{\partial s}\,(t, s) = N(d_1(s)).$$