Lecture 8

Brownian Motion
and
Black & Scholes model

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where $x_1, x_2, x_3 \dots$ are identically, and independently distributed (iid) random variables taking values $+1$ or -1 with equal probability, eg

$$\mathbb{P}(x_i = +1) = \mathbb{P}(x_i = -1) = \frac{1}{2}$$

First, recall the Random Walk

$$Y_k = \sum_{i=1}^{k} x_i \quad , k=0,1,\ldots$$

where $\{x_i\}_{i=1}^{\infty}$ are random variables

$x_i \in \{-1,+1\}$ and probabilities are equal.

$$\mathbb{P}(x_i=-1) = \mathbb{P}(x_i=+1) = \frac{1}{2}$$

$$\Rightarrow \quad \mathbb{E}[x_i] = (-1)\frac{1}{2} + (+1)\frac{1}{2} = 0$$

$$\mathbb{E}[(x_i)^2] = 1 \quad \text{because} \quad x_i^2 = 1$$

$$\Rightarrow \quad \text{variance}(x_i) = \mathbb{E}[x_i^2] - (\mathbb{E}[x_i])^2 = 1$$

$$\mathbb{E}[(Y_k)^2] = k$$

for all k

The proof is as follows:

(X1+X2) * (X1+X2) = ?

$(X_1 + X_2) \cdot (X_1 + X_2)$ 表示一个二次项的展开，可以使用完全平方公式展开：

$$(X_1 + X_2)^2 = X_1^2 + 2X_1X_2 + X_2^2$$

因此，结果是：

$$X_1^2 + 2X_1X_2 + X_2^2$$

(X1+X2+X3) * (X1+X2) = ?

我们通过分配律展开 $(X_1 + X_2 + X_3) \cdot (X_1 + X_2)$，逐项相乘:

$$(X_1 + X_2 + X_3) \cdot (X_1 + X_2) = X_1(X_1 + X_2) + X_2(X_1 + X_2) + X_3(X_1 + X_2)$$

继续展开:

$$= X_1^2 + X_1X_2 + X_2X_1 + X_2^2 + X_3X_1 + X_3X_2$$

合并同类项（如果存在）:

$$= X_1^2 + 2X_1X_2 + X_2^2 + X_3X_1 + X_3X_2$$

最终结果为:

$$X_1^2 + 2X_1X_2 + X_2^2 + X_3X_1 + X_3X_2$$

（X1+X2+...+X4）*（X1+X2+...+X7） = ?

我们将 $(X_1 + X_2 + X_3 + X_4) \cdot (X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7)$ 展开。

根据分配律，每个项相乘并展开：

$$(X_1 + X_2 + X_3 + X_4) \cdot (X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7) = \sum_{i=1}^{4} \sum_{j=1}^{7} X_i X_j$$

展开计算

每一项 X_i 都与 $X_1, X_2, \ldots, X_7$ 相乘，总共有 $4 \times 7 = 28$ 个项，逐一写出：

$$= X_1^2 + X_1X_2 + X_1X_3 + X_1X_4 + X_1X_5 + X_1X_6 + X_1X_7 + X_2X_1 + X_2^2 + X_2X_3 + X_2X_4 + X_2X_5 + X_2X_6 + X_2X_7$$

$$+X_3X_1 + X_3X_2 + X_3^2 + X_3X_4 + X_3X_5 + X_3X_6 + X_3X_7 + X_4X_1 + X_4X_2 + X_4X_3 + X_4^2 + X_4X_5 + X_4X_6 + X_4X_7$$

简化

通常这类展开结果会按照需要来分组或因式分解，但完整表达如下：

$$\text{结果} = \sum_{i=1}^{4} \sum_{j=1}^{7} X_i X_j$$

Conclusion

As in the Binomial model, we want to let n tend to infinity. However, without scaling the process Y explodes. So we rescale and define a new processes by,

$$X_k^{(n)} := \frac{1}{\sqrt{n}} Y_k, \quad k = 0, 1, 2, \dots,$$

and

$$\boxed{W_t^{(n)} := X_{\lfloor nt \rfloor}^{(n)}, \quad t \geq 0,}$$

where $\lfloor a \rfloor$ is the largest integer less than or equal to a.

Then, for any $t \geq 0$,

$$W_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} x_i.$$

As $\mathbf{E}[x_i] = 0$ and $\mathbf{E}[(x_i)^2] = 1$ for each i,

$$\mathbf{E}[W_t^{(n)}] = \mathbf{E}[X_{\lfloor nt \rfloor}^{(n)}] = 0,$$

$$
\begin{aligned}
\mathbf{E}\left[\left(W_t^{(n)}\right)^2\right] &= \frac{1}{n}\,\mathbf{E}[\left(\sum_{i=1}^{\lfloor nt \rfloor} x_i\right)^2\,] \\
&= \frac{1}{n}\,\mathbf{E}\left[\left(\sum_{i=1}^{\lfloor nt \rfloor} x_i \sum_{j=1}^{\lfloor nt \rfloor} x_j\right)\right] \\
&= \frac{1}{n}\,\mathbf{E}[\sum_{i=1}^{\lfloor nt \rfloor}\sum_{j=1}^{\lfloor nt \rfloor} x_i\, x_j\,] \\
&= \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor}\sum_{j=1}^{\lfloor nt \rfloor}\mathbf{E}\,[\,x_i\, x_j\,].
\end{aligned}
$$

When $i \neq j$, x_i is independent of x_j. Hence, $\mathbf{E}[\, x_i \, x_j \,] = \mathbf{E}[\, x_i \,] \; \mathbf{E}[x_j \,] = 0$. On the other hand, when $i = j$, $x_i x_j = x_i^2 = 1$. Hence,

$$
\begin{aligned}
\mathbf{E}\left[\left(W_t^{(n)}\right)^2\right] &= \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=1}^{\lfloor nt \rfloor} \mathbf{E}[\, x_i \, x_j \,] \\
&= \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} 1 \\
&= \frac{\lfloor nt \rfloor}{n} \approx t, \qquad \text{for large } n.
\end{aligned}
$$

Again $u_m = \#$ of $+1$'s in m steps

$\Rightarrow$ there are $m - u_m$ many -1's in m steps

$$\Rightarrow Y_m = (+1)\,u_m + (-1)\,(m - u_m)$$

$$= 2u_m - m$$

$$\mathbb{P}(Y_m = 2k - m) = \mathbb{P}(u_m = k)$$

for $m = 1, \dots$
$k = 0, \dots, k$.

$$= \frac{k \text{ choose from } m}{2^m}$$

$$= \frac{m!}{k!\,(m-k)!} \cdot \frac{1}{2^m}$$

Note that $\ell = 2k - m \Rightarrow k = \dfrac{\ell + m}{2}$

Theorem. For any $0 \le B \le m$,

$$\mathbb{P}(M_m \geq B) = 2\,\mathbb{P}(Y_m > B) + \mathbb{P}(Y_m = B)$$

where

$$M_m = \max\{Y_0 = 0, Y_1, \ldots, Y_m\}$$

= "running maximum"

Solution by dynamic programming.

Original set $\{x_1, \dots, x_n\}$

k = # of bags already collected

i = # of left-most bags collected

$k = 0, \dots, n$, $i = 0, \dots, k$

$v_k(i)$ = maximum amount of items the 'top' player can collect from this point on, assuming that the other player plays optimally.

$w_k(i)$ = maximum amount of items the other player can collect

Then the choices for the top player are

choose left: gets $x_{i+1} + w_{k+1}(i+1)$

choose right: gets $x_{n-k+i} + w_{k+1}(i)$

Then we choose the maximum.

$$V_k(i) = \max\{ x_{i+1} + w_{k+1}(i+1) ; \; x_{n-k+i} + w_{k+1}(i) \}$$

$$w_k(i) = \begin{cases} v_{k+1}(i) & : \text{if right optimal} \\ v_{k+1}(i+1) & : \text{if left optimal} \end{cases}$$

$$k = 0, 1, \ldots, n-1$$
$$i = 0, 1, \ldots, k.$$

Thank you!

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