Introduction to Financial Mathematics

Lecture 10: Mean-Variance problem and CAPM

December, 21 2024

This lecture outlines the approach developed by Harry Markowitz for portfolio allocation under uncertainty in 1952. Essentially, it was the first quantitative approach to the problem and has been immensely popular. He received the Nobel prize in Economics in 1990.

1 Portfolio Selection

We consider a financial market with m many assets and an investment problem with one time horizon. Namely, they decide on a portfolio allocation based on the gains or losses that will be realized in the next time point. Let R_k be the random return of one dollar invested in the k-the stock. Let $S^{(k)}_{next}$ be the random stock value of the k-th asset and $S^{(k)}_{now}$ be its current value. then,

$$R_k := \frac{S^{(k)}_{next} - S^{(k)}_{now}}{S^{(k)}_{now}}, \qquad k = 1, \ldots, m.$$

Let

$$\mathbf{R} := (R_1, \ldots, R_m) \in \mathbb{R}^m$$

be the random return vector. We assume that its mean and covariance is known to the investors and they base their investement decisions only on these statistics. Set

$$\mathbf{r} := \mathbf{E}[\mathbf{R}] = (r_1, \ldots, r_m),$$

and

$$\mathbf{C}_{k,j} := cov(R_k, R_j) = \mathbf{E}\left[(R_k - r_k)(R_j - r_j)\right], \qquad j, k = 1, \ldots, m.$$

Further let r_0 be the risk-free rate and a_k is the excess return of the k-th asset. Set,

$$\mathbf{a} := (a_1, \ldots, a_m), \quad \text{where} \quad a_k := r_k - r_0, \qquad k = 1, \ldots, m.$$

We may rewrite the above as

$$\mathbf{a} = \mathbf{r} - r_0 \mathbf{1}_m, \qquad \text{where} \qquad \mathbf{1}_m := (1, \ldots, 1) \in \mathbb{R}^m.$$

1.1 Portfolio

A portfolio describes the allocation of investor's wealth in the financial assets and the bond. Mathematically, π_k is the fraction of the wealth invested in the k-th asset. Then, $1 - \pi \cdot \mathbf{1}_m$ fraction is invested in the bond, where $\pi := (\pi_1, \ldots, \pi_m)$.

The return of a given portfolio is the following random variable:

$$R^\pi = \pi \cdot \mathbf{R} + (1 - \pi \cdot \mathbf{1}_m) r_0.$$

Then, the excess return of this portfolio is given by,

$$a^\pi := \mathbf{E}[R^\pi] - r_0 = \pi \cdot \mathbf{E}[\mathbf{R}] - \pi \cdot \mathbf{1}_m r_0 = \pi \cdot (\mathbf{r} - r_0 \mathbf{1}_m) = \pi \cdot \mathbf{a}.$$

Its variance is

$$\begin{aligned} var(R^\pi) &= \mathbf{E}\left[(R^\pi - \mathbf{E}[R^\pi])^2\right] \\ &= \sum_{k=1}^{m} \sum_{j=1}^{m} \pi_k \pi_j \, \mathbf{E}\left[(R_k - r_k)(R_j - r_i)\right] \\ &= \mathbf{C}\pi \cdot \pi. \end{aligned}$$

1.2 Preferences

This theory is based on the observation that the investors prefer portfolios with higher average returns and lower risk. This preference characterization is not complete and can only provide a partial ordering among the potential portfolios. More precisely, let $\mathcal{L}^2$ be the set of all square integrable random variables. We think of them as possible return values.

Definition 1.1 (Mean-Variance Preference)**.** *The mean-variance preference relation over the pairs of random variables $X, Y \in \mathcal{L}^2$ is defined as follows:*

$$X \preceq Y \quad \Leftrightarrow \quad \mathbf{E}[X] \geq \mathbf{E}[Y] \ \text{ and } \ var(X) \leq var(Y).$$

We say that Y is (weakly) preferred over X.

It can be directly verified that the above is a preference relation as defined in economics. Following that theory, we say that Y is *strictly preferred* over X and write $X \prec Y$, when $X \preceq Y$ but $Y \not\preceq X$. For the mean-variance preference, $X \prec Y$ id and only if the following holds:

$$\mathbf{E}[X] \geq \mathbf{E}[Y], \ \ var(X) \leq var(Y), \ \text{ and } \ (\mathbf{E}[X], var(X)) \neq (\mathbf{E}[Y], var(Y)).$$

This preference is *not complete*. That is there are pairs of random variables X, Y such that $X \not\preceq Y$ and $Y \not\preceq X$. Then, it is not possible to give a numerical representation for this preference. Therefore, an investor can make a rational choice only among those which has the same mean , or same variance.

1.3 Markowitz Mean-Variance Problem

As discussed above the mean-variance preference allows the investors to choose an optimal portfolio among all which have the same return r. For a given desired return r, the optimal portfolio should minimize the the portfolio variance $var(R^\pi)$ among all $\pi \in \mathbb{R}^m$ satisfying the constraint $\mathbf{E}[R^\pi] = r$. Hence, mathematically the Markowitz mean-variance problem is the following optimization problem:

> Markowitz Mean-Variance Problem.
>
> Given the excess return vector $\mathbf{a}$, the covariance matrix $\mathbf{C}$ and a desired return level r,
>
> $$\begin{aligned} \text{minimize} \quad & \pi \in \mathbb{R}^m \ \mapsto \ \frac{1}{2}\mathbf{C}\pi \cdot \pi, \\ \text{subject to} \quad & \pi \cdot \mathbf{a} = r - r_0. \end{aligned}$$

The solution is easily obtained by using a Lagrange multiplier λ: we minimize the function

$$\pi \in \mathbb{R}^m \ \mapsto \ f(\pi) := \frac{1}{2}\mathbf{C}\pi \cdot \pi + \lambda[(r - r_0) - \pi \cdot \mathbf{a}],$$

without any constraints on π. Then,

$$0 = \nabla f(\pi_r) = \mathbf{C}\pi_r - \lambda_r \mathbf{a} \qquad \Rightarrow \qquad \pi_r = \lambda_r \mathbf{C}^{-1}\mathbf{a}.$$

We compute λ_r by using the return constraint:

$$r - r_0 = \pi_r \cdot \mathbf{a} = \lambda_r \mathbf{C}^{-1}\mathbf{a} \cdot \mathbf{a} \qquad \Rightarrow \qquad \lambda_r = \frac{r - r_0}{\mathbf{C}^{-1}\mathbf{a} \cdot \mathbf{a}}.$$

> Solution.
>
> $$\pi_r = \lambda_r \mathbf{C}^{-1}\mathbf{a} \qquad \text{where} \qquad \lambda_r = \frac{r - r_0}{\mathbf{C}^{-1}\mathbf{a} \cdot \mathbf{a}}.$$

1.4 Market Portfolio

The optimal portfolio invests $(1 - \pi_r \cdot \mathbf{1}_m)$ fraction of the wealth in the bond and the rest in the risky assets. We want to separate the optimal portfolio into two parts accordingly. One containing only the 'risky' assess and one with only bond. Thus, we look for a constant c_r so that

$$\pi^{(m)} = \frac{1}{c_r}\pi_r \ \text{ and } \ \pi^{(m)} \cdot \mathbf{1}_m = 1, \quad \Rightarrow \quad c_r = \pi_r \cdot \mathbf{1}_m = \lambda_r \mathbf{C}^{-1}\mathbf{a} \cdot \mathbf{1}_m \ \text{ and } \ \pi^{(m)} = \frac{\mathbf{C}^{-1}\mathbf{a}}{\mathbf{C}^{-1}\mathbf{a} \cdot \mathbf{1}_m}.$$

The above expression of $\pi^{(m)}$ *depends only on the statistics of the return vector* $\mathbf{R}$ and not the desired return level r. Hence, it is a characteristic of the financial market. In fact, this is interpreted as the set of all financial instruments that are traded in the world. With a bit restricted view of the world, one may consider S&P 500 as the market portfolio.

Market Portfolio. The portfolio

$$\pi^{(m)} = \lambda^{(m)}\, \mathbf{C}^{-1}\mathbf{a}, \qquad \text{where} \qquad \lambda^{(m)} = \frac{1}{\mathbf{C}^{-1}\mathbf{a}\cdot \mathbf{1}_m},$$

is the market portfolio.

Then, the mean return and the excess return of the market portfolio are given by,

$$r^{(m)} := \mathbf{E}[R^{(m)}], \qquad a^{(m)} := r^{(m)} - r_0 = \pi^{(m)} \cdot \mathbf{a}.$$

The following famous separation result follows immediately from these definitions.

Tobin's separation Theorem

Theorem 1.2. *For any desired return level r, it is optimal to invest*

$$c_r = \frac{r - r_0}{r^{(m)} - r_0}$$

fraction of the wealth in the market portfolio and the rest in the bond. In particular, the composition of the market portfolio is independent of r.

Proof. By the definition of the market portfolio, $\pi_r = c_r \pi^{(m)}$. Moreover, the return of π_r is equal to the desired level r. Hence,

$$r - r_0 = \pi_r \cdot \mathbf{a} = c_r \pi^{(m)} \cdot \mathbf{a} = c_r(r^{(m)} - r_0).$$

□

The following properties of the market portfolio are useful. Let

$$R^{(m)} := \pi^{(m)} \cdot \mathbf{R},$$

and its variance is given by,

$$r^{(m)} := \mathbf{E}[R^{(m)}], \qquad a^{(m)} := r^{(m)} - r_0 = \pi^{(m)} \cdot \mathbf{a}.$$

Then,

$$var(R^{(m)}) = \sum_{k=1}^{m}\sum_{j=1}^{m} \pi_k^{(m)}\pi_j^{(m)} cov(R_k, R_j) = \mathbf{C}\pi^{(m)} \cdot \pi^{(m)}.$$

As $\pi^{(m)} = \lambda^{(m)}\mathbf{C}^{-1}\mathbf{a}$, $\mathbf{C}\pi^{(m)} = \lambda^{(m)}\mathbf{a}$ and $\mathbf{C}\pi^{(m)} \cdot \pi^{(m)} = \lambda^{(m)}(\pi^{(m)} \cdot \mathbf{a})$. Since $\pi \cdot \mathbf{a} = r^{(m)} - r_0$,

$$var(R^{(m)}) = \lambda^{(m)}(r^{(m)} - r_0). \tag{1.1}$$

1.5 Security Market Line

We compute the variance of the optimal portfolio π_r:

$$var(R^{\pi_r}) = \mathbf{C}\pi_r \cdot \pi_r = c_r^2 \mathbf{C}\pi^{(m)} \cdot \pi^{(m)}.$$

Let σ_r be the standard deviation of the optimal portfolio and σ_m be the standard deviation of the market portfolio. Then,

$$\sigma_r = c_r \sigma_m = \frac{r - r_0}{r_m - r_0}\, \sigma_m.$$

We express the return in term of the standard deviation:

$$r = r_0 + \frac{r_m - r_0}{\sigma_m}\, \sigma_r.$$

The above line in the return-standard deviation plane is called the security market line and abbreviated by SML. According to the mean-variance theory, all investor portfolios must have returns and standard deviations on the SML.

Security Market Line

The security market line (SML) is the collection of all return-standard deviation pairs (r, σ) satisfying,

$$r = r_0 + \frac{r_m - r_0}{\sigma_m}\, \sigma.$$

2 Beta

The covariance between any asset and the market portfolio contains substantial information. For any $k = 1, \ldots, m$,

$$\begin{aligned}
cov(R_k, R^{(m)}) &= \mathbf{E}\left[(R_k - r_k)(R^{(m)} - r^{(m)})\right] \\
&= \sum_{j=1}^{m} \mathbf{E}\left[(R_k - r_k)(R_j - r_j)\right]\ \pi_j^{(m)} \\
&= \left(\mathbf{C}\pi^{(m)}\right)_k = \lambda^{(m)} \left(\mathbf{C}\mathbf{C}^{-1}\mathbf{a}\right)_k \\
&= \lambda^{(m)} \mathbf{a}_k = \lambda^{(m)}(r_k - r_0).
\end{aligned}$$

This together with (4.1) implies that

$$cov(R_k, R^{(m)})(r^{(m)} - r_0) = \lambda^{(m)}(r^{(m)} - r_0)(r_k - r_0) = var(R^{(m)})(r_k - r_0).$$

Hence,

$$a_k := (r_k - r_0) = \frac{cov(R_k, R^{(m)})}{var(R^{(m)})}\, (r^{(m)} - r_0).$$

The above provides a formula for the ratio between the excess return of the k-th asset and the excess return of the market portfolio. Additionally, this multiplicative factor can be estimated from the historical return data. This is an important connection between the market portfolio and any asset in the market portfolio predicted by this theory called *Capital Asset Pricing Theory* (CAPM). The difference between the returns and the computed a is called the *alpha* and used by the portfolio managers.

We summarize the CPAM findings in the following:

Excess return and Beta of an Asset.

For any asset in the market portfolio

$$\text{excess return of the asset} = \text{beta of the asset} \times \text{market excess return},$$

where if R is the random variable representing the return of this asset, then,

$$\begin{aligned} \text{excess return of the asset} = \text{excess return of the asset} &= \mathbf{E}[R] - r_0, \\ \text{market excess return} &= r^{(m)} - r_0, \\ \text{beta of the asset} =: \beta &= \frac{cov(R, R^{(m)})}{var(R^{(m)})}, \end{aligned}$$

and r_0 is the risk-free rate.

2.1 Market Excess Return

The previous section shoes how to compute the excess return of a stock in term of the market excess return. That has to estimated from data. There is extensive research on this and one can use published data for this return value. For instance following is a study from 2003 using 101 years of data.

3 Efficient Frontier

We again fix the desired rate of return r but invest only in the stocks. That introduces an additional constraint that $\hat{\pi}\mathbf{1}_m = 1$ and the optimization problem is to

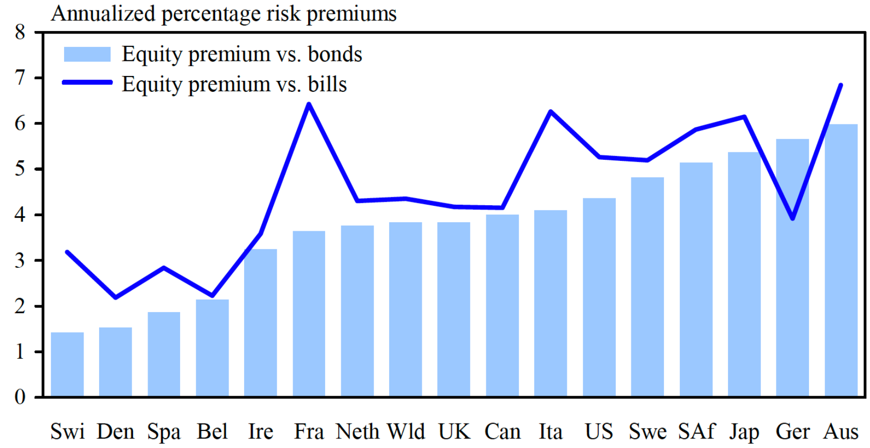
$$\text{minimize} \quad \hat{\pi} \in \mathbb{R}^m \ \mapsto \ \frac{1}{2}\mathbf{C}\hat{\pi} \cdot \hat{\pi}, \tag{3.1}$$

subject to

$$\hat{\pi} \cdot \mathbf{a} = r - r_0 \qquad \text{and} \qquad \hat{\pi} \cdot \mathbf{1}_m = 1.$$

Let $\hat{\pi}_r$ be the optimal solution and $\hat{\sigma}_r$ be its standard deviation. The efficient frontier is the collection of all pairs $(r, \hat{\sigma}_r)$:

FIGURE 3
WORLDWIDE ANNUALIZED EQUITY RISK PREMIUMS 1900–2002



Germany excludes 1922–23.

Source: Dimson, Marsh, and Staunton, *Triumph of the Optimists: 101 Years of Global Investment Returns*, Princeton University Press, 2002 and *Global Investment Returns Yearbook*, ABN AMRO/ London Business School, 2003.

Efficient Frontier

For a given $r > 0$, let $\hat{\sigma}_r$ be the standard deviation of the optimal solution to (6.1). Then, the efficient frontier is the collection of all return-standard deviation pairs,

$$EF := \{(r, \hat{\sigma}_r) \mid r > 0\}.$$

One observes that the market portfolio is on the efficient frontier and also is on the security market line (SML). Recall that SML is given by,

$$r = r_0 + \frac{r_m - r_0}{\sigma_m}\,\sigma, \qquad \forall (r, \sigma) \in SML,$$

where r_m is the market return and σ_m is the standard deviation of the market portfolio.

Theorem 3.1. *The security market line is tangent to the efficient frontier at the market portfolio. That is,*

$$r \le r_0 + \frac{r_m - r_0}{\sigma_m}\,\hat{\sigma}_r, \qquad \forall (r, \hat{\sigma}_r) \in EF.$$

Moreover, the above inequality holds with an equality at (r_m, σ_m), the market portfolio.

Proof. By its definition $\hat{\sigma}_r^2/2$ is the minimum value of the 'efficient frontier' problem (6.1) (EFP). On the other hand, $\sigma_r^2/2$ is the minimum value of the Markowitz mean-variance problem (MVP). MVP minimizes the objective function $C\pi \cdot \pi/2$ over the constraint $\pi \cdot \mathbf{a} = r - r_0$. EFP also minimizes the same objective function (namely, $C\pi \cdot \pi/2$) over the same constraint $\pi \cdot \mathbf{a} = r - r_0$

and an *additional* constraint $\pi \cdot \mathbf{1}_m = 1$. Since the constraint set of EFP is larger, the minimum value obtain by EFP is greater than the minimum value obtained in MVP. Mathematically, this is

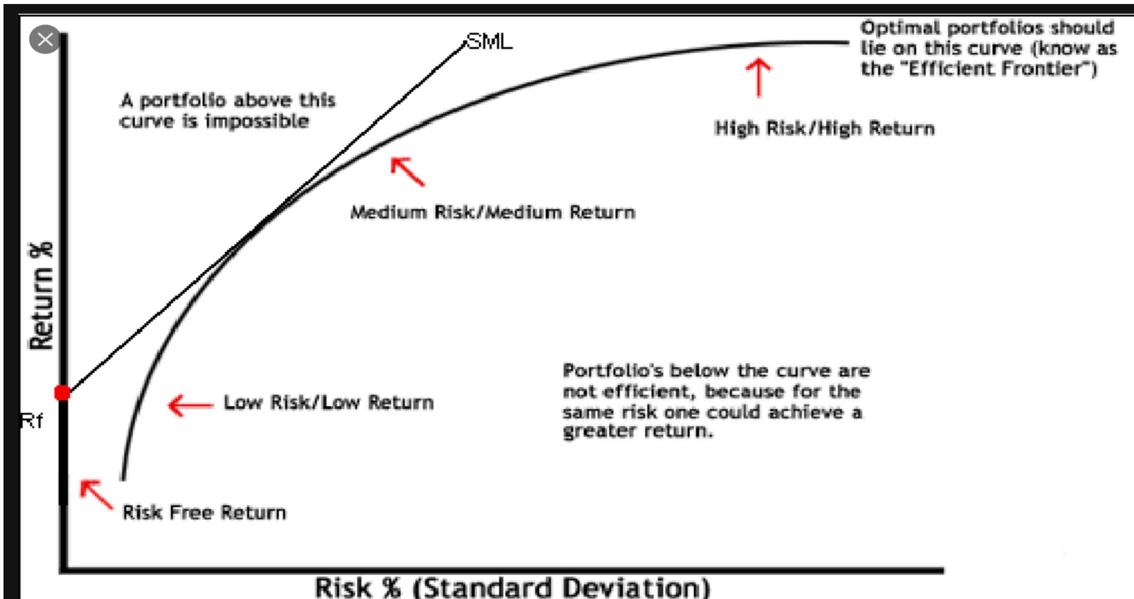
$$\frac{1}{2}\hat{\sigma}_r^2 \geq \frac{1}{2}\sigma_r^2 \quad \Rightarrow \quad \hat{\sigma}_r \geq \sigma_r.$$

As (r, σ_r) is on the SML:

$$\sigma_r = \frac{r - r_0}{r^{(m)} - r_0}\,\sigma^{(m)} \leq \hat{\sigma}_r, \quad \Rightarrow \quad r \leq r_0 + \frac{r_m - r_0}{\sigma_m}\,\hat{\sigma}_r.$$

□

The following is the visualization of the above result:



Source: Busetti (2009:104)

4 Capital Asset Pricing Theory (CPAM)

Markowitz mean-variance problem is the following optimization problem:

Markowitz Mean-Variance Problem.

Given the excess return vector $\mathbf{a}$, the covariance matrix $\mathbf{C}$ and a desired return level r,

$$\begin{aligned} \text{minimize} \quad & \pi \in \mathbb{R}^m \;\mapsto\; \frac{1}{2}\mathbf{C}\pi \cdot \pi, \\ \text{subject to} \quad & \pi \cdot \mathbf{a} = r - r_0. \end{aligned}$$

Solution.

$$\pi_r = \lambda_r \mathbf{C}^{-1}\mathbf{a} \qquad \text{where} \qquad \lambda_r = \frac{r - r_0}{\mathbf{C}^{-1}\mathbf{a}\cdot\mathbf{a}}.$$

4.1 Market Portfolio

Market Portfolio. The portfolio

$$\pi^{(m)} = \lambda^{(m)}\, \mathbf{C}^{-1}\mathbf{a}, \qquad \text{where} \qquad \lambda^{(m)} = \frac{1}{\mathbf{C}^{-1}\mathbf{a}\cdot\mathbf{1}_m},$$

is the market portfolio.

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$$r^{(m)} := \mathbf{E}[R^{(m)}], \qquad a^{(m)} := r^{(m)} - r_0 = \pi^{(m)} \cdot \mathbf{a}.$$

The following famous separation result follows immediately from these definitions.

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Theorem 4.1. *For any desired return level r, it is optimal to invest*

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fraction of the wealth in the market portfolio and the rest in the bond. In particular, the composition of the market portfolio is independent of r.

Proof. By the definition of the market portfolio, $\pi_r = c_r \pi^{(m)}$. Moreover, the return of π_r is equal to the desired level r. Hence,

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$$var(R^{(m)}) = \lambda^{(m)}(r^{(m)} - r_0). \tag{4.1}$$

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We compute the variance of the optimal portfolio π_r:

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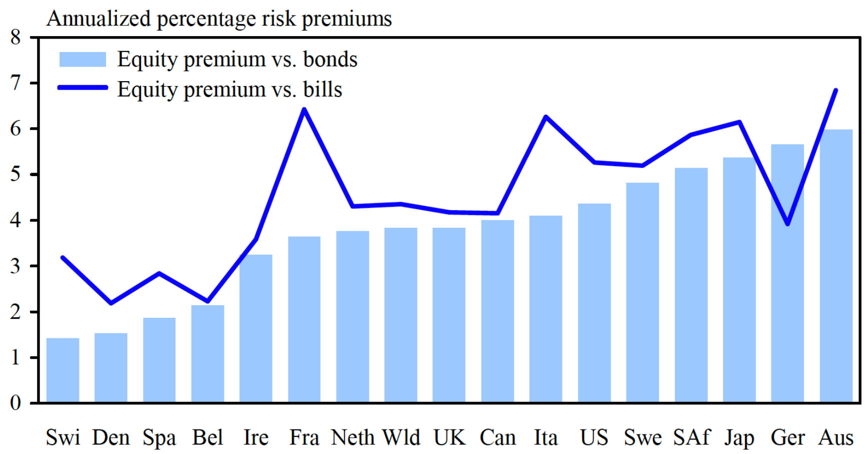

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subject to

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where r_m is the market return and σ_m is the standard deviation of the market portfolio.

Theorem 6.1. *The security market line is tangent to the efficient frontier at the market portfolio. That is,*

$$r \le r_0 + \frac{r_m - r_0}{\sigma_m}\,\hat{\sigma}_r, \qquad \forall (r, \hat{\sigma}_r) \in EF.$$

Moreover, the above inequality holds with an equality at (r_m, σ_m), the market portfolio.

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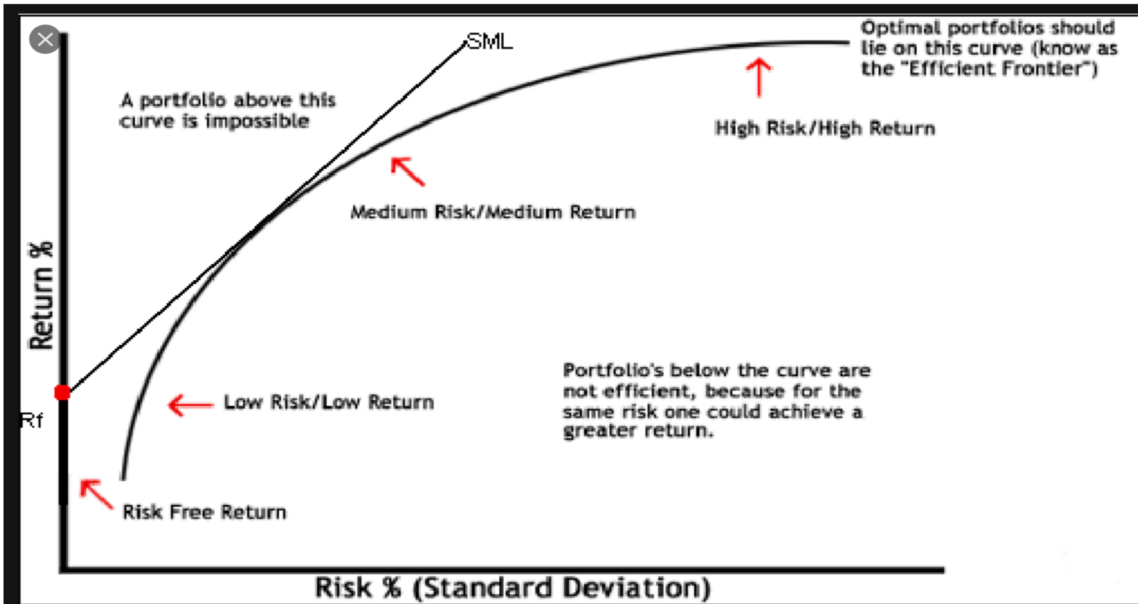

Source: Busetti (2009:104)