Introduction to Financial Mathematics

Lecture 5: American Options.

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1 American Options

American options - like the European ones - are also compactly summarized by a deterministic function φ of the stock value. Indeed, the future pay-off an American option is $\varphi(S_\tau)$ where $\tau \in \{0, 1, \ldots, N\}$ is a *random stopping time chosen by the holder* of the option. The main and the crucial difference is the option given the holder is to collect the reward at any time prior to maturity. We price this in the Binomial model.

The goal is to *price* and *hedge* these options using dynamic programming.

2 Coin-flip games

To motivate the approach that we use to price American options, we first consider a simpler model with coin-flips. Suppose that we flip a fair coin N times. We consider two different 'games' one analogous to a European option and the other to an American. The reward is the number of heads. To describe the games mathematically, we introduce the following notations:

- Let $k = 0, 1, \ldots, N$ be the number of flip-coins,
- let $i = 0, \ldots, k$ be the number of heads in k coin-flips,
- let H_k be the random variable representing the number of heads in the first k coin-flips,
- we discount with $r = 2$,
- and we let $v_{k,i}$ be the expected discounted reward starting from this this node.

In this context, H_k the number of heads is the analogue of the stock price.

First game (European). In this one, we simply flips the coins all the way to the and collect the discounted number of heads as a reward at the end. This is similar to the European options which are settled at maturity.

Clearly

$$v_0 = 3^{-N}\,\mathbb{E}[H_N] = 3^{-N}\,\frac{N}{2}.$$

For comparison, we note that with $N = 2$, $v_0 = \frac{1}{6}$. More generally, for $k = 0, 1, \ldots, N$, and $i = 0, \ldots, k$,

$$v_{k,i} = 3^{k-N}\, \mathbb{E}[H_N \mid H_k = i] = 3^{k-N}\, [i + \frac{N-k}{2}] = 3^{k-N}\, \frac{2i + N - k}{2}.$$

The following recursive equation is the analogue the pricing equation,

$$v_{k,i} = \frac{1}{3}\,(\frac{1}{2}\, v_{k+1,i+1} + \frac{1}{2}\, v_{k+1,i}), \quad k = 0, \ldots, N-1,\ i = 0, \ldots, k,$$

together with terminal condition

$$v_{N,i} = i, \qquad i = 0, \ldots, N.$$

Second game (American). In this game, the player can stop at any time and collect the number of heads. This is similar to the European options which are settled at maturity. We. first solve this for $N = 2$. Then,

$$v_{2,0} = 0, \quad v_{2,1} = 1, \quad v_{2,2} = 2.$$

At $(k, i) = (1, 0)$, if we stop we get 0. So we continue and the expected reward is equal to $v_{1,1}$:

$$v_{1,0} = \frac{1}{3}\,(\frac{1}{2}\, v_{2,1} + \frac{1}{2}\, v_{2,0}) = \frac{1}{6}.$$

At $(1, 1)$ if we stop we get 1 and if we continue the expected rewards is

$$c_{1,1} = \frac{1}{3}\,(\frac{1}{2}\, v_{2,2} + \frac{1}{2}\, v_{2,1}) = \frac{1}{2} < 1.$$

So it is optimal stop here. Hence, $v_{1,1} = 1$.

Initially, we do not stop and therefore,

$$v_0 = \frac{1}{3}(\frac{1}{2}\, v_{1,1} + \frac{1}{2}\, v_{1,0}) = \frac{7}{36} > \frac{1}{6}.$$

As expected, the option of stopping strategically increases the expected reward.

In general,

$$v_{k,i} = \max\{\ i\ ,\ \frac{1}{3}\,(\frac{1}{2}\, v_{k+1,i+1} + \frac{1}{2}\, v_{k+1,i})\ \} \quad k = 0, \ldots, N-1,\ i = 0, \ldots, k,$$

together with terminal condition

$$v_{N,i} = i, \qquad i = 0, \ldots, N.$$

3 An example

Start with a 2-step model with

$$S_0 = 4,\ d = \frac{1}{2},\ 1 + r = \frac{3}{2},\ u = 2, \quad \Rightarrow \quad p^* = \frac{2}{3}.$$

We consider a Put option with $K = 7$. Then, the potential pay-off at any node is given by

$$\varphi_{k,i} = (7 - s_{k,i})^+, \quad k = 0, 1, 2,\ \ i = 0, \ldots, k.$$

The value of the option $v_{k,i}$ is always greater than or equal to $\varphi_{k,i}$ and would be equal only if it is optimal to stop at that node. Moreover at maturity 2, the option has to be exercised and therefore,

$$v_{2,i} = \varphi_{2,i}, \quad i = 0, 1, 2.$$

Then, the stock values (s) and potential pay-offs (φ) are given as,

$$\begin{array}{ccc}
 & & s_{2,2} = 16,\ \varphi_{2,2} = v_{2,2} = 0, \\
 & s_{1,1} = 8,\ \varphi_{1,1} = 0, & \\
s_0 = 4,\ \varphi_0 = 3, & & s_{2,1} = 4,\ \varphi_{2,1} = v_{2,1} = 3, \\
 & s_{1,0} = 2,\ \varphi_{1,0} = 5, & \\
 & & S_{2,0} = 1,\ \varphi_{2,0} = v_{2,0} = 6.
\end{array}$$

The investors have the *option to stop at any node* (k, i) *and collect* $\varphi_{k,i}$. But they also have the *option to continue*. As in the European case, we consider 3 subtrees:

Subtree 1 = node (1,1):

$$\boxed{\begin{array}{cc}
 & s_{2,2} = 16,\ v_{2,2} = 0, \\
s_{1,1} = 8,\ \varphi_{1,1} = 0,\ v_{1,1} =? & \\
 & s_{2,1} = 4,\ v_{2,1} = 3.
\end{array}}$$

Since $\varphi_{1,1} = 0$, the investors do not have any incentive to stop. So it is optimal to continue and the value is as in the European case:

$$v_{1,1} = \frac{1}{1+r}\ [p^* v_{2,2} + (1 - p^*) v_{2,1}] = \frac{2}{3}, \quad \theta_{1,1} = \frac{v_{2,2} - v_{2,1}}{s_{2,2} - s_{2,1}} = -\ \frac{1}{4}.$$

Subtree 2 = node (1,0):

$$\boxed{\begin{array}{cc}
 & s_{2,1} = 4,\ v_{2,2} = 3, \\
s_{1,0} = 2,\ \varphi_{1,0} = 5,\ v_{1,0} =? & \\
 & s_{2,0} = 1,\ v_{2,0} = 6.
\end{array}}$$

At this node $\varphi_{1,0} = 5$, and it is not clear that the investors should stop or continue. If they continue the value is again as in the European case:

$$\frac{1}{1+r}\ [p^* v_{2,1} + (1-p^*) v_{2,0}] = \frac{8}{3} < 5 = \varphi_{1,0}.$$

As continuation would yield in a lower return, the investors should stop and collect $\varphi_{1,0} = 5$. Then,

$$v_{1,0} = 5 = \max\left\{\varphi_{1,0}\ ;\ \frac{1}{1+r}\,[p^* v_{2,1} + (1-p^*) v_{2,0}]\right\}.$$

Subtree 3 = node (0,0):

$$\boxed{\begin{array}{ll} & s_{1,1} = 8,\ v_{1,1} = \dfrac{2}{3}, \\ s_0 = 4,\ \varphi_0 = 3,\ v_0 = ? & \\ & s_{1,0} = 1,\ v_{1,0} = 5. \end{array}}$$

As in subtree 2, we first calculate the continuation value

$$\frac{1}{1+r}\ [p^* v_{1,1} + (1-p^*) v_{1,0}] = \frac{38}{27} < 3 = \varphi_0.$$

So it is optimal to stop at time zero and collect the pay-off:

$$v_0 = 3 = \max\left\{\varphi_0\ ;\ \frac{1}{1+r}\,[p^* v_{1,1} + (1-p^*) v_{1,0}]\right\}.$$

We now have the following solution:

$$\boxed{\begin{array}{lll} & & s_{2,2} = 16,\ v_{2,2} = 0, \\ & s_{1,1} = 8,\ v_{1,1} = \dfrac{2}{3},\ \theta_{1,1} = -\dfrac{1}{4}, & \\ s_0 = 4,\ v_0 = 3,\ \text{exercise} & & s_{2,1} = 4,\ v_{2,1} = 3, \\ & s_{1,0} = 2,\ v_{1,0} = 5,\ \text{exercise} & \\ & & S_{2,0} = 1,\ v_{2,0} = 6. \end{array}}$$

To complete the hedging table, we calculate $\theta_0, \theta_{1,0}, \theta_{1,1}$ as well:

$$\theta_0 = \frac{v_{1,1} - v_{1,0}}{s_{1,1} - s_{1,0}} = -\frac{13}{18}, \qquad \theta_{1,0} = \frac{v_{2,1} - v_{2,0}}{s_{2,1} - s_{2,0}} = -1, \qquad \theta_{1,1} = \frac{v_{2,1} - v_{2,0}}{s_{2,1} - s_{2,0}} = -\frac{1}{4}.$$

As we show below, the above strategies *super-replicate* the American option, if the investors do not exercise optimally.

Hedging.

At the nodes where it is optimal to exercise, it is still needed to build a hedging portfolio. Consider the initial node. At this point a put option is sold for 3. If it is immediately exercised, then the seller gives back this amount and the hedging is done. But if the investors decide to continue (despite the fact that it is not optimal), then the seller of the option shorts $\theta_0 = -\frac{13}{18}$ shares of the stock. Together with the initial proceed of 3 from the sale, the seller now has

$$3 + 4 \times \frac{13}{18} = \frac{53}{9}.$$

So the portfolio position is $\frac{13}{18}$ shares short in the stock and long $\frac{53}{9}$ in the bond. At the next step, the bond amount increases with the interest and becomes

$$\frac{3}{2} \times \frac{53}{9} = \frac{53}{6}.$$

Then the value of this portfolio in each case is given by,

If up: $-\frac{13}{18} \times 8 + \frac{53}{6} > v_{1,1} = \frac{2}{3}$.

If down: $-\frac{13}{18} \times 2 + \frac{53}{6} = \frac{133}{18} > v_{1,0} = 5$.

So in both cases, the sellers has more than needed to cover their position. Additionally, if the investors decide to continue at node $(1,0)$, the investors bring their short position to $\theta_{1,0} = -1$ by shorting an additional $-\theta_{1,0} - (-\theta_0) = 1 - \frac{13}{18} = \frac{5}{18}$ shares. This would give an additional cash amount of $\frac{5}{18} \times 2 = \frac{5}{9}$. So the sellers portfolio now is short one stock and $\frac{53}{6} + \frac{5}{9} = \frac{169}{18}$ in bond. Then, at the next time step, the bond position will become

$$\frac{3}{2} \times \frac{169}{18} = \frac{169}{12}.$$

We again analyze the two movements:

If up: $-1 \times 4 + \frac{169}{12} = \frac{121}{12} > v_{2,1} = 3$.

If down: $-1 \times 1 + \frac{169}{12} = \frac{157}{12} > v_{2,0} = 6$.

Again in both cases, the sellers have more than the liability they have. So the above hedging strategy super-replicates the Put option with initial price of 3.

4 Equation

We have argued in the above example that the pricing equation for the American option with a general pay-off of $\varphi(S_\tau)$ in the Binomial model is

$$\boxed{v_{k,i} = \max\left\{\varphi(s_{k,i})\ ,\ \frac{1}{1+r}\left[p^* v_{k+1,i+1} + (1-p^*) v_{k+1,i}\right]\right\},\ \ k = 0, 1, \dots N-1,\ i = 0, 1 \dots, k}$$

with the final condition

$$\boxed{v_{N,i} = \varphi(s_{N,i}), \quad i = 0, 1 \dots N.}$$

One may rewrite the above equation as

$$V_k = \max\left\{\varphi(S_k)\ ,\ \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left[V_{k+1} \mid S_k\right]\right\}$$

Important to note that, as opposed to European pricing formula, we cannot express $v_{k,i}$ simply in terms of $v_{N,k}$'s.

5 Exercises

1. Consider the following two-step Binomial model,

$$\begin{array}{ccc} & & s_{2,2} = \$484, \\ & s_{1,1} = \$440, & \\ s_0 = \$400, & & s_{2,1} = \$418, \\ & s_{1,0} = \$380, & \\ & & s_{2,0} = \$361. \end{array}$$

   With $r = 0.05$ compute the initial price of an American Put options with $K = 400$.

2. Consider a Binomial model with

$$n = 2, \quad u = 2, \quad d = \frac{1}{2}, \quad r = 0, \quad S_0 = 8.$$

   Compute the initial price of an American Call option with $K = 5$.

3. Consider a Binomial model with

$$n = 2, \quad u = 2, \quad d = \frac{1}{2}, \quad r = \frac{1}{4}, \quad S_0 = 8.$$

   Compute the initial price of an American Put option with $K = 10$.

4. Compute the initial price an American Put option with $K = 110$ on the following two-step Binomial model with $u = 1.2, d = 0.8, r = 0.05$ and $s_0 = 100$:

$$\begin{array}{ccc} & & s_{2,2} = \$144, \\ & s_{1,1} = \$120, & \\ s_0 = \$100, & & s_{2,1} = \$96, \\ & s_{1,0} = \$80, & \\ & & s_{2,0} = \$64. \end{array}$$