Introduction to Financial Mathematics

Lecture 2: Internal Rate of Return, Forwards, Futures, Arbitrage

October 19, 2024

Today we will learn the following topics:

- Different types of interest rates.
- Present value of a cash flow.
- Internal rate of return.
- Forward Contract.
- Forward Price.
- Notion of Arbitrage.
- Future Contracts.

1 Interest Rates

Interest is the amount charged when one takes a loan or the amount earned when one makes a cash investment in a bank or in a bond. For instance, suppose that we take a loan of $\$200,000$ and pay it back 25 days later. In addition to the loan amount, we also pay an interest of $\$685$. Clearly, the amount of interest depends on *length of the loan* called the maturity and the *amount of the loan* called the face value. In almost all cases, interest amount depends *linearly* on the loan amount. For example, if have taken a loan of $\$100,000$ we would have paid an interest of $\$342.5$, and for a loan $\$800,000$ interest would be $\$2,740$. In other words, the ratio of the interest to the loan amount is constant and in this example it is 0.34246% percent. This ratio depends on the duration of the loan. In fact suppose instead we pay the same loan back in 35 days, then the bank would charge a larger interest. Similarly when a cash investment is made, one receives the interest as a payment.

Suppose we take a loan of *one dollars* and pay it back in n days. Then, the face value $F = 1$, and the maturity is $T = \frac{n}{365}$ many years. There are several conventions for reporting the interest rate. In fact, the rate is always annualized but the compounding of the interest may differ.

1. Quote an *annual* interest rate $r \in (0, 1)$, and charge an interest proportional to the duration of the loan. In this case, the interest amount is

$$r \times \frac{n}{365} = rT = r_d\, n, \qquad \text{where } r_d := \frac{r}{365}.$$

2. Again quote an *annual* interest rate r, but every day charge r_d interest to the original loan amount *and to the accrued interest* up to that day. The *accrued interest* is the amount of interest coming from previous days. To make the computations, let us define

$$F_k := \text{loan and the accrued interest at the end of day } k.$$

Note that $F_0 = 1$ is the original loan amount.

- At the end of day one, the accrued interest is r_d and therefore, $F_1 = (1 + r_d)$.
- At the end of day two,

$$\begin{aligned} F_2 &= \text{loan} + \text{interest from day one} + \text{interest for the second day} \\ &= F_1 + r_d\, F_1 = (1 + r_d)\, F_1 = (1 + r_d)^2. \end{aligned}$$

- Similarly, for any $k \geq 0$,

$$\begin{aligned} F_{k+1} &= \text{loan and the accrued interest at the end of day } k + \text{interest for day } (k+1) \\ &= F_k + r_d\, F_k = (1 + r_d)\, F_k = F_{k-1}\, (1 + r_d)^2 = \ldots = (1 + r_d)^{k+1}. \end{aligned}$$

So in this case, at the end of day n, interest payment is equal to

$$(1 + r_d)^n - 1 = (1 + \frac{r}{365})^{365\,T} - 1.$$

The above amount is greater than the amount $r_d\, n$ paid in the first case. This happens because we pay interest on the interest.

3. Quote an *annual* interest rate, and charge interest continuously. This is done to make calculations simpler. To understand the mechanism, imagine that bank collects interest more frequently than every day; say, every hour or even every second and so on.

   Mathematically, we divide a year into M-equal pieces. For instance, in the case of daily interests $M = 365$. Now we assume that M is very, very large. Then, interest rate for each unit time interval is $r_M := \frac{r}{M}$. The loan is paid after T years which corresponds to $M\,T$ time intervals. Then, arguing as in the second case we obtain that the interest that is paid after T-years is

$$(1 + \frac{r}{M})^{M\,T} - 1.$$

   Continuous interest compounding means that the number M is taken to infinity. Therefore,

$$\text{interest paid after } T \text{ years} = \lim_{M \to \infty} (1 + \frac{r}{M})^{M\,T} - 1 = e^{rT} - 1.$$

We summarize this in the following.

Consider a loan or an investment of F dollar. Suppose that the annual interest rate is r and the maturity is T years. Then, the total interest after T years is

- $rT\,F$ if the *interest is compounded once*;
- $[(1+\frac{rT}{n})^n - 1]\,F$ if the *interest is compounded n times*;
- $[e^{rT} - 1]\,F$ if the *interest is compounded continuously*.

We continue with an example.

Example 1.1. Suppose we take a loan of $\$200,000$ and the loan provider charges 5% *annual interest*. Then, $T = 25/365$ and the interest would be

- $\$684.93$ if the interest is compounded once;
- $\$686.06$ if the interest is compounded daily (i.e., 25 times);
- $\$686.10$ if the interest is compounded continuously.

Notice that the differences are very small. Now suppose that $T = 2$ years. Then, the interest after two years would be

- $\$20,000$ if the interest is compounded once;
- $\$21,032.67$ if the interest is compounded daily (i.e., 730 times);
- $\$21,034.19$ if the interest is compounded continuously.

The difference between daily and continuous is still very small. But compounding it only once is substantially smaller. □

1.1 Present Value of a Cash Flow

Suppose that we will receive $¥15,000$ in two years. What would be its value today? Observe that getting this payment is equivalent to a zero coupon Chinese bond with maturity two years. Therefore, its value is exactly equal to the price of this zero coupon bond with face value $¥15,000$. By the above formulae, its price is equal to

$$e^{-2r(2)}15,000 = e^{-2\times 0.02208}15,000 = 14,352.$$

Similarly, the current value of a future cash payment of c dollars at time t, is equal to $ce^{-tr(t)}$ today. This is called *discounting*, and by discounting we can find the present value of a cash flow.

Definition 1.2. Suppose that we are given a yield curve $r(\cdot)$.

a. The price $e^{-tr(t)}$ of the zero-coupon bond with face value zero is called the *discount factor.*

b The *present value the cash flow* $\mathbf{c}\{c(t_i)\}_{i=1,\dots,n}$ is given by,

$$PV(\mathbf{c}) := \sum_{i=1}^{n} c(t_i)\, e^{-t_i r(t_i)}.$$

Example 1.3. Consider the cash flows and the bond introduced in the last lecture Example 1. Suppose that the yield curve is constant 5%, i.e., $r(t) = 0.05$ for every t. Then, we can calculate the present values of each cash flows and the bond:

$$\begin{aligned}
PV(\text{bond}) &= 1,900[e^{-0.025} + e^{-0.05} + e^{-0.075}] + 1,001,900\, e^{-0.1} &&= \$911,979.75\\
PV(\text{cf } 1) &= 1,900[e^{-0.05} + e^{-0.1} + e^{-0.15}] + 1,001,900\, e^{-0.2} &&= \$825,448.21\\
PV(\text{cf } 2) &= 2,200[e^{-0.05} + e^{-0.1} + e^{-0.15}] + 1,002,200\, e^{-0.2} &&= \$826,508.87.
\end{aligned}$$

□

1.2 Internal Rate of Return

Suppose we now *know the prices of the cash flows* and not the yield curve. The following yield definition provides a comparison mechanism.

Definition 1.4.
Let $\mathbf{c} = \{c(t_i)\}_{i=1,\dots,n}$ be a cash flow with a price of $p(\mathbf{c})$. The *internal rate of return* (IRR) of this cash flow $\mathbf{c}$ is the yield value r satisfying the following equation,

$$\boxed{p(\mathbf{c}) = \sum_{i=1}^{n} c(t_i)\, e^{-r\, t_i}.} \tag{1.1}$$

In the case of a bond, the internal rate of return is called *yield-to-maturity*. Form the above definition, it is not clear if the internal rate of return is a well-defined quantity. The following simple result resolves this issue.

Theorem 1.5. *Suppose that $c(t_i) > 0$ for every i and $p(\mathbf{c}) > 0$. Then, the internal rate of return r of $\mathbf{c}$ is uniquely defined. Moreover, it is inversely related to the price and it is strictly positive whenever $p(\mathbf{c}) < \sum_{i=1}^{n} c(t_i)$.*

Proof. Define a function,

$$r \in \mathbb{R} \ \mapsto \ pv(r) := \sum_{i=1}^{n} c(t_i)\, e^{-r\, t_i}.$$

Then,

$$pv'(r) = -\sum_{i=1}^{n} t_i\, c(t_i)\, e^{-r\, t_i} < 0 \qquad \Rightarrow \qquad pv \text{ is strictly decreasing.}$$

Moreover,

$$\lim_{r\to\infty} pv(r) = 0, \quad \text{and} \qquad \lim_{r\to 0} pv(r) = pv(0) = \sum_{i=1}^{n} c(t_i).$$

Hence, the function pv attains any positive value at a unique point. This proves that the internal rate of return is uniquely defined. The positivity of it follows from the monotonicity and the fact that $pv(0) = \sum_{i=1}^{n} c(t_i)$. $\square$

Example 1.6. Consider the bond discussed last lecture and in Example 1.3. The price of this bond is $1,003,100$ In this example, the equation (1.1) is given by,

$$p(\mathbf{c}) = 1,003,100 = pv(r) \; = \; 1,900[e^{-r/2} + e^{-r} + e^{-3r/2} + e^{-2r}] + 1,000,000e^{-2r}.$$

As the price is greater than the face value, we immediately observe that the internal rate of return r is less than the coupon rate. However, they should be close to each other as the difference between the price and the face value is not much.

The solution obtained by a root solver is given by $r = 0.00224$. Hence, this bonds yield-to-maturity is 0.224%. $\square$

Exercises

1. Consider a bond with face value ¥$100,000$, semi-annual coupon of 3.5% and maturity 5 years.

   a. If the price is ¥$100,000$, find the internal rate of return.

   b. If the price is ¥$101,586.07$, find the internal rate of return.

   c. If the price is ¥$98,939.72$, find the internal rate of return.

2. Consider a bond with face value F, semi-annual coupon of $c\%$ and maturity T years. Decide on the folloiwng statement

   a. If the price is exactly equal to F, then internal rate of return is approximately equal to c. Correct? Not correct?

   b. If the price is strictly more than F, then internal rate of return is approximately less than c. Correct? Not correct?

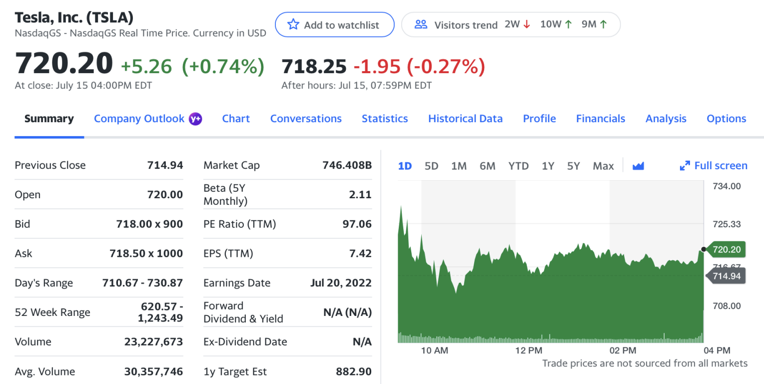
   c. If the price is strictly more than F, then internal rate of return is approximately greater than c. Correct? Not correct?

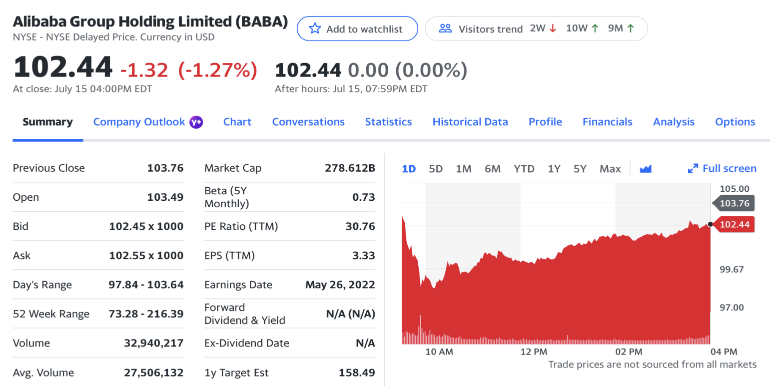
   d. I If the price is strictly less than F, then internal rate of return is approximately greater than c. Correct? Not correct?

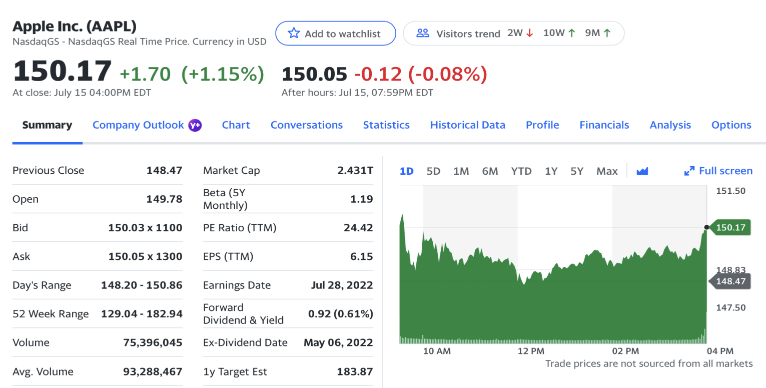
   e. I If the price is strictly less than F, then internal rate of return is approximately less than c. Correct? Not correct?

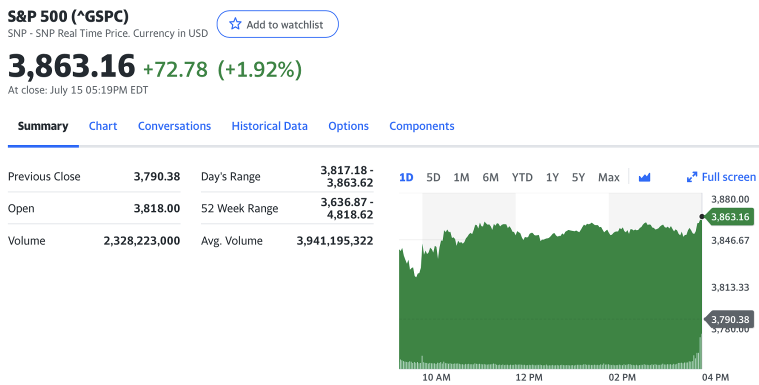
2 Financial Contracts

We consider three most common financial contracts. They are all based on a simple stock such as the price of Tesla, or Apple, or Alibaba, or an index like SP 500. On July 16,2022 the following were their prices. These are from the web site `https://finance.yahoo.com`









2.1 Forwards

Let S_t be the price of the underlying stock at time $t \geq 0$.

Definition 2.1. *The forward contract on a stock with maturity T and strike K is a legally binding sale contract of one stock at time T for a price of K.*

Each forward contract has a buyer and a seller. We will say that the buyer has the long position and the seller the short position. It is important to note that the holder of this contract has then legal obligation to buy the stock at time T for a price K regardless of the actual price of the stock at that time. Also, the seller of the forward contract has the obligation to sell the stock at time T for a price K regardless of the actual price of the stock at that time. Summarizing:

At time T the buyer pays K dollars and get one share of the stock which is worth S_T dollars. Therefore, effectively, this contract pays at maturity $S_T - K$ dollars to the buyer of the forward contract. Of course this amount could be negative, and in that case the buyer loses money.

In general the forward contracts requires an initial settlement fee, depending on K. For instance if K is much smaller than the usual price of the stock, then the seller of the forward contract would require an initial fee. In practice, K is adjusted in such a way that no initial fee is required and this special strike is called the forward price. We formalize this in the following definition.

Definition 2.2 (Forward Price)**.**
The forward price F of a stock with maturity T is the strike that is agreed upon at time zero satisfying the followings:

1. *Holder of the forward contract will buy the stock at time T for a price F regardless of the actual price of the stock at that time.*

2. *The seller of the forward contract will sell the stock at time T for a price F regardless of the actual price of the stock at that time.*

3. *At time zero, no up-front payment is made by either party.*

4. The contract is liquidly traded, i.e., there are investors willing to take the long and the short position of this contract.

□

The above definition fixes the forward price. On the other hand, forward contracts with any strike K may also exist. Next we give the definition of its price.

Definition 2.3 (Price of Forward Contract)**.**
Consider a forward contact with a general strike K. The price $p(K)$ of this contract is paid at time zero. If $p(K) > 0$, the buyer pays this amount to the seller, and if $p(K) < 0$, the seller pays this amount $-p(K)$ to the buyer. Then, the contract is settled at maturity. namely, at time T, the buyer pays K dollars to the seller who gives one share of the stock to the buyer. □

It is important emphasize that the price of a forward contract with a given strike is very different than the forward price.

Another way defining the forward price is this.

Lemma 2.4. *Let $p(K)$ be the price of the price of a forward contract with strike K. Then, the forward price F is the unique solution of $p(F) = 0$.*

Proof. The fact that F solves $p(F) = 0$ follows from the definitions. The only statement to prove is the fact that there exist a unique solution to this equation. As discussed above, the forward contract pays $S_T - K$ dollars. Since this amount decreases with the strike K, it price $p(K)$ must also strictly decrease as we increase K. For $K = 0$, since the payment $S_T - 0 = S_T > 0$, we must have $p(0) > 0$. Also, it is clear that

$$\lim_{K \to \infty} p(K) < 0.$$

Therefore, the graph of the function $p(\cdot)$ crosses the value zero at a unique point. □

3 Pricing

We start with a set of assets and their prices. We assume that any linear combination of these assets are also liquidly traded and the *price of these instruments are linear*. Additionally, we suppose that the cost of trading or transaction costs to zero, and that there is no friction. In particular,

- any financial instrument can be bought or sold for the same price;
- we can find a counter-party for any type of trade and the trades are finalized instantly.

3.1 Arbitrage

These are idealizations of the real financial markets. The main observation is that positive cash flows must be priced positively. Lack of this property is called *arbitrage*. Next we give a formal definition.

Definition 3.1. *An arbitrage* is a linear combination of assets which has non-negative future cash-flows that is not zero all the time and has a price less than or equal to zero.

All efficient markets eliminate arbitrages and result in a pricing system with no arbitrages. We use this as a pricing criteria that we state below.

Definition 3.2. *(Arbitrage Pricing Rule)*
All products must be priced in such a way that the resulting market is *free of arbitrage.* In particular, any price that leads to an arbitrage is not allowed.

Consider a market that has no arbitrages and introduce a new financial contract. We then use the arbitrage pricing rule to determine possible prices for this new instrument so that the market extended with it still has no arbitrages. In the next subsection, we use it to derive the forward price. We then revisit it later in the context of a one-step Binomial model, and then provide theoretical discussion.

Remark 3.3. In complex markets there could be more than one price system that has a no arbitrages. The trinomial model is the easiest example. Uniqueness of the prices is related to the *completeness* of the financial market which is introduced later. $\square$

3.2 Law of One Price

An immediate consequence of no-arbitrage is the *Law of one price* which states that if two contracts result same cash flows, then their prices must be equal. In fact, it applies more generally to assets trading in different currencies as well. We summarize this general statement as a theorem for future references.

Theorem 3.4. *In the absence of trade frictions, manipulations and tariffs, contracts possibly different locations must have the same price, if they produce identical cash-flows when expressed in a common currency.*

Proof. This law is derived from the assumption no-arbitrage and is proved by contradiction. Indeed, suppose to the contrary that same cash flows have different prices. We then sell the expensive one and buy the cheaper one, thus receiving a positive amount. The future cash-flows of our position is zero as two cash-flows cancel each other. This creates an arbitrage. As the markets assumed not to have arbitrages, these cash flows must have the same price. $\square$

4 Forward Price

The pay-off of a forward contract with maturity T and strike K is $S_T - K$.

Theorem 4.1. *Let $p(K)$ be the price of the price of a forward contract with maturity T and strike K. Then,*

$$p(K) = S_0 - KB(T),$$

where S_0 is current price of the stock and $B(T)$ is the price of a government bond with maturity T and face value one. Consequently, the forward price F at time T of a stock is given by

$$F = \frac{S_0}{B(T)}.$$

Proof. The linearity of the prices and the law of one price yield,

$$p(K) = \text{Price of } (S_T - K) = \text{Price of } (S_T) - \text{Price of } (K) = S_0 - K\,\text{Price of } (1) = S_0 - K\,B(T).$$

The forward price F is unique strike that makes the price of that forward contract zero. As we have just shown that the price of a forward contract with maturity T and strike F is $S_0 - FB(T)$,

$$0 = S_0 - FB(T) \quad \Rightarrow \quad F = \frac{S_0}{B(T)}.$$

$\square$

An alternate proof shows how one can use the notion of arbitrage as well. However, we need the concept of short-sales for that proof.

4.1 Short sales.

In idealized markets investors are able to sell financial instruments although they do not possess them. In the actual financial markets, there are limitation and also one is usually required to place a certain amount of cash or other liquid assets as a guarantee. Also in some countries, short-sales are not allowed. However, for large financial institutions these may not apply. We ignore these frictions and assume that one can short-sell stocks.

Conceptually, short-sale corresponds to borrowing money not in units of actual money but in some other quantity such as S&P 500. So if I short-sell 100 shares of S&P 500, this would give me $\$128,176$ today (one share is worth $\$1,281.76$ on January 9, 2020). However, in the future my debt will not be this amount plus interest rate but rather would be the worth of $1,000$ shares of S&P 500 at that time.

Simpler example would be borrowing $1,000$ Euros and converting them to US dollars. Our debt would remain as $1,000$ Euros but it has to be paid back in US dollars. One could also think of borrowing money as shorting an appropriate bond.

Alternate proof of Theorem 4.1

Suppose a forward contract with strike F is offered in the market with no initial fee. Investors have (at least) the following two options:

1. Sign the forward contract with no initial fee.
2. Borrow S_0 dollars by short-selling $S_0/B(T)$ many shares of the bond. With the proceeds buy one share of a stock now.

At time T the financial portfolio of these investors will be as follows:

1. They pay F dollars and get one share of the stock. So the value of their portfolio will be

$$S_T - F,$$

where S_T is future random value of the stock.

2. They have one stock and $S_0/B(T)$ many bond obligations. Since one share of the bond pays one dollar, their total worth is

$$S_T - \frac{S_0}{B(T)}.$$

Therefore the investors will initially buy the forward contract with no initial fee if and only if their proceeds in case 1 exceeds the proceeds from case 2. This is equivalent to

$$F \leq \frac{S_0}{B(T)}.$$

To prove the opposite inequality, consider the two options the investors may compare:

1. Sell one share of the forward contract with no initial fee.
2. Short one stock and collect the current price of the stock. With the proceeds buy $S_0/B(T)$ many shares of the bond now.

At time T the financial portfolio of this investor will be as follows:

1. For F dollars they give one share of the stock. So the value of their portfolio will be

$$-S_T + F.$$

2. They have $S_0/B(T)$ many bonds and short one share of the stock. So their total worth is

$$-S_T + \frac{S_0}{B(T)}.$$

Therefore the investors will sell the forward contract (i.e., consider the case 1 better) only if their proceeds in case 1 exceeds the proceeds from case 2. This is equivalent to

$$F \geq \frac{S_0}{B(T)}.$$

As we need investors both willing to buy and sell, we must have

$$F = \frac{S_0}{B(T)}.$$

$\square$

The above proof shows that for the forward price the investors could replicate this contract by buying and selling appropriate amounts of the stock and the bond. However, it is easier for them to buy the forward contract if they have business reasons for this transactions. The following example illustrates such a reason.

Example 4.2. A German manufacturer signs a contract worth ten million dollars. According to this agreement parts will be delivered four months from now and the payment will be made in US dollars. The company will have to convert the dollars to Euros at the exchange rate of the time. This rate is not known now and it is subject to random fluctuations. The manufacturing is done in Germany and the expenses are in Euros. So if the company does not want to take the exchange rate risk, it can fix the future random exchange rate buy signing a forward contract. This contract fixes the exchange rate to e and in four months the German company pays 10 million US dollars to the bank and receives $10e$ Euros.

Note that in four months, the actual exchange rate will certainly not be exactly equal to e and either the bank or the German company will lose money. But what is gained is the stability. In fact, the bank can always off-set its position by finding another party interested in the opposite transaction. The German company on the other hand, reduces its risk on its revenue. Thus it can take on more contracts.

5 Futures

The futures contracts are very similar to forwards and in standard stock exchanges only futures are traded.

The following is an example of a futures contract from the web page of Barchart:

`https://www.barchart.com/futures/quotes/GC*0/profile`

The spot price for gold on on January 10, 2022 was $1,800.93 per (troy) ounce and the February futures price was $1,801.4 per ounce. The contact is for 100 ounces and it will be executed in February 2022 before February 24, 2022. Until that day, daily payments are made depending on the future price movements keeping the contract neutral at all times. At maturity the futures price is equal to spot price and the holder of the long position would pay the spot price of gold on that day and will receive 100 troy ounces of gold. Gains or losses of the traders are equal to the sum of the daily payments made throughout the course of the contract.

As the value of the contract after the adjustment is zero, theoretically, at any time the futures contract can be given up at no cost.

Gold Feb '22 (GCG22)
1,801.4 **+4.0 (+0.22%)** 15:50 CT [COMEX]
1,801.3 x 50 1,801.4 x 9
CONTRACT SPECIFICATIONS for Mon, Jan 10th, 2022 Alerts Watch Help

Barchart Symbol	GC
Exchange Symbol	GC
Contract	Gold 100-oz
Exchange	COMEX
Tick Size	0.10 (10 cents) per troy ounce ($10.00 per contract)
Margin/Maintenance	$7,150/6,500
Daily Limit	10% above or below previous settlement
Contract Size	100 fine troy ounces
Months	Feb, Apr, Jun, Aug, Oct, Dec (G, J, M, Q, V, Z)
Trading Hours	5:00p.m. - 4:00p.m. (Sun-Fri) (RTH 7:20a.m. - 12:30p.m.) (Settles 12:30p.m.) CST
Value of One Futures Unit	$100
Value of One Options Unit	$100
Last Trading Day	Third last business day of the maturing delivery month

Daily future prices are announced so that one may enter into the futures contracts without an initial payment. Each day the difference between the current futures price and the previous must be paid paid. For instance, the futures price for February 2022 on January 11, 2022 was $\$1,817.4$. So the holder of the long position of this contract would have received 100 times the difference $1,817.4 - 1,801.4 = 16$ which is $\$1,600$.

To make the discussion is easier, we denote the futures price on day k of the contract by F_k. Let M be the number of shares that the contract requires to be bought. In the above example $M = 100$, $F_0 = \$1,801.4$ and $F_1 = \$1,817.4$.

Convergence. As one must pay the spot price at settlement, the futures price on the possible days of delivery are equal to the spot price. That is $F_T = S_T$.

The following payments are made by the holder of the long position:

Day 0: no payment is made and the futures price is F_0.
Day 1: futures price is F_1 and a payment of $M\,(F_0 - F_1)$ is made or received.
....
Day k: futures price is F_k and a payment of $M\,(F_{k-1} - F_k)$ is made or received.
...
Settlement: pay $M\,F_T = K\,S_T$ and receive M shares of the underlying.

There is a margin account that makes sure that both parties have enough money to cover the payments.

Exercises

1. Today (March 14, 2024) the April 2024 future price of gold is \$1,885. We one hold one share of this future contract. Suppose that tomorrow (March 15, 2024) the April 2024 future price of gold becomes \$1,900. Decide which one of the followings is correct:

   a. We have to pay the difference \$15 to the company who sold us the futures contract.

   b. We are paid the difference \$15 by the company who sold us the futures contract.

   c. We don't have to make or receive anything.

   d. If we want we pay the difference \$15 to the company who sold us the futures contract, but we do not have to pay.

2. Today (March 14, 2024) forward price of gold with maturity April 15, 2024 is \$1,885. Suppose we sign a contract for one share. Decide which one of the followings is correct:

   a. We pay a small amount as a deposit.

   b. We place a valuable object as a guarantee (collateral).

   c. We don't have to make or receive anything.

3. As in problem 2, today (March 14, 2024) forward price of gold with maturity April 15, 2024 is \$1,885. Suppose we sign a contract for one share, and tomorrow (March 15, 2024) the forward price of gold with maturity April 15, 2024 becomes \$1,900.

   Decide which one of the followings is correct:

   a. We have to pay the difference \$15 to the company who sold us the futures contract.

   b. We are paid the difference \$15 by the company who sold us the futures contract.

   c. We don't have to make or receive anything.

   d. If we want we pay the difference \$15 to the company who sold us the futures contract, but we do not have to pay.

4. As in problem 2, today (March 14, 2024) forward price of gold with maturity April 15, 2024 is \$1,885. We want to buy a forward contract with strike \$1,800.

   Decide which one of the followings is correct:

   a. We have to pay some positive amount for this.

   b. We can get it for free if we have good credit.

   c. We have to pay more than \$85 depending on the interest rate.

   d. We have to pay something less than \$85 depending on the interest rate.

Below are some theoretical exercises.

1. Argue that the price of a forward contract satisfies $p(0) = S_0$.

2. Consider the forward exchange rate between the US dollars and the euro is denoted by € with maturity T years. Suppose that the forward exchange rate is x_f, i.e., if we sign a forward exchange contract to buy €1 euro with the forward rate x_f, then at maturity we will pay $\$x_f$ dollars and receive €1 euro. The question is to determine x_f so that no initial fee is required.

   Relevant quantities are:
   – the spot rate x, i.e., today we pay $\$x$ dollars to get €1 euro.
   – the current US yield of a zero-coupon bond with maturity T is r_d.
   – the current foreign yield a zero-coupon bond (denominated in euros) with maturity T is r_f.
   – We use the continuous compounding convention.

   Show that
   $$x_f = x\, e^{(r_d - r_f)T}.$$

Numeric Example. On January 11, 2022, the exchange rate between US dollars and Euro was $x = 1.1373$ (EURUSD is the symbol) . March 2022 futures is $x_f = 1.1385$.