Midterm Exam

Review

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Introduction

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Introduction

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01 Lecture 1: Time value of money

1.1 Cash flow

- **Definition:**

  A sequence of future payments. Prices of them depend on the times and sizes of the payments.

- **Comparision:**

  - ✓ If the payment **sizes are the same** at each payment node: people generally prefer the cash flow that is paid earlier.
  - ✓ If the payment **times are the same**: people generally prefer the cash flow with a larger amount.
  - ✓ For cash flows with **different sizes and payment times**: convert the cash flows to their present value at the same payment time for effective comparison (Discounting).

01 # Lecture 1: Time value of money

1.2 Interest Rate

➢ **Definition:**

The amount charged when one takes a loan or the amount earned when one makes a cash investment in a bank or in a bond. It is expressed as a percentage of the principal amount.

➢ **Amount:**

✓ Maturity: length of the loan

✓ Face value: the amount of the loan

➢ **Categories (F dollors, T years, r annual interest rate)**

The total interest after T years:

- $rT\,F$ if the *interest is compounded once*;
- $[(1+\frac{rT}{n})^n - 1]\,F$ if the *interest is compounded* n *times*;
- $[e^{rT} - 1]\,F$ if the *interest is compounded continuously.*

01 Lecture 1: Time value of money

1.3 Bonds.

➢ **Definition:**

Bonds are issued by governments or large corporations. They are contracts promising to pay periodic interest payments called coupons and to repay the face value on the maturity date.

➢ **Characteristics:**

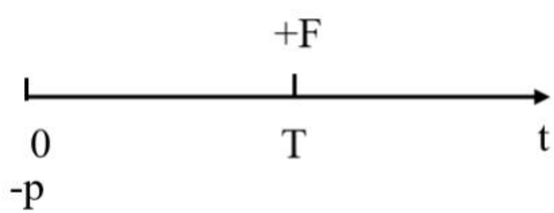
- ✓ **Maturity T:** Maturity T is the length of the bond. It can vary from few weeks to 30/40 years.
- ✓ **Face value F:** Face Value F is the amount of the debt that will be paid in full at maturity.
- ✓ **Coupon rate C:** Coupon rate C is percentage of the face value that will be paid in one year.

1.4 Zero-coupon bonds.

- **Definition:**

  A zero-coupon bond is a type of bond that does not pay periodic interest (coupon payments) to the bondholder. Instead, it is issued at a discount to its face value, and the investor receives the full face value at maturity. The difference between the purchase price (discounted price) and the face value represents the investor's return or yield.

- **Cash Flow:** (Face value F; Maturity T years; Price p)



- **Pricing:**

$$p(t) = e^{-tr(t)}\,F$$

1.5 Yield Curve.

- **Definition:**

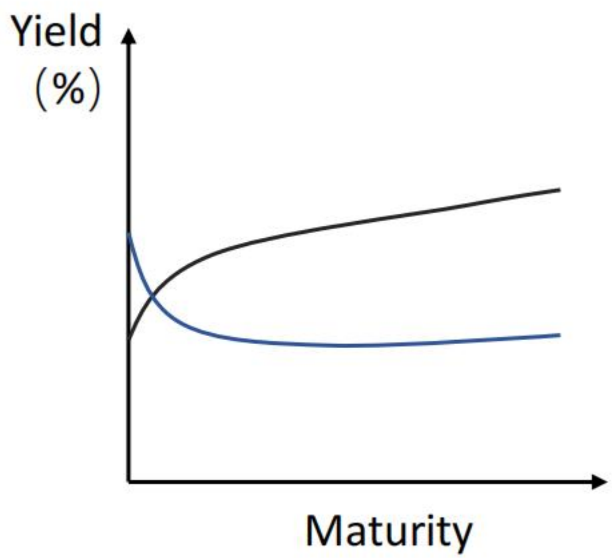
The zero-interest yield for year T is the rate of return that would be earned if the funds were invested today and held for T consecutive years. All principal is paid to the investor at the end of year n, and no interest earnings are paid until the end of the n-year period. The zero-interest yield is also the Yield to Maturity (YTM) on Zero-Coupon Bonds.

- **Calculation:**

$$F = p(t) \cdot e^{tr(t)}$$

$$p(t) = e^{-tr(t)}\, F$$

$$r(t) = -\frac{\ln(p(t)/F)}{t}.$$



01 Lecture 1: Time value of money

1.6 Discounting.

- **Definition:**

Discounting is the process of converting cash flows expected to be received at different points in the future into their present value. This method takes into account the time value of money, which posits that the value of money differs depending on when it is received.

- **Discount factor:**

$$p(t) = e^{-tr(t)}\, F \quad \Rightarrow \quad e^{-tr(t)}$$

- **Bonds pricing (Discounting):**

$$PV(\mathbf{c}) := \sum_{i=1}^{n} c(t_i)\, e^{-t_i r(t_i)}.$$

01 Lecture 1: Time value of money

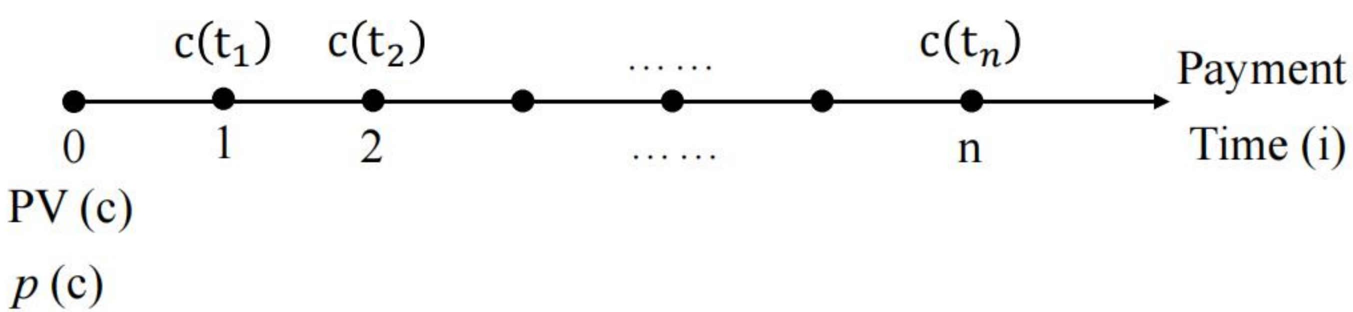
1.7 Internal rate of return (IRR).

➢ **Definition:**

The internal rate of return (IRR) is the yield of a cash flow that makes the present value of the cash flow equal to its cost or purchase price. For bonds, the IRR is often referred to as the yield-to-maturity (YTM).

➢ **Calculaiton:**

$$\boxed{p(\mathbf{c})} = \sum_{i=1}^{n} c(t_i)\, e^{-r\, t_i} = \boxed{PV(\mathbf{c})}$$



01 Lecture 1: Time value of money

1.7 Internal rate of return (IRR).

- **Characteristics:**

  Suppose that $c(t_i) > 0$ for every i and $p(c) > 0$.

  ✓ Uniquely defined

  ✓ Inversely related to the price

  ✓ Strictly positive

Theorem 1.7. *Suppose that $c(t_i) > 0$ for every i and $p(\mathbf{c}) > 0$. Then, the internal rate of return r of $\mathbf{c}$ is uniquely defined. Moreover, it is inversely related to the price and it is strictly positive whenever $p(\mathbf{c}) < \sum_{i=1}^{n} c(t_i)$.*

02 Lecture 2: Forward, Futures, Arbitrage

2.1 Forward contract

- **Definition:**

  - ✓ The forward contract on a stock with maturity T and strike K is a legally binding sale contract of one stock at time T for a price of K

  - ✓ At time T the buyer pays K dollars and get one share of the stock which is worth ST dollars. Therefore, effectively, this contract pays at maturity ST −K dollars to the buyer of the forward contract. Of course this amount could be negative, and in that case the buyer loses money.

2.2 Forward Price.

- The ***forward price*** **F** of a stock with maturity T is the strike that is agreed upon at time zero satisfying the followings:
  - ✓ 1. Holder of the forward contract will buy the stock at time T for a price F regardless of the actual price of the stock at that time.
  - ✓ 2. The seller of the forward contract will sell the stock at time T for a price F regardless of the actual price of the stock at that time.
  - ✓ 3. At time zero, no up-front payment is made by either party.
  - ✓ 4. The contract is liquidly traded, i.e., there are investors willing to take the long and the short position of this contract.

2.3 Price of forward contract.

➢ Definition:

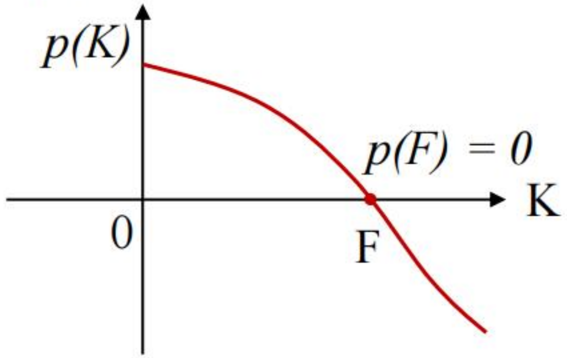
Consider a forward contact with a general strike K. The price *p(K)* of this contract is paid at time zero. If $p(K) > 0$, the buyer pays this amount to the seller, and if If $p(K) < 0$, the seller pays this amount $-p(K)$ to the buyer. Then, the contract is settled at maturity. namely, at time T, the buyer pays K dollars to the seller who gives one share of the stock to the buyer.

➢ Determination:

- ✓ Let *p(K)* be the price of the price of a forward contract with strike K. Then, the forward price F is the unique solution of $p(F) = 0$.
- ✓ **Proof:**

  $p(0) > 0.$

  $$\lim_{K\to\infty} p(K) < 0.$$



02 Lecture 2: Forward, Futures, Arbitrage

2.4 Forward contract pricing.

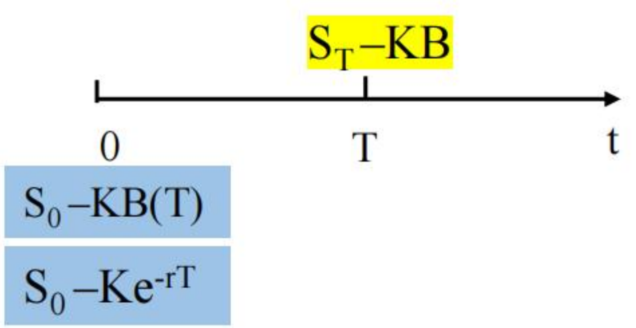
- Let *p(K)* be the price ofthe price of a forward contract with maturity T and strike K. Then,

$$p(K) = S_0 - KB(T) \quad =0$$

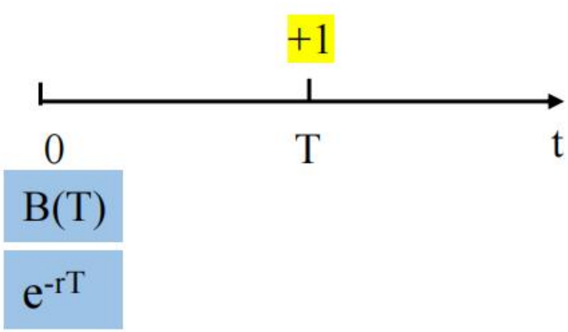
where S_0 is current price ofthe stock and *B(T)* is the price of a government bond with maturity T and face value one. Consequently, the forward price F at time T of a stock is given by

$$F = \frac{S_0}{B(T)}$$

$$F = S_0 e^{rT}$$



Forward Contract



Government Zero-coupon bond

2.5 Arbitrage Pricing Rule.

All products must be priced in such a way that the resulting market is free ofarbitrage. In particular, any price that leads to an arbitrage is not allowed.

2.6 Law of One Price.

- **Definition:**
  - ✓ In the absence of trade frictions, manipulations and tariffs, contracts possibly different locations must have the same price, if they produce identical cash-flows when expressed in a common currency.
- **Significance:**
  - ✓ It emphasizes the importance of price consistency in financial markets.
  - ✓ It is the foundation for the pricing of financial derivatives and risk management, as it ensures fairness in pricing and the efficiency of the market.

03 Lecture 2: Forward, Futures, Arbitrage

2.7 Alternate proof of Theorem 3.1. $F = \frac{S_0}{B(T)}$ $F = S_0 e^{rT}$

- Suppose a forward contract with strike F is offered in the market with no initial fee. Investors have (at least) the following **two options**:

1. Sign the forward contract with no initial fee. (Long position)
2. Borrow S_0 dollars by short-selling $S_0/B(T)$ many shares of the bond. With the proceeds buy one share of a stock now.

- **At time T** the financial portfolio of these investors will be as follows:

1. They pay F dollars and get one share of the stock. So the value of their portfolio will be $\boldsymbol{S_T - F}$, where S_T is future random value of the stock.
2. They have one stock and $S_0/B(T)$ many bond obligations. Sine one share of the bond pays one dollar, their total worth is $\boldsymbol{S_T - S_0/B(T)}$.

- Investors will initially buy the forward contract with no initial fee if and only if their **proceeds in case 1 exceeds the proceeds from case 2**.

$$F \leq \frac{S_0}{B(T)}$$

02 Lecture 2: Forward, Futures, Arbitrage

2.7 Alternate proof of Theorem 3.1.

$$F = \frac{S_0}{B(T)} \qquad F = S_0 e^{rT}$$

- The **two options** the investors may compare:

1. Sell one share of the forward contract with no initial fee. (Short position)
2. Short one stock and collect the current price of the stock. With the proceeds buy $S_0/B(T)$ many shares of the bond now.

- **At time T** the financial portfolio of these investors will be as follows:

1. For F dollars they give one share of the stock. So the value of their portfolio will be $-S_T + F$.
2. They have $S_0/B(T)$ many bonds and short one share of the stock. So their total worth is $-S_T + S_0/B(T)$.

- Investors will initially sell the forward contract only if their **proceeds in case 1 exceeds the proceeds from case 2**.

$$F \geq \frac{S_0}{B(T)}$$

2.7 Alternate proof of $p(K)=S_0-KB(T)$

- The **two options** the investors may compare:

1. Sell one share of the forward contract with strike K and maturity T. (Short position)
2. Short one stock and collect the current price of the stock, S_0. Pay initial fee $p(K)$ to the buyer of the forward contract. With the proceeds buy $[S_0-p(K)]\,/B(T)$ many shares of the bond now.

- **At time T** the financial portfolio of these investors will be as follows:

1. For F dollars they give one share of the stock. So the value of their portfolio will be $\boldsymbol{K-S_T}$.
2. They have $[S_0-p(K)]\,/B(T)$ many bonds and short one share of the stock. So their total worth is $[S_0-p(K)]\,/B(T)$ - $\boldsymbol{S_T}$.

02 Lecture 2: Forward, Futures, Arbitrage

2.8 Future contract.

- **Definition**

  A futures contract is a standardized agreement established by a futures exchange, stipulating the delivery of a specified quantity and quality of goods at a certain time and place in the future.

- **Characteristics**

  - ✓ Contract standardization.
  - ✓ Centralized trading.
  - ✓ Two-way trading and hedging mechanism.
  - ✓ Margin trading (Leverage mechanism).
  - ✓ Daily mark-to-market settlement system.

02 Lecture 2: Forward, Futures, Arbitrage

Comparison of Forward Contracts and Futures Contracts.

Forward Contract	Futures Contract
Over-the-Counter (OTC) Trading	Exchange Trading
Contract Agreement Content (a paper contract)	Standardized (invisible contract)
Negotiation on whether and how much margin to pay	Payment of margin 5%-10%
Used for hedging and physical delivery	Used for risk management and speculation
No trading system, negotiated individually	Trading system is in place
Transactions and prices are not transparent	Transactions and prices are public and transparent
One-to-one	Many-to-many
No clearinghouse	Clearinghouse
Mainly physical delivery	Mainly cash settlement
Credit risk is high	Credit risk is zero
Liquidity is poor, contracts are not easily transferable	Liquidity is strong, contracts are easily transferable

02 Lecture 2: Forward, Futures, Arbitrage

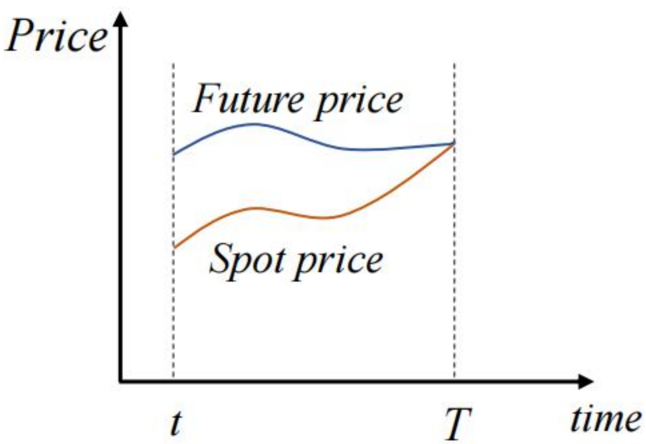
2.9 Futures price

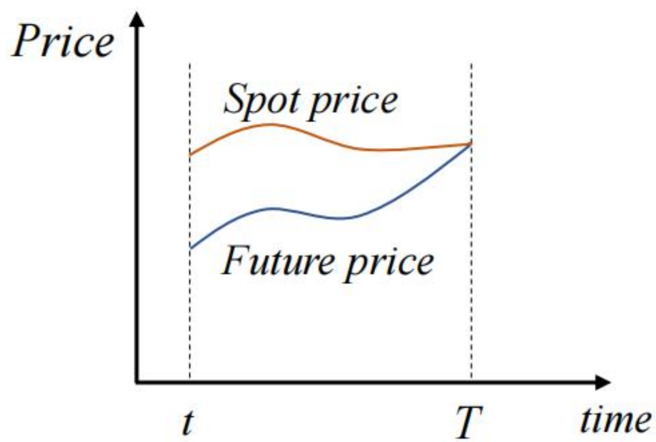
➢ Definition:

Futures price refers to the price at which buyers and sellers agree to trade an underlying asset at a specific time in the future within a futures contract. This price is established at the time the contract is formed and may be adjusted during the contract period based on changes in market conditions.

➢ Convergence:

As one must pay the spot price at settlement, the futures price on the possible days of delivery are equal to the spot price. That is $F_T = S_T$.





Lecture 2: Forward, Futures, Arbitrage

2.9 Futures price

- **Futures price equal to Forward price.**

Consider futures and forward contracts of the same underlying with common maturity T. Assume that the interest rate remains constant during the time period [0,T]. Then, the futures price F_0 is equal to the forwards price F = S_0 /B(T).

$$F = F_0$$

03 Lecture 3: One and Two step binomial model

3.1 Options

- **Definition:**

Options are derivative financial instruments that provide the holder with the right, but not the obligation, to buy (call option) or sell (put option) an underlying asset at a predetermined price within a specific time frame or at expiration.

- **Classification:**

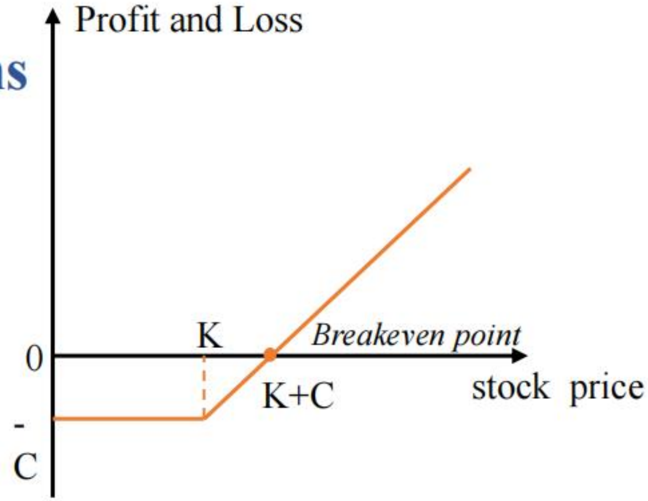
  - ✓ **Call Option:** Grants the holder the right to buy the underlying asset. The future random pay-off of this option is $(S_T-K)+$.
  - ✓ **Put Option:** Grants the holder the right to sell the underlying asset. The future random pay-off of this option is $(K-S_T)+$.

  - ✓ **European Option:** Can only be exercised on the expiry date.
  - ✓ **American Option:** Can be exercised at any time.

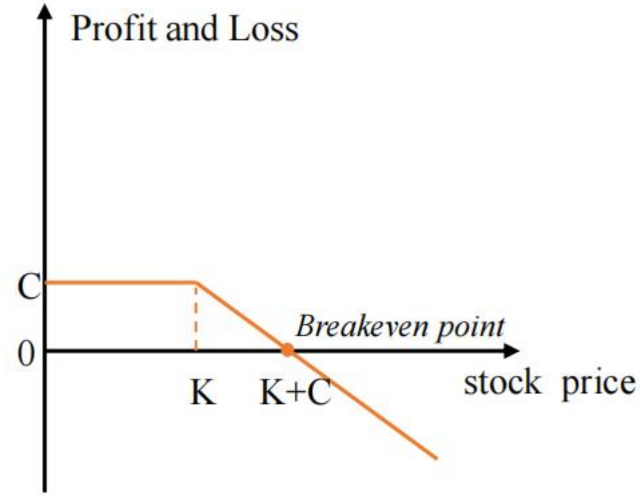
  - ✓ **Physical Option:** Options where the underlying asset is a physical commodity.
  - ✓ **Financial Option:** Options where the underlying asset is a financial instrument.

03 Lecture 3: One and Two step binomial model

3.1 Options

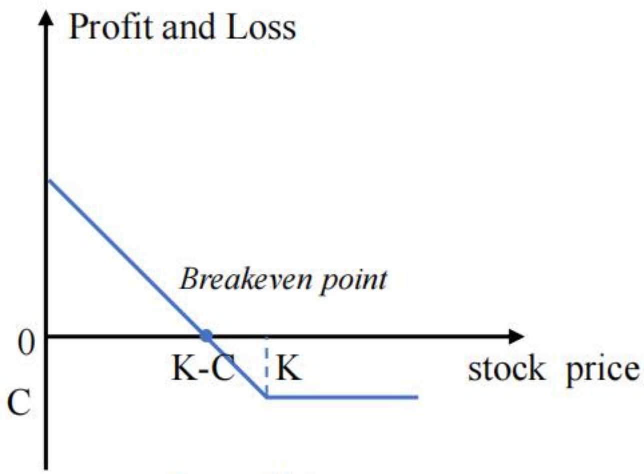


Long Call

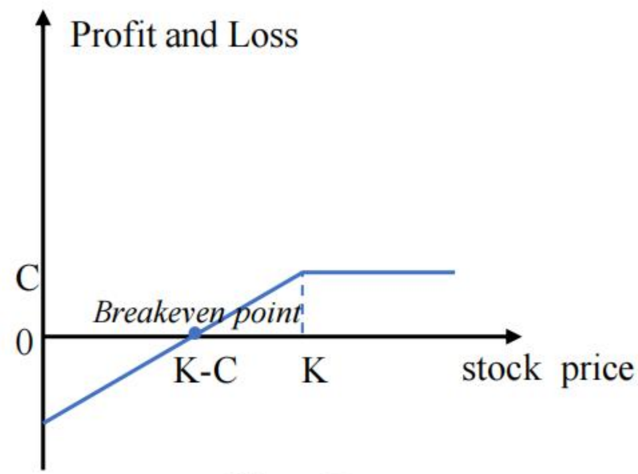


Call option

Short Call



Long Put



Put option

Short Put

3.2 Put-Call Parity (PCP)

➢ Constructing two portfolios

- ✓ Portfolio A: Buy one European Call option and sell one Europeand Put option with same strike K and maturity T.
- ✓ Portfolio B：Buy one Forward contract with strike K and maturity T.

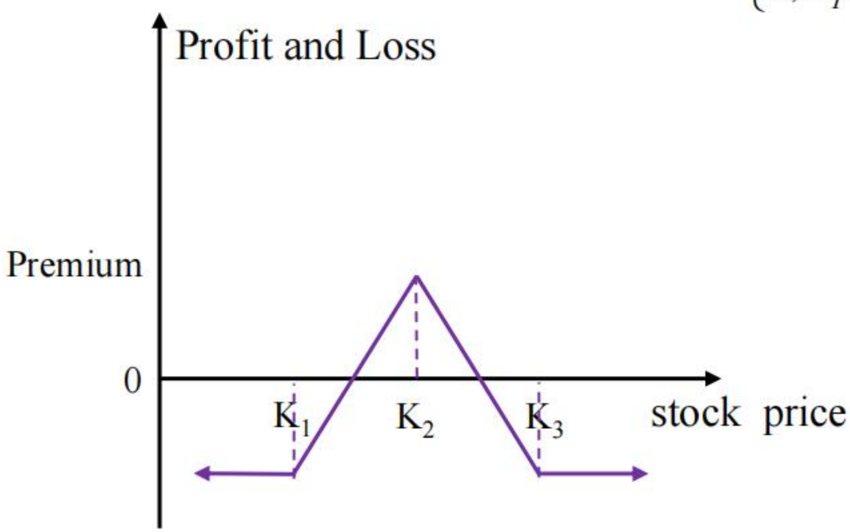
$$(S_T - K)^+ - (K - S_T)^+ = S_T - K.$$

$$\boxed{C(K,T) - P(K,T) = S_0 - KB(T).}$$

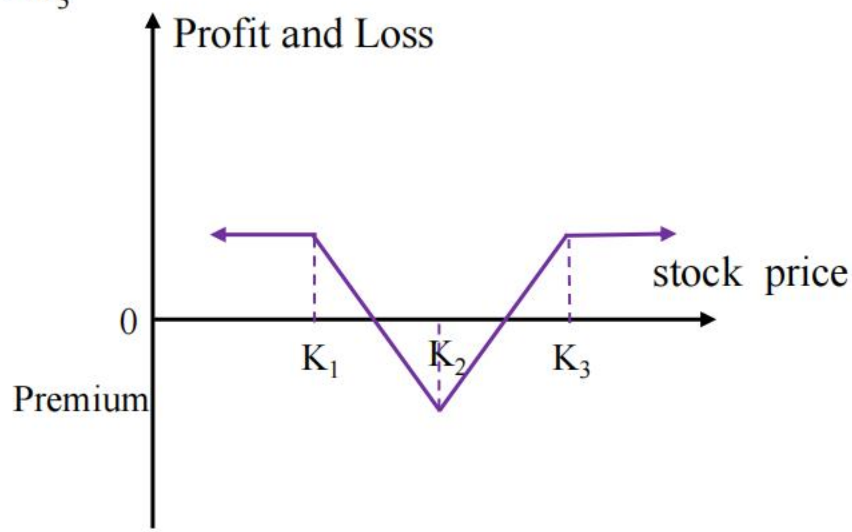
3.3 Butterflies:

- Buy one share of European call (put) option with a lower strike K_1 (in-the-moeny)
- Buy one share of European call (put) option with a higher strike K_3 (out-of-thoeny)
- Sell two shares of European call (put) option with a middle strike K_2 (at-the-me-moeny), $K_2 = (K_1 + K_3)/2$, $K_2 = S_0$.

$$pay-off = \begin{cases} 0, S_T \le K_1 \\ S_T - K_1, K_1 < S_T \le K_2 \\ K_3 - S_T, K_2 < S_T \le K_3 \\ 0, S_T \ge K_3 \end{cases}$$



Long Butterfly



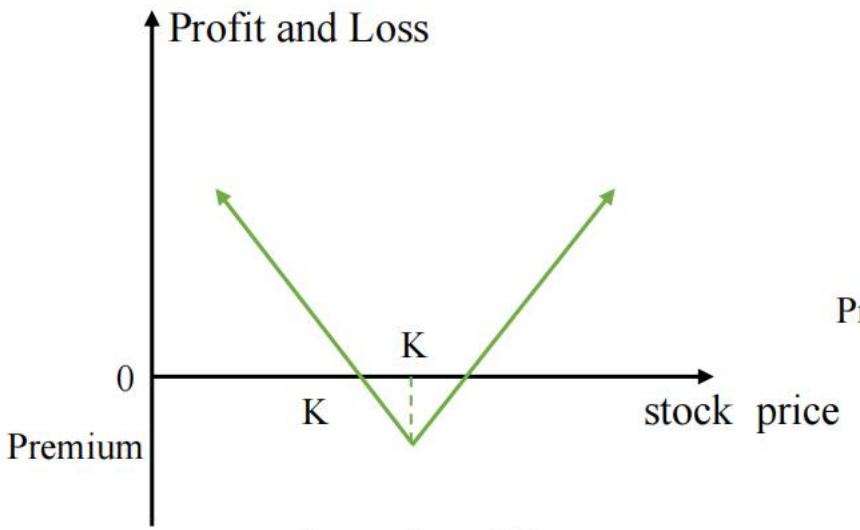
Short Butterfly

03 Lecture 3: One and Two step binomial model

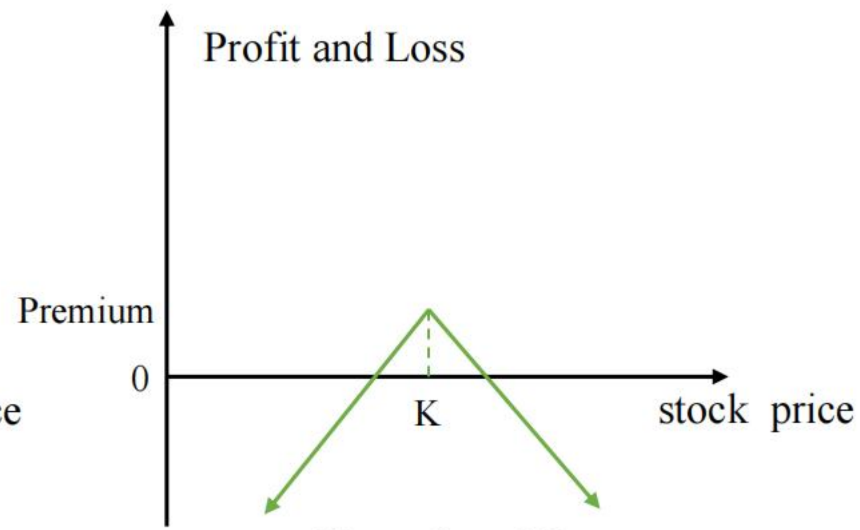
3.3 Straddles:

- Long straddles: Buy one share of European call and put option with strike K and maturity T.
- Short straddels: Sell one share of European call and put option with strike K (out-of-thoeny) and maturity T.

$$pay-off = \begin{cases} K - S_T, S_T \le K \\ S_T - K, S_T > K \end{cases}$$



Long Straddle



Short Straddle

03 Lecture 3: One and Two step binomial model

3.3 One-Step Binomial Model:

Definition:

- The model simulates the future paths of asset prices by dividing them into a series of binary choices (either up or down at each step). At each time step, the asset price can move according to a specific up-factor (u) or down-factor (d).

Core:

- Construct a no-arbitrage or risk-free replicating portfolio

Characteristics:

- Advantages: intuitiveness and flexibility.
- Disadvantages: The significant increase in computational effort required when asset prices are highly volatile or when the option has a long time to maturity.

Applicaiton:

- Options pricing, risk management, and derivative structure design

03 Lecture 3: One and Two step binomial model

3.3 One-Step Binomial Model

Stock

S_0 — up: p → $S_1 = uS_0$

S_0 — down: $1-p$ → $S_1 = dS_0$

$d < 1 + r < u$

European option

V_0? — up: p → v_u

V_0? — down: $1-p$ → v_d

Structured portfolio (option): *buy a shares of the stock and b shares of the zero-coupon bond*

portfolio $aS_0 + bB(T)$ → $auS_0 + b$, $adS_0 + b$

$$aS_0 + bB(T) = V_0, B(T) = \frac{1}{1+r}$$

$$auS_0 + b = v_u$$
$$adS_0 + b = v_d$$

$$\Rightarrow a = \frac{v_u - v_d}{(u-d)S_0} \quad b = \frac{uv_d - dv_u}{u-d}$$

$$\Rightarrow V_0 = \frac{1}{1+r}\left[\frac{(1+r)-d}{u-d}v_u + \frac{u-(1+r)}{u-d}v_d\right]$$

$$p = \frac{(1+r)\text{-}d}{u\text{-}d}$$

e^{-rT}

$$V_0 = \frac{1}{1+r}\left[pv_u + (1-p)v_d\right]$$

Lecture 3: One and Two step binomial model

3.4 Risk-neutral probability:

- **Definition:**

  In financial mathematics, the risk-neutral probability is a hypothetical probability that assumes market participants do not account for risk when evaluating and pricing financial assets.

- **Characteristics of risk-neutral world:**

  - ✓ A risk-neutral world is a theoretical financial environment where all investors have a neutral attitude towards risk.
  - ✓ The current price of an asset is equal to the present value of its expected future value, discounted at the risk-free interest rate.
  - ✓ The expected return on all securities is equal to the risk-free rate, regardless of the actual risk preferences of the market.

03 Lecture 3: One and Two step binomial model

- **Risk Neutral Probability:** $p^* = \dfrac{(1+r)-d}{u-d}$

p^* is the probability of a stock price increase in a risk-neutral world.

- **The price of an European option:** $V_0 = \dfrac{1}{1+r}\left[p^* v_u + (1-p^*)v_d\right] = \dfrac{1}{1+r}E(V_1)$

- **The expected return of the stock at time T=1 is when the probability of rise is p^*:**

$$E(S_1) = p^* u S_0 + (1-p^*) d S_0$$

$$E(S_1) = p^*(u-d)S_0 + dS_0$$

$$E(S_1) = (1+r)S_0 \Rightarrow S_0 = \frac{1}{(1+r)}E(S_1)$$

The above formula illustrates that stock prices increase at the risk-free rate when the probability of stock price increase is the risk-neutral probability **p^***.

04 Lecture 4: Multi step Binomial model

4.1 Binomial model pricing equation (European option):

To develop this recursion, we start with the node $(T-1, i)$. From this node the stock price can make only one up or down movement. After these movements, we know the value of the option. Hence, we have the following diagram:

$$S_{T-1} = s_{T-1,i},\ v_{T-1,i} = ? \quad \begin{array}{l} S_T = s_{T,i+1} = u s_{T-1,i}, \quad v_{T,i+1} = \varphi(s_{T,i+1}) \\ \\ S_T = s_{T,i} = d s_{T-1,i}, \quad v_{T,i} = \varphi(s_{T,i}). \end{array}$$

This is exactly a one-step binomial model with initial stock price $s_{T-1,i}$ and

$$v_u = \varphi(s_{T,i+1}) = v_{T,i+1}, \qquad v_d = \varphi(s_{T,i}) = v_{T,i}.$$

Since d, r, u are time independent, p^* is given by,

$$p^* = \frac{(1+r) - d}{u - d}.$$

Hence,

$$v_{T-1,i} = \frac{1}{1+r}\left[p^*\, v_{T,i+1} + (1-p^*)\, v_{T,i}\right].$$

04 Lecture 4: Multi step Binomial model

4.2 Hedge (European option):

Consider European option with a pay-off φ in a multi-step binomial model with nodes (k,i), $k=0,\ldots,N$, $i=0,\ldots,k$. Let $v_{k,i}$ be the value of the option computed in Theorem 5.2.1. Set

$$\boxed{\theta_{k,i} := \frac{v_{k+1,i+1} - v_{k+1,i}}{s_{k,i}(u-d)}, \qquad k=0,1,\ldots N-1,\ i=0,1\ldots k.} \tag{2.1}$$

$$\boxed{c_{k,i} := v_{k,i} - \theta_{k,i} s_{k,i}, \qquad k=0,1,\ldots N-1,\ i=0,1\ldots k.} \tag{2.2}$$

Theorem 2.1.

The hedging strategy that starts with v0 dollars, and at each node

a. *investing $\theta_{k,i}$ shares in the stock and*

b. *holding $c_{k,i}$ dollars in the bank (borrowing if negative)*

✓ *Perfectly replicates the option pay-off $\varphi(S_N)$ without any additional funds.*

✓ *Additionally, the value of this portfolio at each node is exactly equal to $v_{k,i}$.*

05 Lecture 5: American Options

5.1 American Options:

- **Definition:**
  - ✓ American Option is a type of financial derivative that allows the holder to exercise the option at any time prior to the expiration date, unlike the European Option which can only be exercised on the expiration date. The American Option offers greater flexibility as investors can choose the optimal time to exercise based on market conditions.
- **Characteristics:**
  - ✓ Early Exercise Rights
  - ✓ Diverse Exercise Strategies
  - ✓ Hedging Requirements
  - ✓ Incentive to Exercise Early

05 Lecture 5:American Options

5.2 payoff of American Options

- **European Options:**

  The European Opitons pays: $\varphi(S_T)$

  where T is the pre-determined maturity.

  - ✓ Call option: $\varphi(S_T) = (S_T - K)^+$
  - ✓ Put option: $\varphi(S_T) = (K - S_T)^+$

- **American Options:**

  The American Opitons pays: $\varphi(S_\tau)$, the holder can chose to stop anytime τ between initiation and the maturity,$0 \leq \tau \leq T$.

  - ✓ Call option: $\varphi(S_\tau) = (S_\tau - K)^+$
  - ✓ Put option: $\varphi(S_\tau) = (K - S_\tau)^+$

02 American Options

American Option Binomial Model Pricing:

- **Example:**

$$S_0 = 4,\ d = \frac{1}{2},\ 1 + r = \frac{3}{2},\ u = 2$$

We consider a Put option with K=7. Then, the potential pay-off at any node is given by $\varphi_{k,i} = \left(7 - s_{k,i}\right)^+$, $k = 0,1,2,\ \ i = 0, \ldots, k$

✓ Compute the risk-neutral up probability, $p^* = \frac{(1+r)-d}{u-d} = \frac{2}{3}$

✓ At maturity 2, the option has to be exercised and therefore,

$$v_{2,i} = \varphi_{2,i}, \quad i = 0, 1, 2.$$

✓ Compute the stock values ($s_{k,i}$) and potential pay-offs ($\varphi_{k,i}$):

$$s_{2,2} = 16,\ \varphi_{2,2} = v_{2,2} = 0,$$

$$s_{1,1} = 8,\ \varphi_{1,1} = 0,$$

$$s_0 = 4,\ \varphi_0 = 3, \qquad s_{2,1} = 4,\ \varphi_{2,1} = v_{2,1} = 3,$$

$$s_{1,0} = 2,\ \varphi_{1,0} = 5,$$

$$S_{2,0} = 1,\ \varphi_{2,0} = v_{2,0} = 6.$$

5.3 American Option Binomial Model Pricing:

Subtree 1 = node (1,1):

$$s_{1,1} = 8,\ \varphi_{1,1} = 0,\ v_{1,1} = ?$$

$$s_{2,2} = 16,\ v_{2,2} = 0,$$

$$s_{2,1} = 4,\ v_{2,1} = 3.$$

$$stop\ price = \varphi_{1,1} = (7-8)^{+} = 0$$

$$continuation\ \ price = \frac{1}{1+r}\left[p^{*}v_{2,2} + (1-p^{*})v_{2,1}\right] = \frac{2}{3}$$

$$v_{1,1} = max\{stop\ price; continuation\ price\}$$

$$= max\left\{\varphi_{1,1}; \frac{1}{1+r}\left[p^{*}v_{2,2} + \left(1+p^{*}\right)v_{2,1}\right]\right\}$$

Continue

$$\boldsymbol{v_{1,1} = \frac{2}{3}}$$

$$\theta_{1,1} = \frac{v_{2,2} - v_{2,1}}{s_{2,2} - s_{2,1}} = -\frac{1}{4}$$

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5.3 American Option Binomial Model Pricing:

Subtree 2 = node (1,0):

$$s_{2,1} = 4,\ v_{2,1} = 3,$$

$$s_{1,0} = 2,\ \varphi_{1,0} = 5,\ v_{1,0} = ?$$

$$s_{2,0} = 1,\ v_{2,0} = 6.$$

$$stop\ price = \varphi_{1,0} = (7-2)^{+} = 5$$

$$continuation\ \ price = \frac{1}{1+r}\left[p^{*}v_{2,1} + (1-p^{*})v_{2,0}\right] = \frac{8}{3}$$

$$v_{1,0} = max\{stop\ price; continuation\ price\}$$

Stop

$$v_{1,1} = 5$$

$$= max\left\{\varphi_{1,0}; \frac{1}{1+r}\left[p^{*}v_{2,1} + \left(1+p^{*}\right)v_{2,0}\right]\right\}$$

$$\theta_{1,0} = \frac{v_{2,1} - v_{2,0}}{s_{2,1} - s_{2,0}} = -1$$

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5.3 American Option Binomial Model Pricing:

Subtree 3 = node (0,0):

$$s_{1,1} = 8,\ v_{1,1} = \frac{2}{3},$$

$$s_0 = 4,\ \varphi_0 = 3,\ v_0 = ?$$

$$s_{1,0} = 1,\ v_{1,0} = 5.$$

$$stop\ price = \varphi_0 = (7-4)^+ = 3$$

$$continuation\ \ price = \frac{1}{1+r}\left[p^* v_{1,1} + (1-p^*)v_{1,0}\right] = \frac{38}{27}$$

Stop

$v_0 = 3$

$$v_{1,0} = max\{stop\ price; continuation\ price\}$$

$$=max\left\{\varphi_0; \frac{1}{1+r}\left[p^* v_{1,1} + \left(1+p^*\right)v_{1,0}\right]\right\}$$

$$\theta_0 = \frac{v_{1,1} - v_{1,0}}{s_{1,1} - s_{1,0}} = -\frac{13}{18}$$

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5.4 Hedging (super-replicate the American option):

- **Put Option Seller:** $v_0 = 3$
  - ✓ If it is immediately exercised, then the seller gives back this amount and the hedging is done.
  - ✓ But if the investors decide to continue (despite the fact that it is not optimal), then the seller will hedge.
- **Step 0**
  - ✓ The Put Option seller of the option shorts $\theta_0 = -\frac{13}{18}$ shares of the stock. Together with the initial proceed of 3 from the sale, the seller now has

$$3 + 4 \times \frac{13}{18} = \frac{53}{9}$$

- **Step 1**

If up: $-\frac{13}{18} \times 8 + \frac{53}{6} > v_{1,1} = \frac{2}{3}.$

If down: $-\frac{13}{18} \times 2 + \frac{53}{6} = \frac{133}{18} > v_{1,0} = 5.$

5.4 Hedging (super-replicate the American option):

- **Step 1**
  - ✓ If the investors decide to continue at node (1,0), the investos bring their short position to $\theta_{1,0} = -1$ by shorting an additional $-\theta_{1,0} - (-\theta_0) = 1 - \frac{13}{18} = \frac{5}{18}$ shares.
  - ✓ The cash amout (in bond): $\frac{53}{6} + \frac{5}{18} \times 2 = \frac{169}{18}$
- **Step 2**

$$\textit{If up:} \qquad -1 \times 4 + \frac{169}{12} = \frac{121}{12} > v_{2,1} = 3.$$

$$\textit{If down:} \qquad -1 \times 1 + \frac{169}{12} = \frac{157}{12} > v_{2,0} = 6.$$

The above hedging strategy super-replicates the Put option with initial price of 3.

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5.5 The Equation of Amercian Options Pricing:

We have argued in the above example that the pricing equation for the American option with a general pay-off of $\varphi(S_\tau)$ in the Binomial model is

$$\boxed{v_{k,i} = \max\left\{\varphi(s_{k,i})\ ,\ \frac{1}{1+r}\left[p^* v_{k+1,i+1} + (1-p^*)v_{k+1,i}\right]\right\},\ \ k=0,1,\ldots N-1,\ i=0,1\ldots,k}$$

with the final condition

$$\boxed{v_{N,i} = \varphi(s_{N,i}), \quad i = 0,1\ldots N.}$$

One may rewrite the above equation as

$$V_k = \max\left\{\varphi(S_k)\ ,\ \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left[V_{k+1} \mid S_k\right]\right\}$$

Important to note that, as opposed to European pricing formula, we cannot express $v_{k,i}$ simply in terms of $v_{N,k}$'s.

06 Summary and Q&A

- **Four financial instruments:**
  - ✓ Bond (Time value of money; zero-coupon bond; bond pring)
  - ✓ Forward contract (Forward price; the price of Forward contract)
  - ✓ Future contract (Future price; Daily mark-to-market settlement)
  - ✓ Option (Call and Put; European and American; Put-Call Parity; Path-dependent)

- **Three significant calculation:**
  - ✓ Discounting (Present value, Internal rate of interest)
  - ✓ Binomial Model (One, Two, Multiple; risk-neutral probability; Binomial Pricing Formula)

- **Two significant concept:**
  - ✓ Arbitrage
  - ✓ Hedge

- **One significant law:**
  - ✓ Law of One Price

Thanks!

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