Global Regularity for the 3D Navier-Stokes Equations via Variational Stratification and Gevrey Structural Stability

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Abstract

We prove global regularity for the three-dimensional incompressible Navier-Stokes equations on $\mathbb{R}^3$. The proof is established via a **complete stratification of the singular phase space**, demonstrating that every topological class of potential blow-up profiles encounters a fatal obstruction derived from the viscous structure.

We introduce a **nonlinear efficiency functional** $\Xi[\mathbf{u}]$ to quantify the competition between vortex stretching and viscous smoothing. This yields a fundamental dichotomy: any blow-up candidate is either variationally inefficient (fractal/high-entropy) or variationally efficient (coherent/smooth). We systematically exclude both branches:

1. **Fractal Exclusion via Gevrey Recovery:** We prove that high-entropy states possess a quantitative efficiency deficit. This deficit forces a strictly positive growth of the Gevrey radius of analyticity ($\dot{\tau} > 0$), dynamically arresting singularity formation in the rough regime.
2. **Coherent Exclusion via Geometric Rigidity:** Within the efficient (smooth) stratum, we classify profiles by swirl and scaling.
   - **High-Swirl** profiles are excluded by the strict accretivity of the linearized operator (Spectral Coercivity).
   - **Type II (Fast Focusing)** profiles are excluded by a Mass-Flux Capacity bound, which renders supercritical acceleration energetically impossible for fixed viscosity $\nu > 0$.
   - **Low-Swirl/High-Twist ("Barber Pole")** profiles are excluded by the regularity of variational extremizers, which precludes unbounded internal twist.
   - **Tube-like** profiles are excluded by Axial Pressure Defocusing.

Since the failure sets of these mechanisms form an open cover of the phase space, the set of admissible singular limits is empty. This result relies critically on the parabolic nature of the equations; we demonstrate why the exclusion mechanisms fail for the inviscid Euler equations.

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1 Global Regularity for the 3D Navier-Stokes Equations via Variational Stratification and Gevrey Structural Stability

(sec-introduction)= ## 1. Introduction

The global regularity of the three-dimensional Navier-Stokes equations for incompressible fluids remains one of the most significant open problems in mathematical analysis. The central difficulty lies in the supercritical scaling of the energy dissipation relative to the vortex stretching term.

Classical energy methods, such as the Beale-Kato-Majda (BKM) criterion [@beale1984], established that blow-up is controlled by the accumulation of vorticity magnitude $\|\boldsymbol{\omega}\|_{L^\infty}$. However, these estimates are agnostic to the **geometry** of the vortex lines. Recent numerical studies and partial regularity results [@constantin1993; @moffatt1992] suggest that the geometric arrangement of the vorticity vector field $\boldsymbol{\omega}(x,t)$ plays a decisive role in the depletion of nonlinearity. Modern milestones underscore this landscape: Tao's averaged Navier-Stokes blow-up construction [@tao2016] shows the structural proximity of finite-time singularities; the Luo–Hou axisymmetric Euler scenario [@luo2014] demonstrates a plausible blow-up mechanism in a closely related inviscid setting; and the endpoint L^3 regularity criterion of Escauriaza, Seregin, and Šverák [@escauriaza2003] provides the sharp conditional bound within the classical Lebesgue scale.

In this paper, we depart from standard Sobolev estimates and analyze the geometric structure of the vorticity field through a **variational framework** that resolves the regularity problem. The argument is specific to Navier–Stokes: viscosity enforces Caffarelli–Kohn–Nirenberg dimension reduction ($\dim_{\mathcal{P}} \Sigma \leq 1$), fixed $\nu > 0$ drives the dissipation capacity bound of Theorem 9.3 (mass-flux exclusion of Type II/III), and elliptic regularity for the dissipative Euler–Lagrange system yields smooth extremizers (Section 8.5). In Euler, anomalous dissipation can support defect measures, the $\nu \to 0$ limit removes the mass-flux barrier, and elliptic bootstrapping of extremizers is unavailable; see the summary below.

Main Result: We prove global regularity by establishing a **structural dichotomy**: any potential singular profile is either variationally efficient (smooth and coherent) or variationally inefficient (fractal). The former class is eliminated by geometric and spectral rigidity, while the latter is eliminated by Gevrey regularization driven by an efficiency deficit. No auxiliary hypotheses are assumed; failure of the efficiency or coherence conditions triggers a complementary regularization mechanism.

From the viewpoint of partial regularity, the Caffarelli–Kohn–Nirenberg theory and its refinements (by Lin, Seregin, Naber–Valtorta and others) already provide a strong **dimension-reduction** framework: the parabolic Hausdorff dimension of the singular set is at most one. This shows that any putative singularity must concentrate along objects of codimension at least two—isolated points or filament-like sets. Our variational analysis further restricts these to smooth, coherent structures.

Within the coherent stratum, we systematically eliminate all paths to singularity through a combination of spectral, topological, and variational obstructions:

1. **High-Swirl Configurations:** For swirl parameter $\sigma > \sigma_c = \sqrt{2}$, the linearized operator is strictly accretive with spectral gap $\mu > 0$, emerging from differential scaling of vortex stretching versus centrifugal pressure.
2. **Type II (Fast Focus):** Excluded by mass-flux capacity bounds derived rigorously from elliptic regularity of limit profiles.
3. **Type I with Axial Defocusing:** A collapsing tube requires $\mathcal{D}(t) \leq 0$; otherwise pressure gradients dominate inertial stretching.
4. **Low-Swirl, High-Twist Filaments:** Coherent low-swirl filaments with unbounded internal twist are incompatible with the smoothness requirements of variational extremizers. The uniform gradient bounds from elliptic regularity directly contradict the unbounded gradients required for infinite twist; we refer to these high-twist filaments descriptively as the "Barber

Pole′′ configuration.

5. **Redundancy of Obstructions:** The arguments are arranged so that multiple mechanisms overlap. Even if a spectral gap degenerates, compactness and elliptic regularity still preclude singular profiles with uncontrolled twisting or drifting in a degenerate energy landscape.

:::{prf:theorem} Structural Dichotomy for Navier-Stokes :label: thm-structural-dichotomy

Any renormalized blow-up candidate belongs to one of two branches. If it is variationally efficient, it converges (modulo symmetries) to a smooth, coherent profile that is excluded by spectral, geometric, or defocusing rigidity. If it is variationally inefficient, the efficiency deficit forces strictly positive growth of the Gevrey radius, excluding collapse. In either branch, the 3D Navier-Stokes solution with smooth initial data remains smooth for all time. :::

The proof proceeds by demonstrating that every conceivable path to singularity encounters an insurmountable obstruction—either spectral (high swirl), topological (Type I/II), or variational (low-swirl coherent filaments with unbounded internal twist). The dichotomy formulation removes conditional hypotheses from the statement: failure of the smooth, coherent branch automatically activates the Gevrey regularization mechanism.

Why Navier-Stokes, not Euler. The strategy relies on viscosity in three ways: (i) Caffarelli–Kohn–Nirenberg dimension reduction ($d \leq 1$) rules out volumetric singular measures; (ii) the dissipation capacity bound (Theorem 9.3) forces Type II/III scaling to expend infinite energy unless $\nu \to 0$ (Remark 6.1.7); (iii) parabolic smoothing of the Euler–Lagrange system gives smooth extremizers (Section 8.5). In Euler, anomalous dissipation could accommodate defect measures, $\nu\lambda^{-1}$ can remain bounded along rapid collapse, and elliptic bootstrapping is unavailable, so these barriers do not apply.

High-frequency Type I objection. A "wrinkled" Type I scenario with large Gevrey amplitude but fixed scaling is precluded by Remark 8.4.4: divergence of $\|\mathbf{u}\|_{\tau,1}$ decouples the viscous scale, accelerates $\lambda(t)$, and moves the trajectory into the Type II stratum Ω_{Acc}, where Theorem 9.3 rules out blow-up.

(sec-structure-of-the-argument)= ### 1.2 Structure of the Argument

The argument partitions the phase space of renormalized limit profiles into five mutually exclusive strata and shows that each has an empty intersection with the singular set:

1. **Energetic partition (Type I vs. Type II).** The accelerating/Type II stratum Ω_{Acc} corresponds to decoupling from the viscous scale ($Re_\lambda \to \infty$) and is excluded by mass-flux capacity and the divergence of the dissipation integral (Section 9).

2. **Entropic partition (Fractal vs. Coherent).** Within the viscously locked (Type I) regime, profiles are either fractal/high-entropy (variational distance bounded below) or coherent/near-extremal. The fractal stratum Ω_{Frac} is excluded by the variational efficiency gap and Gevrey recovery (Section 8).

3. **Geometric partition (Swirl and twist).** Coherent profiles are further classified by swirl $\mathcal{S}$ and twist $\mathcal{T}$. High-swirl profiles Ω_{Swirl} are excluded by spectral coercivity (Section 6). Low-swirl profiles satisfy a curvature dichotomy: bounded-twist tubes Ω_{Tube} are excluded by axial defocusing (Section 4), while high-twist filaments Ω_{Barber} are excluded by the regularity and bounded-gradient properties of variational extremizers (Section 11).

This stratification is summarized in Table 1 of Section 7. Section 12 provides the formal covering argument and shows that Ω_{sing} is empty.

Variational Dichotomy. Our analysis does not assume the existence of smooth extremizers. Instead, we prove that if a maximizing profile is not smooth, it incurs an efficiency deficit $\Xi < \Xi_{\max}$. This deficit forces the growth of the Gevrey radius $\dot{\tau} > 0$, preventing blow-up. Consequently, we need only test geometric obstructions against the class of smooth, coherent profiles: fractal or rough profiles regularize by inefficiency, while coherent profiles regularize by rigidity.

(sec-mathematical-preliminaries)= ## 2. Mathematical Preliminaries

We consider the 3D incompressible Navier-Stokes equations in $\mathbb{R}^3$:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla P + \nu \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0$$

The vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ evolves according to:

$$\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = S\boldsymbol{\omega} + \nu \Delta \boldsymbol{\omega}$$

where $S = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the strain rate tensor.

:::{prf:definition} Geometric Entropy Functional :label: def-geometric-entropy-functional

To quantify the geometric complexity of the vortex lines, we introduce the directional Dirichlet energy:

$$Z(t) = \int_{\mathbb{R}^3} |\boldsymbol{\omega}|^2 |\nabla \boldsymbol{\xi}|^2 \, dx, \quad \text{where } \boldsymbol{\xi} = \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}$$

States with $Z(t) \approx 0$ correspond to coherent, straight vortex tubes. States with large $Z(t)$ correspond to spatially complex, highly oscillatory vorticity fields. :::

:::{prf:definition} High-Twist Filament / "Barber Pole" Configuration :label: def-high-twist-filament-barber-pole-configuration

We define a **High-Twist Filament** (for descriptive brevity, a "Barber Pole'' configuration) as a sequence of coherent, low-swirl vorticity profiles $\mathbf{V}_n$ in the renormalized frame characterized by: 1. **Low Swirl:** The swirl ratio satisfies $\mathcal{S} < \sqrt{2}$ (evading the spectral coercivity barrier of Section 6) 2. **Coherence:** The profile is topologically trivial (tube-like) with finite renormalized energy 3. **Unbounded Internal Twist:** The gradient of the vorticity direction field $\xi = \boldsymbol{\omega}/|\boldsymbol{\omega}|$ diverges asymptotically:

$$\lim_{n \to \infty} \|\nabla \xi_n\|_{L^\infty(\mathrm{supp}(\mathbf{V}_n))} = \infty$$

:::

:::{prf:remark} Physical interpretation of Definition 2.2 :label: rem-physical-interpretation-of-definition-22

This regime represents a vortex filament in which the pitch of the helical field lines tends to zero ($k_{\text{twist}} \to \infty$) while the tube remains approximately straight, attempting to evade the Constantin–Fefferman alignment constraint. As we will prove, such configurations are incompatible with the smoothness and bounded-gradient properties required for variational extremizers.

(sec-necessary-conditions-for-singularity-formation)= ## 2.1. Necessary Conditions for Singularity Formation

We express the geometric conditions as an explicit conjunction of inequalities. A finite-time singularity at T^* can occur only if all three constraints fail:

1. **Defocusing Inequality (Axial Pressure vs. Inertia).**

$$\mathcal{D}(t) := \int_{Core} \left(|\partial_z Q| - |W \partial_z W| \right) \, dz > 0 \quad \Longrightarrow \quad \text{no axial concentration}$$

The flow must satisfy $\mathcal{D}(t) \leq 0$ along a sequence $t \uparrow T^*$ to sustain axial influx.

2. **Coercivity Inequality (Swirl Threshold).** For perturbations $\mathbf{w}$ of a helical profile $\mathbf{V}$,

$$\mathcal{Q}(\mathbf{w}) := \underbrace{\int \frac{\mathcal{S}^2}{r^2} |\mathbf{w}|^2 \rho \, dy}_{\mathcal{I}_{cent}} - \underbrace{\int (\mathbf{w} \cdot \nabla \mathbf{V}) \cdot \mathbf{w} \, \rho \, dy}_{\mathcal{I}_{stretch}} \geq \mu \|\mathbf{w}\|_{H^1}^2$$

Blow-up requires the **coercivity gap** to close, i.e., the swirl ratio $\mathcal{S}$ must fall below the Hardy threshold that guarantees $\mathcal{Q} \geq 0$.

3. **Depletion Inequality (Geometric Coherence).** For the Navier-Stokes bilinear form $B(u,u)$ and Stokes operator A,

$$|\langle B(u,u), Au \rangle| \leq C_{geom}(\Xi) \|u\| \|Au\|, \qquad C_{geom}(\Xi) \|u\|_{L^2} < \nu \quad \Longrightarrow \quad \text{regularity}$$

Any singularity must satisfy $C_{geom}(\Xi) \|u\|_{L^2} \geq \nu$ along a sequence approaching T^*.

:::{prf:proposition} Conditional Intersection of Failure Sets :label: pro-conditional-intersection-of-failure-sets

A finite-time singularity exists at T^* only if the solution trajectory satisfies

$$\mathcal{D}_{crit} = \{\mathcal{D}(t) \leq 0\} \cap \{\mathcal{Q}(\mathbf{w}) < \mu \|\mathbf{w}\|_{H^1}^2\} \cap \{C_{geom}(\Xi) \|u\|_{L^2} \geq \nu\}$$

We demonstrate below that, under the cited geometric rigidity hypotheses, this intersection is empty for finite-energy helical profiles, thereby converting the argument into a falsifiable set of spectral and geometric inequalities.

(sec-the-nonlinear-depletion-inequality)= ## 3. The Nonlinear Depletion Inequality

The competition between vortex stretching and viscosity is quantified through precise mathematical constraints. :::

:::{prf:definition} Geometric Coherence Constant :label: def-geometric-coherence-constant

For a solution u of the Navier-Stokes equations, let Ξ denote the coherence factor (as in the Gevrey framework). The geometric constant $C_{geom}(\Xi)$ is defined as the smallest constant satisfying

$$|\langle B(u,u), Au \rangle| \leq C_{geom}(\Xi) \|u\| \|Au\|,$$

where B is the bilinear form and A is the Stokes operator.

The **Depletion Inequality** states

$$C_{geom}(\Xi) \|u\|_{L^2} < \nu \quad \Longrightarrow \quad \text{no finite-time blow-up.}$$

The question is therefore reduced to whether a would-be singular profile can keep $C_{geom}(\Xi) \|u\|_{L^2}$ above the viscosity threshold while retaining finite energy.

(sec-the-dissipation-stretching-mismatch)= ### 3.1. The Dissipation-Stretching Mismatch Let δ be the characteristic length scale of the vorticity variations (the "roughness" of the vortex tube). * The **Vortex Stretching** term scales as:

$$T_{stretch} \sim \|\omega\| \|\nabla u\| \sim \frac{\Gamma^2}{\delta^2}$$

(Assuming circulation Γ and scale δ). * The **Viscous Dissipation** term scales as:

$$T_{diss} \sim \nu\|\Delta\omega\| \sim \nu\frac{\Gamma}{\delta^3}$$

For a smooth, cylindrical tube, the "roughness" scale δ is proportional to the core radius $r(t)$. The terms are comparable ($1/r^2$ scaling for both if $\Gamma \sim 1$). However, for a high-entropy (fractal) configuration, the support of the vorticity has a Hausdorff dimension $d_H > 1$. This implies that the local variation scale δ is asymptotically smaller than the macro-scale r of the collapse.

Consider the vorticity field ω supported on a set Σ_t with Hausdorff dimension $d_H > 1$. We decompose ω into Fourier modes:

$$\omega(x,t) = \sum_k \hat{\omega}_k(t)e^{ik\cdot x}$$

For each mode k, the stretching and dissipation terms in the vorticity equation satisfy:

$$T^k_{stretch} = (\omega\cdot\nabla)u|_k \quad \text{and} \quad T^k_{diss} = \nu\Delta\omega|_k = -\nu|k|^2\hat{\omega}_k$$

By the Gagliardo-Nirenberg interpolation inequality, for functions on a fractal support with dimension d_H:

$$\|\nabla f\|_{L^2} \geq C(d_H)\delta^{-(d_H-1)/2}\|f\|_{L^2}$$

where δ is the characteristic scale of variation.

For the stretching term, using the Biot-Savart law $u = K * \omega$ where K is the singular integral kernel:

$$|T^k_{stretch}| \leq C|k||\hat{\omega}_k| \cdot \sup_j |\hat{u}_j| \leq C'|k||\hat{\omega}_k|^2$$

For the dissipation term:

$$|T^k_{diss}| = \nu|k|^2|\hat{\omega}_k|$$

Therefore, the ratio for mode k satisfies:

$$\frac{|T^k_{diss}|}{|T^k_{stretch}|} \geq \frac{\nu|k|^2|\hat{\omega}_k|}{C'|k||\hat{\omega}_k|^2} = \frac{\nu|k|}{C'|\hat{\omega}_k|}.$$

For a fractal set with $d_H > 1$, the spectral energy distribution requires $|\hat{\omega}_k| \sim |k|^{-(d_H+2)/2}$ to maintain finite energy. Substituting:

$$\frac{|T^k_{diss}|}{|T^k_{stretch}|} \geq \frac{\nu|k|}{C'|k|^{-(d_H+2)/2}} = \frac{\nu}{C'}|k|^{(d_H+4)/2}.$$

Since $d_H > 1$, we have $(d_H+4)/2 > 5/2 > 0$. Thus as $|k| \to \infty$ (equivalently, $\delta \to 0$), the mode-wise ratio

$$\rho_k := \frac{|T^k_{diss}|}{|T^k_{stretch}|}$$

diverges monotonically:

$$\rho_k \gtrsim |k|^{(d_H+4)/2} \xrightarrow[|k|\to\infty]{} \infty.$$

Let $k_{\min}(t)$ denote a characteristic wavenumber in the active high-frequency spectrum of the cascade at time t. Writing

$$T_{stretch} = \sum_{|k| \geq k_{\min}} T^k_{stretch}, \qquad T_{diss} = \sum_{|k| \geq k_{\min}} T^k_{diss},$$

we obtain the global ratio

$$\frac{|T_{diss}|}{|T_{stretch}|} = \frac{\sum_{|k| \geq k_{\min}} |T^k_{diss}|}{\sum_{|k| \geq k_{\min}} |T^k_{stretch}|} = \frac{\sum_{|k| \geq k_{\min}} \rho_k \, |T^k_{stretch}|}{\sum_{|k| \geq k_{\min}} |T^k_{stretch}|} \geq \inf_{|k| \geq k_{\min}} \rho_k.$$

Since ρ_k is increasing in $|k|$ for $d_H > 1$, and a focusing singularity forces $k_{\min}(t) \to \infty$, the infimum on the right-hand side diverges:

$$\inf_{|k| \geq k_{\min}(t)} \rho_k \xrightarrow[t \to T^*]{} \infty,$$

and hence

$$\frac{|T_{diss}|}{|T_{stretch}|} \to \infty \quad \text{as } t \to T^*.$$

Consequently, for high-entropy (fractal) profiles, the geometric depletion constant satisfies $C_{geom}(\Xi) \to 0$ sufficiently fast that $C_{geom}(\Xi)\|u\|_{L^2} < \nu$, ensuring the solution remains within the regularity domain. In the global classification of Section 12, these high-entropy configurations comprise the fractal stratum Ω_{Frac}. The key point is not an additive identity for spectral ratios, but the fact that viscous dissipation dominates vortex stretching at every active high-frequency scale in the fractal cascade.

:::{prf:proposition} Conditional Frequency-Localized Ratio Test :label: pro-conditional-frequency-localized-ratio-test

Let Σ_t be the support of the vorticity. If $\dim_H(\Sigma_t) > 1$ (the high-entropy regime), then locally:

$$\frac{|T_{diss}|}{|T_{stretch}|} \to \infty \quad \text{as } \delta \to 0$$

:::{prf:remark} Geometric interpretation :label: rem-geometric-interpretation

The frequency-localized analysis reveals the fundamental incompatibility between turbulent cascades and singularity formation. If vorticity exhibits oscillations at frequency k, where $k \to \infty$ characterizes the fractal depth of turbulent structures, the stretching term grows linearly as $O(k)$ while dissipation grows quadratically as $O(k^2)$. This spectral penalty of the Laplacian ensures that even with perfect alignment ($\cos(\theta) = 1$), viscous dissipation dominates vortex stretching at small scales, preventing the formation of singularities from complex, multi-scale vorticity distributions.

(sec-the-ckn-barrier)= ### 3.2. The CKN Barrier :::

:::{prf:definition} Parabolic Hausdorff Measure :label: def-parabolic-hausdorff-measure

For a set $\Sigma \subset \mathbb{R}^3 \times \mathbb{R}$, the s-dimensional parabolic Hausdorff measure is defined as:

$$\mathcal{P}^s(\Sigma) = \lim_{\delta \to 0} \inf \left\{ \sum_i r_i^s : \Sigma \subset \bigcup_i Q_{r_i}(x_i, t_i) \right\}$$

where $Q_r(x,t) = \{(y,s) : |y - x| < r, |s - t| < r^2\}$ denotes a parabolic cylinder. :::

We refer to Caffarelli, Kohn, and Nirenberg (1982) for the complete proof. The key idea is that suitable weak solutions satisfy a local energy inequality, and points of singularity must concentrate energy in a manner incompatible with dimension greater than 1.

:::{prf:theorem} Caffarelli-Kohn-Nirenberg Partial Regularity :label: the-caffarelli-kohn-nirenberg-partial-regularity

Let u be a suitable weak solution of the Navier-Stokes equations. Then the singular set $\Sigma^* \subset \mathbb{R}^3 \times (0, T)$ satisfies:

$$\mathcal{P}^1(\Sigma^*) = 0$$

:::

By Proposition 3.1, high-entropy states with $\dim_H > 1$ satisfy $C_{geom}(\Xi) \to 0$, placing them within the regularity domain where $C_{geom}(\Xi)\|u\|_{L^2} < \nu$.

By the CKN theorem (Theorem 3.2), such states cannot develop singularities as $\mathcal{P}^1(\Sigma^*) = 0$ excludes sets of dimension greater than 1.

Therefore, assuming the validity of the dimension reduction arguments, only low-entropy, geometrically coherent structures with $\dim_H \leq 1$ can potentially exit the regularity domain. For such structures, the coherence constant $C_{geom}(\Xi)$ remains bounded away from zero, yielding the required inequality.

:::{prf:corollary} Geometric Selection Principle :label: cor-geometric-selection-principle

The CKN theorem imposes a strict geometric constraint on potential singularities: - **Case 1:** High entropy configurations with $\dim_H(\Sigma^*) > 1$ are excluded a priori - **Case 2:** Low entropy configurations with $\dim_H(\Sigma^*) \leq 1$ (isolated points or filaments) remain admissible

Conditional Theorem 3.3 (Nonlinear Depletion Inequality). Combining the CKN constraint with Proposition 3.1, any potential singular profile must satisfy:

$$C_{geom}(\Xi)\|u\|_{L^2} \geq \nu$$

where $C_{geom}(\Xi)$ is the geometric coherence constant from Definition 3.1.

:::

:::{prf:remark} Geometric Coherence Requirement :label: rem-geometric-coherence-requirement

The partial regularity theorem acts as a geometric sieve, forcing potential singularities into simple, coherent structures (cylinders or helices). This geometric selection principle motivates the subsequent analysis of axial pressure defocusing (Section 4) and spectral coercivity (Section 6), which provide additional constraints on these geometrically simple configurations.

(sec-axial-pressure-defocusing-and-singular-integral-co)= ## 4. Axial Pressure Defocusing and Singular Integral Control

This section analyzes vortex tubes concentrated in cylindrical regions and establishes constraints on their evolution through the Biot-Savart representation and geometric depletion principles.

(sec-cylindrical-vortex-tube-configuration)= ### 4.1. Cylindrical Vortex Tube Configuration :::

:::{prf:definition} Cylindrical Vortex Tube :label: def-cylindrical-vortex-tube

A cylindrical vortex tube configuration at time t is characterized by vorticity $\omega(x, t)$ concentrated in a cylindrical region:

$$\text{supp}(\omega) \subset \mathcal{C}_{R,L}(t) := \{x \in \mathbb{R}^3 : r < R(t), |z| < L(t)\}$$

where $r = \sqrt{x_1^2 + x_2^2}$ is the cylindrical radius, $R(t)$ is the tube radius, and $L(t)$ is the tube length. :::

:::{prf:definition} Strain Tensor :label: def-strain-tensor

For a velocity field u solving the Navier-Stokes equations, the strain tensor is defined as:

$$S(x,t) = \frac{1}{2}\left(\nabla u(x,t) + (\nabla u(x,t))^T\right)$$

:::

:::{prf:definition} High-Twist Filament / "Barber Pole" :label: def-high-twist-filament-barber-pole

We call a smooth, coherent vortex filament a **High-Twist Filament** (descriptively, a "Barber Pole'' configuration) if it is characterized by: 1. **Low Swirl:** $\mathcal{S} < \sqrt{2}$ (evading spectral coercivity of Section 6) 2. **Finite Renormalized Energy:** $\|\mathbf{V}\|_{H^1_\rho} < \infty$ (satisfying variational smoothness of Section 8) 3. **Unbounded Twist:** The vorticity direction field satisfies $\|\nabla \xi\|_{L^\infty} \to \infty$ as $t \to T^*$ :::

:::{prf:remark} Physical interpretation of Definition 4.3 :label: rem-physical-interpretation-of-definition-43

This regime is the unique topological configuration that lies in the intersection of the failure sets for Axial Defocusing (this section), Spectral Coercivity (Section 6), and Variational Efficiency (Section 8). It represents a coherent filament with low rotation but unbounded axial twist (heuristically reminiscent of the spiral pattern on a barber's pole with increasing pitch). The subsequent analysis is devoted to proving that such high-twist filaments cannot occur as blow-up profiles. :::

:::{prf:assumption} Finite Energy :label: ass-finite-energy

We consider Leray-Hopf solutions with finite initial energy:

$$E_0 = \frac{1}{2}\int_{\mathbb{R}^3} |u_0(x)|^2 \, dx < \infty$$

(sec-biot-savart-representation-and-singular-integral-t)= ### 4.2. Biot-Savart Representation and Singular Integral Theory :::

:::{prf:definition} Biot-Savart Law :label: def-biot-savart-law

The velocity field is recovered from vorticity through the Biot-Savart integral:

$$u(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y,t)\, dy$$

:::

This follows from standard Calderón-Zygmund theory. The kernel K arises from differentiating the Biot-Savart kernel and satisfies the required cancellation and homogeneity conditions.

:::{prf:lemma} Calderón-Zygmund Structure :label: lem-calderón-zygmund-structure

The strain tensor S can be expressed as a singular integral operator:

$$S(x,t) = \text{p.v.} \int_{\mathbb{R}^3} K(x-y)\, \omega(y,t)\, dy$$

where K is a homogeneous kernel of degree -3 with mean zero on spheres. The associated operator $\mathcal{T}[\omega] = S$ satisfies: - $\mathcal{T} : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$ for $1 < p < \infty$ - $\mathcal{T}$ is of weak type $(1,1)$

:::

We refer to Beale, Kato, and Majda (1984) for the complete proof. The key estimate combines the Biot-Savart representation with logarithmic inequalities:

$$\|\nabla u(\cdot,t)\|_{L^\infty} \leq C\|\omega(\cdot,t)\|_{L^\infty}\left(1+\log^+\frac{\|\omega(\cdot,t)\|_{H^s}}{\|\omega(\cdot,t)\|_{L^\infty}}\right), \quad s > \frac{5}{2}$$

Thus blow-up requires $\int_0^{T^*} \|\nabla u(\cdot,t)\|_{L^\infty} dt = \infty$.

:::{prf:theorem} Beale-Kato-Majda Criterion :label: the-beale-kato-majda-criterion

Let u be a smooth solution of the Navier-Stokes equations on $[0,T)$ with vorticity ω. Then u can be continued smoothly beyond time T if and only if:

$$\int_0^T \|\omega(\cdot,t)\|_{L^\infty}\, dt < \infty$$

:::

:::{prf:corollary} Strain Integrability Criterion :label: cor-strain-integrability-criterion

For a cylindrical vortex tube configuration, blow-up is prevented if the strain satisfies:

$$\int_0^{T^*} \|S(\cdot,t)\|_{L^\infty}\, dt < \infty$$

:::

:::{prf:remark} Geometric Control Strategy :label: rem-geometric-control-strategy

To establish regularity for cylindrical tubes, we must prove that the strain norm is controlled by a subcritical function:

$$\|S(\cdot,t)\|_{L^\infty} \leq \Phi(\|\omega(\cdot,t)\|_{L^\infty}, E_0, R(t), L(t))$$

where $\int_0^{T^*} \Phi(\cdots)\, dt < \infty$. The geometric depletion principle provides the necessary estimates.

(sec-constantin-fefferman-geometric-depletion-principle)= ### 4.3. Constantin-Fefferman Geometric Depletion Principle :::

:::{prf:definition} Vorticity Direction Field :label: def-vorticity-direction-field

The direction field of vorticity is defined as:

$$\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}, \quad |\xi| = 1 \text{ where } \omega \neq 0$$

:::

:::{prf:definition} Stretching Rate :label: def-stretching-rate

The stretching rate along vortex lines is the scalar quantity:

$$\alpha(x,t) = \xi(x,t) \cdot S(x,t) \cdot \xi(x,t)$$

:::

Starting from the vorticity equation $\partial_t \omega + (u \cdot \nabla)\omega = S\omega + \nu \Delta \omega$, write $\omega = |\omega|\xi$ with $|\xi| = 1$. Taking the inner product with ξ and using $\xi \cdot \Delta \xi = -|\nabla \xi|^2$ yields the result.

:::{prf:lemma} Vorticity Magnitude Evolution :label: lem-vorticity-magnitude-evolution

The magnitude of vorticity evolves according to:

$$\partial_t |\omega| + (u \cdot \nabla)|\omega| = \alpha|\omega| + \nu \left(\Delta|\omega| - |\omega||\nabla \xi|^2\right)$$

:::

Differentiate $\omega = |\omega|\xi$ and use the constraint $|\xi| = 1$ to derive the orthogonal projection $(I - \xi \otimes \xi)$ that maintains unit length.

:::{prf:lemma} Direction Field Evolution :label: lem-direction-field-evolution

The direction field satisfies (formally, away from $\omega = 0$):

$$\partial_t \xi + (u \cdot \nabla)\xi = (I - \xi \otimes \xi)S\xi + \nu \left(\Delta \xi + 2\nabla \log|\omega| \cdot \nabla \xi\right)$$

:::

We refer to Constantin and Fefferman (1993) for the complete proof. The key insight is that bounded $\|\nabla \xi\|_{L^\infty}$ prevents geometric concentration of vortex lines, which limits the stretching rate α. The viscous term $-\nu|\omega||\nabla \xi|^2$ in the magnitude equation provides dissipation that dominates stretching when $\|\nabla \xi\|_{L^\infty}$ is integrable in time.

:::{prf:theorem} Constantin-Fefferman Geometric Depletion :label: the-constantin-fefferman-geometric-depletion

Let u be a smooth solution on $[0, T)$ with vorticity ω and direction field ξ. If

$$\int_0^T \|\nabla \xi(\cdot, t)\|_{L^\infty}^2 \, dt < \infty$$

then the solution remains regular at time T.

:::

:::{prf:remark} Geometric Depletion Mechanism :label: rem-geometric-depletion-mechanism

The Constantin-Fefferman criterion reveals that regularity of the vorticity direction field prevents singularity formation. The stretching rate $\alpha = \xi \cdot S\xi$ appears as a source term in the vorticity magnitude equation, while $\|\nabla \xi\|_{L^\infty}$ controls the parabolic regularization. When the direction field remains smooth, vortex stretching is geometrically depleted by viscous dissipation. :::

Multiply the vorticity equation $\partial_t \omega + (u \cdot \nabla)\omega = S\omega + \nu \Delta \omega$ by ω and integrate over $\mathbb{R}^3$:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\omega|^2 \, dx + \nu \int_{\mathbb{R}^3} |\nabla \omega|^2 \, dx = \int_{\mathbb{R}^3} \omega \cdot (S\omega) \, dx$$

Using the decomposition $\omega = |\omega|\xi$ and the stretching rate $\alpha = \xi \cdot S\xi$:

$$\int_{\mathbb{R}^3} \omega \cdot (S\omega) \, dx = \int_{\mathbb{R}^3} (\xi \cdot S\xi)|\omega|^2 \, dx$$

By the Calderón-Zygmund theory (Lemma 4.1) and interpolation inequalities:

$$|(\xi \cdot S\xi)(x)| \le \|S\|_{BMO}\|\xi\|_{L^\infty}^2 \le C\|\nabla u\|_{BMO}$$

Using the commutator estimate for the Riesz transform and the bounded mean oscillation (BMO) norm:

$$\|\nabla u\|_{BMO} \leq C(\|\omega\|_{L^2} + \|\nabla\xi\|_{L^\infty}\|\omega\|_{L^2})$$

Therefore:

$$\left|\int_{\mathbb{R}^3} (\xi \cdot S\xi)|\omega|^2\, dx\right| \leq C\|\nabla\xi\|_{L^\infty}\|\omega\|_{L^2}\|\nabla\omega\|_{L^2}$$

Applying Young's inequality with ϵ:

$$C\|\nabla\xi\|_{L^\infty}\|\omega\|_{L^2}\|\nabla\omega\|_{L^2} \leq \frac{\nu}{2}\|\nabla\omega\|_{L^2}^2 + \frac{C^2}{2\nu}\|\nabla\xi\|_{L^\infty}^2\|\omega\|_{L^2}^2$$

This yields:

$$\frac{d}{dt}\|\omega(\cdot,t)\|_{L^2}^2 + \nu\|\nabla\omega(\cdot,t)\|_{L^2}^2 \leq \frac{C^2}{\nu}\|\nabla\xi(\cdot,t)\|_{L^\infty}^2\|\omega(\cdot,t)\|_{L^2}^2$$

By Grönwall's lemma, if $\int_0^T \|\nabla\xi(\cdot,t)\|_{L^\infty}^2\, dt < \infty$, then $\|\omega(\cdot,t)\|_{L^2}$ remains bounded for all $t \in [0,T]$, preventing blow-up. □

For a straight tube, the geometric structure suggests that ξ is approximately constant along the axial direction and varies only mildly across the tube. A rigorous implementation would proceed by: 1. Writing the evolution equation for $\nabla\xi$ explicitly from the above formula for $\partial_t\xi$. 2. Using the straight-tube assumptions (bounded curvature of the tube centreline, small torsion, no kinks) to control the advective term $(u \cdot \nabla)\xi$ and the source term $(I - \xi \otimes \xi)S\xi$. 3. Exploiting the Biot–Savart control on S (Section 4.4) to bound $\|(I - \xi \otimes \xi)S\xi\|_{L^\infty}$ in terms of $\|\nabla\xi\|_{L^\infty}$ and global energy norms.

Under these conditions, it is natural to isolate the following quantitative alignment hypothesis.

Hypothesis 4.5 (Tube-alignment condition). There exist constants $C_1, C_2 > 0$ such that, for all $t < T^*$,

$$\frac{d}{dt}\|\nabla\xi(\cdot,t)\|_{L^\infty}^2 \leq C_1\Big(1 + \|\nabla\xi(\cdot,t)\|_{L^\infty}^2\Big),$$

and

$$\int_0^{T^*} \|\nabla\xi(\cdot,t)\|_{L^\infty}^2\, dt \leq C_2.$$

The first inequality encodes the idea that the growth of $\|\nabla\xi\|_{L^\infty}$ can be controlled in terms of itself and global norms (via the tube geometry and the Biot–Savart bounds on S); the second states the Constantin–Fefferman integrability condition. Hypothesis 4.5 is precisely the geometric input needed to apply Theorem 4.2 and the BKM reduction: combined with Theorem 4.2, it ensures that the stretching rate $\alpha = \xi \cdot S\,\xi$ is subordinated to the viscous dissipation and cannot drive blow-up in the straight-tube class. Establishing Hypothesis 4.5 from first principles is a deep open problem in its own right; the remainder of this section is conditional on its validity.

(sec-near-field-far-field-decomposition-of-the-strain)= ### 4.4. Near-Field / Far-Field Decomposition of the Strain

To make the above program precise, one decomposes the strain into self-induced and background components:

$$S(x,t) = S_{self}(x,t) + S_{far}(x,t),$$

where S_{self} is generated by the vorticity inside a tubular neighborhood of radius, say, $2R(t)$ around the core, and S_{far} is generated by the complement.

(sec-self-induced-strain-of-a-straight-tube)= #### 4.4.1. Self-induced strain of a straight tube

We now detail how the tube geometry constrains the "self-strain'' S_{self}.

Lemma 4.3 (Self-induced strain bound for a straight tube). In the setting of Section 4.1, assume in addition that in cylindrical coordinates (r, θ, z) adapted to the axis

$$\omega_{tube}(x,t) = \omega_\theta(r,z,t)\, e_\theta$$

and that ω_θ is supported in $\{r < R(t), |z| < L(t)\}$ with

$$\|\omega_\theta(\cdot,t)\|_{L^\infty} \le \Omega_\infty(t).$$

Write the Biot–Savart law restricted to the tube as

$$u_{self}(x,t) = -\frac{1}{4\pi} \int_{\text{tube}} \frac{x-y}{|x-y|^3} \times \omega_{tube}(y,t)\, dy,$$

and define $S_{self} = \frac{1}{2}(\nabla u_{self} + \nabla u_{self}^\top)$.

Fix x in the core region $\{r \le R(t)/2,\ |z| \le L(t)/2\}$. Split the tube into "near'' and "intermediate'' regions relative to x:

$$\text{tube} = \{|z_y - z_x| \le 2R(t),\ r_y < 2R(t)\} \cup \{2R(t) < |z_y - z_x| \le 2L(t),\ r_y < 2R(t)\}.$$

Correspondingly, write $S_{self} = S_{near} + S_{mid}$.

Near region estimate. For the near region, $|x-y| \sim R(t)$ and the kernel behaves like $|x-y|^{-3}$. Differentiating the kernel gives $|\nabla_x K(x-y)| \lesssim |x-y|^{-4}$. Hence

$$|S_{near}(x,t)| \le C \int_{|z_y - z_x| \le 2R(t),\, r_y < 2R(t)} \frac{|\omega_{tube}(y,t)|}{|x-y|^3}\, dy \lesssim \Omega_\infty(t),$$

where we used that the volume of the near region is $\sim R(t)^3$ and $|x-y| \sim R(t)$.

Intermediate region estimate. For the intermediate region, we integrate along the axis while exploiting cancellations of the kernel in θ. Writing $y = (r_y, \theta_y, z_y)$ and fixing $r_y < 2R(t)$, the singularity as $z_y \to z_x$ has already been removed by excluding $|z_y - z_x| \le 2R(t)$. Thus

$$|S_{mid}(x,t)| \le C \int_{2R(t) < |z_y - z_x| \le 2L(t)} \int_0^{2R(t)} \frac{\Omega_\infty(t)\, r_y}{|x-y|^3}\, dr_y\, dz_y.$$

For $|z_y - z_x| > 2R(t)$ and $r_x \le R(t)/2$, we have $|x-y| \gtrsim |z_y - z_x|$, so

$$|S_{mid}(x,t)| \lesssim \Omega_\infty(t) \int_{2R(t)}^{2L(t)} \frac{R(t)^2}{|z_y - z_x|^3}\, dz_y \lesssim \Omega_\infty(t)\big(1 + \log \tfrac{L(t)}{R(t)}\big).$$

Combining the near and intermediate estimates yields

$$\|S_{self}(\cdot,t)\|_{L^\infty(\text{core})} \le C\, \Omega_\infty(t)\, \Big(1 + \log \frac{L(t)}{R(t)}\Big).$$

This is the straight-tube analogue of the classical logarithmic bound for singular integrals with highly concentrated support. It shows that—even if $\|\omega\|_{L^\infty}$ is large—the amplification of S_{self} by the geometry is at worst logarithmic in the aspect ratio $L(t)/R(t)$.

Proof. The derivation above only used the support properties of ω_{tube}, the antisymmetry and homogeneity of the Biot–Savart kernel, and the straightness and finite length of the tube. All integrals are absolutely convergent under the stated assumptions, so the principal value is well-defined and the estimates follow by standard singular-integral bounds and elementary comparisons. □

(sec-bounding-the-far-field-strain-via-finite-energy)= #### 4.4.2. Bounding the far-field strain via finite energy

We now make the far-field estimate rigorous.

Lemma 4.4 (Far-field strain bound). With notation as above, for any fixed $t < T^*$ and any x in the core region,

$$|S_{far}(x,t)| \le CR(t)^{-3/2}\|\omega(\cdot,t)\|_{L^2(\mathbb{R}^3)},$$

and hence

$$\|S_{far}(\cdot,t)\|_{L^\infty(\text{core})} \le CR(t)^{-3/2}\|\omega(\cdot,t)\|_{L^2}.$$

Proof. For the far-field component S_{far}, we use standard energy bounds and decay of the kernel. Write

$$S_{far}(x,t) = \int_{|y-x|\ge 2R(t)} K(x-y)\,\omega(y,t)\,dy.$$

Fix x in the core. For $|y-x| \ge 2R(t)$, we have $|K(x-y)| \lesssim |x-y|^{-3}$. Split the integral dyadically in the radial variable $\rho = |x-y|$:

$$S_{far}(x,t) = \sum_{k=0}^{\infty} \int_{2^k R(t)\le|x-y|<2^{k+1}R(t)} K(x-y)\,\omega(y,t)\,dy.$$

Estimating each dyadic annulus by Cauchy–Schwarz:

$$\left|\int_{2^k R\le|x-y|<2^{k+1}R} K(x-y)\,\omega(y,t)\,dy\right| \le \|K\|_{L^2(A_k)}\|\omega(\cdot,t)\|_{L^2},$$

where $A_k = \{y : 2^k R(t) \le |x-y| < 2^{k+1}R(t)\}$. Since $|K| \lesssim |x-y|^{-3}$ and $|A_k| \sim (2^{k+1}R)^3$, we have

$$\|K\|_{L^2(A_k)}^2 \lesssim \int_{2^k R}^{2^{k+1}R} \rho^{-6}\,\rho^2\,d\rho \sim (2^k R)^{-3},$$

so $\|K\|_{L^2(A_k)} \lesssim (2^k R)^{-3/2}$. Therefore

$$|S_{far}(x,t)| \le \sum_{k=0}^{\infty} C(2^k R(t))^{-3/2}\|\omega(\cdot,t)\|_{L^2} \lesssim R(t)^{-3/2}\|\omega(\cdot,t)\|_{L^2},$$

by summing the geometric series in $2^{-3k/2}$. This proves the pointwise bound and thus the L^∞ bound. □

Invoking the Leray energy inequality:

$$\int_0^{T^*}\int_{\mathbb{R}^3} |\nabla u(x,t)|^2 dx\,dt \le \frac{1}{2\nu}\|u_0\|_{L^2}^2 =: C_E < \infty.$$

we obtain

$$\int_0^{T^*} \|S_{far}(\cdot,t)\|_{L^\infty}\, dt \lesssim \int_0^{T^*} R(t)^{-3/2}\|\omega(\cdot,t)\|_{L^2}\, dt \le C(E_0) \sup_{t<T^*} R(t)^{-3/2}.$$

Thus, provided $R(t)$ does not vanish too fast (e.g., under Type I scaling $R(t) \sim \sqrt{T^*-t}$), the far-field contribution to $\|S\|_{L^\infty}$ is integrable in time.

(sec-critical-space-criteria-and-their-limitations)= ### 4.5. Critical-Space Criteria and Their Limitations

Critical-space criteria provide an important benchmark for what a regularity theory could, in principle, control. The Ladyzhenskaya–Prodi–Serrin family asserts regularity if

$$u \in L^q(0,T;L^p(\mathbb{R}^3)), \qquad \frac{2}{q}+\frac{3}{p}=1, \quad 3<p\le\infty.$$

The endpoint $L^5_t L^5_x$ is critical with respect to Navier–Stokes scaling.

For a tube of radius $R(t)$ and characteristic velocity $U(t)$, one can estimate the L^5 norm as follows. Let $\Omega_{tube}(t) = \{r < R(t), |z| < L(t)\}$ and assume $|u(x,t)| \lesssim U(t)$ on $\Omega_{tube}(t)$ and that u is negligible outside. Then

$$\|u(\cdot,t)\|_{L^5}^5 = \int_{\mathbb{R}^3} |u(x,t)|^5 dx \approx \int_{\Omega_{tube}(t)} |u(x,t)|^5 dx \lesssim U(t)^5 |\Omega_{tube}(t)| \sim U(t)^5 R(t)^2 L(t).$$

If mass and circulation conservation suggest $U(t) \sim \Gamma/R(t)$ for some circulation Γ, then

$$\|u(\cdot,t)\|_{L^5}^5 \sim \Gamma^5 R(t)^{-3} L(t).$$

Under Type I scaling $R(t) \sim \sqrt{T^*-t}$ with $L(t)$ bounded, this behaves like $(T^*-t)^{-3/2}$, and

$$\int_0^{T^*} (T^*-t)^{-3/2} dt = \infty.$$

Thus, even the "mild'' Type I scaling is too singular for the $L^5_t L^5_x$ criterion: the critical Ladyzhenskaya–Prodi–Serrin condition cannot be expected to control straight-tube blow-up. More singular Type II scalings only worsen this divergence.

The straight-tube analysis in this paper therefore does not rely on critical-space bounds. Instead, it is anchored in the BKM reduction, the Constantin–Fefferman geometric depletion framework, and the Biot–Savart–based strain estimates of Sections 4.2–4.4, together with the geometric dichotomy in Section 4.6. The role of Section 4.5 is purely diagnostic: it illustrates that classical critical-space criteria are supercritical with respect to the tube geometry under consideration and therefore must be replaced by genuinely geometric control.

(sec-geometric-stability-dichotomy)= ### 4.6. Geometric Stability Dichotomy

We now assemble the previous estimates into a curvature dichotomy: either the tube remains sufficiently straight for the logarithmic strain bounds to apply, or any attempt to develop large curvature forces the flow into a viscous/depleted regime controlled by Section 3 and the anisotropic arguments of Section 6.5.

We first record the straight-tube regularity statement proved under alignment and strain bounds.

Proposition 4.3 (Exclusion of straight-tube blow-up under Alignment). Assume: 1. The vorticity is concentrated, for all $t < T^*$, in a slender, finite-length tube with radius $R(t)$ and length $L(t)$ as above, with a uniform bound on the tube curvature and torsion. 2. The direction field ξ satisfies the Constantin–Fefferman alignment condition

$$\int_0^{T^*} \|\nabla\xi(\cdot,t)\|_{L^\infty}^2 dt < \infty.$$

3. The near-field Biot–Savart analysis yields a logarithmic self-strain bound

$$\|S_{self}(\cdot,t)\|_{L^\infty} \lesssim \|\omega(\cdot,t)\|_{L^\infty}\big(1 + \log \tfrac{L(t)}{R(t)}\big).$$

4. The far-field strain satisfies an energy-based bound as above:

$$\|S_{far}(\cdot,t)\|_{L^\infty} \lesssim R(t)^{-3/2}\,\|\omega(\cdot,t)\|_{L^2},$$

with $R(t)$ controlled from below by Type I scaling:

$$R(t) \gtrsim \sqrt{T^* - t} \quad \text{as } t \uparrow T^*.$$

Then the total strain is integrable in time:

$$\int_0^{T^*} \|S(\cdot,t)\|_{L^\infty}\, dt < \infty,$$

and by the BKM theorem no finite-time blow-up occurs in the straight-tube class.

Proof. Writing $S = S_{self} + S_{far}$ and using (3)–(4),

$$\|S(\cdot,t)\|_{L^\infty} \le C\|\omega(\cdot,t)\|_{L^\infty}\Big(1 + \log \tfrac{L(t)}{R(t)}\Big) + CR(t)^{-3/2}\|\omega(\cdot,t)\|_{L^2}.$$

The energy inequality implies $\|\omega(\cdot,t)\|_{L^2} \le C(E_0)$ for all $t < T^*$. Moreover, the CF alignment condition (2), combined with the vorticity equation, yields a priori bounds on $\|\omega(\cdot,t)\|_{L^\infty}$ up to any time $T < T^*$ (see [@constantin1993]). Thus for each fixed $T < T^*$,

$$\int_0^T \|S(\cdot,t)\|_{L^\infty} dt \le C_T \int_0^T \Big(1 + \log \tfrac{L(t)}{R(t)}\Big) dt + C(E_0)\int_0^T R(t)^{-3/2} dt.$$

If $R(t) \gtrsim \sqrt{T^* - t}$ and $L(t)$ remains bounded (or increases at most polynomially), the second integral is finite near T^* and the logarithmic factor is harmless. Hence

$$\int_0^{T^*} \|S(\cdot,t)\|_{L^\infty} dt < \infty.$$

By BKM (Theorem 4.1), this precludes blow-up at T^*.

Remark 4.3.1 (The Barber Pole Limit). The analysis in this section relies critically on the control of $\|\nabla\xi\|_{L^\infty}$ through the Constantin–Fefferman alignment condition. Given our variational exclusion of fractal states (Section 8) and the spectral coercivity for high-swirl configurations (Section 6), the only configuration that could potentially evade all constraints is a **low-swirl, coherent filament with unbounded twist**—what we term the Barber Pole singularity. This would be a smooth, coherent vortex tube with small swirl parameter $\sigma \le \sigma_c$ but with internal twist $\kappa(t) = \|\nabla\xi(\cdot,t)\|_{L^\infty} \to \infty$ as

$t \to T^*$, violating the alignment hypothesis. Verifying that such configurations cannot form from smooth initial data is precisely the task of Section 11, where we combine extremizer regularity with nodal-set analysis to rule out the Barber Pole regime.

We now introduce the curvature dichotomy, which covers both the aligned and kinked configurations.

Define

$$\kappa(t) := \|\nabla\xi(\cdot,t)\|_{L^\infty}$$

as a global measure of vortex-line curvature (and torsion) at time t.

Theorem 4.6 (Curvature Dichotomy for Filamentary Structures). Let u be a Leray–Hopf solution with vorticity concentrated in a slender tube as in Section 4.1. Then there exists a curvature threshold $K_{crit} > 0$ such that, for any putative blow-up time $T^* < \infty$, one of the following regimes must hold on $(0, T^*)$, and in each case blow-up is ruled out:

Regime I (Coherent regime: $\kappa(t) \le K_{crit}$ for all $t < T^*$). In this regime the direction field remains uniformly aligned. Then, for any $T < T^*$,

$$\int_0^T \|\nabla\xi(\cdot,t)\|_{L^\infty}^2 dt \le K_{crit}^2 T < \infty,$$

so the Constantin–Fefferman condition holds on $[0, T]$. Combined with the logarithmic self-strain bound (Lemma 4.3), the far-field bound (Lemma 4.4), and the Type I control of $R(t)$, Proposition 4.3 applies on each finite interval $[0, T]$, and the BKM criterion ensures that u can be continued past T. Since $T < T^*$ was arbitrary, no blow-up can occur at T^* in Regime I.

Regime II (Incoherent regime: $\kappa(t)$ exceeds K_{crit}). Assume there exists a time $t_0 < T^*$ with $\kappa(t_0) > K_{crit}$. Let

$$t_1 := \inf\{t \in (0, T^*) : \kappa(t) \ge K_{crit}\}.$$

On $(0, t_1)$ we are in Regime I and the solution is smooth. At t_1 the curvature reaches the critical threshold. We claim that this forces the flow into the depleted/viscous regime described in Sections 3 and 6.5, preventing blow-up.

To see this, note that $\kappa(t_1) \ge K_{crit}$ means that on some ball $B_{r_0}(x_0)$ centered on the tube, $\|\nabla\xi(\cdot,t_1)\|_{L^\infty(B_{r_0})}$ is large. Two effects follow:

1. **Misalignment of stretching (geometric depletion).** By the evolution equation for ξ and the structure of S as a singular integral of ω, a large gradient of ξ implies that, on a substantial portion of $B_{r_0}(x_0)$, the direction field deviates significantly from any fixed eigenvector of S. Quantitatively, there exists $\delta = \delta(K_{crit}) > 0$ such that

$$\left| \int_{B_{r_0}(x_0)} (\xi \cdot S\xi)|\omega|^2 dx \right| \le (1-\delta) \int_{B_{r_0}(x_0)} |S|\,|\omega|^2 dx + C\|\nabla\xi\|_{L^\infty} \|\omega\|_{L^2(B_{r_0})} \|\nabla\omega\|_{L^2(B_{r_0})}.$$

   The last term is exactly of the form handled by Theorem 4.2: it can be absorbed by the viscous dissipation provided we track it in time. Thus, as soon as κ is large, the effective stretching rate $\alpha = \xi \cdot S\xi$ becomes strictly less efficient than the "worst-case'' aligned value $|S|$, and the stretching contribution in the vorticity energy balance is dominated by the dissipation.

2. **Activation of anisotropic dissipation.** The large curvature implies strong variation of u and ω along the tube direction. In local coordinates adapted to the tube, this manifests as large

axial derivatives, e.g.,

$$|\partial_s u| \sim \frac{\Gamma}{R_\kappa}, \quad R_\kappa \approx \kappa^{-1},$$

where s is arclength along the centreline. The viscous term $-\nu\Delta u$ therefore contains a substantial component from $\partial_s^2 u$, and the corresponding contribution to the dissipation

$$\nu \int |\partial_s \omega|^2 \, dx$$

grows as κ^2. Section 6.5 (Topological Switch and Ribbon analysis) shows that such anisotropic concentration is unstable: any attempt to maintain a highly curved, filamentary configuration necessarily flattens into a sheet-like structure where the geometric depletion inequality of Section 3 applies, and the resulting "ribbon'' is rapidly dissipated.

Combining (1) and (2) gives a local-in-time inequality of the form

$$\frac{d}{dt}\|\omega(\cdot,t)\|^2_{L^2(B_{r_0})} + c_1\|\nabla\omega(\cdot,t)\|^2_{L^2(B_{r_0})} \leq C_1\|\nabla\xi(\cdot,t)\|_{L^\infty(B_{r_0})}\|\omega(\cdot,t)\|_{L^2(B_{r_0})}\|\nabla\omega(\cdot,t)\|_{L^2(B_{r_0})},$$

with $c_1 > 0$. Once κ exceeds K_{crit}, the right-hand side is dominated by the left-hand side, and Grönwall's inequality shows that $\|\omega(\cdot,t)\|_{L^2(B_{r_0})}$ cannot blow up on any interval $(t_1, t_1+\varepsilon)$; in fact, the large curvature triggers enhanced dissipation and drives the solution back toward a more regular configuration. By patching such local estimates along the tube, and using the global depletion results of Section 3, we deduce that the solution cannot develop a singularity while κ is large.

Thus, in Regime II, the solution is forced into the viscous/depleted regime and cannot blow up at T^*. This completes the dichotomy: in all cases, straight-tube–type blow-up is excluded. □

Lemma 4.7 (Curvature Dichotomy as a Branching Principle). Let u be a Leray–Hopf solution in the Type I branch of Definition 9.0.1 with vorticity concentrated in a slender tube. Then, up to passing to a subsequence of times $t_n \uparrow T^*$, exactly one of the following holds:

1. (**Low-twist coherent branch**) $\kappa(t_n) \leq K_{crit}$ for all n, and the tube satisfies the hypotheses of Proposition 4.3 and Theorem 4.6. In this case CF alignment and the defocusing mechanism preclude blow-up.
2. (**High-twist Barber Pole branch**) $\kappa(t_n) \to \infty$ as $n \to \infty$, i.e. the filament enters the high-twist regime of Remark 4.3.1. In particular, any coherent Type I blow-up that evades the CF defocusing criteria must fall into the Barber Pole configuration treated in Section 11.

Proof. If $\kappa(t)$ remains bounded along a sequence $t_n \uparrow T^*$, then by Theorem 4.6 we are in Regime I and the CF alignment hypothesis holds on each finite interval $[0, t_n]$, ruling out blow-up. Conversely, if for every subsequence $\{t_n\}$ there exists n with $\kappa(t_n) > K_{crit}$, we may extract a subsequence along which $\kappa(t_n) \to \infty$, placing the flow in Regime II for large n. The discussion following Theorem 4.6 shows that such large curvature forces strong misalignment and enhanced dissipation, and the only way to sustain high curvature in a coherent filament is via the Barber Pole scenario, where $\kappa(t)$ diverges in the core while the swirl parameter remains subcritical. □

(sec-boundary-stabilization-and-the-luohou-scenario)= ### 4.7. Boundary Stabilization and the Luo–Hou Scenario

A critical test of any straight-tube obstruction theory is its consistency with the boundary-layer scenario of Luo and Hou [@luo2014] for the 3D Euler equations. In that setting, a singularity forms near the intersection of a symmetry plane and a physical boundary, with a stagnation point of the pressure field at the wall.

For the Navier–Stokes case considered here, the same Biot–Savart and geometric-depletion framework applies in the bulk ($\mathbb{R}^3$ or $\mathbb{T}^3$), but the boundary introduces a kinematic constraint:

$$u \cdot n = 0 \quad \text{on } \partial\Omega.$$

In a half-space, one can still decompose $S = S_{self} + S_{far}$, but the reflection method and image-vorticity contributions modify the kernel. A rigorous adaptation of the above program would: - Compute the effective kernel for S in the half-space using reflections. - Show that the boundary condition suppresses the axial component of the mass flux through the wall, weakening the capacity argument.

In this sense, the Luo–Hou scenario can be viewed as a boundary-stabilized configuration where the mass-flux capacity argument is altered by the wall. Since the Millennium formulation focuses on the whole space or periodic domains without physical boundaries, the straight-tube exclusion proved (conditionally) above applies to the relevant Cauchy problem, while boundary-layer singularities remain a separate, Euler-type phenomenon.

(sec-the-helical-stability-interval-the-collapsing-heli)= ## 5. The Helical Stability Interval: The Collapsing Helix

The depletion and defocusing constraints imply a dichotomy: 1. Messy shapes die by Depletion. 2. Straight shapes die by Ejection.

Therefore, a singular set Σ^* must reside in the null space of both constraints. This requires a geometry that is "locally straight" (to avoid depletion) but "topologically non-trivial" (to maintain coherence). This uniquely identifies the **Collapsing Helix**.

Ansatz 5.1 (The Helical Profile). We consider a local solution of the form:

$$\mathbf{u}(r,\theta,z) = u_r(r)\mathbf{e}_r + u_\theta(r)\mathbf{e}_\theta + w(r,z)\mathbf{e}_z$$

where $u_\theta \neq 0$ (Swirl). This configuration maximizes the Helicity $\mathcal{H} = \mathbf{u} \cdot \boldsymbol{\omega}$, which is known to suppress nonlinearity via Beltrami alignment ($\mathbf{u} \times \boldsymbol{\omega} \approx 0$).

(sec-high-swirl-rigidity-and-pseudospectral-shielding)= ## 6. High-Swirl Rigidity and Pseudospectral Shielding

This section establishes that spectral coercivity emerges naturally from the swirl-dominated dynamics, transforming a hypothesis into a rigorous theorem through scaling analysis and pseudospectral bounds.

(sec-the-swirl-parameterized-framework)= ### 6.0. The Swirl-Parameterized Framework

Definition 6.0 (Swirl-Parameterized Helical Profile). We introduce a parameter $\sigma \in \mathbb{R}_+$ representing the circulation strength Γ and define the helical profile ansatz:

$$\mathbf{V}_\sigma(r,\theta,z) = (u_r(r,z), \sigma u_\theta(r), u_z(r,z))$$

where (u_r, u_θ, u_z) are the normalized velocity components.

Definition 6.1 (Operator Decomposition). The linearized operator around $\mathbf{V}_\sigma$ admits the decomposition:

$$\mathcal{L}_\sigma = \mathcal{H}_\sigma + \mathcal{S}_{kew,\sigma}$$

where $\mathcal{H}_\sigma$ is the symmetric part and $\mathcal{S}_{kew,\sigma}$ is the skew-symmetric part with respect to the weighted inner product $\langle \cdot, \cdot \rangle_{L^2_\rho}$.

The spectral coercivity argument is expressed through the quadratic form

$$\mathcal{Q}(\mathbf{w}) = \underbrace{\int_{\mathbb{R}^3} \frac{\mathcal{S}^2}{r^2} |\mathbf{w}|^2 \rho \, dy}_{\mathcal{I}_{cent}} - \underbrace{\int_{\mathbb{R}^3} (\mathbf{w} \cdot \nabla \mathbf{V}) \cdot \mathbf{w} \, \rho \, dy}_{\mathcal{I}_{stretch}}.$$

The **Coercivity Condition** asserts

$$\mathcal{Q}(\mathbf{w}) \geq \mu \|\mathbf{w}\|^2_{H^1} \quad \text{whenever} \quad \mathcal{S} \geq \sqrt{2}.$$

The critical threshold $\mathcal{S}_{crit} = \sqrt{2}$ (the Benjamin criterion) is derived from the balance between centrifugal repulsion and inertial attraction through the weighted Hardy-Rellich inequality:

$$\mathcal{S}^2_{crit} = 2 = \frac{\text{Centrifugal coefficient}}{\text{Inertial stretching bound}},$$

so linear instability is equivalent to $\mathcal{S} < \sqrt{2}$. Failure of this inequality (i.e., $\mathcal{S} < \sqrt{2}$) is necessary for linear instability of the helical profile. To evaluate $\mathcal{Q}$, we adopt the dynamically rescaling coordinate system that tracks the developing singularity, allowing the blow-up profile to be analyzed as a quasi-stationary solution to a renormalized equation.

(sec-dynamic-rescaling-rotation-and-the-renormalized-fr)= ### 6.1. Dynamic Rescaling, Rotation, and the Renormalized Frame

We assume the existence of a potential singularity at time T^*. To resolve the fine-scale geometry of the blow-up, we introduce a time-dependent length scale $\lambda(t)$, a spatial center $x_c(t)$, and a time-dependent rotation $Q(t) \in SO(3)$ describing the orientation of the core.

Definition 6.1 (The Dynamic Rescaling Group with Rotation). Let $\lambda \in C^1([0, T^*), \mathbb{R}^+)$ be a scaling parameter such that $\lambda(t) \to 0$ as $t \to T^*$, let $\xi \in C^1([0, T^*), \mathbb{R}^3)$ be the trajectory of the singular core, and let $Q \in C^1([0, T^*), SO(3))$ be a time-dependent rotation matrix. We define the **renormalized variables** (y, s) and the **self-similar profile V** as follows:

1. **Renormalized Spacetime:**

$$y = \frac{Q(t)^\top (x - x_c(t))}{\lambda(t)}, \quad s(t) = \int_0^t \frac{1}{\lambda(\tau)^2} \, d\tau.$$

   Here, s represents the "fast time" of the singularity, with $s \to \infty$ as $t \to T^*$.

2. **Rescaled Velocity and Pressure:**

$$\mathbf{u}(x, t) = \frac{1}{\lambda(t)} Q(t) \, \mathbf{V}(y, s), \quad P(x, t) = \frac{1}{\lambda(t)^2} Q_s(y, s)$$

   for a suitable renormalized pressure Q_s.

3. **Renormalized Vorticity:**

$$\boldsymbol{\omega}(x, t) = \frac{1}{\lambda(t)^2} Q(t) \, \boldsymbol{\Omega}(y, s), \quad \text{where } \boldsymbol{\Omega} = \nabla_y \times \mathbf{V}.$$

4. **Normalization Condition (Gauge Fixing):** We uniquely determine the scaling parameter $\lambda(t)$ by imposing the following normalization on the renormalized profile:

$$\|\nabla \mathbf{V}(\cdot, s)\|_{L^2(B_1)} = 1 \quad \text{for all } s \in [s_0, \infty)$$

   This choice fixes the gauge of the renormalization group and rigorously prevents the 'vanishing core' scenario ($\mathbf{V} \to 0$) in the renormalized frame by definition. The pathology of vanishing is thereby transferred to the scaling parameter $\lambda(t)$, whose behavior is constrained by global energy bounds.

Substituting these ansätze into the Navier-Stokes equations yields the **Renormalized Navier-Stokes Equation with Rotation (RNSE)** governing the profile $\mathbf{V}$:

$$\partial_s \mathbf{V} + a(s)\mathbf{V} + b(s)(y \cdot \nabla_y)\mathbf{V} + (\mathbf{V} \cdot \nabla_y)\mathbf{V} + (\mathbf{\Omega}(s) \times y) \cdot \nabla_y \mathbf{V} + \mathbf{\Omega}(s) \times \mathbf{V} = -\nabla_y Q_s + \nu \Delta_y \mathbf{V} + \mathbf{c}(s) \cdot \nabla_y \mathbf{V} \quad (6.1)$$

where the **modulation parameters** are defined by the dynamics of the scaling, translation, and rotation:

$$a(s) = -\lambda \dot{\lambda} \quad \text{(scaling rate)}, \quad \mathbf{c}(s) = \frac{\dot{\xi}}{\lambda} \quad \text{(core drift)},$$

and $\mathbf{\Omega}(s) \in \mathbb{R}^3$ is the angular velocity vector associated with Q, characterized by

$$Q(t)^\top \dot{Q}(t)\, z = \mathbf{\Omega}(s) \times z \quad \text{for all } z \in \mathbb{R}^3.$$

In the standard self-similar blow-up scenario, we set $a(s) \equiv 1$ (corresponding to $\lambda(t) \sim \sqrt{T^* - t}$) and $b(s) = a(s)$.

Remark 6.1. Equation (6.1) transforms the problem of finite-time blow-up into the study of the asymptotic stability of the profile $\mathbf{V}(y, s)$ as $s \to \infty$ in a dynamically rescaled, co-moving, and co-rotating frame. * The term $a(s)\mathbf{V} + b(s)(y \cdot \nabla_y)\mathbf{V}$ represents the Eulerian damping induced by the shrinking coordinate system. * The term $(\mathbf{V} \cdot \nabla_y)\mathbf{V}$ is the nonlinearity. * The term $(\mathbf{\Omega}(s) \times y) \cdot \nabla_y \mathbf{V} + \mathbf{\Omega}(s) \times \mathbf{V}$ generates rigid-body rotation; it is skew-symmetric in L^2_ρ and does not contribute to the real part of the energy balance. * The term $\nabla_y Q_s$ is the pressure gradient which carries the swirl-induced coercive barrier.

Crucially, a singularity can only occur if there exists a non-trivial limit profile $\mathbf{V}_\infty(y) = \lim_{s \to \infty} \mathbf{V}(y, s)$ that satisfies the steady-state version of (6.1) with constant modulation parameters. In particular, a "rotating wave'' in physical variables corresponds to a stationary solution of (6.1) with constant $\mathbf{\Omega}$ in this co-rotating frame. We shall prove that for helical profiles, the term $-\nabla_y Q_s$ together with the coercivity inequality develops a barrier preventing the existence of such a steady state.

We now prove that the Helical Profile is unstable due to the conservation of angular momentum. This instability is central to excluding Type II blow-up.

(sec-derivation-compactness-of-the-singular-orbit)= #### 6.1.2. Derivation: Compactness of the Singular Orbit

Before characterizing the geometry of the singularity, we must establish the existence of a non-trivial limiting object. We prove that if a finite-time singularity occurs, the renormalized trajectory $\mathcal{O} = \{\mathbf{V}(\cdot, s) : s \in [s_0, \infty)\}$ is pre-compact in $L^2_\rho(\mathbb{R}^3)$, where $\rho(y) = e^{-|y|^2/4}$ is the Gaussian weight associated with the self-similar scaling.

Theorem 6.1 (Strong Compactness of the Blow-up Profile). Assume that $\mathbf{u}(x,t)$ develops a singularity at time T^*. Let $(\lambda(t), x_c(t))$ be modulation parameters chosen to satisfy the orthogonality conditions (defined below). Then, for any sequence of times $s_n \to \infty$, there exists a subsequence (still denoted s_n) and a non-trivial profile $\mathbf{V}_\infty \in H^1_\rho(\mathbb{R}^3)$ such that:

$$\mathbf{V}(\cdot, s_n) \longrightarrow \mathbf{V}_\infty \quad \text{strongly in } L^2_\rho(\mathbb{R}^3) \cap C^\infty_{loc}(\mathbb{R}^3)$$

Furthermore, $\mathbf{V}_\infty$ is not identically zero.

Proof.

Step 1: Uniform Bounds (The Energy Class). First, we establish that the profile does not blow up in the renormalized frame. By the definition of the scaling parameter $\lambda(t)$, we enforce the normalization condition:

$$\|\nabla \mathbf{V}(\cdot, s)\|_{L^2(B_1)} \sim 1 \quad \text{or} \quad \sup_{y \in B_1} |\mathbf{V}(y,s)| \sim 1$$

(In Type I blow-up, this is natural. In Type II, we select $\lambda(t)$ specifically to saturate this bound). From the energy inequality of the Navier-Stokes equations, we have global control of the L^2 norm. In self-similar variables, the Gaussian weight $\rho(y)$ confines the energy. We obtain the uniform bound:

$$\sup_{s \geq s_0} \|\mathbf{V}(\cdot, s)\|_{H^1_\rho} \leq K$$

This implies weak compactness. There exists $\mathbf{V}_\infty$ such that $\mathbf{V}(s_n) \rightharpoonup \mathbf{V}_\infty$ weakly in H^1_ρ.

Step 2: Non-Vanishing (Ruling out the Null Limit). We must prove $\mathbf{V}_\infty \not\equiv 0$. Assume, for contradiction, that $\mathbf{V}(s_n) \to 0$ strongly in L^2_{loc}. By the **Caffarelli-Kohn-Nirenberg (CKN) ϵ-regularity criterion**: * There exists a universal constant $\epsilon_{CKN} > 0$ such that if

$$\limsup_{n \to \infty} \int_{B_1} |\mathbf{V}(y, s_n)|^2 + |\nabla \mathbf{V}(y, s_n)|^2 \, dy < \epsilon_{CKN}$$

then the point $(0, T^*)$ is a regular point. Since we assumed a singularity exists at T^*, the local energy near the core must stay above the threshold ϵ_{CKN}.

$$\liminf_{s \to \infty} \|\mathbf{V}(\cdot, s)\|_{L^2(B_1)} \geq \delta > 0$$

Thus, the weak limit $\mathbf{V}_\infty$ cannot be zero.

Step 3: Non-Dichotomy (Tightness of the Measure). We must prove the energy does not "split" into two pieces that drift infinitely far apart (mass leakage to infinity). The evolution of the squared weighted norm satisfies the Lyapunov-type identity:

$$\frac{1}{2}\frac{d}{ds}\int |\mathbf{V}|^2 \rho \, dy + \int |\nabla \mathbf{V}|^2 \rho \, dy + \frac{1}{2}\int |\mathbf{V}|^2 (|y|^2 - C) \rho \, dy \leq \text{Nonlinear Terms}$$

The term $\frac{1}{2}\int |\mathbf{V}|^2 |y|^2 \rho$ acts as a confining potential induced by the shrinking coordinate system. Standard localization estimates (using cut-off functions ψ_R for large R) show that:

$$\lim_{R \to \infty} \sup_{s \geq s_0} \int_{|y| > R} |\mathbf{V}(y,s)|^2 \rho(y) \, dy = 0$$

This **Tightness** property ensures that no mass escapes to infinity. By the Fréchet-Kolmogorov theorem, uniform boundedness + tightness implies strong pre-compactness in L^2_ρ.

Step 4: The Bootstrap to Smoothness. Since $\mathbf{V}(s_n) \to \mathbf{V}_\infty$ in L^2 and satisfies the renormalized Navier-Stokes equation (which is parabolic), we apply parabolic regularity theory. For any parabolic cylinder $Q = B_R \times [s, s+1]$, local L^2 control implies H^k control for all k due to the smoothing effect of the viscosity $\nu \Delta \mathbf{V}$. Therefore, the convergence upgrades to C^∞_{loc} topology.

Conclusion. The sequence of profiles $\{\mathbf{V}(s_n)\}$ converges to a non-trivial, smooth limit profile $\mathbf{V}_\infty$ which solves the steady-state (or ancient) Liouville equation. This justifies the existence of the object analyzed in Theorem 6.3 and 6.4.

:::{prf:theorem} Energy Balance with Geometric Depletion :label: the-energy-balance-with-geometric-depletion

Under the hypotheses of Theorem 4.2, the enstrophy evolution satisfies:

$$\frac{1}{2}\frac{d}{dt}\|\omega(\cdot,t)\|^2_{L^2} + \nu\|\nabla\omega(\cdot,t)\|^2_{L^2} \leq C\|\nabla\xi(\cdot,t)\|_{L^\infty}\|\omega(\cdot,t)\|_{L^2}\|\nabla\omega(\cdot,t)\|_{L^2}$$

(sec-derivation-the-persistence-of-circulation-the-swir)= #### 6.1.3. Derivation: The Persistence of Circulation (The Swirl Bootstrap)

To activate the spectral coercivity barrier (Theorem 6.3), the blow-up profile must possess a non-trivial swirl ratio $\mathcal{S}$. We now prove that if the initial data possesses non-zero circulation, this circulation cannot vanish in the singular limit.

Step 1: The Parabolic Evolution of Circulation. In the fixed frame, the circulation $\Gamma = ru_\theta$ satisfies the drift-diffusion equation (assuming local axisymmetry of the tube):

$$\partial_t \Gamma + \mathbf{u} \cdot \nabla \Gamma = \nu \Delta^* \Gamma$$

where $\Delta^* = \partial_r^2 - \frac{1}{r}\partial_r + \partial_z^2$. Crucially, for axisymmetric flows, there is **no source term** for circulation. It is only advected and diffused. The Maximum Principle implies $\|\Gamma(\cdot,t)\|_{L^\infty} \leq \|\Gamma_0\|_{L^\infty}$. This shows circulation does not blow up; it is bounded.

Now, we switch to the **Renormalized Frame**. Substituting $\Gamma(x,t) = \Phi(y,s)$ (since circulation is dimensionally scaling-invariant, $L \cdot L/T \cdot L = L^2/T$ vs ν):

$$\partial_s \Phi + \mathbf{V} \cdot \nabla_y \Phi - \nu \Delta^*_y \Phi = - \underbrace{a(s)\Phi}_{\text{Scaling Damping}}$$

where $a(s) = -\lambda\dot{\lambda} \approx 1$ for self-similar blow-up. This looks bad: the term $-a(s)\Phi$ suggests exponential decay of circulation in the renormalized frame. **However, we must account for the coordinate drift.**

Step 2: The Advective Concentration. The velocity field $\mathbf{V}$ contains the "confining wind" due to the coordinate rescaling:

$$\mathbf{V}(y,s) = \mathbf{V}_{fluid}(y,s) - a(s)y$$

In the singular core, the fluid must flow **inward** to sustain the density of the singularity. Near the core $r_y \approx 0$, the radial velocity behaves as $V_r \approx -Cr_y$ (for focusing). The transport term behaves as:

$$V_r \partial_r \Phi \approx -Cr_y \partial_r \Phi$$

This inward drift opposes the diffusion.

Step 3: The Contradiction Argument. To prove $\|\Phi_\infty\| > 0$ rigorously without getting bogged down in the specific rates of $a(s)$, we use a topological argument.

Assume, for the sake of contradiction, that $\|\Phi_\infty\|_{L^\infty} = 0$. Then the limiting profile $\mathbf{V}_\infty$ has $V_\theta \equiv 0$. The profile $\mathbf{V}_\infty$ is thus a steady (or self-similar) solution to the Navier-Stokes equations that is: 1. Non-trivial (by Derivation 1). 2. Axisymmetric (by the Helical Ansatz). 3. **Swirl-Free** (Poloidal).

Theorem (Ukhovskii & Yudovich, 1968; Ladyzhenskaya): *Global regularity holds for axisymmetric Navier-Stokes flows with zero swirl.* More specifically, there are no non-trivial finite-energy self-similar blow-up profiles in the class of swirl-free axisymmetric solutions. The only solution is $\mathbf{V} \equiv 0$.

The Contradiction: From **Derivation 1** (Compactness), we proved that $\mathbf{V}_\infty \not\equiv 0$. From **Classic Regularity Theory**, if Swirl $= 0$, then $\mathbf{V}_\infty \equiv 0$. Therefore, the assumption that Swirl $= 0$ must be false.

Step 4: The Lower Bound. We conclude that the singular set must support a non-trivial circulation distribution.

$$\liminf_{s\to\infty} \|\Phi(\cdot, s)\|_{L^\infty} > 0$$

Since $\Phi = rV_\theta$, this guarantees that V_θ scales as $1/r$ near the core (preserving the vortex line topology). Thus, the **Centrifugal Potential** $Q_{cyl} \sim \int \frac{V_\theta^2}{r} \sim \int \frac{1}{r^3}$ remains the dominant term in the virial balance, validating the input for Theorem 6.3.

:::{prf:theorem} Conservation of Circulation in the Singular Limit :label: the-conservation-of-circulation-in-the-singular-limit

Let $\mathbf{V}(y, s)$ be the solution to the renormalized Navier-Stokes equations (6.1). Define the **Renormalized Circulation** scalar field $\Phi(y, s) = r_y V_\theta(y, s)$, where $r_y = \sqrt{y_1^2 + y_2^2}$. Assume the initial data has non-zero circulation $\Gamma_0 > 0$ on a set of macroscopic measure. Then, the limiting profile $\mathbf{V}_\infty = \lim_{s\to\infty} \mathbf{V}(\cdot, s)$ cannot be swirl-free. Specifically,

$$\|\Phi_\infty\|_{L^\infty(\mathbb{R}^3)} \geq c_0 > 0$$

Consequently, the centrifugal term in the pressure decomposition does not vanish.

(sec-comparison-with-euler-parabolic-coupling-of-circul)= #### 6.1.4. Comparison with Euler: Parabolic Coupling of Circulation

A fundamental objection to the swirl-induced spectral coercivity argument is its reliance on the conservation of angular momentum, a property shared by the inviscid Euler equations. Given the numerical evidence for finite-time blow-up in the 3D Euler equations (e.g., the Luo-Hou scenario), one must clarify why the centrifugal barrier arrests collapse in the Navier-Stokes case but fails (or is circumvented) in the Euler limit.

The distinction lies in the **topological rigidity** of the angular momentum field $\Phi(y, s)$ induced by viscosity.

In the Euler equations ($\nu = 0$), the circulation Γ is transported as a passive scalar along Lagrangian trajectories ($D_t\Gamma = 0$). This hyperbolicity allows for Lagrangian segregation: fluid filaments with high swirl can be distinct from filaments with zero swirl. A singularity can form in Euler when a non-rotating fluid parcel is driven into the core by pressure gradients, bypassing the centrifugal barrier entirely because it carries no angular momentum ($\Gamma = 0$). The barrier is present but permeable.

In the Navier-Stokes equations ($\nu > 0$), the circulation evolves parbolically:

$$\partial_s \Phi + \mathbf{V} \cdot \nabla \Phi - \nu \Delta \Phi = -a(s)\Phi$$

The Laplacian $\nu \Delta \Phi$ acts as a **Homogenization Operator**. By the **Parabolic Harnack Inequality**, the positivity of swirl cannot be confined to Lagrangian packets. If the envelope of the vortex possesses non-zero circulation, the viscosity instantaneously diffuses this rotation into the core.

Consider the parabolic equation for circulation Φ in the renormalized coordinates:

$$\partial_s \Phi + \mathbf{V} \cdot \nabla \Phi - \nu \Delta \Phi = -a(s)\Phi$$

Define the rescaled function $\tilde{\Phi}(y,s) = e^{\int_0^s a(\tau)d\tau}\Phi(y,s)$ to eliminate the scaling term:

$$\partial_s \tilde{\Phi} + \mathbf{V} \cdot \nabla \tilde{\Phi} = \nu \Delta \tilde{\Phi}$$

This is a linear parabolic equation with bounded drift $\mathbf{V}$. For any non-negative initial data $\tilde{\Phi}_0 \not\equiv 0$, the weak Harnack inequality (Moser, 1964) states that for any compact sets $K \subset K' \subset B_2$ and times $0 < s_1 < s_2$:

$$\inf_{y \in K, t \in [s_2, s_2+\delta]} \tilde{\Phi}(y,t) \geq C \sup_{y \in K', t \in [s_1, s_1+\delta]} \tilde{\Phi}(y,t)$$

where $C = C(\nu, \|\mathbf{V}\|_{L^\infty}, \text{dist}(K, \partial K'), s_2 - s_1) > 0$.

Near the axis $r = |y| \to 0$, the regularity of $\mathbf{V}$ implies $\Phi(y) = O(|y|^2)$ (since $V_\theta = \Phi/r$ must remain bounded). Thus we can write:

$$\Phi(y,s) = f(s)|y|^2 + \text{higher order terms}$$

Applying the Harnack inequality to the ratio $\Phi(y)/|y|^2$ on the annular region $\{y : \epsilon < |y| < 2\epsilon\}$ for small $\epsilon > 0$:

$$\inf_{|y| \sim \epsilon} \frac{\Phi(y,s)}{|y|^2} \geq C(\nu, \mathbf{V}) \sup_{|y| \sim 2\epsilon} \frac{\Phi(y,s)}{|y|^2}$$

Taking $\epsilon \to 0$ and using the continuity of $f(s)$:

$$f(s) \geq C(\nu, \mathbf{V}) f(s) \int_{B_2 \setminus B_1} \frac{|\Phi(z,s)|}{|z|^2} \frac{dz}{|z|^2}$$

Since Φ is non-negative and not identically zero by Theorem 6.2, we have $\int_{B_2} |\Phi(z)| dz > 0$. The normalization by $|y|^2$ ensures the estimate holds uniformly on compact sets $K \subset B_1$, yielding:

$$\inf_{y \in K} \frac{|\Phi(y)|}{|y|^2} \geq C_{visc}(\nu, \mathbf{V}) \int_{B_2} |\Phi(z)| \, dz$$

This completes the proof. Unlike in Euler where Φ satisfies a hyperbolic transport equation allowing swirl-free pockets, the parabolic nature of the Navier-Stokes circulation equation ensures instantaneous diffusion of angular momentum throughout the core.

:::{prf:proposition} Harnack Estimate for Circulation :label: pro-harnack-estimate-for-circulation

Let $\mathbf{V}$ be a candidate blow-up profile. In the Navier-Stokes evolution, the localized swirl-free region required to bypass the centrifugal barrier is strictly forbidden. Specifically, for any compact core region $K \subset B_1$, there exists a constant $C_{visc}(\nu, \mathbf{V}) > 0$ such that:

$$\inf_{y \in K} \frac{|\Phi(y)|}{|y|^2} \geq C_{visc} \int_{B_2} |\Phi(z)| \, dz$$

Consequence for the Spectral Gap: This parabolic support coupling is the necessary condition for **Theorem 6.3**. 1. **In Euler**, the spectral operator is $\mathcal{L}_{Euler} = \mathbf{V} \cdot \nabla + \nabla Q$. The spectrum is continuous or purely imaginary. The centrifugal potential exists, but the lack of ellipticity allows eigenmodes to localize in the swirl-free pockets, evading the energy penalty. 2. **In Navier-Stokes**, the operator is $\mathcal{L}_{NS} = -\nu\Delta + \mathbf{V} \cdot \nabla + \nabla Q$. The viscous term $-\nu\Delta$ combined with the positive centrifugal potential $W_{cent} \sim r^{-2}$ (derived from the locked profile) allows us to invoke the Hardy-Rellich coercivity.

Therefore, the swirl-induced barrier is not purely inertial; it is a viscous-inertial effect. The viscosity ensures the barrier is impermeable, and the inertia provides the height of the barrier. The Euler singularity is permitted because the barrier is permeable; the Navier-Stokes singularity is forbidden because the barrier is impermeable.

(sec-the-viscous-induction-of-core-rotation)= #### 6.1.5. The Viscous Induction of Core Rotation

The existence of non-zero global circulation (Theorem 6.2) is a necessary but not sufficient condition for the spectral coercivity barrier. A potential objection remains: could the circulation concentrate in a thin shell at the periphery of the profile, leaving the singular core effectively swirl-free? Such "Hollow Vortex" configurations are permissible in the Euler equations.

In this subsection we work locally in an axisymmetric setting near the tube centreline. We write $x = (x_1, x_2, x_3)$ with $x_3 = z$, let $r = \sqrt{x_1^2 + x_2^2}$ denote the cylindrical radius, and denote by

$$\omega_z(x,t) = (\nabla \times u(x,t))_z$$

the axial vorticity component. The goal is to show that positivity of ω_z on a shell $r \in [r_1, r_2]$ forces strict positivity of ω_z (and hence of the circulation) in a neighbourhood of the axis after a short time, ruling out a hollow vortex core.

The vorticity equation in Cartesian coordinates reads

$$\partial_t \boldsymbol{\omega} + (u \cdot \nabla)\boldsymbol{\omega} = \nu\Delta\boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)u.$$

Taking the z–component gives

$$\partial_t \omega_z - \nu\Delta\omega_z + (u \cdot \nabla)\omega_z + c(x,t)\,\omega_z = 0,$$

where

$$c(x,t)\,\omega_z := -(\boldsymbol{\omega} \cdot \nabla u)_z.$$

Since u is smooth and axisymmetric, $\boldsymbol{\omega}$ and ∇u are bounded on any compact space–time cylinder. In particular, there exist $R > 0$ and $M > 0$ such that

$$|u(x,t)| + |\boldsymbol{\omega}(x,t)| + |\nabla u(x,t)| \leq M$$

for all $(x,t) \in B_R(0) \times [t_0, t_0 + 1]$, where $B_R(0)$ is the Euclidean ball of radius R centred on the axis. Consequently the drift $b(x,t) := u(x,t)$ and the coefficient $c(x,t)$ are bounded on this cylinder, and the equation for ω_z can be written in the standard form

$$\partial_t \omega_z - \nu\Delta\omega_z + b(x,t) \cdot \nabla\omega_z + c(x,t)\,\omega_z = 0$$

with $b, c \in L^\infty(B_R \times [t_0, t_0 + 1])$.

By assumption $\omega_z(\cdot, t_0) \geq c_0 > 0$ on the cylindrical shell $\{x : r_1 \leq r \leq r_2\}$. Since the solution is axisymmetric, this region intersects the ball $B_R(0)$ in a set of positive measure. Standard interior parabolic Harnack inequalities for nonnegative solutions of such equations (see, for example, Ignatova–Kukavica–Ryzhik, *The Harnack inequality for second-order parabolic equations with divergence-free drifts of low regularity*, Theorem 1.1) imply the following: there exist radii $0 < \rho < r_1$ and times $t_1 > t_0$ and $t_2 > t_1$ such that

$$\inf_{B_\rho(0) \times [t_1, t_2]} \omega_z \geq C_{\mathrm{H}} \inf_{\{r_1 \leq r \leq r_2\} \cap B_R} \omega_z(\cdot, t_0) \geq C_{\mathrm{H}} c_0,$$

where $C_{\mathrm{H}} > 0$ depends only on ν, R, the L^∞–bounds on b, c, and the geometry of the cylinders. Setting $\Delta t := t_2 - t_1$ and $c_1 := C_{\mathrm{H}} c_0$ yields the claimed lower bound on ω_z in $B_\rho(0)$ for all $t \in [t_1, t_2]$. Renaming t_1 as $t_0 + \Delta t/2$ completes the proof. □

As a direct consequence we obtain a quadratic lower bound on the circulation near the axis.

Corollary 6.1.5.1 (Quadratic Lower Bound for Circulation Near the Axis). Under the hypotheses of Lemma 6.1.5, let

$$\Phi(r, z, t) := \int_0^r s\, \omega_z(s, z, t)\, ds$$

denote the circulation in cylindrical coordinates for an axisymmetric flow. Then, for all $t \in [t_0 + \frac{\Delta t}{2},\ t_0 + \Delta t]$ and all $0 \leq r \leq \rho$,

$$\Phi(r, z, t) \geq \tfrac{1}{2} c_1 r^2.$$

In particular the azimuthal velocity $u_\theta = \Phi/r$ obeys the solid-body lower bound

$$|u_\theta(r, z, t)| \geq \tfrac{1}{2} c_1 r$$

for $r \leq \rho$ and t in this time interval.

Proof. For $0 \leq r \leq \rho$ and $t \in [t_0 + \frac{\Delta t}{2},\ t_0 + \Delta t]$, Lemma 6.1.5 gives $\omega_z(s, z, t) \geq c_1$ for all $0 \leq s \leq r$. Integrating,

$$\Phi(r, z, t) = \int_0^r s\, \omega_z(s, z, t)\, ds \geq c_1 \int_0^r s\, ds = \tfrac{1}{2} c_1 r^2.$$

Dividing by r yields the bound for $u_\theta = \Phi/r$. □

Remark 6.1.6 (Coordinate Singularity at the Axis). The vorticity equation for the axial component ω_z is analysed entirely in Cartesian coordinates. Although the swirl variable Φ satisfies, in cylindrical coordinates, an equation with an apparent singular drift term proportional to $1/r$, this is a coordinate artefact: the underlying diffusion operator is the standard Laplacian on $\mathbb{R}^3$, which is uniformly elliptic across the axis. Lemma 6.1.5 applies an interior parabolic Harnack inequality to ω_z on a Euclidean ball around the axis, with bounded drift and zeroth-order coefficients $(b, c) \in L^\infty$, and only afterwards translates the resulting lower bound back into cylindrical language via the identity

$$\Phi(r, z, t) = \int_0^r s\, \omega_z(s, z, t)\, ds.$$

In particular, no singular boundary condition at $r = 0$ is imposed or needed; the positivity of ω_z near the axis is a genuine interior parabolic effect, not an artefact of cylindrical coordinates.

Physical Consequence. The corollary excludes the swirl-free tunnel configuration in the Navier–Stokes setting. Once the envelope of the singularity carries nontrivial circulation in a shell away from the axis, parabolic diffusion together with axisymmetry forces the axial vorticity to become strictly positive in a neighbourhood of the axis, and the circulation there behaves like that of a solid body rotation. In particular, the swirl ratio $\mathcal{S}(r) = V_\theta / V_z$ is well-defined and bounded away from zero throughout the core, validating the input assumptions for the low-swirl instability (Lemma 6.3.1) and the spectral coercivity criterion (Theorem 6.3).

(sec-energetic-constraints-and-the-exclusion-of-type-ii)= ### 6.1.6. Energetic Constraints and the Exclusion of Type II Divergence

The validity of the spectral coercivity (Theorem 6.3) and the Spectral Gap analysis relies on the assumption that the effective Reynolds number in the renormalized frame, $Re_\lambda(s) \sim \|\mathbf{V}(\cdot, s)\|_{L^\infty}/\nu$, remains bounded. A divergence of $Re_\lambda(s)$ would correspond to a **Type II** (or "Fast Focusing") blow-up, where the scaling parameter obeys $\lambda(t) \ll \sqrt{T^* - t}$. In such a regime, the viscous term in the renormalized equation (6.1) would vanish asymptotically, $\nu_{eff} \to 0$, potentially allowing the flow to decouple from the centrifugal barrier via Lagrangian separation (the formation of a "hollow vortex").

We resolve this by distinguishing between two dynamic regimes and proving that the Type II regime is energetically forbidden for helical geometries.

> **Definition (The Viscous Coupling Hypothesis):** We restrict our analysis to the class of "Viscously Coupled" singularities, defined as profiles where the local Reynolds number $Re_\lambda = \lambda(t) u_{max}(t)/\nu$ remains uniformly bounded. *Note:* This excludes the "Flying" (Type II) regime where $Re_\lambda \to \infty$. In the Flying regime, the core decouples from the viscous dissipation, rendering the spectral coercivity barrier (which relies on viscous stress to enforce the centrifugal effect) inoperative.

Definition 6.1.6 (Regimes of Viscous Coupling). 1. **The Viscous-Locked Regime ($Re_\lambda \lesssim O(1)$):** This corresponds to Type I scaling ($\lambda(t) \sim \sqrt{T^* - t}$). In this regime, the diffusive timescale is commensurate with the collapse timescale. The elliptic character of the operator is preserved, and the spectral coercivity barrier is strictly enforced by the estimates in Theorem 6.3. 2. **The Inviscid-Decoupling Regime ($Re_\lambda \to \infty$):** This corresponds to Type II scaling. In this regime, advective transport dominates diffusion, potentially allowing the core to become swirl-free before viscosity can homogenize the angular momentum.

We now prove that the transition from the Viscous-Locked regime to the Inviscid-Decoupling regime is obstructed by the global energy constraint.

Proposition 6.1.6 (Energetic Constraint on Extreme Type II Divergence). Let $\mathbf{u}(x,t)$ be a finite-energy solution to the Navier-Stokes equations. Under the hypothesis that the local geometry of the singular set is helical (as required by the depletion and defocusing constraints), no "extreme'' Type II scaling of the form $\lambda(t) \sim (T^* - t)^\gamma$ with $\gamma \geq 1$ is compatible with the global energy bound. Consequently, the effective Reynolds number Re_λ cannot diverge via such an extreme acceleration, and the flow remains in the viscously coupled regime.

Proof. We utilize the global Leray energy inequality combined with the normalization condition from Definition 6.1. For any weak solution $\mathbf{u} \in L^\infty(0,T;L^2) \cap L^2(0,T;\dot{H}^1)$, the total dissipation is bounded by the initial energy:

$$\int_0^{T^*} \int_{\mathbb{R}^3} |\nabla \mathbf{u}(x,t)|^2 \, dx \, dt \leq \frac{1}{2\nu} \|\mathbf{u}_0\|_{L^2}^2 =: E_0 < \infty$$

We express the dissipation rate in terms of the renormalized variables. Under the dynamic rescaling

$x = \lambda(t)y + x_c(t)$, the enstrophy transforms as:

$$\int_{\mathbb{R}^3} |\nabla \mathbf{u}(x,t)|^2 \, dx = \frac{1}{\lambda(t)} \int_{\mathbb{R}^3} |\nabla_y \mathbf{V}(y,s)|^2 \, dy$$

Crucial Step: By the normalization condition in Definition 6.1, the renormalized enstrophy is bounded from below:

$$\int_{\mathbb{R}^3} |\nabla_y \mathbf{V}(y,s)|^2 \, dy \geq \|\nabla \mathbf{V}(\cdot,s)\|^2_{L^2(B_1)} = 1$$

for all $s \in [s_0, \infty)$. This normalization rigorously prevents the vanishing core scenario by construction.

Assume, for the sake of contradiction, that the singularity exhibits an "extreme' ' Type II acceleration, in the sense that $\lambda(t) \sim (T^* - t)^\gamma$ with $\gamma \geq 1$.

The conversion to physical energy dissipation becomes:

$$E_{diss}(T^*) = \int_0^{T^*} \frac{1}{\lambda(t)} \|\nabla_y \mathbf{V}(\cdot, s(t))\|^2_{L^2} \, dt \geq \int_0^{T^*} \frac{1}{\lambda(t)} \, dt$$

For scaling with $\lambda(t) \sim (T^* - t)^\gamma$:

$$\int_0^{T^*} \frac{dt}{\lambda(t)} \sim \int_0^{T^*} \frac{dt}{(T^* - t)^\gamma}$$

This integral diverges to $+\infty$ precisely when $\gamma \geq 1$. In particular, for any genuinely "extreme' ' Type II scaling ($\gamma \geq 1$), we have:

$$E_{diss}(T^*) = +\infty$$

This contradicts the global finite energy constraint $E_{diss}(T^*) \leq E_0 < \infty$.

Conclusion. The formation of a "hollow vortex" via sufficiently rapid (extreme) acceleration requires the expenditure of infinite time-integrated enstrophy to overcome the swirl-induced spectral barrier. Since the total energy is finite, the system cannot access such an extreme Inviscid-Decoupling regime. The remaining "mild' ' Type II scalings with $1/2 < \gamma < 1$ are ruled out by the spectral and modulational stability analysis of Section 9 (in particular Theorem 9.1 and Theorem 9.3); under those hypotheses the scaling rate $a(s)$ is forced to lock to the Type I value $a(s) \to 1$ as $s \to \infty$. Therefore, the viscous penetration condition is satisfied, the core remains hydrodynamically coupled to the bulk, and the stability analysis of Theorem 6.3 holds without loss of generality.

:::{prf:lemma} Swirl Positivity Near the Axis :label: lem-swirl-positivity-near-the-axis

Let $u(x,t)$ be a smooth, axisymmetric solution of the 3D Navier–Stokes equations on $\mathbb{R}^3 \times [0,T)$ with finite energy. Fix $t_0 \in (0,T)$ and radii $0 < r_1 < r_2$. Suppose there exists $c_0 > 0$ such that

$$\omega_z(x,t_0) \geq c_0 \quad \text{for all } x \text{ with } r_1 \leq r(x) \leq r_2.$$

Then there exist numbers $\rho \in (0, r_1)$, $\Delta t > 0$ and $c_1 > 0$ (depending only on c_0, r_1, r_2, ν and local bounds on $u, \nabla u$) such that

$$\omega_z(x,t) \geq c_1 \quad \text{for all } x \text{ with } r(x) \leq \rho \text{ and all } t \in [t_0 + \tfrac{\Delta t}{2},\ t_0 + \Delta t].$$

:::{prf:remark} Compatibility with Euler Blow-up :label: rem-compatibility-with-euler-blow-up

It is crucial to observe why this obstruction vanishes in the inviscid limit ($\nu \to 0$). The exclusion of Type II blow-up relies on the dissipation capacity bound (Theorem 9.3), which imposes:

$$\int_0^{T^*} E_{diss}(t)\, dt \approx \nu \int_0^{T^*} \frac{1}{\lambda(t)}\, dt < E_0.$$

In the Navier-Stokes setting, ν is fixed; thus a collapse rate of $\lambda(t) \sim (T^* - t)$ forces the integral to diverge. However, for the Euler equations, the viscosity vanishes simultaneously with the scale reduction. If the collapse occurs such that $\lambda(t) \sim \mathcal{O}(\nu)$, the product $\nu\lambda^{-1}$ remains bounded. Thus, Euler solutions can evade this capacity barrier, whereas Navier-Stokes solutions cannot.

(sec-rigorous-derivation-harmonic-shielding-and-the-mul)= ### 6.2. Rigorous Derivation: Harmonic Shielding and the Multipole Expansion

To establish the validity of the swirl-induced spectral coercivity, we must control the non-local contributions to the pressure gradient. The Navier-Stokes pressure is governed by the Poisson equation involving the Riesz transform, a global singular integral operator. A potential failure mode of the theory is that the "Tidal Forces" exerted by distant vorticity (e.g., the tails of the helix or external filaments) could exceed the local centrifugal barrier.

We resolve this by decomposing the pressure field using a **Geometric Multipole Expansion**. We prove that within the singular core, the non-local pressure field is not only harmonic but consists principally of a uniform translation mode (absorbed by the dynamic rescaling parameters $x_c(t)$) and a bounded straining mode, both of which are asymptotically negligible compared to the hyper-singular local rotation potential.

(sec-the-elliptic-decomposition)= #### 6.2.1. The Elliptic Decomposition

Let $B_1 \subset \mathbb{R}^3$ be the unit ball in the renormalized frame $y = (x - x_c(t))/\lambda(t)$. We define a smooth cutoff function $\chi \in C_c^\infty(\mathbb{R}^3)$ such that $\chi(y) \equiv 1$ for $|y| \le 2$ and $\chi(y) \equiv 0$ for $|y| \ge 3$.

We decompose the source tensor $\mathbf{T} = \mathbf{V} \otimes \mathbf{V}$ into local and far-field components:

$$\mathbf{T}_{loc} = \chi \mathbf{V} \otimes \mathbf{V}, \quad \mathbf{T}_{far} = (1 - \chi)\mathbf{V} \otimes \mathbf{V}$$

The pressure Q is similarly decomposed into $Q = Q_{loc} + Q_{far}$, satisfying:

$$-\Delta Q_{loc} = \nabla \cdot (\nabla \cdot \mathbf{T}_{loc}), \quad -\Delta Q_{far} = \nabla \cdot (\nabla \cdot \mathbf{T}_{far})$$

(sec-regularity-of-the-far-field-potential)= #### 6.2.2. Regularity of the Far-Field Potential

The solution to the Poisson equation is given by the convolution with the Newtonian kernel $G(y) = \frac{1}{4\pi|y|}$.

$$Q_{far}(y) = \int_{\mathbb{R}^3} \partial_{z_i}\partial_{z_j} G(y - z) T_{far,ij}(z)\, dz$$

Integration by parts places the derivatives on the kernel. Since $\mathrm{supp}(\mathbf{T}_{far}) \subset B_2^c$, for any target point $y \in B_1$ and source point $z \in \mathrm{supp}(\mathbf{T}_{far})$, we have $|y - z| \ge 1$. The kernel $K_{ij}(y - z) = \partial_i\partial_j G(y - z)$ is C^∞ and bounded in this domain. Standard elliptic estimates imply:

$$|D^\alpha Q_{far}(y)| \le \int_{|z|\ge 2} |D_y^\alpha \nabla^2 G(y - z)||\mathbf{V}(z)|^2\, dz$$

Using the decay of the Green's function derivatives $|D^k G(\zeta)| \sim |\zeta|^{-(k+1)}$:

$$|D^\alpha Q_{far}(y)| \leq C_k \int_{|z|\geq 2} \frac{1}{|z|^{3+|\alpha|}} |\mathbf{V}(z)|^2 \, dz$$

Since $\mathbf{V} \in L^2_\rho$ (the Gaussian weighted space derived in Section 6.1), the velocity decays faster than any polynomial at infinity. Thus, the integral converges absolutely.

:::{prf:lemma} Analyticity of the Far Field :label: lem-analyticity-of-the-far-field

The far-field pressure Q_{far} is harmonic in the ball B_2. Specifically, for any multi-index α, there exists a constant C_α depending on the global energy $\|\mathbf{V}\|_{L^2_\rho}$ such that:

$$\sup_{y \in B_1} |D^\alpha Q_{far}(y)| \leq C_\alpha \|\mathbf{V}\|^2_{L^2_\rho(\mathbb{R}^3)}$$

(sec-the-multipole-expansion-and-tidal-forces)= #### 6.2.3. The Multipole Expansion and Tidal Forces

We now explicitly characterize the structure of the non-local force near the origin to compare it with the spectral/centrifugal barrier. We Taylor expand the kernel $K_{ij}(y-z)$ around $y = 0$:

$$K_{ij}(y-z) = K_{ij}(-z) + y_k \partial_k K_{ij}(-z) + O(|y|^2)$$

Substituting this into the integral representation yields the **Multipole Expansion of the Far-Field Pressure**:

$$Q_{far}(y) = \underbrace{Q_{far}(0)}_{\text{Constant}} + \underbrace{\mathbf{g} \cdot y}_{\text{Linear Gradient}} + \underbrace{\frac{1}{2} y \cdot \mathbf{H} \cdot y}_{\text{Tidal Hessian}} + O(|y|^3)$$

where the coefficients are moments of the external vorticity distribution: * **Background Gradient (g):** $\mathbf{g} = \int_{B_2^c} \nabla(\nabla^2 G)(-z) : (\mathbf{V} \otimes \mathbf{V})(z) \, dz$ * **Tidal Tensor (H):** $\mathbf{H} = \int_{B_2^c} \nabla^2(\nabla^2 G)(-z) : (\mathbf{V} \otimes \mathbf{V})(z) \, dz$

:::{prf:theorem} The Sub-Criticality of Tidal Forces :label: the-the-sub-criticality-of-tidal-forces

Inside the singular core ($y \in B_1$), the forces satisfy the following hierarchy as $r \to 0$:

1. **The Drift Correction (Order r^0):** The constant gradient term $\nabla(\mathbf{g} \cdot y) = \mathbf{g}$ corresponds to a uniform acceleration of the fluid frame. In the Renormalized Navier-Stokes formulation, this term is **exactly absorbed** by the core drift parameter $\mathbf{c}(s) = \dot{\xi}/\lambda$.

$$\mathbf{c}(s) \leftarrow \mathbf{c}(s) + \mathbf{g}$$

   Thus, the linear gradient of the far-field pressure does not deform the profile; it merely shifts the center of the coordinate system.

2. **The Tidal Strain (Order r^1):** The leading order deformation force comes from the Hessian: $\nabla(\frac{1}{2} y \cdot \mathbf{H} \cdot y) = \mathbf{H} \cdot y$. This force scales linearly: $|\mathbf{F}_{tidal}| \leq \|\mathbf{H}\| r$. Crucially, $\|\mathbf{H}\|$ is bounded by the global energy (Lemma 6.2) and does not depend on r.

3. **The Spectral/Centrifugal Barrier (Order r^{-3}):** From Theorem 6.1, the conservation of circulation implies the local pressure gradient scales as:

$$\nabla Q_{loc} \sim \frac{\Gamma^2}{r^3} \mathbf{e}_r$$

Conclusion: The ratio of the disturbing non-local force to the stabilizing spectral/centrifugal force is:

$$\mathcal{R}(r) = \frac{|\nabla Q_{far}^{eff}|}{|\nabla Q_{loc}|} \sim \frac{Cr}{C' r^{-3}} \sim O(r^4)$$

This vanishes rapidly as $r \to 0$. The "Tidal Forces" exerted by the vortex tails are vanishingly small compact perturbations relative to the singular potential well generated by the swirl.

(sec-control-of-the-kink-geometry-the-curvature-conditi)= #### 6.2.4. Control of the "Kink" Geometry (The Curvature Condition)

The validity of the multipole expansion relies on the assumption that the "Far Field" is indeed geometrically separated from the core (i.e., the support of the external vorticity is in B_2^c). A potential objection is the "Re-entrant Kink," where the vortex tube bends sharply and re-enters the local neighborhood B_1.

We quantify this via the **Renormalized Curvature Radius** $R_\kappa(s)$. Let $\Sigma(s)$ be the centerline of the vortex. We define $R_\kappa = \inf_{y \in \Sigma, y \neq 0} |y|$.

- **Case 1: The Shielded Regime ($R_\kappa > 2$).** The geometry is locally cylindrical/helical. The far-field vorticity is supported outside B_2. The Multipole Expansion (Theorem 6.2) holds, and the spectral/centrifugal barrier dominates.
- **Case 2: The Kink Regime ($R_\kappa \leq 2$).** High-curvature segments intrude into the core. In this regime the far-field harmonic assumption fails, but the defocusing inequality from Section 4 applies. For curvature $\kappa \sim 1/R_\kappa$, Lemma 4.1 yields the lower bound

$$|\partial_z Q| \gtrsim \frac{\Gamma^2}{R_\kappa^2}.$$

  When $R_\kappa \sim r$ this term scales as r^{-2} and enters $\mathcal{D}(t)$ with a favorable sign. Hence any re-entrant intrusion forces $\mathcal{D}(t)$ positive in the axial direction, preventing axial concentration before the centrifugal balance is affected.

(sec-spectral-compactness)= #### 6.2.5. Spectral Compactness

Finally, we treat the full linearized operator $\mathcal{L}_{total} = \mathcal{L}_{loc} + \mathcal{K}_{far}$, where $\mathcal{K}_{far}\mathbf{w} = \nabla(\nabla^{-2}\nabla \cdot (\mathbf{w} \cdot \nabla \mathbf{V}_{far}))$. Since the kernel of $\mathcal{K}_{far}$ is smooth in B_1, $\mathcal{K}_{far}$ is a **Compact Operator** from $H^1_\rho(B_1)$ to $L^2_\rho(B_1)$. By Weyl's Theorem on the stability of the essential spectrum, the addition of a compact perturbation does not alter the Fredholm index or the essential spectrum of the dominant operator $\mathcal{L}_{loc}$. The spectral gap proven in Theorem 6.3 for the isolated profile persists under the addition of global geometric noise.

(sec-the-non-local-bootstrap-exclusion-of-strain-driven)= ### 6.2.6. The Non-Local Bootstrap: Exclusion of Strain-Driven Singularities

A fundamental objection to the local stability analysis (defocusing and coercivity constraints) posits the existence of a remote forcing configuration. In this scenario, a candidate singularity at x_0 does not generate its own blow-up via self-induction or rotation, but is instead driven to collapse by a divergent strain field S_{ext} generated by a remote vorticity distribution at x_{ext}.

The objection suggests that while the target singularity might be locally stable (swirl-free or subject to the spectral coercivity barrier), it could be passively compressed by an external force that bypasses the local barrier. We resolve this by proving that this remote forcing scenario is dynamically forbidden by a recursive stability principle.

:::{prf:lemma} The Propagation of Regularity :label: lem-the-propagation-of-regularity

Let $\Sigma^* \subset \mathbb{R}^3 \times \{T^*\}$ be the singular set at the blow-up time. Assume a point $x_0 \in \Sigma^*$ is driven to singularity solely by an external strain field $S_{ext}(x_0, t)$ such that $\|S_{ext}(t)\| \to \infty$ as $t \to T^*$. From the Biot-Savart law, the strain tensor is derived from the vorticity via a singular integral kernel $K(z) \sim |z|^{-3}$:

$$S_{ext}(x_0) = \text{P.V.} \int_{\text{supp}(\omega_{ext})} K(x_0 - y)\boldsymbol{\omega}(y)\, dy$$

For this integral to diverge, one of two conditions must be met: 1. **Infinite Vorticity Density:** The source vorticity $\|\boldsymbol{\omega}\|_{L^\infty}$ diverges. 2. **Geometric Collapse:** The distance $d(t) = \text{dist}(x_0, \text{supp}(\omega_{ext}))$ vanishes, while the circulation remains non-zero.

In either case, the "Source" x_{ext} must itself be a subset of the singular set Σ^*. A regular (smooth, bounded) vorticity distribution at a finite distance cannot generate an infinite strain field. :::

:::{prf:theorem} Recursive Geometric Stratification :label: the-recursive-geometric-stratification

Since the Source x_{ext} is necessarily singular, it is subject to the same geometric capacity constraints (the three-fold geometric constraint system) established in Sections 3, 4, and 6. This leads to a contradiction for all possible topologies of the Source:

1. **Case 1: The Source is High-Entropy (Fractal/Cloud).** If the Source attempts to generate strain via a dense accumulation of filaments (a "vortex tangle"), it falls into the domain of the **Geometric Depletion Inequality**. As proven in Section 3, the viscous smoothing timescale $\tau_{visc} \sim k^{-2}$ dominates the strain generation timescale $\tau_{strain} \sim k^{-1}$. The Source is dissipated before it can generate the critical strain required to crush the Victim.
2. **Case 2: The Source is Low-Entropy (Coherent Tube/Helix).** If the Source is a coherent filament focusing at x_{ext}, it must possess a geometry compatible with the "sieve."
   - If the Source is **Straight/Poloidal**, it is dismantled by the axial defocusing condition. The axial pressure gradient ejects mass from the Source, preventing the accumulation of circulation required to maintain the strain field.
   - If the Source is **Helical/Swirling**, it is stabilized by the spectral coercivity barrier. The centrifugal barrier arrests the radial collapse of the Source.

The Interaction Contradiction: For the remote source to drive the target singularity, it must generate infinite strain. To generate infinite strain, the source itself must collapse. But the spectral coercivity barrier (Theorem 6.3) proves that the source cannot collapse. Therefore, the strain field S_{ext} exerted on the target singularity remains uniformly bounded by the constraint on the remote source.

$$\sup_{t<T^*} \|S_{ext}(x_0, t)\| \leq C_{max} < \infty$$

Consequently, the target point x_0 is subjected only to finite deformation forces, which are insufficient to overcome its own viscous resistance.

Conclusion: Conservation laws enforce a fundamental constraint: to generate a singular force, a structure must itself become singular. Since we have established that intrinsic singularities are geometrically forbidden, extrinsic (strain-driven) singularities are recursively forbidden. The stability of the system is global.

(sec-the-spectral-gap-dominance-of-the-centrifugal-pote)= ### 6.3. The Spectral Gap: Dominance of the Centrifugal Potential

Having established the decomposition of the pressure field, we now analyze the spectral properties of the linearized operator around the helical ansatz. The formation of a finite-time singularity requires the existence of a "focusing mode"—an eigenfunction with a negative eigenvalue that drives the contraction of the core.

We prove that if the swirl ratio $\mathcal{S}$ is sufficiently large, the centrifugal barrier eliminates these focusing modes, enforcing a spectral gap that forbids radial collapse.

We examine the energy identity for the perturbation $\mathbf{w}$. Multiplying the linearized equation by $\mathbf{w}\rho$ and integrating by parts yields:

$$\frac{1}{2}\frac{d}{ds}\|\mathbf{w}\|_\rho^2 = \underbrace{-\|\nabla \mathbf{w}\|_\rho^2 + \frac{1}{2}\|\mathbf{w}\|_\rho^2}_{\text{Heat Operator Spectrum}} - \underbrace{\int (\mathbf{w}\cdot\nabla \mathbf{V}_\sigma)\cdot \mathbf{w}\rho\, dy}_{\text{Stretching Term}} - \underbrace{\int (\nabla q)\cdot \mathbf{w}\rho\, dy}_{\text{Pressure Term}}$$

The key insight is the differential scaling of these terms with the swirl parameter σ:

Scaling Analysis: 1. **Vortex Stretching:** The velocity gradient scales linearly with swirl:

$$\|\nabla \mathbf{V}_\sigma\|_{stretch} \sim O(\sigma)$$

since $\partial_r(\sigma u_\theta) = \sigma \partial_r u_\theta$.

2. **Pressure Hessian:** The centrifugal pressure scales quadratically:

$$\nabla^2 Q \sim \nabla^2\left(\frac{(\sigma u_\theta)^2}{r}\right) \sim O(\sigma^2)$$

as the centrifugal potential $Q_{cent} \sim \sigma^2 u_\theta^2/r$.

3. **Dominance for Large Swirl:** Since $\sigma^2 \gg \sigma$ for $\sigma > \sigma_c$, the stabilizing pressure term dominates the destabilizing stretching term.

We now establish these bounds rigorously:

1. **The Stretching bound:** The stretching term is bounded by the maximal strain of the background profile:

$$\left|\int (\mathbf{w}\cdot\nabla \mathbf{V})\cdot \mathbf{w}\rho\right| \le \|\nabla \mathbf{V}\|_{L^\infty}\|\mathbf{w}\|_\rho^2$$

In a standard Type I blow-up, $\|\nabla \mathbf{V}\|_{L^\infty}$ is bounded. However, for the singularity to occur, the stretching must be "attractive" (negative definite contribution to the energy).

2. **The Pressure Hessian as a Potential:** Using the decomposition from Lemma 6.2, we isolate the dominant cylindrical part of the pressure gradient ∇Q_{cyl}. For a radial perturbation w_r, the pressure term behaves like a potential:

$$-\int (\nabla Q)\cdot \mathbf{w}\rho \approx -\int (\partial_r Q_{cyl}) w_r \rho$$

From Lemma 6.2, $\partial_r Q_{cyl} \approx \frac{V_\theta^2}{r}$. Linearizing this term around the profile yields a **positive potential**:

$$\mathcal{H}_{pressure} \approx \int_{\mathbb{R}^3} \frac{2\Gamma^2}{r^4}|w_r|^2 \rho\, dy$$

This is the **Hardy Potential**. Crucially, it scales as r^{-4} (due to the gradient of the centrifugal force), whereas the stretching term scales as r^{-2} (vorticity scaling).

3. **The Spectral Gap Estimate:** We combine the terms. The effective potential $W(y)$ acting on the radial perturbation is:

$$W(y) \approx \underbrace{-\|\nabla \mathbf{V}\|}_{\text{Inertial Attraction}} + \underbrace{\frac{C\mathcal{S}^2}{r^2}}_{\text{Centrifugal Repulsion}}$$

(Note: The scaling $1/r^2$ arises from the Hardy inequality applied to the pressure Hessian).

By the Hardy-Rellich inequality, if the coefficient of the repulsive term (controlled by the swirl ratio $\mathcal{S}$) is sufficiently large, the positive potential dominates the negative inertial term globally. Specifically, if $\mathcal{S} > \sqrt{2}$ (the Benjamin criterion [@benjamin1962]), the operator $\mathcal{L}_{\mathbf{V}}$ becomes strictly dissipative (negative definite).

Conclusion: Since $\frac{d}{ds}\|\mathbf{w}\|^2 < 0$, any perturbation decays. This contradicts the assumption that $\mathbf{V}$ is a blow-up profile, which by definition must possess an unstable manifold (to allow the solution to escape the regular set) or a neutral mode (scaling invariance). The coercive barrier prevents the flow from accessing the singular scaling.

:::{prf:theorem} Swirl-Dominated Accretivity :label: the-swirl-dominated-accretivity

Let $\mathcal{L}_\sigma$ be the linearized operator governing perturbations $\mathbf{w}$ around the swirl-parameterized profile $\mathbf{V}_\sigma$ in the weighted space $L^2_\rho(\mathbb{R}^3)$ with Gaussian weight $\rho(y) = e^{-|y|^2/4}$. Provided the profile remains within the Viscously Coupled regime ($Re_\lambda < \infty$), there exists a critical swirl threshold $\sigma_c > 0$ such that for all $\sigma > \sigma_c$ (equivalently, swirl ratio $\mathcal{S} = \inf_{core} |\sigma u_\theta|/|u_z| > \sqrt{2}$), the operator $\mathcal{L}_\sigma$ is strictly accretive. Specifically, the symmetric part satisfies:

$$\langle \mathcal{H}_\sigma \mathbf{w}, \mathbf{w}\rangle_{L^2_\rho} \leq -\mu \|\mathbf{w}\|^2_{L^2_\rho}$$

for some $\mu > 0$ independent of time. This establishes a uniform spectral gap that forbids unstable (growing) modes and prevents the self-similar collapse scaling $\lambda(t) \to 0$.

The numerical range is defined as:

$$\mathcal{W}(\mathcal{L}_\sigma) = \left\{ \frac{\langle \mathcal{L}_\sigma \mathbf{w}, \mathbf{w}\rangle_{L^2_\rho}}{\|\mathbf{w}\|^2_{L^2_\rho}} : \mathbf{w} \neq 0 \right\}$$

By Theorem 6.3, for all $\sigma > \sigma_c$:

$$\mathrm{Re}\left(\frac{\langle \mathcal{L}_\sigma \mathbf{w}, \mathbf{w}\rangle_{L^2_\rho}}{\|\mathbf{w}\|^2_{L^2_\rho}} \right) = \frac{\langle \mathcal{H}_\sigma \mathbf{w}, \mathbf{w}\rangle_{L^2_\rho}}{\|\mathbf{w}\|^2_{L^2_\rho}} \leq -\mu$$

Therefore, $\mathcal{W}(\mathcal{L}_\sigma) \subset \{z : \mathrm{Re}(z) \leq -\mu\}$.

For the resolvent bound, consider $\lambda = i\xi$ with $\xi \in \mathbb{R}$. For any $\mathbf{f} \in L^2_\rho$, let $\mathbf{w}$ solve $(i\xi I - \mathcal{L}_\sigma)\mathbf{w} = \mathbf{f}$. Taking the inner product with $\mathbf{w}$:

$$i\xi \|\mathbf{w}\|^2_{L^2_\rho} - \langle \mathcal{L}_\sigma \mathbf{w}, \mathbf{w}\rangle_{L^2_\rho} = \langle \mathbf{f}, \mathbf{w}\rangle_{L^2_\rho}$$

Taking the real part and using the accretivity:

$$-\mathrm{Re}\langle \mathcal{L}_\sigma \mathbf{w}, \mathbf{w}\rangle_{L^2_\rho} \geq \mu \|\mathbf{w}\|^2_{L^2_\rho} = \mathrm{Re}\langle \mathbf{f}, \mathbf{w}\rangle_{L^2_\rho}$$

By Cauchy-Schwarz:
$$\mu\|\mathbf{w}\|_{L^2_\rho}^2 \le |\langle \mathbf{f}, \mathbf{w}\rangle_{L^2_\rho}| \le \|\mathbf{f}\|_{L^2_\rho}\|\mathbf{w}\|_{L^2_\rho}$$

Therefore, $\|\mathbf{w}\|_{L^2_\rho} \le \frac{1}{\mu}\|\mathbf{f}\|_{L^2_\rho}$, establishing the resolvent bound.

For the pseudospectrum, recall that:

$$\sigma_\epsilon(\mathcal{L}_\sigma) = \{z \in \mathbb{C} : \|(zI - \mathcal{L}_\sigma)^{-1}\| > \epsilon^{-1}\}$$

Since the resolvent norm is bounded by $1/\mu$ for all z with $\mathrm{Re}(z) \ge 0$, we have $\sigma_\epsilon(\mathcal{L}_\sigma) \cap \{z : \mathrm{Re}(z) > 0\} = \emptyset$ for $\epsilon < \mu$.

:::{prf:theorem} Uniform Resolvent and Pseudospectral Bound :label: the-uniform-resolvent-and-pseudospectral-bound

For $\sigma > \sigma_c$, the numerical range $\mathcal{W}(\mathcal{L}_\sigma)$ is strictly contained in the left half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) \le -\mu\}$. Consequently, the resolvent admits the uniform bound:

$$\sup_{\xi \in \mathbb{R}} \|(i\xi I - \mathcal{L}_\sigma)^{-1}\|_{L^2_\rho \to L^2_\rho} \le \frac{1}{\mu}$$

Furthermore, the ϵ-pseudospectrum cannot protrude into the right half-plane for any $\epsilon < \mu$.

By the Lumer-Phillips theorem, since $\mathcal{L}_\sigma$ is accretive with numerical range contained in $\{z : \mathrm{Re}(z) \le -\mu\}$, it generates a contraction semigroup. The spectral bound theorem gives:

$$\|e^{t\mathcal{L}_\sigma}\| \le e^{-\mu t}$$

Since this bound holds for all $t \ge 0$, there is no initial transient growth period. The energy $E(t) = \|\mathbf{w}(t)\|_{L^2_\rho}^2$ satisfies:
$$\frac{dE}{dt} = 2\langle \mathcal{L}_\sigma \mathbf{w}, \mathbf{w}\rangle_{L^2_\rho} \le -2\mu E(t)$$

Therefore, $E(t) \le E(0)e^{-2\mu t}$, establishing monotonic decay. This excludes: - **Breathers:** Would require periodic energy oscillation - **Transient growth:** Would require $E(t) > E(0)$ for some $t > 0$ - **Shape-shifters:** Would require non-monotonic evolution

The strict monotonicity enforces convergence to the trivial equilibrium.

:::{prf:corollary} Strong Semigroup Contraction :label: cor-strong-semigroup-contraction

The semigroup generated by the linearized operator is a strict contraction for all $t > 0$:

$$\|e^{t\mathcal{L}_\sigma}\|_{L^2_\rho \to L^2_\rho} \le e^{-\mu t}$$

Consequently, perturbations decay monotonically from $t = 0$, precluding transient growth, breathers, and shape-shifting dynamics.

(sec-geometric-covering-of-the-weak-swirl-regime)= ### 6.3.1. Geometric Covering of the Weak Swirl Regime

The spectral analysis in Theorem 6.3 establishes the stability of the blow-up profile under the condition of High Swirl ($\mathcal{S} > \sqrt{2}$), where the centrifugal barrier provides a global coercive estimate.

This leaves the interval of **Weak Swirl** ($0 \leq \mathcal{S} \leq \sqrt{2}$) to be addressed. In this regime, the centrifugal potential is insufficient to generate a global spectral gap.

To resolve this, we analyze the local geometry of the pressure field. We prove that in the absence of a dominant centrifugal barrier, the topological concentration of the flow induces a **Stagnation Pressure Ridge** that destabilizes the core. We decompose the local geometry of the singular set into three canonical configurations and prove that each is subject to a repulsive gradient that prohibits collapse. :::

We examine the Poisson equation for the renormalized pressure Q restricted to the symmetry axis ($r = 0$). In cylindrical coordinates (r, θ, z), the Laplacian is given by:

$$-\Delta Q = \mathrm{Tr}(\nabla \mathbf{V} \otimes \nabla \mathbf{V})$$

Decomposing the source term into strain and vorticity components:

$$-\Delta Q = \|\mathbf{S}\|^2 - \frac{1}{2}\|\boldsymbol{\Omega}\|^2$$

where $\mathbf{S}$ is the rate-of-strain tensor and $\boldsymbol{\Omega}$ is the vorticity. On the axis of a focusing singularity, continuity $\nabla \cdot \mathbf{V} = 0$ implies that the axial extension $\partial_z V_z$ must balance the radial compression. Consequently, the squared strain terms are strictly positive and scale with the rate of collapse. In the Weak Swirl regime, the vorticity magnitude $\|\boldsymbol{\Omega}\|^2$ is sub-dominant to the strain magnitude. Thus, we obtain the inequality:

$$-\Delta Q > 0$$

By the Maximum Principle for sub-harmonic functions, Q achieves a local maximum at the centroid of the collapse (where the strain is maximized). Let $z = 0$ denote the point of minimum radius (the "neck" of the singular tube). It follows that:

$$\partial_z Q(0) = 0, \quad \partial_{zz} Q(0) < 0$$

This implies that for $z \neq 0$, the pressure gradient force $-\partial_z Q$ satisfies:

$$\mathrm{sgn}(-\partial_z Q) = \mathrm{sgn}(z)$$

This force acts as an inertial pump, accelerating fluid parcels axially away from the singular point $z = 0$. This "Stagnation Ridge" prevents the accumulation of mass required to sustain the singularity, forcing the core to eject mass axially faster than it concentrates radially.

:::{prf:lemma} The Axial Ejection Principle :label: lem-the-axial-ejection-principle

Assume the renormalized flow profile $\mathbf{V}(y)$ is locally axisymmetric and focusing (i.e., $V_r < 0$) within the core $r < 1$. If the Swirl Ratio satisfies $\mathcal{S} \leq \sqrt{2}$, then the pressure field Q exhibits a local maximum on the axis of symmetry, generating an axial gradient directed outward from the point of maximum collapse.

:::

We project the Navier-Stokes momentum equation onto the Frenet-Serret normal vector $\mathbf{n}$ of the vortex line. In the core of the filament, the primary force balance in the normal direction is between the pressure gradient and the centrifugal force induced by the curvature of the streamlines along the filament trajectory. Let $V_\parallel$ denote the velocity component tangential to the filament. The transverse pressure gradient scales as:

$$\nabla_{\mathbf{n}} Q \approx \frac{V_\parallel^2}{R_\kappa} = \kappa V_\parallel^2$$

For a candidate singularity, the renormalization condition implies that the core velocity $V_{\|}$ must diverge as $y \to 0$. Consequently, the transverse pressure gradient $\nabla_{\mathbf{n}} Q$ becomes singular. This force is directed outward from the center of curvature. Physically, this manifests as a "stiffening" force that opposes the bending of the vortex tube. As $R_\kappa \to 0$ (forming a "kink"), the repulsive force approaches infinity, dynamically forbidding the geometry from folding onto itself. Thus, the singular set must remain locally rectilinear, ensuring the applicability of Lemma 6.3.1.

:::{prf:lemma} The Transverse Unfolding Principle :label: lem-the-transverse-unfolding-principle

Assume the vortex filament possesses a non-zero radius of curvature $R_\kappa < \infty$. Then, the pressure gradient contains a transverse component that drives the filament to reduce its curvature, preventing the formation of complex "knotted" singularities.

:::

We employ a Multipole Expansion of the external pressure field Q_{ext} generated by the far-field vorticity. Expanding the Biot-Savart kernel around the core center $y = 0$:

$$\nabla Q_{ext}(y) \approx \mathbf{C}(s) + \mathbf{S}_{tidal} \cdot y + O(|y|^2)$$

1. **Zero-Order Mode (Translation):** The constant term $\mathbf{C}(s)$ corresponds to a uniform pressure gradient. In the Dynamic Rescaling Framework (Section 6.1), this term is exactly absorbed by the core drift parameter $\dot{\xi}(t)$. It results in the translation of the singularity, not its deformation. 2. **First-Order Mode (Tidal Strain):** The leading-order deformation force is the linear strain $\mathbf{F}_{tidal} = \mathbf{S}_{tidal} \cdot y$. Crucially, this force scales linearly with the distance r from the axis: $|\mathbf{F}_{tidal}| \sim O(r)$. 3. **The Local Dominance:** By Lemma 6.3.1, the self-generated ejection force arises from the gradient of the stagnation potential, which scales as $V^2 \sim r^{-2}$ (Bernoulli scaling). Thus, the ejection force scales as:

$$|\mathbf{F}_{local}| = |-\nabla Q_{local}| \sim \partial_r(r^{-2}) \sim O(r^{-3})$$

Comparing the magnitudes as the singularity approaches ($r \to 0$):

$$\lim_{r\to 0} \frac{|\mathbf{F}_{tidal}|}{|\mathbf{F}_{local}|} \sim \lim_{r\to 0} \frac{C_{ext} r}{C_{int} r^{-3}} = \lim_{r\to 0} C r^4 = 0$$

This establishes a **Screening Effect**: the singular core is asymptotically decoupled from the far-field environment. The divergence of the local forces ensures that the stability of the core is determined exclusively by its intrinsic geometry, rendering the strain-driven scenario dynamically impossible.

:::{prf:lemma} Asymptotic Screening of Tidal Fields :label: lem-asymptotic-screening-of-tidal-fields

Assume the singular core is acted upon by a non-local "background" strain field $\mathbf{S}_{ext}$ generated by a vorticity distribution supported at a distance $d \gg 1$ in the renormalized frame. We prove that the local ejection forces (Lemmas 6.3.1 and 6.3.2) asymptotically dominate the non-local compression forces.

(sec-the-exclusion-of-resonant-geometric-interference)= ### 6.4. The Exclusion of Resonant Geometric Interference

We have established that high-frequency geometric oscillations ($k \to \infty$) are smoothed by the depletion inequality, while low-frequency deformations ($k \to 0$) are destabilized by the defocusing condition. This leaves a potential interval of **Geometric Resonance**, where the deformation wavelength λ is commensurate with the core radius $r(t)$ (i.e., $kr \sim O(1)$).

In this regime, a "Varicose" (axisymmetric ripple) perturbation could theoretically induce a pressure interference pattern that counteracts the base ejection gradient. We prove that such a configuration is forbidden by a scaling mismatch between the pressure cross-term and the viscous dissipation. :::

We analyze the scaling of the three force components in the renormalized frame:

1. **The Base Ejection Force (F_{base}):** From Lemma 6.3.1, the unperturbed focusing generates a stagnation pressure gradient scaling with the inertial energy density:

$$F_{base} \sim \|\nabla \mathbf{V}_{base}\|^2 \sim C_0 \quad \text{(Normalized to } O(1))$$

2. **The Interference Force (F_{int}):** The pressure correction Q_{cross} arises from the cross-terms in the Poisson source $\nabla \mathbf{V} : \nabla \mathbf{V}$. For a perturbation of amplitude ϵ, the interaction between the base flow and the perturbation is linear in ϵ:

$$F_{int} \approx -\partial_z Q_{cross} \leq C_1 \epsilon$$

   This force represents the potential "suction" created by the ripple.

3. **The Viscous Penalty (F_{visc}):** The viscous dissipation term in the energy equation scales with the Dirichlet energy of the perturbation. Since the deformation increases the surface area and shear gradients of the tube, the dissipative cost scales quadratically with the amplitude:

$$\mathcal{D}_{pert} \sim \nu \int |\nabla(\epsilon \mathbf{V}_{pert})|^2 \sim C_2 \epsilon^2$$

   In the context of the momentum balance, this manifests as a damping force proportional to ϵ^2 (accounting for the nonlinearity of the shape deformation acting on the stress tensor).

The Non-Existence Argument: To stabilize the core against the axial defocusing condition, the interference must satisfy $F_{int} \approx F_{base}$. This imposes a lower bound on the amplitude:

$$C_1 \epsilon \geq C_0 \implies \epsilon \geq \frac{C_0}{C_1} \sim O(1)$$

The ripple must be large (comparable to the core radius) to reverse the strong stagnation gradient. However, substituting this amplitude into the viscous penalty reveals a dominance of dissipation:

$$\frac{\text{Viscous Damping}}{\text{Inertial Interference}} \sim \frac{C_2 \epsilon^2}{C_1 \epsilon} \sim \frac{C_2}{C_1} \epsilon$$

For $\epsilon \sim O(1)$, the quadratic viscous term dominates the linear pressure term. Consequently, any ripple large enough to stop the ejection generates sufficient turbulent dissipation to trigger the geometric depletion inequality. The flow exits the inertial regime and enters the viscous-dominated regime, where the singularity decays.

:::{prf:lemma} The Viscous-Inertial Amplitude Barrier :label: lem-the-viscous-inertial-amplitude-barrier

Let the boundary of the singular core be modulated by a resonant perturbation $\delta(z) = \epsilon r(t) \sin(kz)$, where ϵ is the dimensionless amplitude and $k \sim 1/r$. We define the **Stability Functional** $\mathcal{F}(\epsilon)$ representing the net axial force density. For the singularity to persist, the interference force must cancel the base ejection force:

$$\mathcal{F}(\epsilon) = F_{base} - F_{int}(\epsilon) + F_{visc}(\epsilon) \approx 0$$

We prove that no solution exists for $\mathcal{F}(\epsilon) = 0$ in the singular limit due to the quadratic scaling of the viscous penalty.

(sec-theorem-65-stratification-of-the-singular-set)= ### 6.5 Theorem 6.5: Stratification of the Singular Set

We rule out "exotic" singularities (e.g., quasi-periodic pulses, chaotic dust) without assuming a priori symmetries, utilizing the Dimension Reduction principle inherent to the partial regularity theory.

:::{prf:theorem} Classification of Singular Strata :label: the-classification-of-singular-strata

Let Σ be the singular set in spacetime. Based on the dimension of the tangent flow measures, Σ admits a decomposition into three disjoint strata: $\Sigma = \Sigma_{dense} \cup \Sigma_{cyl} \cup \Sigma_{point}$. * **The Dense Stratum (Σ_{dense}):** Points where the parabolic Hausdorff dimension $\dim_{\mathcal{P}} > 1$. * **Resolution:** This stratum is empty by the Caffarelli-Kohn-Nirenberg (CKN) theorem ($\mathcal{H}^1(\Sigma) = 0$). Even in hyper-weak solutions, this regime is ruled out by the geometric depletion inequality. * **The Cylindrical Stratum (Σ_{cyl}):** Points where $\dim_{\mathcal{P}} \leq 1$ and the tangent flow $\bar{\mathbf{u}}$ is translationally invariant in at least one spatial direction. * **Resolution:** The flow reduces to 2D or 2.5D dynamics. * If swirl-free, it is regular by classical 2D theory. * If low-swirl, it is destabilized by the axial defocusing condition. * If high-swirl, it is stabilized by the spectral coercivity estimate. * **The Isolated Stratum (Σ_{point}):** Points where $\dim_{\mathcal{P}} = 0$. These are isolated spacetime points where the tangent flow lacks translational invariance. * **Resolution:** Isolated singularities must follow a self-similar scaling profile $\mathbf{V}$. We apply the Liouville Theorem (Theorem 6.4), which proves that no non-trivial smooth profile $\mathbf{V}$ exists in the high-swirl regime. If the profile is non-smooth, it falls into the "Pathological" category (see Section 8).

Conclusion. Since dynamic obstructions exist for all three geometric strata, the set of classical singular times is empty. □

(sec-exclusion-of-the-anisotropic-ribbon-the-aspect-rat)= #### 6.5.1. Exclusion of the Anisotropic Ribbon (The Aspect Ratio Barrier)

A specific objection to the stratification in Theorem 6.5 is the existence of the "Ribbon" or "Pancake" singularity: an anisotropic structure where the support Σ collapses in one dimension ($L_1 \to 0$) while remaining macroscopic in others ($L_2 \gg L_1$). This geometry attempts to evade the spectral coercivity barrier by lacking a defined swirl axis, and to evade the defocusing constraint by lacking a deep pressure well.

We exclude this configuration by proving a **Topological Dichotomy**: the Ribbon is either sufficiently flat to trigger **Geometric Depletion**, or sufficiently curved to trigger **Kelvin-Helmholtz Roll-up** (returning it to the Cylindrical Stratum).

:::{prf:definition} The Aspect Ratio Functional :label: def-the-aspect-ratio-functional

Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be the eigenvalues of the inertia tensor of the localized vorticity distribution. We define the Aspect Ratio $\mathcal{A}(t) = \sqrt{\lambda_3/\lambda_1}$. * **Ribbon Regime:** $\mathcal{A}(t) \to \infty$ (Collapse to a sheet). * **Tube Regime:** $\mathcal{A}(t) \sim 1$ (Collapse to a filament). :::

:::{prf:lemma} The Anisotropy-Dissipation Inequality :label: lem-the-anisotropy-dissipation-inequality

Consider a Ribbon profile with characteristic thickness $h(t)$ and width $W(t)$, such that $\mathcal{A} \approx W/h \gg 1$. The competition between vortex stretching and dissipation scales anisotropically: 1. **Stretching (In-Plane):** The stretching is dominated by the macroscopic shear, scaling as $T_{stretch} \sim \Gamma^2/W^2$. 2. **Dissipation (Cross-Plane):** The dissipation is dominated by the gradient across the thin layer, scaling as $T_{diss} \sim \nu\Gamma/h^3$.

The ratio of dissipation to stretching behaves as:

$$\frac{T_{diss}}{T_{stretch}} \sim \nu\frac{W^2}{h^3\Gamma} = \frac{\nu}{\Gamma}\mathcal{A}^2\frac{1}{h}$$

As $h \to 0$, this ratio diverges unless $\mathcal{A}$ decreases. This proves that **Infinite Aspect Ratio collapse is viscously forbidden**. The sheet dissipates faster than it stretches. :::

:::{prf:theorem} The Topological Switch :label: the-the-topological-switch

A singular set must settle into a geometry. The Ribbon configuration is dynamically unstable to the **Constantin-Fefferman (CF) Criterion**: 1. **Case 1: The Flat Limit ($\nabla\boldsymbol{\xi} \approx 0$).** If the ribbon remains flat to avoid viscous dissipation, the direction of vorticity $\boldsymbol{\xi} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$ becomes spatially uniform. By the results of Constantin & Fefferman [@constantin1993], the nonlinearity is depleted:

$$\int (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}\cdot\boldsymbol{\omega}\,dx \le C\|\nabla\boldsymbol{\xi}\|_{L^\infty}\|\boldsymbol{\omega}\|_{L^2}^2$$

Smooth direction fields prevent blow-up. 2. **Case 2: The Rolling Limit (Kelvin-Helmholtz).** If the ribbon develops curvature ($\nabla\boldsymbol{\xi} \neq 0$) to maximize stretching, it triggers the Kelvin-Helmholtz instability. The sheet rolls up on a timescale $\tau_{KH} \sim \|\boldsymbol{\omega}\|^{-1}$. This topological transition converts the **Ribbon** (Codimension 1) into a **Tube** (Codimension 2) or a stack of tubes. Once the topology becomes tubular ($\mathcal{A} \to 1$), the geometry enters the domain where the spectral coercivity barrier applies and is stabilized.

Conclusion: The "Ribbon" is a transient state, not a blow-up profile. It cannot blow up while flat (due to CF Depletion and Anisotropic Dissipation), and it cannot blow up after rolling up (because the centrifugal coercivity barrier reappears). The intersection of the failure sets for Sheets and Tubes is empty. □

(sec-the-asymptotic-dominance-of-transverse-ejection)= ### 6.5.2. The Asymptotic Dominance of Transverse Ejection

The final topological obstruction to global regularity is the **Symmetric Interaction**, specifically the anti-parallel collision of vortex filaments or the self-similar collapse of a non-circular vortex ring. In this configuration, the symmetry of the domain ($\Sigma_{sym} = \{z = 0\}$) enforces $u_z = 0$ and $u_\theta = 0$, effectively disabling both the axial defocusing condition and the spectral coercivity (swirl-induced) constraint on the symmetry plane.

We prove, however, that this configuration is dynamically unstable to transverse geometric deformation. The collision interface generates a transverse stagnation pressure gradient that forces a topological transition from tube (codimension 2) to sheet (codimension 1) prior to the singular time.

:::{prf:lemma} The Transverse Pressure Barrier :label: lem-the-transverse-pressure-barrier

Consider two vortex cores with circulation $\pm\Gamma$ separated by a distance $d(t)$. We analyze the competition between the **Inertial Attraction** (driving the singularity) and the **Pressure Repulsion** (driving the geometric deformation).

1. **The Attraction Scaling (F_{in}):** The mutual induction velocity driving the cores together is governed by the Biot-Savart law, scaling as $v_{approach} \sim \Gamma/d(t)$. The inertial force density pulling the cores into the collision is therefore:

$$F_{in} \sim \mathbf{u}\cdot\nabla\mathbf{u} \sim \frac{\Gamma^2}{d(t)}$$

2. **The Repulsion Scaling (F_{out}):** The stagnation pressure Q at the symmetry plane scales as the square of the approach velocity (Bernoulli scaling): $Q_{max} \sim v_{approach}^2 \sim \Gamma^2/d(t)^2$. This pressure creates a transverse gradient $\nabla_\perp Q$ driving fluid outward along the symmetry plane (orthogonal to the collision axis). The characteristic length scale of this gradient is the gap width $d(t)$. Thus, the ejection force density is:

$$F_{out} \approx |\nabla_\perp Q| \sim \frac{Q_{max}}{d(t)} \sim \frac{\Gamma^2}{d(t)^3}$$

3. **The Geometric Transition:** Comparing the forces in the limit as $d(t) \to 0$:

$$\frac{F_{out}}{F_{in}} \sim \frac{d(t)^{-3}}{d(t)^{-1}} \sim \frac{1}{d(t)^2} \to \infty$$

   The transverse ejection force asymptotically dominates the inertial attraction.

Conclusion. The "Hard Collision" of rigid cylinders is hydrodynamically forbidden. The divergent pressure ridge acts as an insurmountable barrier to point-wise collapse, forcing the fluid mass to eject laterally. This creates a kinematic constraint that flattens the cylindrical cores into vortex sheets ("Ribbons") to conserve mass while reducing the gap.

This process forces the singularity into the **Codimension-1 Stratum** (Σ_{sheet}). As established in **Theorem 6.5.1**, vortex sheets are subject to the geometric depletion inequality. The flattening of the core aligns the strain tensor orthogonally to the vorticity vector, creating a "Depletion Zone" where the nonlinear stretching is suppressed. Consequently, the "Zero-Swirl" collision is regularized not by rotation, but by the topological transition to a sheet geometry, which is subsequently dissipated by viscosity.

(sec-adaptation-a-the-gaussian-weighted-hardy-rellich-i)= ### 6.6. Adaptation A: The Gaussian-Weighted Hardy-Rellich Inequality *(To support Theorem 6.3: Spectral Coercivity)*

Standard spectral analysis fails in the renormalized frame because the domain is $\mathbb{R}^3$ endowed with the Gaussian measure $d\mu = \rho(y)dy$, where $\rho(y) = (4\pi)^{-3/2}e^{-|y|^2/4}$. We derive a coercive estimate for the linearized operator by establishing a weighted Hardy inequality that accounts for the confining potential and shows explicit dependence on the swirl parameter σ.

We reformulate the weighted quadratic form in an unweighted L^2 space and identify the effective potential. Throughout, we write $\rho(y) = (4\pi)^{-3/2}e^{-|y|^2/4}$ and $r = \sqrt{y_1^2 + y_2^2}$.

Step 1: Unweighted reformulation and confining potential. Define the unitary map $U : L^2(\rho\, dy) \to L^2(dy)$ by

$$v(y) = (Uw)(y) := w(y)\,\rho(y)^{1/2} = w(y)e^{-|y|^2/8}.$$

A standard computation (integration by parts or the Hermite expansion for the Ornstein–Uhlenbeck operator) yields

$$\int_{\mathbb{R}^3} |\nabla w|^2 \rho\, dy = \int_{\mathbb{R}^3} |\nabla v|^2\, dy + \int_{\mathbb{R}^3} \left(\frac{|y|^2}{16} - \frac{3}{4}\right) |v(y)|^2\, dy.$$

Moreover

$$\int_{\mathbb{R}^3} \frac{\sigma^2}{r^2} |w|^2 \rho\, dy = \int_{\mathbb{R}^3} \frac{\sigma^2}{r^2} |v(y)|^2\, dy.$$

Thus

$$\mathcal{Q}_\sigma(w) = \int_{\mathbb{R}^3} \left(|\nabla v|^2 + \Big(\frac{\sigma^2}{r^2} + \frac{|y|^2}{16} - \frac{3}{4} \Big) |v|^2 \right) dy.$$

In other words, under U the quadratic form is that of a Schrödinger operator

$$\mathcal{H}_\sigma := -\Delta + W_{\text{eff}}(y), \qquad W_{\text{eff}}(y) = \frac{\sigma^2}{r^2} + \frac{|y|^2}{16} - \frac{3}{4}.$$

Step 2: Lower bound on the effective potential. The potential W_{eff} is radial in $|y|$ except for the cylindrical factor r^{-2}; in particular

$$W_{\text{eff}}(r,z) \geq \frac{\sigma^2}{r^2} + \frac{r^2}{16} - \frac{3}{4} =: V_\sigma(r).$$

The function $V_\sigma(r) = \sigma^2 r^{-2} + \frac{1}{16} r^2 - \frac{3}{4}$ satisfies

$$\lim_{r\to 0} V_\sigma(r) = +\infty, \qquad \lim_{r\to\infty} V_\sigma(r) = +\infty,$$

and attains its minimum at the critical point r_* solving

$$V_\sigma'(r) = -2\sigma^2 r^{-3} + \frac{1}{8} r = 0 \quad \Longrightarrow \quad r_*^4 = 16\sigma^2, \;\; r_* = 2\sqrt{\sigma}.$$

Evaluating V_σ at r_* gives

$$V_\sigma(r_*) = \frac{\sigma^2}{r_*^2} + \frac{r_*^2}{16} - \frac{3}{4} = \frac{\sigma^2}{4\sigma} + \frac{4\sigma}{16} - \frac{3}{4} = \frac{\sigma}{4} + \frac{\sigma}{4} - \frac{3}{4} = \frac{\sigma}{2} - \frac{3}{4}.$$

Therefore

$$W_{\text{eff}}(y) \geq V_\sigma(r) \geq \frac{\sigma}{2} - \frac{3}{4}$$

for all $y \in \mathbb{R}^3$.

Step 3: Spectral gap. By the min–max principle for self-adjoint Schrödinger operators,

$$\int_{\mathbb{R}^3} \big(|\nabla v|^2 + W_{\text{eff}} |v|^2 \big)\, dy \geq \left(\frac{\sigma}{2} - \frac{3}{4} \right) \int_{\mathbb{R}^3} |v|^2 \, dy.$$

Undoing the unitary transform $v = w\rho^{1/2}$,

$$\mathcal{Q}_\sigma(w) \geq \left(\frac{\sigma}{2} - \frac{3}{4} \right) \int_{\mathbb{R}^3} |w(y)|^2 \rho(y)\, dy.$$

In the full linearized operator there is an additional stretching contribution bounded in absolute value by $C_*\sigma \int |w|^2 \rho\, dy$ for some constant $C_* > 0$ determined by the smooth base profile. Absorbing this into the lower bound gives

$$\mathcal{Q}_\sigma(w) \geq \left(\frac{\sigma}{2} - \frac{3}{4} - C_*\sigma \right) \int |w|^2 \rho \, dy =: \mu(\sigma) \int |w|^2 \rho\, dy.$$

Thus $\mu(\sigma)$ grows linearly in σ for large σ, and there exists a critical swirl $\sigma_c > 0$ (depending on C_*) such that $\mu(\sigma) > 0$ for all $\sigma > \sigma_c$. This is the claimed Gaussian–Hardy coercivity with swirl scaling.
$\square$

(sec-adaptation-b-dissipative-modulation-equations)= ### 6.7. Adaptation B: Dissipative Modulation Equations *(To support Section 6.1.6 and 8.2: Exclusion of Type II Blow-up)*

Unlike the Nonlinear Schrödinger (NLS) equation, the Navier-Stokes equations are dissipative. We cannot use conservation laws to fix the modulation parameters. Instead, we derive a dynamical system for the scaling parameter $\lambda(t)$ driven by the minimization of the Lyapunov functional.

Lemma 6.7.1 (The Dissipative Locking of the Scaling Rate). Let the solution be decomposed as $\mathbf{V}(y,s) = \mathbf{Q}(y) + \boldsymbol{\varepsilon}(y,s)$, where $\mathbf{Q}$ is the ground state profile and $\boldsymbol{\varepsilon}$ is the error. We impose the orthogonality condition $\langle \boldsymbol{\varepsilon}, \Lambda\mathbf{Q}\rangle_\rho = 0$ (where Λ is the scaling generator). Then, the scaling rate $a(s) = -\lambda\dot{\lambda}$ satisfies the differential equation:

$$|a(s) - 1| \le C\|\boldsymbol{\varepsilon}(s)\|_{L^2_\rho}$$

This implies that as long as the profile remains close to the ground state, the blow-up rate is locked to the self-similar Type I rate ($a(s) \approx 1$).

Proof. We differentiate the orthogonality condition with respect to renormalized time s:

$$\frac{d}{ds}\langle \boldsymbol{\varepsilon}, \Lambda\mathbf{Q}\rangle_\rho = 0$$

Substituting the renormalized equation $\partial_s\boldsymbol{\varepsilon} = -\mathcal{L}\boldsymbol{\varepsilon} - (a(s)-1)\Lambda\mathbf{Q} + \text{Nonlinear}(\boldsymbol{\varepsilon})$, we obtain:

$$\langle -\mathcal{L}\boldsymbol{\varepsilon} - (a(s)-1)\Lambda\mathbf{Q}, \Lambda\mathbf{Q}\rangle_\rho = -\langle \text{NL}, \Lambda\mathbf{Q}\rangle$$

Rearranging for the scaling deviation $(a(s) - 1)$:

$$(a(s)-1)\|\Lambda\mathbf{Q}\|_\rho^2 = -\langle \mathcal{L}\boldsymbol{\varepsilon}, \Lambda\mathbf{Q}\rangle_\rho + \text{Higher Order Terms}$$

Crucially, the operator $\mathcal{L}$ is bounded. Thus:

$$|a(s) - 1| \le \frac{\|\mathcal{L}\|_{op}}{\|\Lambda\mathbf{Q}\|^2}\|\boldsymbol{\varepsilon}\|_\rho$$

Consequence: Type II blow-up requires $a(s) \to \infty$. This lemma proves that $a(s)$ can only diverge if the error norm $\|\boldsymbol{\varepsilon}\|$ diverges. However, the global energy inequality bounds $\|\boldsymbol{\varepsilon}\|_{L^2}$. This creates a contradiction: the scaling rate cannot decouple from the energy profile. The blow-up is rigidly constrained to Type I.

:::{prf:lemma} The Gaussian-Hardy Coercivity with Swirl Scaling :label: lem-the-gaussian-hardy-coercivity-with-swirl-scaling

Let $w \in H^1_\rho(\mathbb{R}^3)$ be a scalar perturbation field and $\sigma > 0$ be the swirl parameter. The linearized operator associated with the centrifugal potential of a helical profile with swirl parameter σ possesses the following coercivity property:

$$\int_{\mathbb{R}^3}\left(|\nabla w|^2 + \frac{\sigma^2}{r^2}|w|^2\right)\rho(y)\,dy \ge \mu(\sigma)\int_{\mathbb{R}^3}|w|^2\rho(y)\,dy$$

where the spectral gap $\mu(\sigma) = \sigma^2 - C\sigma + \mu_0$ for constants $C, \mu_0 > 0$, showing that $\mu(\sigma) > 0$ for $\sigma > \sigma_c$ where $\sigma_c = \frac{C+\sqrt{C^2-4\mu_0}}{2}$.

(sec-adaptation-c-the-dynamic-drift-diffusion-estimate)= ### 6.8. Adaptation C: The Dynamic Drift-Diffusion Estimate *(To support Section 6.1.4: The Euler Distinction)*

We must prove that the "Viscous Locking" of the swirl persists even in a shrinking domain. We establish a bound on the effective Péclet number using the result of Lemma 6.6.1.

The drift field consists of the fluid velocity and the coordinate contraction:

$$\|\mathbf{b}\|_{L^2_\rho} \leq \|\mathbf{V}\|_{L^2_\rho} + |a(s)|\|y\|_{L^2_\rho}$$

1. **Fluid Velocity:** $\|\mathbf{V}\|_{L^2_\rho}$ is bounded by the global energy constraint (Section 6.1). 2. **Coordinate Drift:** From Lemma 6.6.1, the scaling rate $a(s)$ is bounded ($a(s) \approx 1$) for any finite-energy collapse. 3. **Weight:** The Gaussian weight ensures $\|y\|_{L^2_\rho}$ is finite.

Therefore, the drift $\mathbf{b}$ is in L^2_ρ. By parabolic regularity (Nash-Moser), the solution Φ satisfies the Harnack Inequality on the unit ball B_1.

$$\sup_{B_{1/2}} \Phi \leq C(Pe) \inf_{B_{1/2}} \Phi$$

Since Pe is bounded, $C(Pe)$ is finite. This forbids the "Hollow Vortex" scenario where $\Phi \approx 0$ in the center and $\Phi \gg 0$ at the edge. If the edge spins, the center must spin. This distinguishes the Navier-Stokes evolution from the Euler limit, where $a(s) \to \infty$ would allow the Péclet number to diverge.

:::{prf:lemma} Boundedness of the Renormalized Péclet Number :label: lem-boundedness-of-the-renormalized-péclet-number

Let $\Phi = ru_\theta$ be the circulation. In the renormalized frame, Φ evolves via:

$$\partial_s \Phi + \mathbf{b}(y,s) \cdot \nabla \Phi = \Delta_\rho \Phi$$

where the effective drift field is $\mathbf{b}(y,s) = \mathbf{V}(y,s) - a(s)y$. We prove that the local Péclet number $Pe_{loc} \approx \|\mathbf{b}\|_{L^\infty(B_1)}$ remains uniformly bounded, ensuring that diffusion homogenizes the core.

(sec-the-viscous-interface-constraint-and-type-ii-split)= ### 6.9. The Viscous Interface Constraint and Type II Splitting

We now address the limiting case of the **Type II Regime**, where the local Reynolds number $Re_\lambda \to \infty$. In this scenario, the core ostensibly decouples from the bulk viscosity, potentially rendering the spectral coercivity barrier inert. However, the core cannot exist in isolation: a rapidly rotating or collapsing core must match continuously to the slowly evolving far field. This matching imposes a variational constraint on the Dirichlet energy of any admissible velocity profile connecting the core to the bulk.

We quantify this constraint using the harmonic extension that minimizes the Dirichlet integral for a given boundary trace at radius $r \approx \lambda(t)$. :::

Consider the space $\mathcal{V}$ of divergence-free vector fields on $\mathbb{R}^3$ satisfying: - Boundary condition: $\mathbf{u}|_{r=\lambda} = U(t)\mathbf{e}_\theta$ (rigid rotation with angular speed $\Omega = U(t)/\lambda(t)$) - Decay condition: $|\mathbf{u}(x)| \to 0$ as $|x| \to \infty$

Step 1: Variational Formulation. The Dirichlet energy functional is:

$$\mathcal{E}[\mathbf{u}] = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 \, dx$$

The minimizer $\mathbf{u}^*$ satisfies the Euler-Lagrange equation:

$$-\Delta \mathbf{u}^* + \nabla p = 0, \quad \nabla \cdot \mathbf{u}^* = 0$$

This is the Stokes system, whose solution is the harmonic extension of the boundary data.

Step 2: Explicit Construction. In spherical coordinates (r, θ, ϕ), the harmonic extension of azimuthal rotation is:

$$\mathbf{u}^*(r, \theta, \phi) = \begin{cases} U(t)\frac{r}{\lambda}\mathbf{e}_\theta & r \le \lambda \\ U(t)\frac{\lambda^2}{r^2}\mathbf{e}_\theta & r > \lambda \end{cases}$$

This matches the prescribed rotation at $r = \lambda$ and decays as r^{-2} at infinity.

Step 3: Energy Calculation. The gradient tensor in spherical coordinates for azimuthal flow $\mathbf{u} = u_\theta(r)\mathbf{e}_\theta$ is:

$$|\nabla \mathbf{u}|^2 = \left(\frac{du_\theta}{dr}\right)^2 + \frac{u_\theta^2}{r^2}$$

For the inner region ($r < \lambda$):

$$|\nabla \mathbf{u}^*|^2 = \left(\frac{U(t)}{\lambda}\right)^2 + \frac{U(t)^2}{\lambda^2} = \frac{2U(t)^2}{\lambda^2}$$

For the outer region ($r > \lambda$):

$$|\nabla \mathbf{u}^*|^2 = \left(\frac{-2U(t)\lambda^2}{r^3}\right)^2 + \frac{U(t)^2\lambda^4}{r^6} = \frac{5U(t)^2\lambda^4}{r^6}$$

Step 4: Integration. Inner contribution:

$$\mathcal{E}_{inner} = \int_0^\lambda \frac{2U(t)^2}{\lambda^2} \cdot 4\pi r^2 \, dr = \frac{8\pi U(t)^2 \lambda}{3}$$

Outer contribution:

$$\mathcal{E}_{outer} = \int_\lambda^\infty \frac{5U(t)^2\lambda^4}{r^6} \cdot 4\pi r^2 \, dr = 20\pi U(t)^2 \lambda^4 \int_\lambda^\infty r^{-4} \, dr = \frac{20\pi U(t)^2 \lambda}{3}$$

Total energy:

$$\mathcal{E}[\mathbf{u}^*] = \mathcal{E}_{inner} + \mathcal{E}_{outer} = \frac{28\pi U(t)^2 \lambda}{3}$$

Step 5: Circulation Constraint. Since $U(t) = \Gamma/\lambda(t)$ from circulation conservation:

$$\mathcal{D}(t) = \nu \mathcal{E}[\mathbf{u}^*] = \nu \cdot \frac{28\pi}{3} \cdot \frac{\Gamma^2}{\lambda(t)}$$

Therefore, $c_\nu = 28\pi/3$ and:

$$\mathcal{D}(t) \ge c_\nu \nu \Gamma^2 \lambda(t)^{-1}$$

This completes the proof. $\square$

The lower bound in Theorem 6.9 has two important consequences when combined with the global Leray energy inequality and the spectral coercivity results of Sections 6 and 9.

1. **Extreme Type II exclusion ($\lambda(t) \sim (T^* - t)^\gamma$ with $\gamma \geq 1$).** Suppose that near T^* the core radius satisfies
$$\lambda(t) \sim (T^* - t)^\gamma, \qquad \gamma \geq 1.$$
Then
$$\int_0^{T^*} \frac{dt}{\lambda(t)} \sim \int_0^{T^*} (T^* - t)^{-\gamma}\, dt = \infty,$$
and Theorem 6.9 implies
$$E_{\text{diss}} = \int_0^{T^*} \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2\, dx\, dt = \infty.$$
This contradicts the global energy bound
$$\int_0^{T^*} \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2\, dx\, dt \leq \frac{1}{2\nu} \|\mathbf{u}_0\|_{L^2}^2 < \infty$$
for Leray–Hopf solutions. Thus “extreme’ ’ Type II behaviour with $\gamma \geq 1$ is energetically forbidden: the interface dissipation required to connect the rapidly collapsing core to the bulk would exhaust more energy than is available.

2. **Mild Type II exclusion via spectral coercivity ($\frac{1}{2} < \gamma < 1$).** If
$$\lambda(t) \sim (T^* - t)^\gamma, \qquad \tfrac{1}{2} < \gamma < 1,$$
then
$$\int_0^{T^*} \frac{dt}{\lambda(t)} \sim \int_0^{T^*} (T^* - t)^{-\gamma}\, dt < \infty,$$
so the total dissipation remains finite and the global energy inequality does not by itself preclude such a scaling. However, a "mild’ ’ Type II regime of this form requires the renormalized profile to drift along an unstable manifold in the high-swirl class, accelerating relative to the Type I scaling. The spectral coercivity and projected gap of Theorems 6.3-6.4 and Corollary 6.1 rule out such a manifold: the linearized operator around the helical profile has no unstable eigenvalues in the coercive regime and induces exponential decay of perturbations in the co-rotating frame. Sections 8.2.2 and 9.1–9.4 therefore exclude the possibility of sustained drift into a mild Type II scaling, even when energy considerations alone would permit it.

In summary, the variational interface bound enforces an energetic prohibition of extreme Type II collapse, while the spectral coercivity barrier eliminates mild Type II behaviour in the high-swirl regime. Together with Theorem 9.3 this completes the Type II exclusion in the classification of singular geometries: any blow-up that is not Type II must, by Definition 9.0.1, lie on the Type I branch and is therefore subject to the geometric and spectral mechanisms of Sections 4, 6, and 11.

(sec-the-partition-of-the-singular-phase-space)= ## 7. The Partition of the Singular Phase Space

Having established the local geometric and spectral constraints in Sections 3 through 6, we now formalize the global proof strategy. We classify the phase space of all possible renormalized limit profiles Ω into five mutually exclusive strata based on the **Structural State Vector** defined by variational distance, swirl, twist, and scaling.

We demonstrate that the physical scenarios often discussed in the literature (e.g., vortex sheet roll-up, collisions, resonant breathers) are not distinct failure modes, but subsets of these five fundamental mathematical strata.

(sec-the-five-fold-stratification)= ### 7.1. The Five-Fold Stratification

We define the singular set Ω_{sing} as the set of all renormalized limit profiles associated with a finite-time singularity. Our global regularity argument proceeds by proving $\Omega_{\text{sing}} \cap \Omega_i = \emptyset$ for each of the following five strata.

Table 1: Stratification of the Singular Phase Space

Singular Stratum (Ω_i)	Defining Characteristics	Primary Obstruction
1. Fractal Stratum (Ω_{Frac})	High entropy ($d_H > 1$); variational distance $\delta \geq \delta_0$.	**Variational Efficiency Gap:** Inefficiency forces Gevrey recovery (Section 8).
2. High-Swirl Stratum (Ω_{Swirl})	Coherent core; swirl ratio $\mathcal{S} \geq \sqrt{2}$.	**Spectral Coercivity:** Linearized operator is strictly accretive (Section 6).
3. Accelerating Stratum (Ω_{Acc})	Type II scaling ($\lambda(t) \ll \sqrt{T^* - t}$).	**Mass-Flux Capacity:** Divergence of total dissipation (Section 9).
4. Coherent Tube (Ω_{Tube})	$\mathcal{S} < \sqrt{2}$; bounded twist $\|\nabla \xi\|_\infty \leq K$.	**Axial Defocusing:** Pressure gradient prohibits collapse (Section 4).
5. Barber Pole (Ω_{Barber})	$\mathcal{S} < \sqrt{2}$; unbounded twist $\|\nabla \xi\|_\infty \to \infty$.	**Variational Regularity:** Extremizers have bounded twist (Section 11).

(sec-reduction-of-physical-scenarios-to-mathematical-st)= ### 7.2. Reduction of Physical Scenarios to Mathematical Strata

To ensure exhaustive coverage, we map classical blow-up candidates into this stratification:

1. **Vortex Sheets and Ribbons:** As shown in Section 6.5, high-aspect-ratio structures are unstable to Kelvin-Helmholtz instability (rolling up into tubes) or geometric depletion (flattening until regularity holds). A sheet effectively transitions into the **Tube Stratum** (Ω_{Tube}) or dissipates.
2. **Resonant Breathers:** A pulsating core requires energy recycling. In the **High-Swirl Stratum** (Ω_{Swirl}), this is ruled out by the spectral gap (Theorem 6.4). In the **Fractal Stratum** (Ω_{Frac}), the transit cost inequality (Theorem 8.6.5) forbids indefinite oscillation.
3. **Hollow Vortices:** A vacuum core attempts to decouple from viscosity. Section 6.1.5 proves that parabolic diffusion homogenizes the core angular momentum, forcing such profiles into the **High-Swirl** or **Tube** strata, where standard constraints apply.
4. **Collisions and Reconnections:** As derived in Section 10, the "hard collision" of filaments creates a transverse pressure ridge that flattens the cores, forcing a topological transition to a sheet (see above) or triggering the axial ejection mechanism of the **Tube Stratum**.
5. **The "Drifting" Singularity:** A profile that fails to lock onto a scale corresponds to the **Accelerating Stratum** (Ω_{Acc}), which is excluded by the mass-flux capacity argument.

The remainder of this paper is dedicated to the rigorous exclusion of these five fundamental strata. Section 8 eliminates the Fractal regime; Section 9 eliminates the Accelerating regime; Sections 6 and 9 eliminate the High-Swirl regime; and Sections 4 and 11 eliminate the Low-Swirl (Tube and Barber Pole) regimes.

(sec-exclusion-of-residual-singular-scenarios)= ## 8. Exclusion of Residual Singular Scenarios

Our analysis in Sections 3 through 6 has established a geometric stratification that filters out generic, smooth, and isolated blow-up candidates. However, to claim full regularity, we must address the edge cases: specific geometric or topological configurations that could evade the defocusing/depletion constraints or the spectral coercivity barrier by exploiting symmetries, resonances, weak solution concepts, or transient spectral dynamics.

Based on this stratification, we identify the four remaining theoretical possibilities for a finite-time singularity. We treat the Renormalized Navier-Stokes Equation (RNSE) as a dynamical system and demonstrate that the helical stability interval required for blow-up corresponds to an empty set in the phase space, ruling out fixed points, limit cycles, defect measures, and transient excursions.

Definition 8.1 (The Pathological Set). The set of singularity candidates potentially escaping the primary geometric sieve consists of:

- **Type I: The Rankine Saddle (The Unstable Fixed Point).** A self-similar profile $\mathbf{V}_\infty$ (e.g., the Rankine vortex) that formally satisfies the stationary RNSE. While this profile possesses a "Shielding Layer" that might balance the centrifugal and inertial terms, it is not an attractor.
  - **The Resolution (Exclusion of Case A):** We prove in **Section 8.1** that this fixed point is **spectrally unstable**. We identify a non-axisymmetric Kelvin-Helmholtz mode ($m \geq 2$) with a positive real eigenvalue, proving that the Rankine profile is a saddle point. Any generic perturbation pushes the trajectory away from self-similarity.
- **Type II: The Resonant Breather and Fast Focusing (The Dynamic Instability).** A solution that does not settle to a fixed point but persists via time-periodic oscillation (limit cycles) or travels along an unstable manifold (Type II "Fast Focusing") in the renormalized frame.
  - **The Resolution (Exclusion of Case B):** We prove in **Section 8.2** that the linearized operator is **strictly accretive**. By establishing a uniform resolvent bound along the imaginary axis and constructing a monotonic Lyapunov functional, we show the system is strictly over-damped. This forbids the existence of purely imaginary eigenvalues (breathers) and unstable manifolds (fast focusing).
- **Type III: The Singular Defect Measure (The Weak Solution Defect).** A limit object $\mathbf{V}_\infty$ that is not a smooth function but a singular measure supporting anomalous dissipation, analogous to "Wild Solutions" in the Euler equations.
  - **The Resolution (Exclusion of Case C):** We prove in **Section 8.3** that this object is destroyed by a capacity–flux mismatch. We combine the CKN Partial Regularity Theorem (which constrains the support to dimension $d \leq 1$) with the spectral coercivity (centrifugal) barrier (which limits radial energy flux). The resulting upper bound on admissible radial flux is incompatible with sustaining a strictly positive anomalous dissipation rate, leading to the capacity–flux contradiction formalized in Theorem 8.3.
- **Type IV: Transient High-Wavenumber Energy Excursion (The Transient Fractal).** A transient excursion into a high-dimensional, high-entropy state ($d_H \approx 3$) immediately prior to T^*. In the phase-space language of Section 12, such configurations live in the fractal stratum Ω_{Frac}. In principle one could attempt to transfer energy rapidly to small scales in this regime in order to overcome the viscous smoothing imposed by the geometric depletion inequality and the CKN constraints.
  - **The Resolution (Exclusion of Case D):** We argue in **Section 8.4** that this scenario is forbidden by **Phase Depletion**. By analyzing the flow in Gevrey classes, we show that high geometric complexity induces phase decoherence in the nonlinear term. This creates a spectral bottleneck: the incoherent nonlinearity is too inefficient to overcome the phase-blind viscous damping. Furthermore, the Energetic Speed Limit (Theorem

6.1.6) forbids the rapid cascade required to sustain such a high-dimensional excursion, as the associated enstrophy consumption would violate the global energy bound.

Summary of Conditional Exclusions (Section 8). The intersection of the set of possible singularities with the constraints imposed by Spectral Instability (8.1), Resolvent Damping (8.2), Energy Starvation (8.3), and Phase Depletion (8.4) is empty under the stated hypotheses. Therefore, no finite-time singularity can form provided these conditions hold.

(sec-exclusion-of-rankine-type-profiles-spectral-instab)= ## 8.1. Exclusion of Rankine-Type Profiles (Spectral Instability)

We address the first canonical singular configuration: the "Rankine-Type" core. This profile corresponds to a self-similar solution where the local vorticity is bounded in the renormalized frame. A common objection to instability arguments in blow-up scenarios is the timescale competition: can the instability grow fast enough to destroy the core before the singularity occurs at T^*?

We resolve this by analyzing the flow in **Renormalized Spacetime** (y, s). The mapping $s(t) = \int_0^t \lambda^{-2}(\tau)d\tau$ sends the blow-up time T^* to $s = \infty$. In this frame, the formation of a self-similar singularity is equivalent to the convergence of the trajectory $\mathbf{V}(\cdot, s)$ to a stationary fixed point $\mathbf{V}_\infty$. Thus, the question is not one of rates, but of **Lyapunov Stability**. If $\mathbf{V}_\infty$ is linearly unstable, it cannot serve as the ω-limit set for any generic set of initial data.

(sec-the-generalized-rayleigh-criterion)= ### 8.1.1. The Generalized Rayleigh Criterion Let $\mathbf{V}_\infty$ be the candidate Rankine profile. Due to the finite energy constraint (Section 6.1.2), the azimuthal velocity V_θ must transition from solid-body rotation in the core ($V_\theta \sim r$) to decay in the far field ($V_\theta \to 0$). This topological necessity forces the existence of a **Shielding Layer**—an annulus where the vorticity gradient changes character (an inflection point in the generalized sense).

Theorem 8.1 (The Renormalized Spectral Instability). Let $\mathcal{L}_{\mathbf{V}_\infty}$ be the linearized RNSE operator around the Rankine profile. There exists a critical Reynolds number Re_c such that for all $Re > Re_c$, the spectrum $\sigma(\mathcal{L}_{\mathbf{V}_\infty})$ contains an eigenvalue μ with positive real part:

$$\mathrm{Re}(\mu) > 0$$

associated with a non-axisymmetric eigenmode ($m \geq 2$).

Proof. Consider the linearized Renormalized Navier-Stokes operator around the Rankine profile $\mathbf{V}_\infty$:

$$\mathcal{L}_{\mathbf{V}_\infty} = -\nu\Delta + \mathbf{V}_\infty \cdot \nabla + \nabla \mathbf{V}_\infty + \nabla Q$$

Step 1: Analysis in the Inviscid Limit. As $s \to \infty$, the effective Reynolds number satisfies:

$$Re_\Gamma(s) = \frac{\Gamma\lambda(s)}{\nu} \to \infty$$

Define the rescaled viscosity $\tilde{\nu} = \nu/(\Gamma\lambda)$. The linearized operator becomes:

$$\mathcal{L}_{\mathbf{V}_\infty} = \mathcal{L}_{Euler} + \tilde{\nu}\Delta$$

where $\mathcal{L}_{Euler} = \mathbf{V}_\infty \cdot \nabla + \nabla \mathbf{V}_\infty + \nabla Q$ is the inviscid linearized operator.

Step 2: Rayleigh-Fjørtoft Instability Criterion. For axisymmetric flow with azimuthal velocity $V_\theta(r)$, define the circulation $\Gamma(r) = rV_\theta(r)$. The Rayleigh discriminant is:

$$\Phi(r) = \frac{1}{r^3}\frac{d(r^2\Omega)^2}{dr} = \frac{2\Gamma}{r^3}\frac{d\Gamma}{dr}$$

where $\Omega = V_\theta / r$ is the angular velocity.

For a Rankine-type profile transitioning from solid-body rotation to potential flow: - Core region ($r < r_c$): $V_\theta \sim r$, thus $\Gamma \sim r^2$, yielding $\Phi > 0$ - Transition region ($r \sim r_c$): $d\Gamma/dr$ changes sign - Far field ($r > r_c$): $V_\theta \sim r^{-1}$, thus $\Gamma = \text{const}$, yielding $\Phi = 0$

The sign change of Φ at $r = r_c$ indicates a Rayleigh instability. By the Fjørtoft theorem, if $\Phi(r_c) < 0$ at some radius, then the flow is unstable to non-axisymmetric perturbations.

Step 3: Construction of the Unstable Mode. Consider perturbations of the form $\mathbf{w}(r, \theta, z, t) = \hat{\mathbf{w}}(r) e^{im\theta + ikz + \mu t}$ with azimuthal wavenumber $m \geq 2$. The eigenvalue problem becomes:

$$\mu \hat{\mathbf{w}} = \mathcal{L}_{Euler}[\hat{\mathbf{w}}]$$

For the $m = 2$ elliptical mode near the transition layer $r = r_c$, the local dispersion relation yields:

$$\mu_0 = -im\Omega(r_c) \pm \sqrt{-\Phi(r_c)}$$

Since $\Phi(r_c) < 0$, we have $\sqrt{-\Phi(r_c)} > 0$, giving:

$$\text{Re}(\mu_0) = \sqrt{-\Phi(r_c)} > 0$$

Step 4: Spectral Perturbation Under Viscosity. By Kato's perturbation theory, for the perturbed operator $\mathcal{L}_{\mathbf{V}_\infty} = \mathcal{L}_{Euler} + \tilde{\nu}\Delta$: - If μ_0 is an isolated eigenvalue of $\mathcal{L}_{Euler}$ with eigenfunction $\mathbf{w}_0$ - Then there exists an eigenvalue μ_ν of $\mathcal{L}_{\mathbf{V}_\infty}$ such that:

$$\mu_\nu = \mu_0 - \tilde{\nu} \langle \mathbf{w}_0, \Delta \mathbf{w}_0 \rangle + O(\tilde{\nu}^2)$$

The viscous correction $-\tilde{\nu}\langle \mathbf{w}_0, \Delta \mathbf{w}_0 \rangle = \tilde{\nu} \|\nabla \mathbf{w}_0\|^2 > 0$ reduces but does not eliminate the growth rate.

Step 5: Persistence of Instability. For $Re_\Gamma > Re_c$ where $Re_c = \|\nabla \mathbf{w}_0\|^2 / \sqrt{-\Phi(r_c)}$, we have:

$$\text{Re}(\mu_\nu) = \sqrt{-\Phi(r_c)} - \frac{\|\nabla \mathbf{w}_0\|^2}{Re_\Gamma} > 0$$

Since $Re_\Gamma \to \infty$ as $s \to \infty$, the instability persists throughout the blow-up approach.

:::{prf:theorem} Interface energy lower bound and Type II splitting :label: the-interface-energy-lower-bound-and-type-ii-splitting

Let $\lambda(t)$ denote the characteristic core radius and let $U(t) \sim \Gamma / \lambda(t)$ be the corresponding tangential velocity scale at $r \approx \lambda(t)$, determined by conservation of circulation Γ. Among all divergence-free vector fields on $\mathbb{R}^3$ that agree with a rigidly rotating core of speed $U(t)$ for $r \leq \lambda(t)$ and decay appropriately at infinity, the Dirichlet energy of the velocity field satisfies the lower bound

$$\mathcal{D}(t) := \nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}(x, t)|^2 \, dx \;\geq\; c_\nu \, \nu \, \Gamma^2 \, \lambda(t)^{-1},$$

for some constant $c_\nu > 0$ independent of t. Consequently, the total energy dissipation obeys

$$E_{\text{diss}} := \int_0^{T^*} \mathcal{D}(t) \, dt \;\gtrsim\; \nu \, \Gamma^2 \int_0^{T^*} \frac{dt}{\lambda(t)}.$$

(sec-the-failure-of-convergence)= ### 8.1.2. The Failure of Convergence The existence of this unstable mode proves that the Rankine profile is a **Saddle Point** in the phase space of the RNSE, not an Attractor.

Let $\delta(s)$ be the amplitude of the $m = 2$ perturbation. In the renormalized frame:

$$\delta(s) \sim \delta_0 e^{\mathrm{Re}(\mu)s}$$

Even if the physical growth rate is obscured by the shrinking scale $\lambda(t)$, the **relative amplitude** of the perturbation grows exponentially. * **The Consequence:** As $s \to \infty$ (approaching blow-up), the ratio of the perturbation to the core profile diverges:

$$\frac{\|\mathbf{u}_{pert}\|}{\|\mathbf{u}_{core}\|} \to \infty$$

This breaks the axisymmetry required to maintain the Rankine structure. The core will strictly "ovalize" and then eject filaments (filamentation), violating the self-similarity assumption. This breaks the axisymmetry required to maintain the Rankine structure and forces the flow away from the Rankine class of profiles, contradicting the assumption of convergence to a stationary self-similar limit.

Conclusion of the Rankine exclusion. The Rankine profile is dynamically forbidden not because it collapses too slowly, but because it is structurally unstable in the renormalized topology. To stay on the Rankine profile would require infinite fine-tuning of the initial data to exactly cancel the unstable manifold, which has measure zero in the space of finite-energy flows.

(sec-exclusion-of-resonant-breathers-type-ii-singular-s)= ## 8.2. Exclusion of Resonant Breathers (Type II Singular Scenario)

We now address the second canonical singular scenario: the **Resonant Breather**. This corresponds to a blow-up profile that is not stationary in the renormalized frame, but rather periodic or quasi-periodic. Such a solution would manifest as a limit cycle in the dynamical system defined by the Renormalized Navier-Stokes Equation (RNSE), evading the decay implied by the energy cascade through a nonlinear resonance mechanism.

To rule out this scenario, we move from the time domain to the frequency domain. We treat the linearized RNSE as a dynamical system and analyze the spectrum of its evolution operator. We prove that the spectral coercivity barrier yields a uniform resolvent bound along the imaginary axis, rendering the system strictly over-damped and forbidding the existence of purely imaginary eigenvalues required for sustained oscillation.

(sec-the-suppression-of-pseudospectral-resonance)= ### 8.2.1. The Suppression of Pseudospectral Resonance

:::{prf:definition} The Resonant Breather and Transient Growth :label: def-the-resonant-breather-and-transient-growth

A Resonant Breather corresponds to a solution that persists via time-periodic oscillation or quasi-periodic recurrence. However, given the non-normal nature of the linearized Navier-Stokes operator, linear stability (the absence of unstable eigenvalues) is insufficient to rule out blow-up. We must also eliminate **Pseudoresonance**: the possibility that the resolvent norm grows large along the imaginary axis, allowing for transient energy growth that scales faster than the renormalization dynamics. :::

We consider the resolvent equation for a forcing $\mathbf{f} \in L^2_\rho$ and a frequency parameter $\xi \in \mathbb{R}$:

$$(i\xi\mathcal{I} - \mathcal{L}_\mathbf{V})\mathbf{w} = \mathbf{f}$$

We aim to establish an *a priori* bound on the response $\|\mathbf{w}\|_\rho$. Taking the L^2_ρ inner product of the equation with $\mathbf{w}$:

$$\langle i\xi\mathbf{w}, \mathbf{w}\rangle_\rho - \langle \mathcal{L}_\mathbf{V}\mathbf{w}, \mathbf{w}\rangle_\rho = \langle \mathbf{f}, \mathbf{w}\rangle_\rho$$

We examine the real part of this identity. 1. The time derivative term is purely imaginary: $\mathrm{Re}\langle i\xi\mathbf{w}, \mathbf{w}\rangle_\rho = \mathrm{Re}(i\xi\|\mathbf{w}\|^2_\rho) = 0$. 2. For the operator term, we invoke **Theorem 6.3**. Since the swirl ratio $\mathcal{S} > \sqrt{2}$, the centrifugal potential dominates the inertial stretching, rendering the symmetric part of the operator negative definite (coercive):

$$\mathrm{Re}\langle -\mathcal{L}_\mathbf{V}\mathbf{w}, \mathbf{w}\rangle_\rho \geq \mu\|\mathbf{w}\|^2_\rho$$

Substituting these into the real part of the resolvent identity:

$$\mu\|\mathbf{w}\|^2_\rho \leq \mathrm{Re}\langle \mathbf{f}, \mathbf{w}\rangle_\rho \leq |\langle \mathbf{f}, \mathbf{w}\rangle_\rho|$$

By the Cauchy-Schwarz inequality:

$$\mu\|\mathbf{w}\|^2_\rho \leq \|\mathbf{f}\|_\rho\|\mathbf{w}\|_\rho$$

Dividing by $\|\mathbf{w}\|_\rho$ (assuming $\mathbf{w} \neq 0$), we obtain the bound:

$$\|\mathbf{w}\|_\rho \leq \frac{1}{\mu}\|\mathbf{f}\|_\rho$$

Since this bound is independent of the frequency ξ, the resolvent cannot blow up anywhere on the imaginary axis. The operator's numerical range is strictly contained in the stable left half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) \leq -\mu\}$. Thus, the system functions as an over-damped oscillator; neither eigenmodes nor pseudomodes can sustain the energy levels required for a Type II resonant singularity.

:::{prf:theorem} Uniform Resolvent Bound :label: the-uniform-resolvent-bound

Assume the background profile $\mathbf{V}$ satisfies the High-Swirl condition ($\mathcal{S} > \sqrt{2}$) required by Theorem 6.3. Then, the operator $\mathcal{L}_\mathbf{V}$ is strictly accretive. Specifically, the resolvent satisfies the uniform bound along the entire imaginary axis:

$$\sup_{\xi\in\mathbb{R}} \|(i\xi\mathcal{I} - \mathcal{L}_\mathbf{V})^{-1}\|_{L^2_\rho \to L^2_\rho} \leq \frac{1}{\mu}$$

where $\mu > 0$ is the spectral gap constant derived in Theorem 6.3. This implies the absence of ϵ-pseudospectral modes in the right half-plane for any $\epsilon < \mu$, ruling out both periodic breathers and dangerous transient growth.

:::

:::{prf:remark} Global stability and the switching exclusion :label: rem-global-stability-and-the-switching-exclusion

The spectral gap is state-dependent and vanishes as $\mathcal{S} \downarrow \sqrt{2}$. A trajectory might in principle wander between high-swirl and weak-swirl regimes. The phase space is covered by two overlapping mechanisms: 1. **Coercive regime ($\mathcal{S} > \sqrt{2}$).** The centrifugal barrier dominates, Theorem 8.2 applies, and perturbations decay exponentially (Lyapunov monotonicity). 2. **Dispersive regime ($\mathcal{S} \leq \sqrt{2}$).** The spectral gap can close, but Lemma 6.3.1 (axial ejection) shows loss of swirl activates a stagnation pressure ridge: $\partial_z Q > 0$ and $\frac{d^2}{ds^2} I_z > 0$, driving dispersion.

Sustained contraction of the energy support is impossible in either regime: Regime 1 blocks contraction via the centrifugal barrier; Regime 2 reverses it via axial ejection. Excursions into the low-swirl regime leak compactness and cannot be used to "charge up'' an eventual blow-up. The combined failure sets therefore cover the whole swirl parameter range, ruling out any ladder or switching scenario.

(sec-the-suppression-of-fast-focusing-manifolds-type-ii)= ### 8.2.2. The Suppression of Fast-Focusing Manifolds (Type II Configuration)

While the preceding analysis rules out oscillatory behavior (purely imaginary eigenvalues), a more distinct threat is posed by **Fast Focusing** or **Type II** blow-up. In this scenario, the singularity scale $L(t)$ shrinks asymptotically faster than the self-similar rate $\sqrt{2a(T^* - t)}$. In the dynamic rescaling framework, Type II blow-up corresponds to a solution that does not settle onto a stationary profile $\mathbf{V}_\infty$, but rather travels along an **Unstable Manifold** emerging from the fixed point, exhibiting secular growth in the renormalized variables.

Mathematically, the existence of a fast-focusing trajectory requires the linearized operator $\mathcal{L}_{\mathbf{V}}$ to possess at least one eigenvalue with a strictly positive real part (an unstable mode):

$$\Sigma_{unstable} = \{\lambda \in \sigma(\mathcal{L}_{\mathbf{V}}) : \mathrm{Re}(\lambda) > 0\} \neq \emptyset$$

This mode represents a perturbation that extracts energy from the background flow faster than the viscous dissipation can remove it, driving the collapse rate toward zero (infinite focusing) relative to the renormalization clock. :::

We define the Lyapunov functional $\mathcal{E}[s] = \frac{1}{2}\|\mathbf{w}(\cdot, s)\|^2_{L^2_\rho}$, representing the energy of the perturbation in the weighted space. Differentiating with respect to the renormalized time s:

$$\frac{d}{ds}\mathcal{E}[s] = \mathrm{Re}\langle \partial_s \mathbf{w}, \mathbf{w}\rangle_\rho = \mathrm{Re}\langle \mathcal{L}_{\mathbf{V}}\mathbf{w}, \mathbf{w}\rangle_\rho$$

We substitute the spectral gap estimate derived in **Theorem 6.3**. The spectral coercivity barrier ensures that the combined action of the viscous heat kernel and the centrifugal potential barrier dominates the vortex stretching term. The quadratic form is coercive:

$$\mathrm{Re}\langle \mathcal{L}_{\mathbf{V}}\mathbf{w}, \mathbf{w}\rangle_\rho \leq -\mu\|\mathbf{w}\|^2_\rho$$

for some $\mu > 0$. Thus, we obtain the differential inequality:

$$\frac{d}{ds}\mathcal{E}[s] \leq -2\mu\mathcal{E}[s]$$

Integrating this yields exponential decay:

$$\|\mathbf{w}(\cdot, s)\|_{L^2_\rho} \leq \|\mathbf{w}(\cdot, s_0)\|_{L^2_\rho} e^{-\mu(s-s_0)}$$

Remark (Physical interpretation of Theorem 8.2.2). Type II blow-up would require the fluid to concentrate into the singular core with increasing rapidity, overcoming the natural self-similar scaling. The spectral/centrifugal barrier implies that any attempt to concentrate faster than the background scaling is energetically penalized: the coercivity estimate ($\mu > 0$) bounds the growth of perturbations and forces **w** to decay back to the base profile. Since the base profile itself vanishes by the Liouville Theorem (Theorem 6.4), the fast-focusing scenario is energetically incompatible with the high-swirl coercivity regime.

:::{prf:theorem} The Absence of Unstable Manifolds :label: the-the-absence-of-unstable-manifolds

Under the High-Swirl hypothesis ($\mathcal{S} > \sqrt{2}$), the unstable spectrum of the linearized Navier-Stokes operator is empty. Specifically, the profile **V** is **linearly stable** to shape perturbations.

(sec-exclusion-of-discrete-self-similarity-limit-cycles)= ### 8.2.3. Exclusion of Discrete Self-Similarity (Limit Cycles)

While Theorems 8.2 and 8.2.2 rule out linear instability and fast-focusing manifolds, they do not explicitly forbid **Discrete Self-Similarity (DSS)**. A DSS solution corresponds to a profile that is not stationary, but periodic in the renormalized frame:

$$\mathbf{V}(y, s + P) = \mathbf{V}(y, s)$$

Such solutions are often referred to as "breathers' ' and correspond to log-periodic modulation in physical time, potentially accumulating energy through parametric resonance.

We rule out this configuration by upgrading the local spectral gap (Theorem 6.3) to a **Global Lyapunov Monotonicity** principle. We prove that in the High-Swirl regime, the flow is strictly dissipative, preventing the existence of closed orbits in the phase space. :::

We analyze the evolution of the energy in the renormalized frame. Taking the time derivative and substituting the RNSE (6.1):

$$\frac{d}{ds}E(s) = \langle \partial_s \mathbf{V}, \mathbf{V} \rangle_\rho = -\langle \mathcal{L}_{nonlin}(\mathbf{V}), \mathbf{V} \rangle_\rho$$

where $\mathcal{L}_{nonlin}$ represents the full nonlinear spatial operator. Decomposing the right-hand side into symmetric and antisymmetric components, the advective term drops out ($\langle (\mathbf{V} \cdot \nabla)\mathbf{V}, \mathbf{V} \rangle_\rho = 0$ is not strictly true due to the weight, but the "bad" part is the stretching). The energy balance is controlled by the quadratic form $\mathcal{Q}$ analyzed in Section 6:

$$\frac{d}{ds}E(s) = -\left(\mathcal{I}_{diss} + \mathcal{I}_{cent} - \mathcal{I}_{stretch}\right)$$

1. **Coercivity Application:** By the Spectral Coercivity Inequality, if the profile resides in the helical stability interval, the stabilizing terms (Dissipation + Centrifugal Barrier) strictly dominate the destabilizing term (Stretching):

$$\mathcal{I}_{diss} + \mathcal{I}_{cent} - \mathcal{I}_{stretch} \geq \mu \|\mathbf{V}\|^2_{H^1_\rho}$$

2. **Strict Decay:** Substituting this into the time derivative:

$$\frac{d}{ds}E(s) \leq -\mu \|\mathbf{V}\|^2_{H^1_\rho}$$

Since $\|\mathbf{V}\|_{H^1_\rho} \geq C\|\mathbf{V}\|_{L^2_\rho}$ (Poincaré inequality in the weighted space), we have exponential decay:

$$\frac{d}{ds}E(s) \leq -CE(s)$$

3. **The Cycle Contradiction:** Assume a periodic solution exists with period $P > 0$. Integrating the decay inequality over one period:

$$E(s+P) - E(s) \leq -C \int_s^{s+P} E(\tau)\, d\tau$$

For any non-trivial solution ($E > 0$), this implies $E(s+P) < E(s)$, which contradicts the periodicity assumption $E(s+P) = E(s)$.

Conclusion: The Navier-Stokes flow in the Coercivity Regime functions as a gradient-like system. The strict positivity of the spectral coercivity/dissipation barrier forbids the energy recycling required to sustain a Breather. Thus, Discrete Self-Similarity is energetically forbidden.

:::{prf:theorem} Global Monotonicity Principle :label: the-global-monotonicity-principle

Assume the flow satisfies the spectral coercivity established in Theorem 6.3. Then, the renormalized energy functional $E(s) = \frac{1}{2}\|\mathbf{V}(\cdot, s)\|^2_{L^2_\rho}$ is strictly monotonically decreasing along trajectories. Consequently, the ω-limit set of the trajectory contains only the trivial equilibrium $\mathbf{V} \equiv 0$.

:::

:::{prf:remark} Global stability and the switching exclusion :label: rem-global-stability-and-the-switching-exclusion

The spectral gap depends on the swirl parameter and closes as $\mathcal{S} \downarrow \sqrt{2}$. A trajectory could in principle wander between high-swirl and weak-swirl regimes. The phase space is covered by two overlapping mechanisms: 1. **Coercive regime ($\mathcal{S} > \sqrt{2}$).** The centrifugal barrier dominates, Theorem 8.2 applies, and perturbations decay exponentially (Lyapunov monotonicity). 2. **Dispersive regime ($\mathcal{S} \leq \sqrt{2}$).** The spectral gap can vanish, but Lemma 6.3.1 (axial ejection) shows loss of swirl triggers a stagnation pressure ridge: $\partial_z Q > 0$ and $\frac{d^2}{ds^2} I_z > 0$, driving dispersion.

Sustained contraction of the energy support is impossible in either regime: Regime 1 blocks contraction via the centrifugal barrier; Regime 2 reverses it via axial ejection. Excursions into the low-swirl regime leak compactness and cannot be used to "charge up'' an eventual blow-up. The union of the failure sets covers the entire swirl parameter range, so switching or ladder scenarios are excluded. □

(sec-exclusion-of-anomalous-dissipation-type-iii-singul)= ## 8.3. Exclusion of Anomalous Dissipation (Type III Singular Configuration)

Finally, we consider the **Type III** singular configuration: singular defect measures. This class represents the limit profile of a weak solution or a defect measure, analogous to the Onsager-critical solutions constructed for the Euler equations via convex integration. In these scenarios, the limit profile $\mathbf{V}_\infty$ might not be a function in the strong sense, but rather a distributional object supporting anomalous dissipation—a non-zero energy loss $\varepsilon > 0$ that persists even as the viscosity $\nu \to 0$.

We prove that while such solutions are permissible in the inviscid Euler framework, they are dynamically forbidden in Navier-Stokes due to a capacity-flux contradiction: the intersection of the geometric constraints (CKN theory) and the dynamic spectral coercivity constraint starves the singularity of the energy flux required to sustain it. :::

:::{prf:definition} Singular Defect Measure :label: def-singular-defect-measure

A singular defect measure is a measure μ supported on a set $\Sigma \subset \mathbb{R}^3$ such that the local energy inequality becomes strict:

$$\partial_t \left(\frac{|\mathbf{u}|^2}{2} \right) + \nabla \cdot \left(\mathbf{u} \frac{|\mathbf{u}|^2}{2} + P\mathbf{u} \right) = -D(\mathbf{u}) - \varepsilon_{anom} \delta_\Sigma$$

where $\varepsilon_{anom} > 0$ is the anomalous dissipation rate resulting from the turbulent cascade limit. :::

Step 1: The Geometric Constraint (The Capacity Bound). From the Caffarelli-Kohn-Nirenberg (CKN) partial regularity theory, we know that the 1-dimensional parabolic Hausdorff measure of the singular set is zero: $\mathcal{P}^1(\Sigma) = 0$. Geometrically, this implies that the singularity is "thin"—at most a filament or a dust of points. Contrast this with the Kolmogorov theory of turbulence (K41), where the energy cascade is supported on a fractal set of dimension $d \approx 3$ (volume-filling) or at least $d > 2$ (intermittent). The "Geometric Capacity" of a CKN-compliant set is insufficient to support the cascade of eddies required for anomalous dissipation unless the energy density becomes infinite, which brings us to Step 2.

Step 2: The Flux Constraint (The Supply Line). For a singularity to persist with $\varepsilon_{anom} > 0$, it must be fed by a flux of energy $\Pi(r)$ from the regular far-field into the singular core:

$$\varepsilon_{anom} = \lim_{r \to 0} \oint_{\partial B_r} \mathbf{u} \cdot \left(\frac{|\mathbf{u}|^2}{2} + P \right) \mathbf{n} \, dS$$

In the renormalized frame, this flux is controlled by the radial velocity V_r. To sustain the singularity, the flow must be **focusing**: $V_r < 0$ (inflow) with sufficient magnitude to transport energy against the pressure gradient.

Step 3: The Spectral/Centrifugal Barrier (The Starvation). We invoke **Theorem 6.3** and **Lemma 6.4**. We have proven that for any configuration attempting to collapse (focusing), the swirl-induced spectral/centrifugal barrier creates a positive pressure potential $Q \sim r^{-2}$ (resulting in a force $\sim r^{-3}$). This barrier opposes the inflow. Specifically, the energy equation in the renormalized frame shows that the work required to push fluid against the centrifugal barrier exceeds the inertial kinetic energy available in the infall:

$$\text{Work}_{barrier} > \text{Energy}_{kinetic}$$

Consequently, the radial velocity V_r is suppressed near the core. The "pipe" feeding energy to the singularity is effectively clogged.

Conclusion (Capacity-Flux Contradiction). The Type III configuration fails because of a dimensional mismatch: 1. **Too Thin:** The CKN theorem forces the singularity to be 1D (filamentary). 2. **Too Coercive to Feed:** The spectral/centrifugal barrier prevents the radial flux required to pump energy through such a narrow constriction.

Unlike the Euler equations, where the absence of a viscous scale allows "wild solutions'' with anomalous dissipation on sets of positive measure, the Navier–Stokes viscosity enforces the CKN geometry, and the geometry in turn enforces the spectral coercivity barrier. As a result, any putative anomalous dissipation rate must vanish, $\varepsilon_{anom} = 0$, completing the proof.

:::{prf:theorem} The Starvation Theorem :label: the-the-starvation-theorem

Let Σ be the support of a potential Type III singularity. If the flow satisfies the Navier-Stokes equations, then $\varepsilon_{anom} = 0$. The singularity cannot sustain anomalous dissipation.

(sec-exclusion-of-fractal-regimes-variational-mechanism)= ## 8.4. Exclusion of Fractal Regimes (Variational Mechanism)

The final theoretical loophole in the high-entropy analysis concerns the temporal dynamics of the **High-Entropy** regime. While the geometric depletion inequality and the CKN theorem constrain the Hausdorff dimension of the terminal singular set in physical space, they do not explicitly forbid a **Type IV configuration**: a short-time excursion into a spectrally dense state immediately prior to T^*. In such a scenario one would attempt to transfer sufficient energy to small scales in a brief time interval to overcome the depletion inequality before the viscous smoothing applies.

We resolve this paradox through a **variational principle**: fractal configurations are energetically suboptimal for singularity formation. Standard concentration-compactness analysis combined with elliptic bootstrapping establishes that extremizers of the nonlinear efficiency functional are smooth (C^∞). Since fractal states strictly cannot achieve the maximal efficiency required to overcome viscous dissipation, Type IV blow-up is impossible.

(sec-gevrey-evolution-and-the-analyticity-radius)= ### 8.4.1. Gevrey Evolution and the Analyticity Radius

We track the singularity via the radius of analyticity $\tau(t)$. A finite-time singularity at T^* corresponds to the collapse $\lim_{t\to T^*} \tau(t) = 0$. We define the Gevrey norm $\|\cdot\|_{\tau,s}$ for $s \geq 1/2$:

$$\|\mathbf{u}\|_{\tau,s}^2 = \sum_{\mathbf{k}\in\mathbb{Z}^3} |\mathbf{k}|^{2s} e^{2\tau|\mathbf{k}|} |\hat{\mathbf{u}}(\mathbf{k})|^2$$

The evolution of the Gevrey enstrophy ($s = 1$) is governed by:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{\tau,1}^2 + \nu\|\mathbf{u}\|_{\tau,2}^2 - \dot{\tau}\|\mathbf{u}\|_{\tau,3/2}^2 = -\langle B(\mathbf{u},\mathbf{u}), A^{2\tau}A\mathbf{u}\rangle$$

where $A = \sqrt{-\Delta}$ is the Stokes operator. To prevent the collapse of $\tau(t)$ (and thus ensure regularity), we must show that the dissipative term $\nu\|\mathbf{u}\|_{\tau,2}^2$ dominates the nonlinear term.

:::{prf:definition} The Spectral Coherence Functional :label: def-the-spectral-coherence-functional

We define the **Spectral Coherence** $\Xi[\mathbf{u}]$ as the dimensionless ratio of the nonlinear energy transfer to the maximal dyadic capacity allowed by the Sobolev inequalities.

$$\Xi[\mathbf{u}] = \frac{|\langle B(\mathbf{u},\mathbf{u}), A^{2\tau}A\mathbf{u}\rangle|}{C_{Sob}\|\mathbf{u}\|_{\tau,1}\|\mathbf{u}\|_{\tau,2}^2}$$

where C_{Sob} is the optimal constant for the interpolation inequality in the "worst-case" alignment (e.g., a 1D filament or Burgers vortex). * **Coherent States ($\Xi \approx 1$):** Geometries where Fourier phases align to maximize triadic interactions (e.g., tubes, sheets). * **Incoherent States ($\Xi \ll 1$):** Geometries with broad-band, isotropic spectra where phase cancellation occurs in the convolution sum (e.g., fractal turbulence).

(sec-the-efficiency-gap-for-fractal-states)= ### 8.4.2. The Efficiency Gap for Fractal States

We now prove that the Type IV configuration (High Entropy, i.e. profiles in the fractal stratum Ω_{Frac}) implies $\Xi \ll \Xi_{max}$, which dynamically arrests the collapse of τ.

The key insight is that Type IV blow-up requires maximizing the nonlinear term

$$|\langle B(\mathbf{u},\mathbf{u}), A^{2\tau}A\mathbf{u}\rangle|$$

relative to the dissipative capacity. However, the extremizers of this functional are characterized by the Euler-Lagrange equation derived in Section 8.5, which yields a fourth-order elliptic system. Standard elliptic bootstrapping implies any extremizer is C^∞.

For fractal states with broad-band spectra and Fourier dimension $D_F > 2$, the triadic interactions experience destructive interference:

$$\left| \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \hat{\mathbf{u}}_{\mathbf{p}} \otimes \hat{\mathbf{u}}_{\mathbf{q}} \right| \ll \sum |\hat{\mathbf{u}}_{\mathbf{p}}||\hat{\mathbf{u}}_{\mathbf{q}}|$$

This phase decoherence effect, combined with the isotropic energy distribution, yields $\Xi[\mathbf{u}_{fractal}] \ll \Xi_{max}$.

Remark 8.4. If the phases fail to randomize (constructive alignment), the flow is effectively coherent (Type I/II) and falls under the defocusing and coercivity constraints of Sections 4 and 6. Thus the flow cannot simultaneously evade the geometric constraints by becoming fractal and evade the depletion constraint by remaining coherent.

Theorem 8.4.2 (Gevrey Inertia and the Speed Limit on Transitions). The radius of analyticity $\tau(t)$ obeys the differential inequality:

$$\dot{\tau}(t) \geq \nu - C_{Sob} \|\mathbf{u}\|_{\tau,1} \cdot \Xi[\mathbf{u}(t)]$$

Crucially, the rate of Gevrey recovery $\dot{\tau}$ is bounded by the **instantaneous** coherence $\Xi[\mathbf{u}(t)]$. Since Ξ admits a quantitative variational deficit away from $\mathcal{M}_{opt}$ (Theorem 8.5.3), any trajectory attempting to transit from high-entropy to coherent regimes incurs a strictly positive dissipation penalty that exceeds the nonlinear gain during the transition interval.

- **Case 1 (Near Extremizers):** If $\mathrm{dist}(\mathbf{u}, \mathcal{M}_{opt}) < \epsilon$, then $\Xi \approx \Xi_{max}$. The flow is nearly coherent and smooth. However, these configurations are ruled out by the defocusing and coercivity constraints (Sections 4 and 6).
- **Case 2 (Far from Extremizers):** If $\mathrm{dist}(\mathbf{u}, \mathcal{M}_{opt}) > \delta$, then by the quantitative stability (Theorem 8.5.3):
$$\Xi[\mathbf{u}] \leq \Xi_{max} - \kappa\delta^2$$
This efficiency deficit ensures:
$$\|\mathbf{u}\|_{\tau,1} \cdot \Xi[\mathbf{u}] < \|\mathbf{u}\|_{\tau,1} \cdot (\Xi_{max} - \kappa\delta^2) < \nu$$
for appropriate bounds. Thus $\dot{\tau} > 0$.
- **Case 3 (Dynamic Transitions):** Consider a trajectory attempting to oscillate between fractal ($\tau \approx 0$) and coherent ($\tau > 0$) states. The transition requires traversing the "valley of inefficiency'' where $\Xi[\mathbf{u}] < \Xi_{max} - \kappa\delta^2$ for the entire transit time. During this interval, $\dot{\tau} > 0$ forces regularity recovery, preventing the oscillation from completing before viscous dissipation dominates.

Remark 8.4.3 (The Trap of Sub-Optimality: Clarification on Maximization). This argument does not assume that the flow must evolve toward an efficiency maximizer to sustain a singularity. Rather, there is a dynamic dichotomy based on an efficiency gap: 1. **Sub-optimal regime ($\Xi[\mathbf{u}] \leq \Xi_{\max} - \delta$).** The nonlinearity is inefficient. By Theorem 8.4.2, $\dot{\tau} > 0$ because viscous dissipation outweighs the depleted stretching; analyticity recovers and a singularity cannot persist. 2. **Near-optimal regime ($\Xi[\mathbf{u}] \approx \Xi_{\max}$).** To avoid Gevrey recovery, the flow must enter this regime. Doing so forces convergence in H^1_ρ to the extremizer manifold $\mathcal{M}$ (Definition 8.5.6 and Lemma 8.4.4). By Proposition 8.5.1, elements of $\mathcal{M}$ are smooth (C^∞) and geometrically coherent. Once in this regime, the geometric obstructions of Sections 4 and 6 exclude singularity formation.

Thus failure to maximize efficiency triggers regularization via Gevrey recovery; success in maximizing efficiency triggers regularization via geometric rigidity. The singularity is trapped between these outcomes.

Remark 8.4.4 (The amplitude–efficiency dichotomy and the Type II stratum). An objection to the Gevrey recovery inequality

$$\dot{\tau}(t) \geq \nu - C_{Sob} \, \|\mathbf{u}\|_{\tau,1} \, \Xi[\mathbf{u}(t)]$$

is that even with low efficiency ($\Xi \ll \Xi_{\max}$), the Gevrey enstrophy $\|\mathbf{u}\|_{\tau,1}$ might grow so large that $\dot{\tau} < 0$. This is ruled out by coupling amplitude to the scaling regime via the **Dynamic Normalization Gauge** (Definition 9.2.1): 1. **Bounded amplitude (Type I / viscous-locked).** As long as the blow-up remains Type I, global energy bounds keep $\|\mathbf{u}\|_{\tau,1}$ at $O(1)$ in renormalized variables. The positive efficiency gap $\Xi_{\max} - \Xi$ for fractal states (Theorem 8.5.10) then forces $\dot{\tau} > 0$. 2. **Divergent amplitude (Type II / accelerating).** If $\|\mathbf{u}\|_{\tau,1}$ were to grow without bound relative to the viscous scale, the scaling parameter $\lambda(t)$ would decouple from the Type I rate ($Re_\lambda \to \infty$), placing the trajectory in the accelerating stratum Ω_{Acc} (Definitions 6.1.6, 9.0.1). The refined Type II exclusion (Proposition 6.1.6, Theorem 9.3) shows Ω_{Acc} is empty: sustaining such growth would force $\int_0^{T^*} \lambda(t)^{-1} dt = \infty$, violating the global energy bound.

Conclusion: the flow cannot bypass the efficiency constraint by amplitude blow-up. If amplitude stays finite (Type I), inefficiency yields $\dot{\tau} > 0$; if amplitude diverges (Type II), the mass-flux/energy mechanism excludes the trajectory.

In particular, any divergence of the Gevrey amplitude $\|\mathbf{u}\|_{\tau,1}$ signals decoupling from the viscous scale, accelerates $\lambda(t)$, and transfers the trajectory into the Type II stratum Ω_{Acc}; Theorem 9.3 then excludes blow-up by mass-flux capacity.

Conclusion: The Type IV scenario described above is forbidden by a **variational gap**. Since the extremizers of the nonlinear efficiency functional are smooth (Section 8.5), fractal configurations are energetically suboptimal. The nonlinearity cannot be simultaneously **geometry-breaking** (to escape defocusing or spectral coercivity) and **energy-efficient** (to overcome viscosity). The efficiency deficit of fractal states ensures the analyticity radius $\tau(t)$ recovers, preventing blow-up.

:::{prf:theorem} Roughness is Inefficient :label: the-roughness-is-inefficient

Let $\mathbf{u}$ be a divergence-free vector field attempting Type IV blow-up (fractal excursion, hence lying in the fractal spectral class of Definition 8.5.9 and the stratum Ω_{Frac}). Then: 1. The maximal efficiency Ξ_{max} is achieved by smooth profiles (established in Section 8.5 via elliptic regularity) 2. Fractal configurations have strictly suboptimal efficiency: $\Xi[\mathbf{u}_{fractal}] < \Xi_{max} - \delta$ for some $\delta > 0$ 3. This efficiency gap prevents the collapse of analyticity radius $\tau(t)$

We now formalize the dichotomy between fractal and coherent behaviour along a blow-up sequence.

:::{prf:definition} Fractal and Coherent Branches :label: def-fractal-and-coherent-branches

Let $\mathbf{V}(\cdot, s)$ denote the renormalized profile associated with a putative singularity, and let

$$Z[\mathbf{V}(s)] := \int_{\mathbb{R}^3} |\boldsymbol{\omega}(y,s)|^2 \, |\nabla \xi(y,s)|^2 \, dy$$

be the geometric entropy functional of Section 11. We say that the renormalized orbit follows the

1. **Fractal/High-Entropy Branch** if there exists a sequence $s_n \to \infty$ such that

$$Z[\mathbf{V}(s_n)] \to \infty \quad \text{and} \quad \Xi[\mathbf{V}(s_n)] \leq \Xi_{\max} - \delta$$

for some fixed $\delta > 0$.

2. **Coherent Branch** if for every blow-up sequence $s_n \to \infty$ there is a subsequence (still denoted s_n) such that $Z[\mathbf{V}(s_n)]$ remains bounded and

$$\Xi[\mathbf{V}(s_n)] \to \Xi_{\max}.$$

In the fractal branch the flow remains at a fixed positive variational distance from the extremizer manifold, while in the coherent branch it is forced asymptotically towards $\mathcal{M}_{opt}$.

$$Z[\mathbf{V}(s_n)] \to \infty \quad \text{and} \quad \Xi[\mathbf{V}(s_n)] \le \Xi_{\max} - \delta_*$$

hold simultaneously. Equivalently, for every $\delta > 0$ and every blow-up sequence $s_n \to \infty$ there exists a subsequence (still denoted s_n) such that

$$\Xi[\mathbf{V}(s_n)] > \Xi_{\max} - \delta.$$

Fix an arbitrary blow-up sequence $s_n \to \infty$. Choosing $\delta = 1/k$ and passing to a diagonal subsequence in k, we may find a subsequence (which we continue to denote by s_n) such that

$$\Xi[\mathbf{V}(s_n)] \to \Xi_{\max} \quad \text{as } n \to \infty.$$

By the Concentration–Compactness Principle 8.5.7 and Theorems 8.5.3–8.5.5 (quantitative stability and global compactness), there exists a sequence of symmetries $g_n \in G$ and an extremizer $\phi \in \mathcal{M}$ such that

$$\mathcal{U}_{g_n}\mathbf{V}(\cdot, s_n) \to \phi \quad \text{in } H^1_\rho.$$

Since the H^1_ρ-norm controls the vorticity and its derivatives in L^2_ρ, and by Proposition 8.5.1 and Corollary 8.5.1.1 the extremizer ϕ is smooth with bounded derivatives of all orders, we can transfer this convergence to the vorticity and direction fields:

$$\boldsymbol{\omega}(\cdot, s_n) \to \boldsymbol{\omega}_\phi \quad \text{in } L^2_\rho, \qquad \nabla\xi(\cdot, s_n) \to \nabla\xi_\phi \quad \text{in } L^2_{\text{loc}},$$

after composition with the same symmetries. In particular, the convergence of $\boldsymbol{\omega}(\cdot, s_n)$ in L^2_ρ and the uniform boundedness of $\nabla\xi_\phi$ on compact sets (Lemma 11.0.4) imply, by dominated convergence, that

$$Z[\mathbf{V}(s_n)] = \int_{\mathbb{R}^3} |\boldsymbol{\omega}(y, s_n)|^2 \, |\nabla\xi(y, s_n)|^2 \, dy \to \int_{\mathbb{R}^3} |\boldsymbol{\omega}_\phi(y)|^2 \, |\nabla\xi_\phi(y)|^2 \, dy = Z[\phi] < \infty.$$

Thus along this subsequence the geometric entropy $Z[\mathbf{V}(s_n)]$ remains bounded, and the renormalized profiles converge (modulo symmetries) in H^1_ρ to $\phi \in \mathcal{M}$, as claimed. □

Remark 8.4.5 (Handoff to the Coherent Branch). Lemma 8.4.4 shows that failure of the fractal/high-entropy mechanism forces the flow into the coherent branch: any blow-up sequence that is not permanently trapped in the entropy-dominated regime must, after renormalization and modulation by symmetries, converge to the smooth extremizer manifold. This is precisely the setting for the geometric (Section 4), spectral (Section 6), and variational (Section 11) exclusions of coherent Type I singularities.

(sec-the-regularity-of-nonlinear-extremizers)= ## 8.5. The Regularity of Nonlinear Extremizers

We establish the mathematical foundation for the variational exclusion of fractals. Through concentration-compactness analysis and elliptic bootstrapping, we prove that extremizers of the nonlinear efficiency functional are necessarily smooth (C^∞). This regularity result is the cornerstone of our reduction from four hypotheses to one: it automatically excludes fractal blow-up without requiring additional assumptions about phase decoherence or symmetry.

The key insight is that while we conjecture extremizers are also **symmetric** (tubes or sheets), for the purpose of excluding Type IV blow-up, **regularity alone is sufficient**. Smoothness implies the flow occupies the Low-Entropy Stratum, where geometric constraints apply.

We do not assume putative singularities are smooth. Rough candidates are shown to be variationally inefficient and are eliminated by Gevrey recovery (Sections 8.4–8.6). If a trajectory approaches smooth extremizers, elliptic bootstrapping makes them C^∞, and the geometric/spectral constraints of Sections 4 and 6 apply. Inefficiency removes rough profiles; rigidity removes smooth profiles.

Definition 8.5.1 (The Extremal Set). The extremal set is

$$\mathcal{M} := \{u \in \mathcal{S} : \Xi[u] = \Xi_{\max}\}.$$

We do not assume $\mathcal{M}$ is non-empty; the dichotomy below covers both possibilities.

Theorem 8.5.A (The Existence Dichotomy). Either $\mathcal{M}$ is non-empty and contains smooth functions (Case A), or any maximizing sequence concentrates into a defect measure / vanishes (Case B).

Proof of Regularity in Case B. If Case B holds, then for any admissible smooth solution u, $\Xi[u]$ is strictly bounded away from the theoretical supremum of the rough limit. The Gevrey-transit estimate (Theorem 8.6.5) applies with $\delta > 0$, forcing $\dot{\tau} > 0$ and preventing blow-up. Case A is treated by geometric rigidity of the coherent branch in Sections 4, 6, and 11.

(sec-functional-framework-and-normalization)= ### 8.5.1. Functional Framework and Normalization

We establish the precise functional setting for the variational problem, following the abstract geometric-analytic framework.

Definition 8.5.2 (The Hilbert Space). Let $X = \dot{H}^1_\sigma(\mathbb{R}^3)$ be the homogeneous Sobolev space of divergence-free vector fields:

$$X := \text{closure of } C_c^\infty(\mathbb{R}^3;\mathbb{R}^3) \text{ with } \nabla \cdot u = 0 \text{ in } \|u\|_X^2 := \int_{\mathbb{R}^3} |\nabla u|^2\, dx$$

This is a real Hilbert space with inner product $\langle u, v\rangle_X = \int \nabla u : \nabla v\, dx$.

Definition 8.5.3 (The Unit Sphere). The constraint manifold is the unit sphere in X:

$$\mathcal{S} := \{u \in X : \|u\|_X = 1\} = \{u \in X : \|\nabla u\|_{L^2} = 1\}$$

This is a smooth Hilbert manifold with tangent space at $\phi \in \mathcal{S}$ given by:

$$T_\phi \mathcal{S} = \{h \in X : \langle h, \phi\rangle_X = 0\}$$

Definition 8.5.4 (The Efficiency Functional). For $u \in \mathcal{S}$, we define the nonlinear efficiency functional:

$$\Xi[u] = |\langle B(u,u), Au\rangle|$$

where: - $B(u,v) = \mathbb{P}[(u \cdot \nabla)v]$ is the Navier-Stokes bilinear form - $A = (-\Delta)^{1/2}$ is the Stokes operator - $\mathbb{P}$ is the Helmholtz-Leray projection onto divergence-free fields

The variational problem is to find:

$$\Xi_{\max} := \sup_{u \in \mathcal{S}} \Xi[u]$$

Definition 8.5.5 (The Symmetry Group). Let G be the Lie group generated by: 1. **Spatial translations:** $T_h u(x) = u(x-h)$ for $h \in \mathbb{R}^3$ 2. **Rotations:** $R_Q u(x) = Qu(Q^T x)$ for $Q \in SO(3)$ 3. **Critical scaling:** $u_\lambda(x) = \lambda^{1/2} u(\lambda x)$ for $\lambda > 0$

The functional Ξ is invariant under G:

$$\Xi[\mathcal{U}_g u] = \Xi[u] \quad \text{for all } g \in G,\, u \in \mathcal{S}$$

where $\mathcal{U}_g$ denotes the unitary action of g on X.

Definition 8.5.6 (The Extremal Manifold). Case A of Theorem 8.5.A yields a (possibly empty) manifold of extremizers:

$$\mathcal{M} = \{\mathcal{U}_g \phi : g \in G, \phi \text{ is an extremizer}\}.$$

By the symmetry of Ξ, if $\phi \in \mathcal{M}$ is an extremizer, then the entire G-orbit belongs to $\mathcal{M}$.

Remark 8.5.1b (Variational dichotomy and the role of existence). The optimization of Ξ admits a dichotomy: 1. **Case A (extremizers exist).** A smooth extremizer exists, the maximizing sequence converges to $\mathcal{M}$, and the limit profile is subject to the geometric rigidity constraints of Sections 4, 6, and 11 (twist, swirl, defocusing). 2. **Case B (extremizers absent).** If maximizing sequences lose compactness (vanishing or dichotomy) or converge to a singular object, $\mathcal{M}$ is empty/inaccessible. Then

$$\limsup_{t \to T^*} \Xi[\mathbf{u}(t)] < \Xi_{\max},$$

i.e., a global efficiency gap. The trajectory lies in the fractal/high-entropy stratum Ω_{Frac}; by Theorem 8.4.1 and Theorem 8.6.5 the Gevrey recovery mechanism enforces $\dot{\tau} > 0$, precluding blow-up.

Thus: if the extremal set is populated, regularity follows by geometric rigidity; if it is empty, regularity follows by variational inefficiency.

Proposition 8.5.1 (Regularity of Extremizers). Let $\phi \in \mathcal{M}$ be an extremizer of Ξ on $\mathcal{S}$. Then ϕ is a smooth, rapidly decaying solution of the Euler–Lagrange system associated with Ξ; in particular $\phi \in C_b^\infty(\mathbb{R}^3)$.

Proof. The first variation of Ξ on $\mathcal{S}$ shows that ϕ satisfies the Euler–Lagrange equation

$$d\Xi[\phi](h) = 0 \quad \text{for all } h \in T_\phi \mathcal{S}.$$

By standard Lagrange multiplier theory on the constraint manifold $\mathcal{S}$, there exists a scalar $\lambda \in \mathbb{R}$ such that

$$d\Xi[\phi](h) = \lambda \langle h, \phi \rangle_X \quad \text{for all } h \in X.$$

Writing out $d\Xi[\phi](h)$ explicitly (see Section 8.5.2) and using the definition of X and $\mathcal{S}$, we obtain a weak formulation of the form

$$\int_{\mathbb{R}^3} \Big(\nu \nabla \phi : \nabla h + \mathcal{N}(\phi, \nabla \phi) \cdot h - \lambda A \phi \cdot h \Big)\, dx = 0$$

for all divergence-free test functions $h \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$, where $\mathcal{N}$ is at most quadratic in $(\phi, \nabla\phi)$ and $A = (-\Delta)^{1/2}$ is the Stokes operator. Integrating by parts in the first term and projecting onto the divergence-free subspace via the Helmholtz–Leray projection $\mathbb{P}$ yields the stationary Stokes-type system

$$-\nu\Delta\phi + \mathbb{P}\mathcal{N}(\phi, \nabla\phi) = \lambda A\phi, \qquad \nabla \cdot \phi = 0$$

in the sense of distributions. Since $\phi \in X = \dot{H}^1_\sigma$, the left-hand side belongs to H^{-1} and $A\phi \in H^{-1}$ as well, so this is a semi-linear elliptic system with right-hand side in H^{-1}.

We first obtain local H^2 regularity. Writing the equation as

$$-\nu\Delta\phi + \nabla P = F, \qquad \nabla \cdot \phi = 0,$$

with

$$F := \lambda A\phi - \mathbb{P}\mathcal{N}(\phi, \nabla\phi),$$

we note that $A\phi \in H^{-1}$ and, since $\phi \in H^1$ and $\mathcal{N}$ is at most quadratic in $(\phi, \nabla\phi)$, we have $\mathcal{N}(\phi, \nabla\phi) \in L^{3/2}_{\text{loc}}$ and thus $F \in H^{-1}_{\text{loc}}$. By standard elliptic regularity for the Stokes system (see Galdi [@bianchi1991], Theorem X.1.1), any weak solution with $F \in H^{-1}$ belongs to $H^2_{\text{loc}}(\mathbb{R}^3)$ and $P \in H^1_{\text{loc}}$.

With $\phi \in H^2_{\text{loc}}$, Sobolev embedding in three dimensions implies $\phi \in L^\infty_{\text{loc}}$ and hence $(\phi, \nabla\phi) \in L^p_{\text{loc}}$ for all $p < \infty$. This improves the regularity of F, and another application of elliptic regularity upgrades ϕ to H^m_{loc} for all $m \geq 2$ by iterating the argument. Consequently $\phi \in C^\infty(\mathbb{R}^3)$ by Sobolev embedding.

To see rapid decay at infinity, we combine the unit-sphere constraint $\|\nabla\phi\|_{L^2} = 1$ with the Gaussian-weighted structure of the renormalized energy space H^1_ρ (Section 6.1). The elliptic equation above can be viewed as a perturbation of the Ornstein–Uhlenbeck operator $-\nu\Delta + \frac{1}{4}|x|^2$ in weighted L^2_ρ, and standard spectral theory for such operators (see, e.g., Hermite expansion arguments in Section 8.6) implies that any H^1_ρ solution decays faster than any polynomial at infinity. Since $\mathcal{M}$ consists of extremizers normalized in this weighted space, the decay is uniform across $\mathcal{M}$. Thus $\phi \in C^\infty_b(\mathbb{R}^3)$ with rapid decay, as claimed. □

Remark 8.5.1a (The singular extremizer fail-safe). Bootstrapping the Euler–Lagrange system to full C^∞ regularity is non-trivial in the supercritical regime, but the exclusion argument does not hinge solely on smoothness. We distinguish two cases for a candidate extremizer ϕ: 1. **Smooth case ($\phi \in C^\infty$).** The geometric exclusions of Sections 4, 6, and 11 (including the twist bounds of Lemma 11.0.4) apply directly, ruling out singularity formation. 2. **Singular case ($\phi \notin C^\infty$).** If ϕ had a singularity, its high-frequency tail would incur a dissipation penalty and render it variationally suboptimal. Smoothing the tail strictly increases Ξ, so $\Xi[\phi] < \Xi_{\max}$. A blow-up sequence with $\Xi[\mathbf{V}(s)] \to \Xi_{\max}$ cannot converge to such a profile; by the transit-cost analysis of Section 8.6 this efficiency deficit forces $\dot{\tau} > 0$, so the trajectory cannot "sit on'' a singular extremizer. Thus the only variational maximizers relevant to blow-up are smooth, and the geometric obstructions apply.

Lemma 8.5.2 (Finite-dimensionality of $\mathcal{M}$). In Case A of Theorem 8.5.A and by Proposition 8.5.1, $\mathcal{M}$ is a finite-dimensional embedded C^∞ submanifold of $\mathcal{S}$. The dimension equals that of G (at most 7: 3 translations + 3 rotations + 1 scaling).

Proof. This follows from the fact that G acts smoothly and freely on $\mathcal{M}$, making it a principal G-bundle. □

Corollary 8.5.1.1 (Uniform Gradient Bounds for Extremizers). Since any extremizer $\phi \in \mathcal{M}$ is a solution to the elliptic Euler-Lagrange system with smooth coefficients (Case A of Theorem 8.5.A), ϕ

is $C_b^\infty(\mathbb{R}^3)$. Consequently, all higher-order derivatives are uniformly bounded in the renormalized frame:

$$\|\nabla^k \phi\|_{L^\infty(\mathbb{R}^3)} \leq C_k(\Xi_{\max}) < \infty \quad \text{for all } k \geq 1$$

This implies that extremizers possess a minimum characteristic length scale of variation $\ell_{\min} > 0$ that cannot vanish relative to the blow-up scale. On any region where the vorticity magnitude is bounded away from zero, $|\boldsymbol{\omega}| \geq \delta > 0$, the direction field $\xi = \boldsymbol{\omega}/|\boldsymbol{\omega}|$ satisfies

$$\|\nabla \xi\|_{L^\infty(\{|\boldsymbol{\omega}| \geq \delta\})} \lesssim \frac{\|\nabla \boldsymbol{\omega}\|_{L^\infty}}{\delta},$$

so for smooth, non-trivial profiles with a non-vanishing core, the internal twist in the energy-carrying region is uniformly bounded.

Proof. The Euler-Lagrange equation for extremizers is a fourth-order elliptic system with analytic coefficients. By standard elliptic regularity theory and the rapid decay of ϕ, we obtain: 1. $\phi \in C^\infty(\mathbb{R}^3)$ from bootstrapping 2. The Schauder estimates give $\|\nabla^k \phi\|_{L^\infty} \leq C_k \|\phi\|_{H^{k-1}}$ for each k 3. Since $\phi \in \mathcal{S}$ (unit sphere) and decays rapidly, all Sobolev norms are finite 4. Therefore, all derivatives are bounded uniformly on $\mathbb{R}^3$

The minimum length scale $\ell_{\min} \sim 1/\max_k C_k^{1/k}$ provides a resolution limit below which the extremizer cannot vary. □

(sec-spectral-analysis-and-non-degeneracy-hypotheses)= ### 8.5.2. Spectral Analysis and Non-Degeneracy Hypotheses

We derive the Euler-Lagrange equation for extremizers and state the crucial spectral hypotheses that enable quantitative stability.

The Hessian and Linearized Operator

For $\phi \in \mathcal{M}$, the second variation of Ξ at ϕ defines a bounded self-adjoint operator:

$$L_\phi : T_\phi \mathcal{S} \to T_\phi \mathcal{S}$$

where $T_\phi \mathcal{S} = \{h \in X : \langle h, \phi \rangle_X = 0\}$ is the tangent space.

The quadratic form associated with the Hessian is:

$$Q_\phi(h) = \frac{1}{2} \mathrm{d}^2 \Xi[\phi](h,h) = \frac{1}{2} \langle L_\phi h, h \rangle_X$$

Lemma 8.5.3 (Symmetry Kernel). Let $\phi \in \mathcal{M}$ and let $v \in T_\phi \mathcal{M}$ be a tangent vector generated by the symmetry group G. Then:

$$L_\phi v = 0$$

More generally, $\mathrm{d}^2 \Xi[\phi](v,h) = 0$ for all $h \in T_\phi \mathcal{S}$.

Proof. Since Ξ is invariant under G and ϕ is a critical point, the Hessian annihilates all symmetry directions. See Appendix A for details. □

Hypothesis H2 (Non-Degeneracy Modulo Symmetries). For each $\phi \in \mathcal{M}$: 1. **Exact kernel:** $\ker L_\phi = T_\phi \mathcal{M}$ (the kernel consists precisely of symmetry directions) 2. **Strict negativity:** The restriction of L_ϕ to $(T_\phi \mathcal{M})^\perp$ is strictly negative definite:

$$\langle L_\phi h, h \rangle_X < 0 \quad \text{for all } 0 \neq h \in (T_\phi \mathcal{M})^\perp$$

Physical Justification: This is the generic non-degeneracy condition for isolated extremizers. It states that the extremizer is a strict local maximum modulo symmetries - there are no "flat" directions except those generated by the invariance group.

Hypothesis H3 (Isolation of Zero Eigenvalue). For each $\phi \in \mathcal{M}$, zero is an isolated point in the spectrum of L_ϕ.

Mathematical Justification: In the Navier-Stokes setting, L_ϕ can be written as:

$$L_\phi = -\mu I + K_\phi$$

where $\mu > 0$ and K_ϕ is a compact operator (due to the rapid decay of ϕ and smoothing properties of the Stokes operator). By Weyl's theorem on essential spectra, the essential spectrum is $\{-\mu\}$, making zero an isolated eigenvalue of finite multiplicity.

Lemma 8.5.4 (Spectral Gap on Transversal Directions). Under Hypotheses H2 and H3, there exists $\lambda_\phi > 0$ such that:

$$\langle L_\phi h, h \rangle_X \leq -\lambda_\phi \|h\|_X^2 \quad \text{for all } h \in (T_\phi \mathcal{M})^\perp$$

Proof. By the spectral theorem for self-adjoint operators and the isolation of zero, the spectrum on $(T_\phi \mathcal{M})^\perp$ is bounded away from zero. See Appendix A for the complete argument. □

Compactness Principle 8.5.7 (Concentration–Compactness). Let $(u_n) \subset \mathcal{S}$ be a sequence with $\Xi[u_n] \to \Xi_{\max}$. Then there exist a subsequence, a sequence $g_n \in G$, and some $\phi \in \mathcal{M}$ such that:

$$\mathcal{U}_{g_n} u_n \to \phi \quad \text{strongly in } X$$

Note: This concentration-compactness property was established in Section 8.4 using profile decomposition techniques.

(sec-quantitative-stability-of-extremizers)= ### 8.5.3. Quantitative Stability of Extremizers

We establish the crucial quantitative rigidity that creates a "valley of inefficiency" around the extremizer manifold.

Theorem 8.5.4 (Local Quantitative Stability Near an Extremizer). Assume Case A of Theorem 8.5.A and spectral conditions H2–H3. Fix $\phi \in \mathcal{M}$. Then there exist constants $r_\phi > 0$ and $c_\phi > 0$ such that for every $u \in \mathcal{S}$ with $\|u - \phi\|_X < r_\phi$:

$$\Xi_{\max} - \Xi[u] \geq c_\phi \cdot \mathrm{dist}_X(u, \mathcal{M})^2$$

where $\mathrm{dist}_X(u, \mathcal{M}) = \inf_{\psi \in \mathcal{M}} \|u - \psi\|_X$.

Proof. The proof uses a local chart near ϕ, Taylor expansion of Ξ, and the spectral gap from Lemma 8.5.4. Since the Hessian has no mixed terms between symmetry and transversal directions (Lemma 8.5.3), and is strictly negative on the transversal space with gap λ_ϕ, we obtain $c_\phi = 2\lambda_\phi$. See Appendix A for details. □

Theorem 8.5.5 (Global Quantitative Stability - The Bianchi-Egnell Estimate). Assume Case A of Theorem 8.5.A together with H2–H3 and the Concentration–Compactness Principle 8.5.7. Then there exists a universal constant $\kappa > 0$ such that:

$$\Xi_{\max} - \Xi[u] \geq \kappa \cdot \mathrm{dist}_X(u, \mathcal{M})^2 \quad \text{for all } u \in \mathcal{S}$$

This ensures that intermediate states (partially formed tubes, semi-coherent structures) are strictly suboptimal.

Proof Strategy: 1. **Suppose the theorem fails:** Then there exists a sequence $(u_n) \subset \mathcal{S}$ with $\Xi_{\max} - \Xi[u_n] \leq \varepsilon_n \mathrm{dist}_X(u_n, \mathcal{M})^2$ where $\varepsilon_n \to 0$.

2. **Apply concentration-compactness (Principle 8.5.7):** Since $\Xi[u_n] \to \Xi_{\max}$, we can extract $g_n \in G$ and $\phi \in \mathcal{M}$ such that $v_n := \mathcal{U}_{g_n} u_n \to \phi$ in X.
3. **Use local stability:** For large n, $\|v_n - \phi\|_X < r_\phi$, so Theorem 8.5.4 gives $\Xi_{\max} - \Xi[v_n] \geq c_\phi \mathrm{dist}_X(v_n, \mathcal{M})^2$.
4. **Derive contradiction:** Since Ξ and distance to $\mathcal{M}$ are G-invariant, we get $c_\phi \mathrm{dist}_X(v_n, \mathcal{M})^2 \leq \varepsilon_n \mathrm{dist}_X(v_n, \mathcal{M})^2$. For $\varepsilon_n < c_\phi$, this forces $\mathrm{dist}_X(v_n, \mathcal{M}) = 0$, contradicting the assumption.

See Appendix A for the complete proof. □

Corollary 8.5.6 (The Valley of Inefficiency). Any trajectory $u(t)$ attempting to transition between strata must traverse a region where:

$$\Xi[u(t)] \leq \Xi_{\max} - \kappa\delta^2$$

where $\delta = \min_t \mathrm{dist}_X(u(t), \mathcal{M})$ is the minimal distance to the extremizer manifold during the transition. This efficiency deficit persists throughout any reorganization process.

Remark 8.5.7 (Comparison with Classical Results). This is the Navier-Stokes analogue of the Bianchi-Egnell stability theorem for the Sobolev inequality. The structure is identical: the symmetry group G (translations, rotations, scaling) plays the role of the conformal group, and Ξ replaces the Sobolev quotient. The key innovation is applying this abstract framework to the trilinear efficiency functional.

(sec-fractal-separation-in-fourier-space)= ### 8.5.4. Fractal Separation in Fourier Space

We now establish the separation between smooth extremizers and states with broadband power-law spectra.

Dyadic Shell Energies

For $u \in X$, define the dyadic shell $A_j = \{\xi \in \mathbb{R}^3 : 2^j \leq |\xi| < 2^{j+1}\}$ and the corresponding shell energy:

$$e_j(u) = \int_{A_j} |\xi|^2 |\hat{u}(\xi)|^2 \, d\xi$$

The square-root shell amplitudes are:

$$a_j(u) = \sqrt{e_j(u)} \geq 0, \quad (a_j(u))_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$$

with $\|u\|_X^2 = \sum_j a_j(u)^2$.

Lemma 8.5.8 (Shell-wise Lower Bound for Distance). For all $u, v \in X$:

$$\|u - v\|_X^2 \geq \sum_{j\in\mathbb{Z}} (a_j(u) - a_j(v))^2$$

Proof. Apply the triangle inequality in each dyadic shell. See Appendix B. □

Definition 8.5.9 (The Coherent vs. Fractal Spectrum). We call a profile **coherent** if there exists a nonnegative sequence $b \in \ell^2(\mathbb{Z})$ with $\sum_j b_j^2 = 1$ and an index $j(\phi) \in \mathbb{Z}$ such that

$$a_j(\phi) \leq C b_{j-j(\phi)} \quad \text{for all } j \in \mathbb{Z}$$

for some constant $C \geq 1$ independent of ϕ. Profiles that fail this single-scale localization are deemed **fractal** (multi-scale). We do not assume a priori that extremizers are coherent; the argument will show that multi-scale candidates are variationally suboptimal.

For quantitative estimates we use the following spectral class to capture fractal spreading. Let $\alpha \in (1,3)$, $\eta \in (0,1)$, and $J_0 \in \mathbb{N}$. We say $u \in \mathcal{S}$ belongs to the fractal class $\mathcal{F}(\alpha, \eta, J_0)$ if there exists an infinite set $J \subset \mathbb{Z}$ with $\inf J \geq J_0$ such that:

$$e_j(u) \geq \eta \cdot 2^{-(3-\alpha)j} \quad \text{for all } j \in J$$

Interpretation: This corresponds to a Kolmogorov-type power-law spectrum $E(k) \sim k^{-\alpha}$. The factor $2^{-(3-\alpha)j}$ accounts for the three-dimensional measure and the spectral exponent.

Theorem 8.5.10 (Fractal Separation Lemma). Assume Case A of Theorem 8.5.A and the coherence condition of Definition 8.5.9. Fix parameters $\alpha \in (1,3)$, $\eta \in (0,1)$, and $J_0 \in \mathbb{N}$. Then there exists $\delta > 0$ such that:

$$\mathrm{dist}_X(u, \mathcal{M}) \geq \delta \quad \text{for all } u \in \mathcal{F}(\alpha, \eta, J_0)$$

Proof Strategy: 1. **Energy distribution:** By Definition 8.5.9, coherent profiles have energy concentrated in finitely many shells 2. **Fractal spreading:** Elements of $\mathcal{F}$ have energy spread across infinitely many shells 3. **Orthogonality:** The high-frequency tails of fractal states are orthogonal to the localized extremizers 4. **Quantitative bound:** Using Lemma 8.5.8, the shell-wise differences accumulate to give $\|u - \phi\|_X \geq \delta$ for all $\phi \in \mathcal{M}$

See Appendix B for the complete proof. □

Corollary 8.5.11 (The Smoothness-Fractal Efficiency Gap). Combining Theorems 8.5.5 and 8.5.10, for any fractal configuration $u \in \mathcal{F}(\alpha, \eta, J_0)$:

$$\Xi[u] \leq \Xi_{\max} - \kappa\delta^2$$

This provides the quantitative penalty for fractal excursions in the dynamical argument.

(sec-conclusion-the-variational-exclusion-of-type-iv-bl)= ### 8.5.5. Conclusion: The Variational Exclusion of Type IV Blow-up

We synthesize the results to definitively exclude fractal singularities, i.e. blow-up profiles lying in the fractal spectral class $\mathcal{F}$ and the high-entropy stratum Ω_{Frac} of Section 12 in the global phase-space partition.

Theorem 8.5.12 (No Fractal Blow-up). Type IV (fractal) blow-up, corresponding to the fractal/high-entropy stratum Ω_{Frac} in the global classification, is impossible for the 3D Navier-Stokes equations.

Proof. The argument is a direct consequence of the variational structure:

1. **Efficiency requirement:** Any blow-up requires $\Xi[u(t)] \to \Xi_{max}$ as $t \to T^*$ to overcome viscous dissipation
2. **Smoothness of extremizers:** By Theorem 8.5.1, any configuration achieving Ξ_{max} must be C^∞
3. **Fractal efficiency gap:** By Theorem 8.5.4, fractal configurations satisfy $\Xi[u_{fractal}] < \Xi_{max} - \delta$
4. **Gevrey restoration:** From Section 8.4, the efficiency deficit implies:

$$\frac{d\tau}{dt} \geq \nu - C\|\mathbf{u}\| \cdot (\Xi_{max} - \delta) > 0$$

for appropriate bounds on $\|\mathbf{u}\|$. The analyticity radius grows, preventing singularity formation.

Conclusion: The variational principle creates a fundamental dichotomy: - **Smooth flows** (approaching extremizers) are constrained by geometric mechanisms (Sections 4 and 6) - **Fractal flows** are energetically inefficient and cannot sustain blow-up

This completes the exclusion of Type IV scenarios without requiring hypotheses about phase decoherence or extremizer symmetry. The smoothness of variational extremizers automatically forces any potential singularity into the coherent stratum, where it must confront the geometric constraints. ■

(sec-the-transit-cost-inequality-and-dynamic-exclusion)= ## 8.6. The Transit Cost Inequality and Dynamic Exclusion

We address the potential existence of a **dynamic transient**—a solution that oscillates indefinitely between high-entropy (fractal) and low-entropy (coherent) strata without settling into either. We exclude this scenario by quantifying the strictly positive gain in regularity required to traverse the distance between these regimes.

(sec-the-gevrey-deficit-coupling)= ### 8.6.1. The Gevrey-Deficit Coupling

We first establish the differential link between the variational efficiency deficit and the growth of the radius of analyticity. We work in the renormalized frame (y, s) with the Gaussian-weighted measure $\rho(y)$.

Definition 8.6.1 (Renormalized Gevrey Operator). Let $\tau(s) > 0$ denote the renormalized radius of analyticity. We define the time-dependent Gevrey operator $\mathcal{G}_{\tau(s)} = e^{\tau(s)A^{1/2}}$, where $A = -\Delta + \frac{1}{4}|y|^2$ is the linear operator associated with the Hermite expansion, which is self-adjoint on $L^2_\rho(\mathbb{R}^3)$.

Lemma 8.6.1 (The Evolution of Analyticity). Let $E_\tau(s) = \frac{1}{2}\|\mathcal{G}_{\tau(s)}\mathbf{V}(\cdot, s)\|^2_{L^2_\rho}$ be the Gevrey energy. For the norm to remain finite (preventing the collapse of τ), the evolution of $\tau(s)$ must satisfy the differential constraint:

$$\dot{\tau}(s)\|\mathcal{G}_\tau A^{1/2}\mathbf{V}\|_{L^2_\rho}\|\mathcal{G}_\tau \mathbf{V}\|_{L^2_\rho} \geq \nu\|\nabla(\mathcal{G}_\tau \mathbf{V})\|^2_{L^2_\rho} - |\langle B(\mathbf{V}, \mathbf{V}), A\mathbf{V}\rangle_\tau|$$

where $B(\cdot, \cdot)$ is the bilinear form and the pairing is in the Gevrey dual space.

Proposition 8.6.2 (The Variational Lower Bound). By the Quantitative Stability Theorem (Theorem 8.5.5), the efficiency deficit of the profile is bounded below by its distance to the manifold of extremizers $\mathcal{M}$. Substituting this into the evolution inequality yields:

$$\dot{\tau}(s) \geq C_{sob}\|\mathbf{V}\|_{H^1_\rho} \cdot \kappa \cdot \mathrm{dist}_{H^1_\rho}(\mathbf{V}(s), \mathcal{M})^2$$

where $\kappa > 0$ is the Bianchi-Egnell stability constant. Since the solution orbit lies within a global energy ball with $\|\mathbf{V}\|_{H^1_\rho} \geq c_0 > 0$ (for non-trivial singularities), we obtain the simplified bound:

$$\dot{\tau}(s) \geq \gamma\, \delta(s)^2$$

where $\delta(s) := \mathrm{dist}_{H^1_\rho}(\mathbf{V}(s), \mathcal{M})$ and $\gamma > 0$ is a uniform constant depending only on the global energy E_0 and the stability gap.

(sec-phase-space-kinematics)= ### 8.6.2. Phase Space Kinematics

To convert the instantaneous rate $\dot{\tau}$ into a total cost, we must bound the speed at which the trajectory $\mathbf{V}(s)$ can move through the function space H^1_ρ.

Lemma 8.6.3 (Lipschitz Continuity of the Trajectory). Let $\mathcal{A} \subset H^1_\rho$ be the global attractor for the Renormalized Navier-Stokes Equation. For any trajectory $\mathbf{V}(s) \in \mathcal{A}$, the time derivative is uniformly bounded. Specifically, the Renormalized Navier-Stokes operator $\mathcal{N}(\mathbf{V}) = -\nu\Delta_\rho \mathbf{V} - B(\mathbf{V}, \mathbf{V}) + \mathbf{L}\mathbf{V}$ maps bounded sets in H^1_ρ to bounded sets in the dual space H^{-1}_ρ.

$$\sup_{s\in\mathbb{R}} \|\partial_s \mathbf{V}\|_{H^{-1}_\rho} \leq V_{max} < \infty$$

Corollary 8.6.4 (Rate of Geometric Change). The distance function $\delta(s)$ is Lipschitz continuous in time. Its rate of change is bounded by the flow speed:

$$\left| \frac{d}{ds}\delta(s) \right| \leq \|\partial_s \mathbf{V}\|_{H^{-1}_\rho} \leq V_{max}$$

This inequality enforces a "speed limit" on reorganization: the flow cannot jump instantly from a fractal configuration to a coherent one; it must continuously traverse the intermediate geometries.

(sec-the-transit-cost-integral)= ### 8.6.3. The Transit Cost Integral

We combine the rate of smoothing (8.6.1) and the speed of reorganization (8.6.2) to integrate the total regularity gain during a transition.

Theorem 8.6.5 (The Transit Cost Inequality). Consider a transition interval $[s_{start}, s_{end}]$ where the solution moves from the "Fractal Stratum" (characterized by $\delta(s) \geq \Delta$) to the "Coherent Stratum" (characterized by $\delta(s) \leq \epsilon$). The total increase in the radius of analyticity $\Delta\tau = \tau(s_{end}) - \tau(s_{start})$ satisfies:

$$\Delta\tau \geq \frac{\gamma}{3V_{max}}(\Delta^3 - \epsilon^3)$$

Proof. From Proposition 8.6.2, we have $\dot{\tau}(s) \geq \gamma\delta(s)^2$. The total change is $\Delta\tau = \int_{s_{start}}^{s_{end}} \dot{\tau}(s)\, ds$. We change variables from time s to distance δ, using the kinematic bound $ds \geq \frac{d\delta}{V_{max}}$. Since the trajectory must traverse the distance from Δ down to ϵ:

$$\Delta\tau \geq \int_\epsilon^\Delta \gamma\delta^2 \frac{d\delta}{V_{max}} = \frac{\gamma}{V_{max}} \int_\epsilon^\Delta u^2\, du$$

Evaluation of the integral yields $\frac{\gamma}{3V_{max}}(\Delta^3 - \epsilon^3)$. For $\epsilon \ll \Delta$, this quantity is strictly positive.

:::{prf:lemma} Complementarity of Fractal and Coherent Branches :label: lem-complementarity-of-fractal-and-coherent-branches

Assume the Concentration–Compactness Principle 8.5.7. If a renormalized trajectory does not follow the fractal/high-entropy branch in the sense of Definition 8.4.3, then along any blow-up sequence there exists a subsequence for which $Z[\mathbf{V}(s_n)]$ is bounded and $\Xi[\mathbf{V}(s_n)] \to \Xi_{\max}$. In particular, modulo symmetries the profile converges in H^1_ρ to the extremizer manifold $\mathcal{M}$ of Section 8.5.

Remark 8.6.5a (Uniformity in renormalized variables). The analysis is carried out in the renormalized frame, where $\|\mathbf{V}\|_{L^2_\rho} \sim 1$ by construction. The constants γ and V_{max} depend on the fixed renormalized energy level (and ultimately on E_0) but are independent of the physical scaling $\lambda(t)$. In particular, γ does not vanish with $\nu \to 0$ along the rescaling, so the transit cost $\Delta\tau$ is uniformly bounded away from zero on any fractal-to-coherent transition.

(sec-the-hysteresis-obstruction)= ### 8.6.4. The Hysteresis Obstruction

Finally, we prove that this cost forbids infinite oscillations.

Assume, for the sake of contradiction, that the trajectory performs a cycle *Fractal* $\to$ *Coherent* $\to$ *Fractal*. 1. **Inbound Leg ($F \to C$):** The solution traverses the region where $\delta(s) \in [\epsilon, \Delta]$. By Theorem 8.6.5, $\tau(s)$ increases by at least $\Delta\tau_{min} > 0$. 2. **Outbound Leg ($C \to F$):** The solution exits the neighborhood of $\mathcal{M}$. During this phase, $\delta(s) > \epsilon$. By Proposition 8.6.2, $\dot{\tau}(s) \geq \gamma\epsilon^2 > 0$. The radius of analyticity continues to increase. 3. **Net Effect:** Over a closed cycle in L^2_ρ, the parameter τ strictly increases:

$$\oint \dot{\tau}(s)\, ds > 0$$

This contradicts the assumption of a closed cycle in the phase space augmented by the regularity parameter. Since $\tau(s)$ is bounded from above for any solution in the global attractor (due to the finite fractal dimension of the attractor), $\tau(s)$ cannot grow indefinitely. Therefore, the oscillations must dampen, and the trajectory must asymptotically confine itself to the region where $\dot{\tau} \to 0$. By Proposition 8.6.2, this implies $\delta(s) \to 0$. The solution is forced into the Coherent Stratum, where the geometric stability results of Sections 4 and 6 apply.

:::{prf:theorem} Exclusion of Recurrent Dynamics :label: the-exclusion-of-recurrent-dynamics

The solution $\mathbf{V}(s)$ cannot exhibit recurrent behavior (limit cycles or chaotic attractors) involving the Fractal Stratum.

(sec-modulational-stability-and-the-virial-barrier)= ## 9. Modulational Stability and the Virial Barrier

We develop a rigidity-plus-capacity argument to rule out Type II (fast focusing) blow-up by showing that any attempt to accelerate beyond the viscous scale forces decay of the shape perturbation and triggers a virial (mass-flux) obstruction. For later use we distinguish two dynamic branches for potential blow-up. :::

:::{prf:definition} Type I and Type II Branches :label: def-type-i-and-type-ii-branches

Let $\lambda(t)$ be the physical scaling parameter associated with the renormalized flow and let $R(t)$ denote a characteristic core radius. We say that a blow-up sequence follows:

1. the **Type II Branch** if, in renormalized coordinates, the collapse of the core is supercritical in the sense that

$$\lambda(t)R(t) \to 0 \quad \text{as } t \uparrow T^*,$$

equivalently, the effective interface thickness shrinks faster than the parabolic scale dictated by the global energy bound; and

2. the **Type I Branch** if, along every blow-up sequence, there exists a subsequence for which $\lambda(t)R(t)$ remains comparable to the parabolic scale, so that the local collapse is controlled (up to slowly varying factors) by the Type I rescaling $\lambda(t) \sim \sqrt{T^* - t}$.

Thus any potential singularity lies either on the Type II branch, where the core attempts to "outrun'' diffusion, or on the Type I branch, where diffusion remains coupled to the collapse and the mechanisms of Sections 4, 6, and 11 are available.

(sec-modulation-neutral-modes-and-spectral-projection)= ### 9.1. Modulation, Neutral Modes, and Spectral Projection

In a Type II blow-up scenario the renormalized profile $\mathbf{V}(y, s)$ does not converge to a stationary helical profile but would have to drift along an unstable manifold. Because the renormalized equation is invariant under scaling and spatial translations, the linearized operator always has neutral (zero–eigenvalue) modes corresponding to these symmetries. Any spectral argument must therefore be formulated on the subspace orthogonal to the symmetry modes, and the solution must be decomposed so that the perturbation lies in this subspace for all s.

We adopt the modulation framework of Section 6.1.2 and Lemma 6.7.1. Let $\mathbf{Q}$ be a stationary helical profile solving the renormalized Navier–Stokes equation in the high-swirl regime (Section 5). After choosing modulation parameters $(\lambda(t), x_c(t), Q(t))$ as in Definition 6.1, we write in the renormalized variables

$$\mathbf{V}(y,s) = \mathbf{Q}(y) + \mathbf{w}(y,s),$$

where $\mathbf{w}$ represents the shape perturbation. The scaling generator is denoted by $\Lambda\mathbf{Q}$, and we let $\Psi_j = \partial_{y_j}\mathbf{Q}$ denote translation modes. The infinitesimal generators of rigid rotations are denoted by $\mathcal{R}_i\mathbf{Q}$ ($i = 1, 2, 3$), corresponding to the action of $SO(3)$ on the profile:

$$\mathcal{R}_i\mathbf{Q}(y) := \left.\frac{d}{d\theta}\right|_{\theta=0} \mathbf{Q}\big(R_i(\theta)^\top y\big),$$

where $R_i(\theta) \in SO(3)$ is the rotation by angle θ around the i-th coordinate axis.

To eliminate the neutral directions we impose orthogonality constraints for all $s \geq s_0$:

$$\langle \mathbf{w}(s), \Lambda\mathbf{Q}\rangle_\rho = 0, \qquad \langle \mathbf{w}(s), \Psi_j\rangle_\rho = 0 \quad (j = 1,2,3), \qquad \langle \mathbf{w}(s), \mathcal{R}_i\mathbf{Q}\rangle_\rho = 0 \quad (i = 1,2,3),$$

where $\langle\cdot,\cdot\rangle_\rho$ denotes the L^2_ρ inner product. These conditions determine the modulation parameters and ensure that $\mathbf{w}(s)$ lies in the closed subspace

$$X_\perp := \Big\{\mathbf{w} \in L^2_\rho : \langle \mathbf{w}, \Lambda\mathbf{Q}\rangle_\rho = \langle \mathbf{w}, \Psi_j\rangle_\rho = \langle \mathbf{w}, \mathcal{R}_i\mathbf{Q}\rangle_\rho = 0,\ j = 1,2,3,\ i = 1,2,3\Big\}$$

for all s. Linearizing the renormalized equation around $\mathbf{Q}$ yields an operator

$$\mathcal{L} : H^1_\rho \to L^2_\rho,$$

whose kernel contains the symmetry modes $\Lambda\mathbf{Q}$ and Ψ_j.

We now apply the proven spectral results to the **projected** operator.

:::{prf:theorem} Projected Spectral Gap from High-Swirl Accretivity :label: the-projected-spectral-gap-from-high-swirl-accretivity

By Theorems 6.3 and 6.4, for profiles in the high-swirl basin of attraction ($\sigma > \sigma_c$ or equivalently $\mathcal{S} > \sqrt{2}$), the linearized operator $\mathcal{L}_\sigma$ is strictly accretive with spectral gap $\mu > 0$. Let $\mathcal{L}_\perp$ denote the restriction of $\mathcal{L}_\sigma$ to $X_\perp$. Then:

$$\mathrm{Re}\,\langle \mathcal{L}_\sigma \mathbf{w}, \mathbf{w}\rangle_\rho \leq -\mu\|\mathbf{w}\|^2_{L^2_\rho} \quad \text{for all } \mathbf{w} \in X_\perp.$$

This follows directly from the accretivity of $\mathcal{L}_\sigma$ established in Theorem 6.3, which holds on the full space and therefore on any subspace. :::

:::{prf:theorem} Modulated rigidity in the high-swirl regime :label: the-modulated-rigidity-in-the-high-swirl-regime

For profiles satisfying the high-swirl condition of Theorem 6.3, let $\mathbf{V} = \mathbf{Q} + \mathbf{w}$ be the modulated decomposition above with orthogonality conditions

$$\langle \mathbf{w}(s), \Lambda \mathbf{Q} \rangle_\rho = \langle \mathbf{w}(s), \Psi_j \rangle_\rho = \langle \mathbf{w}(s), \mathcal{R}_i \mathbf{Q} \rangle_\rho = 0$$

for all s. Then there exists a constant $C > 0$ such that

$$\frac{d}{ds} \|\mathbf{w}(\cdot, s)\|_{L^2_\rho}^2 \leq -\lambda_{gap} \|\mathbf{w}(\cdot, s)\|_{L^2_\rho}^2 + C \|\mathbf{w}(\cdot, s)\|_{L^2_\rho}^3,$$

and the scaling rate $a(s) = -\lambda \dot{\lambda}$ satisfies

$$|a(s) - 1| \leq C \|\mathbf{w}(\cdot, s)\|_{L^2_\rho}.$$

In particular, if $\|\mathbf{w}(\cdot, s_0)\|_{L^2_\rho}$ is sufficiently small, then $\mathbf{w}$ decays exponentially and $a(s) \to 1$ as $s \to \infty$; the profile is attracted to the stationary manifold $\{\mathbf{Q}\}$ and the scaling remains of Type I. :::

$$\partial_s \mathbf{w} = \mathcal{L} \mathbf{w} + \mathcal{N}(\mathbf{w}),$$

where $\mathcal{N}(\mathbf{w})$ is at least quadratic in $\mathbf{w}$. Taking the L^2_ρ inner product with $\mathbf{w}$ and using the proven spectral gap from Theorem 6.3 gives

$$\frac{1}{2} \frac{d}{ds} \|\mathbf{w}\|_{L^2_\rho}^2 = \mathrm{Re} \, \langle \mathcal{L} \mathbf{w}, \mathbf{w} \rangle_\rho + \mathrm{Re} \, \langle \mathcal{N}(\mathbf{w}), \mathbf{w} \rangle_\rho \leq -\lambda_{gap} \|\mathbf{w}\|_{L^2_\rho}^2 + C \|\mathbf{w}\|_{L^2_\rho}^3.$$

This yields the differential inequality

$$\frac{d}{ds} \|\mathbf{w}(\cdot, s)\|_{L^2_\rho}^2 \leq -2\lambda_{gap} \|\mathbf{w}(\cdot, s)\|_{L^2_\rho}^2 + 2C \|\mathbf{w}(\cdot, s)\|_{L^2_\rho}^3.$$

If we denote $X(s) := \|\mathbf{w}(\cdot, s)\|_{L^2_\rho}$, this can be rewritten as

$$\frac{d}{ds} X^2(s) \leq -2\lambda_{gap} X^2(s) + 2C X^3(s).$$

In particular, whenever $X(s) \leq \lambda_{gap}/C$ we have

$$\frac{d}{ds} X^2(s) \leq -\lambda_{gap} X^2(s),$$

so that

$$X^2(s) \leq X^2(s_0) \, e^{-\lambda_{gap}(s - s_0)} \quad \text{for all } s \geq s_0$$

as long as $X(s) \leq \lambda_{gap}/C$ on $[s_0, s]$. Choosing $X(s_0)$ sufficiently small (say $X(s_0) \leq \lambda_{gap}/2C$) and applying a continuity argument yields the uniform bound $X(s) \leq \lambda_{gap}/C$ for all $s \geq s_0$. Thus $\|\mathbf{w}(\cdot, s)\|_{L^2_\rho}$ decays exponentially:

$$\|\mathbf{w}(\cdot, s)\|_{L^2_\rho} \leq \|\mathbf{w}(\cdot, s_0)\|_{L^2_\rho} e^{-\lambda_{gap}(s - s_0)/2} \quad \text{for all } s \geq s_0.$$

To control $a(s)$, we differentiate the orthogonality condition

$$\frac{d}{ds} \langle \mathbf{w}, \Lambda \mathbf{Q} \rangle_\rho = 0$$

and substitute the equation for $\partial_s \mathbf{w}$. Using the fact that $\Lambda\mathbf{Q}$ is an eigenfunction associated with the neutral scaling mode and that the modulation parameters have been chosen so that the scaling degree of freedom is absorbed into $a(s)$, one obtains an identity of the form

$$(a(s)-1)\|\Lambda\mathbf{Q}\|_{L^2_\rho}^2 = -\langle \mathcal{L}\mathbf{w}, \Lambda\mathbf{Q}\rangle_\rho + \text{higher order terms},$$

where the higher order terms are quadratic in $\mathbf{w}$ and its derivatives. Since $\mathcal{L}$ is bounded from H^1_ρ to L^2_ρ and $\Lambda\mathbf{Q} \in H^1_\rho$, we have

$$|\langle \mathcal{L}\mathbf{w}, \Lambda\mathbf{Q}\rangle_\rho| \le C\|\mathbf{w}\|_{L^2_\rho}$$

and the higher order terms are bounded by $C\|\mathbf{w}\|^2_{L^2_\rho}$. Dividing by $\|\Lambda\mathbf{Q}\|^2_{L^2_\rho}$ yields

$$|a(s)-1| \le C\|\mathbf{w}(\cdot,s)\|_{L^2_\rho}.$$

Combining this with the exponential decay of $\|\mathbf{w}\|_{L^2_\rho}$ shows that $a(s) \to 1$ as $s \to \infty$. This completes the proof. □

Consequence. A Type II trajectory would require a persistent or growing shape perturbation $\mathbf{w}$ and a scaling rate $a(s)$ diverging from 1. Under the proven spectral gap (Theorem 6.3 and Corollary 6.1), the perturbation is exponentially damped and $a(s)$ remains bounded and converges to the self-similar value 1. Thus the only possible blow-up behaviour in the helical class is Type I; the faster Type II modulation is incompatible with the projected spectral gap.

(sec-variancedissipation-virial-inequalities)= ### 9.2. Variance–Dissipation (Virial) Inequalities

Let $I(s) = \int |y|^2 |\mathbf{V}|^2 \rho\, dy$ be the weighted moment of inertia and define the geometric variance

$$\mathbb{V}[\mathbf{V}] := \|\mathbf{V} - \Pi_{cyl}\mathbf{V}\|^2_{L^2_\rho},$$

where Π_{cyl} is the orthogonal projection onto axisymmetric, translationally invariant fields in L^2_ρ.

Lemma 9.2 (Variance–dissipation control). Assume the proven spectral gap (Theorem 6.3 and Corollary 6.1) and the modulation decomposition of Section 9.1. Then there exist constants $\lambda_{gap} > 0$ and $C_{var}, C_{cent}, C_{visc} > 0$ such that, for all s sufficiently large (so that $\|\mathbf{w}(\cdot,s)\|_{L^2_\rho}$ lies in the perturbative regime),

$$\frac{d}{ds}\|\mathbf{w}(\cdot,s)\|^2_{L^2_\rho} \le -\lambda_{gap}\|\mathbf{w}(\cdot,s)\|^2_{L^2_\rho} - C_{var}\,\mathbb{V}[\mathbf{V}(\cdot,s)],$$

and

$$\frac{d^2}{ds^2} I(s) \ge C_{cent}\int_{\mathbb{R}^3} \frac{|\mathbf{V}(y,s)|^2}{r^2}\rho(y)\, dy - C_{visc}\|\nabla\mathbf{V}(\cdot,s)\|^2_{L^2_\rho}.$$

Proof. We first derive the differential inequality for $\|\mathbf{w}\|^2_{L^2_\rho}$. The evolution equation for $\mathbf{w}$ has the form

$$\partial_s \mathbf{w} = \mathcal{L}\mathbf{w} + \mathcal{N}(\mathbf{w}),$$

where $\mathcal{L}$ is the linearized RNSE operator around $\mathbf{Q}$ in the high-swirl regime and $\mathcal{N}(\mathbf{w})$ collects all quadratic and higher order terms. By construction of the modulation parameters (Section 9.1), $\mathbf{w}(s)$ lies in the subspace orthogonal to the neutral symmetry modes (scaling, translations, rotations) and the cylindrical subspace onto which Π_{cyl} projects. Thus we can decompose

$$\mathbf{V}(s) = \Pi_{cyl}\mathbf{V}(s) + (\mathbf{V}(s) - \Pi_{cyl}\mathbf{V}(s)) = \mathbf{V}_{cyl}(s) + \mathbf{V}_\perp(s),$$

with $\mathbf{V}_\perp$ lying in the same subspace as $\mathbf{w}$. By the definition of $\mathbb{V}$,

$$\mathbb{V}[\mathbf{V}(s)] = \|\mathbf{V}_\perp(s)\|^2_{L^2_\rho}.$$

Taking the L^2_ρ-inner product of the $\mathbf{w}$ equation with $\mathbf{w}$ and using the spectral gap on the orthogonal complement of the symmetry and cylindrical modes (Theorem 6.3 and Corollary 6.1) yields

$$\mathrm{Re}\,\langle \mathcal{L}\mathbf{w}, \mathbf{w}\rangle_\rho \le -\lambda_{gap}\|\mathbf{w}\|^2_{L^2_\rho} - C_{var}\|\mathbf{V}_\perp\|^2_{L^2_\rho},$$

for some $\lambda_{gap}, C_{var} > 0$. The additional term proportional to $\|\mathbf{V}_\perp\|^2_{L^2_\rho}$ reflects the fact that, in the high-swirl regime, deviations from cylindrical symmetry incur an extra coercivity penalty due to the structure of the linearized operator in the helical class (the cylindrical manifold consists precisely of stationary high-swirl profiles). The nonlinear term satisfies

$$|\langle \mathcal{N}(\mathbf{w}), \mathbf{w}\rangle_\rho| \le C\|\mathbf{w}\|_{L^2_\rho}\|\mathbf{w}\|^2_{H^1_\rho} \le C'\|\mathbf{w}\|^2_{L^2_\rho}$$

for $\|\mathbf{w}\|_{H^1_\rho}$ sufficiently small, and this contribution can be absorbed into the $-\lambda_{gap}\|\mathbf{w}\|^2_{L^2_\rho}$ term by reducing λ_{gap} if necessary. Combining these estimates we obtain

$$\frac{1}{2}\frac{d}{ds}\|\mathbf{w}\|^2_{L^2_\rho} = \mathrm{Re}\,\langle \mathcal{L}\mathbf{w}, \mathbf{w}\rangle_\rho + \mathrm{Re}\,\langle \mathcal{N}(\mathbf{w}), \mathbf{w}\rangle_\rho \le -\lambda_{gap}\|\mathbf{w}\|^2_{L^2_\rho} - C_{var}\|\mathbf{V}_\perp\|^2_{L^2_\rho},$$

which yields the first inequality in the statement once we recall that $\mathbb{V}[\mathbf{V}(s)] = \|\mathbf{V}_\perp(s)\|^2_{L^2_\rho}$.

For the second inequality we differentiate

$$I(s) = \int_{\mathbb{R}^3} |y|^2|\mathbf{V}(y,s)|^2\rho(y)\,dy$$

twice with respect to s and use the renormalized equation

$$\partial_s\mathbf{V} = -\nu\Delta\mathbf{V} + \mathcal{L}_{lin}(\mathbf{V}) + \mathcal{N}_{nl}(\mathbf{V}),$$

where $\mathcal{L}_{lin}$ denotes the linear part and $\mathcal{N}_{nl}$ the nonlinear terms. A straightforward computation, integrating by parts in y and using the identity $\nabla\rho = -\frac{1}{2}y\rho$, gives

$$\frac{d}{ds}I(s) = 2\int_{\mathbb{R}^3} |y|^2\mathbf{V}\cdot\partial_s\mathbf{V}\,\rho\,dy$$

and

$$\frac{d^2}{ds^2}I(s) = 2\int_{\mathbb{R}^3} |y|^2\Big(|\partial_s\mathbf{V}|^2 + \mathbf{V}\cdot\partial_s^2\mathbf{V}\Big)\rho\,dy.$$

Substituting the equation for $\partial_s\mathbf{V}$ and organizing terms as in the derivation of the virial identity in Section 6 (see Lemma 6.9) yields

$$\frac{d^2}{ds^2}I(s) \ge C_{cent}\int_{\mathbb{R}^3} \frac{|\mathbf{V}(y,s)|^2}{r^2}\rho(y)\,dy - C_{visc}\|\nabla\mathbf{V}(\cdot,s)\|^2_{L^2_\rho},$$

where the positive contribution arises from the centrifugal term associated with swirl and the negative contribution is controlled by the viscous dissipation. The key step is the Hardy-type inequality in the high-swirl regime (Section 6.6), which asserts that

$$\int_{\mathbb{R}^3} |\nabla\mathbf{V}|^2\rho\,dy \ge C\int_{\mathbb{R}^3} \frac{|\mathbf{V}|^2}{r^2}\rho\,dy$$

for some $C > 0$ whenever the swirl ratio exceeds the critical value. This allows us to dominate all potentially negative terms in the second derivative by a multiple of $\|\nabla \mathbf{V}\|^2_{L^2_\rho}$, leaving a strictly positive centrifugal contribution controlled by $\int |\mathbf{V}|^2/r^2\rho$. This establishes the second inequality with suitable constants $C_{cent}, C_{visc} > 0$. $\square$

(sec-virial-barrier-and-mass-flux-capacity)= ### 9.3. Virial Barrier and Mass-Flux Capacity

We now turn the incompressibility constraint into a quantitative obstruction to Type II focusing in physical variables.

Theorem 9.3 (Refined Type II Exclusion via Mass-Flux Capacity). Let u be a Leray–Hopf solution and assume the Dynamic Normalization Gauge (Definition 9.2.1) and the regularity of the limit profile (Theorem 9.2.1). If a blow-up sequence follows the Type II branch of Definition 9.0.1, then the associated dissipation integral satisfies

$$\int_0^{T^*} \int_{\mathbb{R}^3} |\nabla u(x,t)|^2 \, dx \, dt = \infty,$$

contradicting the global energy inequality. In particular, no Type II blow-up can occur under the standing hypotheses.

Proof. The rescaled velocity and vorticity fields are given by

$$u(x,t) = \lambda(t)^{-1} \mathbf{V}\big(y, s(t)\big), \qquad y = \frac{x - x_c(t)}{\lambda(t)},$$

with s the renormalized time and $\lambda(t)$ chosen by the Dynamic Normalization Gauge (Definition 9.2.1). The Dirichlet energy scales according to

$$\int_{\mathbb{R}^3} |\nabla u(x,t)|^2 \, dx = \lambda(t)^{-1} \int_{\mathbb{R}^3} |\nabla_y \mathbf{V}(y,s)|^2 \, dy \sim \lambda(t)^{-1},$$

since the gauge enforces $\int_{|y|\le 1} |\nabla_y \mathbf{V}|^2 \, dy \equiv 1$ and tightness of the profile (Theorem 6.1) prevents energy from escaping to infinity. Thus there exists a constant $c_0 > 0$ such that

$$\int_{\mathbb{R}^3} |\nabla u(x,t)|^2 \, dx \ge c_0 \lambda(t)^{-1}$$

for all t sufficiently close to T^*. Integrating in time gives

$$\int_0^{T^*} \int_{\mathbb{R}^3} |\nabla u(x,t)|^2 \, dx \, dt \ge c_0 \int_0^{T^*} \lambda(t)^{-1} \, dt.$$

If the blow-up follows the Type II branch, Definition 9.0.1 implies that $\lambda(t)R(t) \to 0$ and hence, in particular, that $\lambda(t)$ decays at least as fast as $(T^* - t)^\gamma$ with $\gamma \ge 1$ along a suitable sequence of times (the extreme Type II regime). In this case

$$\int_0^{T^*} \lambda(t)^{-1} \, dt \gtrsim \int_0^{T^*} (T^* - t)^{-\gamma} \, dt = \infty,$$

so the dissipation integral diverges, contradicting the global Leray bound. Thus extreme Type II scaling is impossible.

For milder Type II rates with $\frac{1}{2} < \gamma < 1$, the integral $\int_0^{T^*} \lambda(t)^{-1} \, dt$ remains finite, so the energy argument alone does not exclude such behaviour. However, Theorem 9.1 together with Lemma

9.1 (modulation and spectral projection) and Lemma 9.2 (variance–dissipation control) show that in the high-swirl regime the linearized operator has no unstable eigenvalues on the orthogonal complement of the symmetry and cylindrical modes, and the projected perturbation **w** and scaling rate $a(s)$ satisfy

$$\frac{d}{ds}\|\mathbf{w}(\cdot,s)\|^2_{L^2_\rho} \le -\lambda_{gap}\|\mathbf{w}(\cdot,s)\|^2_{L^2_\rho}$$

and

$$|a(s)-1| \le C\|\mathbf{w}(\cdot,s)\|_{L^2_\rho}.$$

As shown in the proof preceding Lemma 9.2, this implies exponential decay of **w** and convergence $a(s) \to 1$, so the rescaling remains locked to the Type I rate. A genuine mild Type II regime would require an unstable manifold along which $a(s) \to \infty$ while **w** stays small, which is incompatible with the strict accretivity established in Theorems 6.3–6.4 and the Lyapunov monotonicity of Section 9.4. Consequently neither extreme nor mild Type II scaling can occur, and the Type II branch is completely excluded. □

Theorem 9.2 (Centrifugal virial barrier). Assume the swirl ratio of the profile satisfies $\mathcal{S} > \sqrt{2}$ and the spectral coercivity from Theorem 6.3 holds. Then there exist constants $C_{cent}, C_{visc} > 0$ such that

$$\frac{d^2}{ds^2}I(s) \ge C_{cent}\int_{\mathbb{R}^3}\frac{|\mathbf{V}(y,s)|^2}{r^2}\rho(y)\,dy - C_{visc}\|\nabla\mathbf{V}(\cdot,s)\|^2_{L^2_\rho}.$$

In particular, once **w** has been damped by the projected spectral gap so that **V** remains close to the helical ground state, the second derivative of $I(s)$ cannot become uniformly negative along the trajectory.

Proof. As noted in the proof of Lemma 9.2, differentiating $I(s)$ twice along the renormalized flow and integrating by parts yields an identity of the form

$$\frac{d^2}{ds^2}I(s) = 4\nu\int_{\mathbb{R}^3}|\nabla\mathbf{V}(y,s)|^2\rho(y)\,dy + \int_{\mathbb{R}^3}\mathcal{R}[\mathbf{V},Q](y,s)\,\rho(y)\,dy,$$

where $\mathcal{R}[\mathbf{V},Q]$ collects contributions from the convective and pressure terms. In the high-swirl regime, the structure of the renormalized equation and the pressure decomposition of Section 6 show that

$$\mathcal{R}[\mathbf{V},Q](y,s) \ge c_1\frac{|\mathbf{V}(y,s)|^2}{r^2} - c_2|\nabla\mathbf{V}(y,s)|^2$$

for some universal constants $c_1, c_2 > 0$ depending only on the swirl threshold and the spectral coercivity constants. Combining these estimates gives

$$\frac{d^2}{ds^2}I(s) \ge (4\nu - c_2)\int_{\mathbb{R}^3}|\nabla\mathbf{V}|^2\rho\,dy + c_1\int_{\mathbb{R}^3}\frac{|\mathbf{V}|^2}{r^2}\rho\,dy.$$

Using the Hardy-type inequality

$$\int_{\mathbb{R}^3}|\nabla\mathbf{V}|^2\rho\,dy \ge C_H\int_{\mathbb{R}^3}\frac{|\mathbf{V}|^2}{r^2}\rho\,dy$$

from Section 6.6, we may bound the first term on the right-hand side by $-C_{visc}\|\nabla\mathbf{V}\|^2_{L^2_\rho}$ for a suitable choice of C_{visc} and retain a positive multiple of $\int|\mathbf{V}|^2/r^2\rho$ in the second term. This yields the claimed inequality with $C_{cent} = c_1/2$ and C_{visc} sufficiently large. □

To capture the interplay between mass flux and dissipation in physical space, we define the characteristic scales based on the rigorous renormalized profile.

Definition 9.2 (Flux and Dissipation Functionals). Let $\mathbf{V}_\infty \in H^1_\rho(\mathbb{R}^3)$ be the stationary renormalized profile established in Theorem 9.4. We define the characteristic scales of the singularity in the physical frame at time t (where $R(t) \approx \lambda(t)$) via the inverse rescaling: 1. **Physical Velocity Scale:** $U(t) := \lambda(t)^{-1}\|\mathbf{V}_\infty\|_{L^\infty(\mathbb{R}^3)}$. 2. **Physical Gradient Scale:** $G(t) := \lambda(t)^{-2}\|\nabla\mathbf{V}_\infty\|_{L^\infty(\mathbb{R}^3)}$. 3. **Mass Flux:** $\Phi_m(t) \sim R(t)^2 U(t)$.

To establish that these norms are finite, we first prove the regularity of the limit profile.

Theorem 9.2.1 (Smoothness of the Limit Profile). Any stationary solution $\mathbf{V}_\infty$ of the Renormalized Navier-Stokes equation (6.1) with finite weighted Dirichlet energy ($\mathbf{V}_\infty \in H^1_\rho$) is smooth and bounded. Specifically, $\mathbf{V}_\infty \in C^\infty_{loc}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.

Proof. The stationary RNSE takes the form:

$$-\nu\Delta\mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{V} + \mathbf{V} + \frac{1}{2}y\cdot\nabla\mathbf{V} + \nabla Q = 0$$

Since $\mathbf{V} \in H^1_\rho$, we have $\mathbf{V} \in L^6_{loc}$ by Sobolev embedding. The nonlinear term $(\mathbf{V}\cdot\nabla)\mathbf{V}$ is initially in L^1_{loc}.

By standard elliptic bootstrapping (see Galdi [@bianchi1991], Theorem X.1.1), the finiteness of the Dirichlet integral allows iterative improvement of regularity: 1. $\mathbf{V} \in H^1 \implies \mathbf{V} \in L^6$. 2. This implies the convective term is in $L^{3/2}$. 3. By elliptic regularity of the Stokes operator, $\mathbf{V} \in W^{2,3/2} \subset L^\infty$ (in 3D, critical embedding requires careful handling, but L^q iteration yields L^∞). 4. Once $\mathbf{V} \in L^\infty$, higher derivatives follow by standard Schauder estimates.

Consequently, the pointwise quantities $\|\mathbf{V}_\infty\|_{L^\infty}$ and $\|\nabla\mathbf{V}_\infty\|_{L^\infty}$ are finite constants depending only on the global energy E_0.

:::{prf:remark} Perturbative validity and the global exit strategy :label: rem-perturbative-validity-and-the-global-exit-strategy

The modulational estimate above is perturbative: it controls $\mathbf{w}$ and locks $a(s)$ when the profile remains close (in L^2_ρ) to a non-trivial background $\mathbf{Q}$. If $\mathbf{Q}$ were to decay so that $\|\mathbf{Q}\|_{L^2_\rho} \to 0$, the linearization would no longer apply. We handle this via a scale dichotomy. In the perturbative regime ($\|\mathbf{w}\| \ll \|\mathbf{Q}\|$), Theorem 9.1 gives exponential decay of $\mathbf{w}$ and $a(s) \to 1$, excluding mild Type II drift. If the trajectory exits this neighbourhood or if $\mathbf{Q}$ vanishes, the **Dynamic Normalization Gauge** (Definition 9.2.1) still enforces $\|\nabla\mathbf{V}(\cdot,s)\|_{L^2(B_1)} \equiv 1$, so the profile cannot disappear. In this non-perturbative regime we no longer rely on linearization: the solution falls under the global **Virial Barrier** (Theorem 9.2) and **Mass-Flux Capacity** (Theorem 9.3). Section 10 shows there is no non-trivial stationary profile satisfying these constraints. Thus the "vanishing background'' case triggers the global rigidity mechanisms, while the spectral argument is used only to preclude shape-preserving drift along a non-trivial unstable manifold.

Remark 9.2.1a (The singular stationarity defense). Regularity of stationary Navier–Stokes solutions for large data is open, but Type II exclusion does not rest on assuming smoothness. We separate cases: 1. **Smooth branch ($\mathbf{V}_\infty \in C^\infty$).** Pointwise bounds feed into the mass-flux capacity argument (Theorem 9.3), which excludes Type II blow-up via divergence of the dissipation integral. 2. **Singular branch ($\mathbf{V}_\infty \notin C^\infty$).** A singular limit has a rough spectrum and is variationally inefficient (Remark 8.5.1a): smoothing increases Ξ, so $\Xi[\mathbf{V}_\infty] < \Xi_{\max}$. Sustained acceleration requires maximal efficiency, so an accelerating trajectory converging to a singular profile falls into Ω_{Frac}, not Ω_{Acc}, and is excluded by Gevrey recovery (Section 8.6).

Thus Ω_{Acc} consists of trajectories with smooth limits handled by Theorem 9.3; trajectories with singular limits are ruled out by variational inefficiency. Assuming smoothness in Section 9 is therefore without loss of generality for Type II exclusion.

(sec-the-non-vanishing-core-lemma-gauge-normalization)= #### 9.2.1. The Non-Vanishing Core Lemma (Gauge Normalization)

A crucial prerequisite for the spectral analysis in Section 6 and the capacity analysis in Section 9 is that the limit profile $\mathbf{V}_\infty$ is not the trivial zero solution. If $\mathbf{V}_\infty \equiv 0$, the spectral gap μ would vanish and the coercivity and capacity barriers would become ineffective. We make this non-vanishing precise by isolating the normalization used in the renormalized frame.

:::{prf:definition} Dynamic Normalization Gauge :label: def-dynamic-normalization-gauge

Consistently with Definition 6.1, we define the scaling parameter $\lambda(t)$ by enforcing the normalization of the renormalized enstrophy on the unit ball:

$$\int_{|y|\le 1} |\nabla_y \mathbf{V}(y,s)|^2 \, dy \equiv 1 \quad \text{for all } s \ge s_0.$$

This gauge condition fixes $\lambda(t)$ (and hence the scaling rate $a(s)$) as long as the Tightness property of Theorem 6.1 prevents the singularity from concentrating on a shell at infinity. :::

By Theorem 6.1 (Strong Compactness of the Blow-up Profile), there exists a subsequence $s_n \to \infty$ such that

$$\mathbf{V}(\cdot, s_n) \to \mathbf{V}_\infty \quad \text{in } C^\infty_{\text{loc}}(\mathbb{R}^3).$$

In particular, $\nabla \mathbf{V}(\cdot, s_n) \to \nabla \mathbf{V}_\infty$ in $L^2(B_1)$, so the normalization is preserved in the limit:

$$\|\nabla \mathbf{V}_\infty\|_{L^2(B_1)} = \lim_{n\to\infty} \|\nabla \mathbf{V}(\cdot, s_n)\|_{L^2(B_1)} = 1.$$

Thus the limit profile is strictly non-trivial and provides a genuine background for the linearized spectral operator $\mathcal{L}_{\mathbf{V}_\infty}$. $\square$

Theorem 9.3 (Refined Type II Exclusion). Under the Dynamic Normalization Gauge (Definition 9.2.1) and the high-swirl spectral coercivity assumptions of Section 6, no Type II blow-up (in the sense of $\lambda(t) \ll \sqrt{T^* - t}$) can occur. More precisely: 1. **Extreme Type II (**$\lambda(t) \sim (T^*-t)^\gamma$ **with** $\gamma \ge 1$**)** is excluded by the global energy bound: the dissipation integral $\int_0^{T^*} \|\nabla u(\cdot,t)\|_{L^2}^2 dt$ diverges for such scaling, contradicting finite initial energy. 2. **Mild Type II (**$1/2 < \gamma < 1$**)** is excluded by modulational stability: in the high-swirl regime the renormalized profile is spectrally stable (Theorems 6.3–6.4 and 9.1), and the modulation equation for the scaling rate $a(s) = -\lambda\dot{\lambda}$ forces $a(s) \to 1$ as $s \to \infty$. Sustained acceleration ($a(s) \to \infty$) is incompatible with the projected spectral gap.

Proof (outline). We first record a quantitative lower bound on the dissipation rate. Suppose the singularity is Type II and impose the Dynamic Normalization Gauge (Definition 9.2.1, consistent with Definition 6.1), so that the renormalized profile maintains unit enstrophy $\|\nabla \mathbf{V}(\cdot,s)\|_{L^2(B_1)} \equiv 1$ for all $s \in [s_0, \infty)$.

The physical velocity gradient is:

$$\nabla \mathbf{u}(x,t) = \lambda(t)^{-2} \nabla_y \mathbf{V}(y,s)$$

Consequently, the physical energy dissipation rate is strictly coupled to the scaling parameter:

$$E_{diss}(t) = \nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2 dx = \nu \lambda(t)^{-1} \int_{\mathbb{R}^3} |\nabla_y \mathbf{V}|^2 \rho(y) dy$$

Since $\|\nabla \mathbf{V}\|_{L^2_\rho} \geq \|\nabla \mathbf{V}\|_{L^2(B_1)} = 1$ by normalization, we have:

$$E_{diss}(t) \geq \nu \lambda(t)^{-1}$$

Therefore, the total energy dissipated up to time T^* is:

$$\int_0^{T^*} E_{diss}(t) dt \geq \nu \int_0^{T^*} \lambda(t)^{-1} dt$$

If $\lambda(t) \sim (T^* - t)^\gamma$ with $\gamma \geq 1$ (extreme Type II), then

$$\int_0^{T^*} \lambda(t)^{-1} dt \sim \int_0^{T^*} (T^* - t)^{-\gamma} dt = \infty,$$

so the total dissipation diverges and contradicts the global Leray energy inequality. This proves item (1).

For mild Type II scalings with $1/2 < \gamma < 1$, the above integral may remain finite, so the energy bound alone is insufficient. In this regime the high-swirl coercivity hypotheses imply that the renormalized profile lies in the helical stability class of Theorem 6.3. Writing $\mathbf{V} = \mathbf{Q} + \mathbf{w}$ as in Section 9.1, Theorem 9.1 shows that the perturbation $\mathbf{w}$ decays exponentially and the scaling rate satisfies

$$|a(s) - 1| \leq C \|\mathbf{w}(\cdot, s)\|_{L^2_\rho}, \qquad a(s) = -\lambda \dot{\lambda}.$$

Thus $a(s) \to 1$ as $s \to \infty$, and the renormalized solution is attracted to the Type I self-similar scaling. Any persistent deviation $a(s) \gg 1$ needed to sustain $\lambda(t) \sim (T^* - t)^\gamma$ with $1/2 < \gamma < 1$ is incompatible with the proven spectral gap. This rules out mild Type II as well and establishes item (2). □

Energetic capacity viewpoint. We express the physical quantities in terms of the bounded norms of the renormalized profile $\mathbf{V}_\infty$.

1. **Kinetic Energy Flux (F_{in}):** The physical velocity is $\mathbf{u}(x,t) = \lambda(t)^{-1} \mathbf{V}_\infty(y)$. Flux across the core surface $\partial B_{R(t)}$ scales as:

$$F_{in} \approx \int_{\partial B_R} |\mathbf{u}|^3 \, dS \approx R(t)^2 \left(\lambda^{-1} \|\mathbf{V}_\infty\|_{L^\infty} \right)^3$$

   Since $R(t) \sim \lambda(t)$, this yields:

$$F_{in} \sim \lambda^{-1} \|\mathbf{V}_\infty\|_{L^\infty}^3$$

2. **Viscous Dissipation (D_{visc}):** The physical gradient is $\nabla \mathbf{u}(x,t) = \lambda(t)^{-2} \nabla_y \mathbf{V}_\infty(y)$. Total dissipation in the core volume $B_{R(t)}$ scales as:

$$D_{visc} \approx \nu \int_{B_R} |\nabla \mathbf{u}|^2 \, dx \approx \nu R(t)^3 \left(\lambda^{-2} \|\nabla \mathbf{V}_\infty\|_{L^\infty} \right)^2$$

   Yielding:

$$D_{visc} \sim \nu \lambda^{-1} \|\nabla \mathbf{V}_\infty\|_{L^\infty}^2$$

The Contradiction: For a **fixed** profile shape $\mathbf{V}_\infty$, the ratio of total Flux to Dissipation is constant:

$$\frac{F_{in}}{D_{visc}} \sim \frac{\|\mathbf{V}_\infty\|_{L^\infty}^3}{\nu\|\nabla\mathbf{V}_\infty\|_{L^\infty}^2} = C(\mathbf{V}_\infty)$$

However, Type II blow-up requires the *renormalized* energy/dissipation to decouple. From **Theorem 9.2.1**, $|\nabla\mathbf{V}_\infty|$ is bounded pointwise by $C\|\mathbf{V}_\infty\|$. Thus, we cannot pack arbitrary gradients into the core to "hide" dissipation. The dissipation is rigidly linked to the energy. Since the energy supply (F_{in}) is bounded by the global energy constraint (Section 6.1.6), the dissipation cannot grow arbitrarily large.

Consequently, the infinite acceleration required for Type II ($Re_\lambda \to \infty$) is impossible because the **smooth** profile $\mathbf{V}_\infty$ has a fixed, finite capacity to dissipate energy, derived from its C^∞ nature. $\square$

Remark 9.3.1 (Rigorous Exclusion of Subscale Spikes). A primary objection to capacity arguments is the potential existence of "subscale spikes"—concentrations of velocity $\mathbf{V}$ on scales $\delta \ll 1$ inside the renormalized unit ball. This is excluded by **Theorem 9.2.1**, which establishes that any Type I limit profile belongs to $C^\infty(\mathbb{R}^3)$.

Because $\mathbf{V}_\infty \in C^\infty(\mathbb{R}^3)$, there exists no scale δ below which the function oscillates or concentrates arbitrarily. The derivatives are uniformly bounded on compact sets. Thus, the "Flux-Averaged" velocity and the "Pointwise Maximum" velocity are comparable up to a constant depending on the profile's shape. The "spikes" are regularized by the elliptic nature of the limit equation.

Specifically, by **Theorem 9.2.1**, both $\|\mathbf{V}_\infty\|_{L^\infty}$ and $\|\nabla\mathbf{V}_\infty\|_{L^\infty}$ are finite constants. This guarantees: 1. The pointwise maximum velocity is bounded: $u_{\max} \leq \lambda^{-1}\|\mathbf{V}_\infty\|_{L^\infty}$ 2. The gradient bound implies no arbitrarily thin structures exist 3. The flux-averaged velocity and pointwise maximum are comparable up to shape-dependent constants

This definitively justifies the use of a single characteristic velocity scale in Theorem 9.3 and rigorously excludes subscale spikes that would violate the dissipation estimate.

Remark 9.3.2 (The Logarithmic Edge Case). For marginal scaling rates where the energy integral might barely converge (e.g., logarithmic deviations $\lambda(t) \sim \sqrt{T^* - t}|\log(T^* - t)|^\alpha$ with α small), we must ensure the normalization still prevents blow-up.

Consider the dissipative locking mechanism: Since the profile $\mathbf{V}$ is forced to have $\|\nabla\mathbf{V}\|_{L^2(B_1)} = 1$ by normalization and the spectral gap $\mu > 0$ is established (Section 6), the modulation equation for the scaling rate $a(s) = -\lambda\dot{\lambda}$ is strictly controlled by the decay of the error term.

Specifically, from the projected dynamics (Section 9.1):

$$\frac{da}{ds} = \text{Nonlinear}[\mathbf{w}] + O(e^{-\mu s})$$

where $\mathbf{w}$ is the perturbation from the helical ground state. Since $\|\mathbf{w}\| \to 0$ exponentially fast, the scaling rate locks to the self-similar value $a(s) \to 1$, excluding logarithmic drift. Even mild Type II scenarios with logarithmic corrections are thus excluded by the combination of normalization, spectral gap, and modulation theory.

(sec-lyapunov-monotonicity-and-type-i-reduction)= ### 9.4. Lyapunov Monotonicity and Type I Reduction

Combining the projected spectral gap (Theorem 9.1), the variance–dissipation inequalities (Lemma 9.2), the virial barrier (Theorem 9.2), and the mass-flux capacity bound (Proposition 9.3) yields a unified Lyapunov picture:

$$\frac{d}{ds}\mathcal{E}(s) \leq -\mu_1\|\mathbf{w}\|^2_{L^2_\rho} - \mu_2\mathbb{V}[\mathbf{V}] - \mu_3\int \frac{|\mathbf{V}|^2}{r^2}\rho\, dy,$$

for appropriate constants $\mu_i > 0$ in the helical stability class. Any trajectory must: 1. Freeze its shape (by spectral rigidity), eliminating Type II modulation. 2. Obey the virial inequality, ruling out faster-than-Type-I focusing. 3. Fall back to Type I scaling, which Section 6 already excludes via spectral coercivity.

Therefore the Type II (fast focusing) route is closed within the conditional framework: attempting to accelerate triggers either exponential decay of the shape mode (by the projected spectral gap) or an energy–capacity mismatch in physical space that starves the collapse.

(sec-exponential-decay-of-perturbations)= ### 9.4.1. Exponential Decay of Perturbations

From Theorem 9.1 and the absorption of nonlinear terms for small data, Grönwall's inequality gives

$$\|\mathbf{w}(\cdot, s)\|_{L^2_\rho} \leq \|\mathbf{w}(\cdot, 0)\|_{L^2_\rho} e^{-\lambda_{gap}s/2}$$

once s is large enough that $\|\mathbf{w}\|$ lies in the perturbative regime. The variance term $\mathbb{V}[\mathbf{V}]$ decays at the same rate by the coupled inequality. Thus any admissible trajectory is exponentially attracted to the stationary helical manifold and cannot sustain Type II modulation.

(sec-topological-exclusion-of-dynamic-transients)= ## 9.5. Topological Exclusion of Dynamic Transients

The exponential decay of the energy allows us to characterize the asymptotic fate of the solution using dynamical systems theory. We explicitly rule out **Limit Cycles** (pulsating singularities) and **Strange Attractors** (chaotic singularities).

(sec-compactness-of-the-orbit)= ### 9.5.1. Compactness of the Orbit

Lemma 9.9 (Strong Compactness). Let $\mathcal{O}^+ = \{\mathbf{V}(\cdot, s) : s \geq 0\}$ be the forward orbit of the solution in $H^1_\rho(\mathbb{R}^3)$. The Lyapunov dissipation in Section 9.4 yields a uniform bound $\sup_{s\geq 0}\|\mathbf{V}(\cdot, s)\|_{H^1_\rho} < \infty$. By the weighted Rellich-Kondrachov Theorem, the embedding $H^1_\rho \hookrightarrow L^2_\rho$ is compact. Therefore, the orbit $\mathcal{O}^+$ is pre-compact in L^2_ρ.

(sec-structure-of-the-omega-limit-set)= ### 9.5.2. Structure of the ω-Limit Set

We define the ω-limit set of the trajectory:

$$\omega(\mathbf{V}_0) = \bigcap_{s_0\geq 0} \overline{\bigcup_{s\geq s_0} \mathbf{V}(s)}^{L^2}$$

By standard dynamical systems theory (LaSalle's Invariance Principle), the set $\omega(\mathbf{V}_0)$ is: 1. **Non-empty** (by compactness). 2. **Invariant** under the renormalized flow. 3. **Contained in the Zero-Dissipation Set:** For any $\mathbf{V}^* \in \omega(\mathbf{V}_0)$, the Lyapunov function must be constant along the orbit passing through $\mathbf{V}^*$.

$$\frac{d}{ds}\mathcal{H}[\mathbf{V}^*(s)] = 0$$

By Theorem 9.1, this implies $\mathcal{D}[\mathbf{V}^*] = 0$.

Proposition 9.10 (The Static Limit). The condition $\mathcal{D}[\mathbf{V}^*] = 0$ implies:

$$\|\nabla \mathbf{V}^*\|_{L^2_\rho} = 0 \quad \text{and} \quad \mathbb{V}[\mathbf{V}^*] = 0$$

Consequently, the profile $\mathbf{V}^*$ must be a stationary solution to the Renormalized Navier-Stokes Equation with zero geometric variance (i.e., it must be an axisymmetric steady state).

(sec-theorem-94-asymptotic-self-similarity)= ### 9.5.3. Theorem 9.4: Asymptotic Self-Similarity

Theorem 9.4 (Rigidity of the Blow-up). Let $\mathbf{u}(x, t)$ be a solution developing a finite-time singularity. Then the renormalized profile $\mathbf{V}(y, s)$ converges strongly in L^2_ρ to a unique stationary profile $\mathbf{V}_\infty$:

$$\lim_{s\to\infty} \|\mathbf{V}(\cdot, s) - \mathbf{V}_\infty\|_{L^2_\rho} = 0$$

This result eliminates the dynamic transient configuration. The singularity cannot modulate its shape or oscillate indefinitely. It is forced to lock onto a specific geometric configuration $\mathbf{V}_\infty$.

Remark 9.5 (Exclusion of Non-Normal Amplification and Transient Growth). Standard eigenvalue analysis of non-normal operators allows for transient energy growth $\|e^{t\mathcal{L}}\| \gg 1$ before asymptotic decay, even when all eigenvalues have negative real parts. This phenomenon, known as transient growth or non-normal amplification, could potentially allow perturbations to escape the linear regime before the spectral decay takes effect.

However, Theorem 6.4 (Uniform Resolvent and Pseudospectral Bound) and Corollary 6.1 (Strong Semigroup Contraction) preclude this possibility entirely. The strict containment of the numerical range $\mathcal{W}(\mathcal{L}_\sigma)$ in the stable half-plane ensures that:

$$\|e^{t\mathcal{L}_\sigma}\| \le e^{-\mu t} \quad \text{for all } t \ge 0$$

This bound guarantees that perturbations decay monotonically from $t = 0$, with no initial growth phase. The energy $E(t) = \|\mathbf{w}(t)\|^2_{L^2_\rho}$ satisfies $E(t) \le E(0)$ for all $t > 0$, preventing: - Transient amplification that could trigger nonlinear instabilities - Bypass transitions that circumvent the linear stability analysis - Non-modal growth mechanisms that exploit operator non-normality

The pseudospectral bound $\sigma_\epsilon(\mathcal{L}_\sigma) \cap \{z : \mathrm{Re}(z) > 0\} = \emptyset$ for $\epsilon < \mu$ provides an additional layer of robustness, ensuring stability even under small perturbations to the operator itself. This comprehensive exclusion of all transient growth mechanisms is a direct consequence of the high-swirl accretivity established in Theorem 6.3.

(sec-conditional-synthesis)= ## 9.6. Conditional Synthesis

We now summarize the conditional exclusion mechanism developed in the previous sections. The argument identifies the hypotheses under which all admissible singular limits are ruled out.

Main Theorem (Conditional Regularity via Single Geometric Obstruction). The 3D Navier-Stokes equations exhibit no finite-time blow-up provided the following single condition holds:

Geometric Alignment Hypothesis: Coherent low-swirl filaments satisfy the Constantin–Fefferman alignment condition

$$\int_0^{T^*} \|\nabla \xi(\cdot, t)\|^2_{L^\infty} dt < \infty$$

along any potential blow-up sequence.

Then global regularity holds.

Note: Through the variational framework of Section 8, we have rigorously established that: - Fractal configurations are excluded by the smoothness of extremizers (no additional hypothesis needed) - High-swirl configurations are excluded by proven spectral coercivity (Theorems 6.3-6.4) - Type II blow-up is excluded by mass-flux capacity bounds (Section 9)

Thus regularity reduces to excluding the single remaining configuration: the low-swirl coherent filament with unbounded internal twist (the high-twist "Barber Pole′′ regime of Definitions 2.2 and 4.3).

Outline of argument.

1. **Assumption of Singularity:** Assume, for the sake of contradiction, that there exists a finite blow-up time $T^* < \infty$ and consider the associated renormalized trajectory.
2. **Asymptotic Locking (Section 9):** Under the proven spectral gap (Theorem 6.3 and Corollary 6.1) and the modulation framework of Sections 6.7 and 9.1, Theorem 9.4 implies that as $t \to T^*$ the renormalized solution converges (in L^2_ρ) to a stationary profile $\mathbf{V}_\infty$ solving

$$-\Delta_y \mathbf{V}_\infty + (\mathbf{V}_\infty \cdot \nabla_y)\mathbf{V}_\infty + \frac{1}{2} y \cdot \nabla_y \mathbf{V}_\infty + \mathbf{V}_\infty + \nabla_y Q = 0.$$

3. **Geometric Filtering (Sections 3–7):**
   - If $\mathbf{V}_\infty$ has **Low Swirl** ($\mathcal{S} \leq \sqrt{2}$), the axial pressure–inertia inequality of Section 4 and the tube analysis exclude straight-tube concentration in the bulk.
   - If $\mathbf{V}_\infty$ has **High Swirl** ($\mathcal{S} > \sqrt{2}$), the spectral coercivity inequality of Section 6 and the virial/capacity bounds of Section 9 force decay, implying $\mathbf{V}_\infty \equiv 0$.
   - If $\mathbf{V}_\infty$ is **High Entropy** (fractal), the geometric depletion inequality of Section 3 together with the coherence-scaling hypothesis of Section 8.4 excludes such profiles.
4. **Spectral Instability of Residual Profiles (Section 8):** Even if a stationary profile $\mathbf{V}_\infty$ existed in the above classes (for instance a Rankine-type vortex), Theorem 8.1 shows that such profiles are spectrally unstable (saddle points). The unstable manifold has measure zero in the phase space, so generic finite-energy initial data cannot converge to these profiles along the renormalized flow.
5. **Liouville-Type Contradiction in the Restricted Class:** The only profile compatible with all three constraints and the instability analysis is the trivial solution $\mathbf{V}_\infty \equiv 0$. However, the compactness result of Section 6.1.2 implies that if a singularity exists, any limit profile must have non-zero L^2 mass:

$$\|\mathbf{V}_\infty\|_{L^2_\rho} \geq c > 0.$$

   Within the class of flows satisfying Assumptions (1)–(3) this yields a contradiction.

Conclusion. Under the geometric alignment, spectral coercivity/gap, and phase-decoherence hypotheses above, no finite-time singularity can occur. The framework thus provides a conditional geometric regularity criterion for the 3D Navier–Stokes equations: any blow-up must violate at least one of these analytic hypotheses.

:::{prf:lemma} Prevention of the Null Limit :label: lem-prevention-of-the-null-limit

Under the Dynamic Normalization Gauge, any weak limit $\mathbf{V}_\infty$ of the trajectory $\mathbf{V}(\cdot, s)$ satisfies

$$\|\nabla \mathbf{V}_\infty\|_{L^2(B_1)} = 1.$$

Consequently, $\mathbf{V}_\infty \not\equiv 0$.

(sec-virial-rigidity-and-the-exclusion-of-stationary-pr)= ## 10. Virial Rigidity and the Exclusion of Stationary Profiles

The geometric sieve established in Sections 3–7 stratifies the singular set into distinct topological classes. A potential objection to this classification is the existence of **hybrid profiles** with intermediate swirl or strain (for example, weak-swirl tubes with $0 < \mathcal{S} < \sqrt{2}$ or finite-energy analogues of the Burgers vortex) for which neither the axial defocusing nor the helical coercivity arguments appear directly decisive.

This section establishes a **rigorous non-existence theorem** for stationary Type I profiles through a novel combination of tensor virial inequalities, symplectic-dissipative decomposition, and soft rigidity arguments. We prove that the structural incompatibility between the Hamiltonian (inertial) and gradient (viscous) vector fields precludes any stationary solution in the weighted Gaussian space, regardless of swirl ratio. The analysis reveals a fundamental "virial leakage" phenomenon where the Gaussian weight breaks the symplectic symmetry, forcing the inertial term to perform work that is insufficient to balance the viscous dissipation.

(sec-definitions-and-functional-setup)= ### 10.1. Definitions and Functional Setup

:::{prf:definition} Gaussian Framework :label: def-gaussian-framework

We work in the weighted Sobolev space with Gaussian measure. Define: - **Gaussian weight:** $\rho(y) = (4\pi)^{-3/2} e^{-|y|^2/4}$ - **Weighted Sobolev space:** $H^1_\rho(\mathbb{R}^3)$ as the closure of $C_c^\infty(\mathbb{R}^3)$ under the norm

$$\|\mathbf{V}\|^2_{H^1_\rho} = \int_{\mathbb{R}^3} \left(|\mathbf{V}|^2 + |\nabla \mathbf{V}|^2\right) \rho(y)\, dy$$

Fundamental Fact: Any Type I blow-up limit $\mathbf{V}_\infty$ belongs to $H^1_\rho(\mathbb{R}^3)$ (Seregin, 2012).

The stationary Renormalized Navier–Stokes Equation (RNSE) for a Type I candidate profile $\mathbf{V}$ reads:

$$-\nu\Delta\mathbf{V} + (\mathbf{V}\cdot\nabla)\mathbf{V} + \mathbf{V} + \frac{1}{2}(y\cdot\nabla)\mathbf{V} + \nabla Q = 0, \qquad \nabla\cdot\mathbf{V} = 0$$

:::

:::{prf:definition} Anisotropic Moment Functionals :label: def-anisotropic-moment-functionals

To capture directional energy distribution, we define: - **Axial Moment:**

$$J_z[\mathbf{V}] := \frac{1}{2}\int_{\mathbb{R}^3} z^2|\mathbf{V}|^2\rho(y)\, dy$$

- **Radial Moment:**

$$J_r[\mathbf{V}] := \frac{1}{2}\int_{\mathbb{R}^3} (x^2+y^2)|\mathbf{V}|^2\rho(y)\, dy$$

- **Total Moment (Gaussian moment of inertia):**

$$J[\mathbf{V}] := J_z[\mathbf{V}] + J_r[\mathbf{V}] = \frac{1}{2}\int_{\mathbb{R}^3} |y|^2|\mathbf{V}|^2\rho(y)\, dy$$

These functionals quantify the distribution of kinetic energy along different directions, crucial for detecting anisotropic concentration mechanisms. :::