计算机视觉
§0 预备知识

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几何变换

平面点

- $\boldsymbol{x} = (x \quad y)^T \in \mathcal{R}^2$， $\mathcal{R}^2$：2D欧氏平面
- 用齐次坐标(homogeneous coordinates)表示$\boldsymbol{x} = (x \quad y)^T$：
  - $\widetilde{\boldsymbol{x}} = (\tilde{x} \quad \tilde{y} \quad \widetilde{w})^T \in \mathcal{P}^2$
  - $x = \frac{\tilde{x}}{\widetilde{w}}, y = \frac{\tilde{y}}{\widetilde{w}}, (\widetilde{w} \neq 0)$
  - $\mathcal{P}^2 = \mathcal{R}^3 - (0 \quad 0 \quad 0)^T$:2D投影平面(2D Projective Space)
  - $\bar{\boldsymbol{x}} = (x \quad y \quad 1)^T$: $\boldsymbol{x}$的增广向量
- 无穷远点（理想点）：
  - $\widetilde{\boldsymbol{x}} = (\tilde{x} \quad \tilde{y} \quad 0)^T$
- 例：
  - $\boldsymbol{x} = (2 \quad 3)^T \rightarrow \bar{\boldsymbol{x}} = (2 \quad 3 \quad 1)^T, \widetilde{\boldsymbol{x}} = (4 \quad 6 \quad 2)^T, \ldots \widetilde{\boldsymbol{x}} = \alpha(2 \quad 3 \quad 1)^T, \alpha \neq 0$
  - $\widetilde{\boldsymbol{x}} = (3 \quad 4.5 \quad 1.5)^T \rightarrow \boldsymbol{x} = (2 \quad 3)^T$

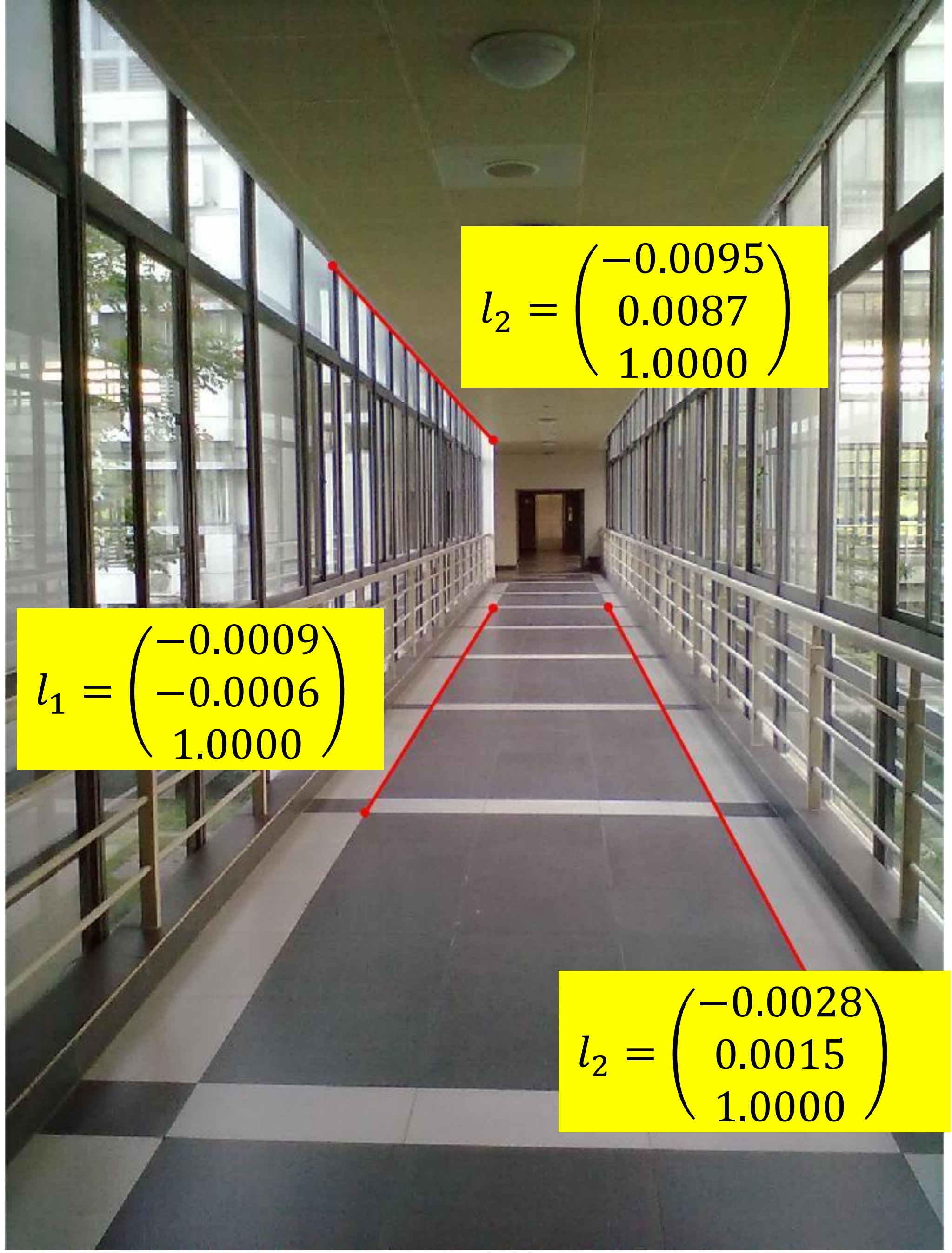
平面直线

- 直线方程：$l: ax + by + c = 0$
  - 齐次坐标表示：$ax_1 + bx_2 + cx_3 = 0 \Leftrightarrow (a\ b\ c)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$
  - 直线的齐次坐标表示：$l = (a, b, c)^T$
  - $\forall \boldsymbol{x} = (x, y)^T \in l \Leftrightarrow l^T \tilde{\boldsymbol{x}} = \tilde{\boldsymbol{x}}^T l = 0$
- 两点$p_1 = (x_1, y_1)^T, p_2 = (x_2, y_2)^T$确定一条直线$l$:

$$\begin{vmatrix} i & j & k \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} i - \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k$$

  - $l = \tilde{p}_1 \times \tilde{p}_2 = \left(\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}, -\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}, \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \right) = (y_1 - y_2, x_2 - x_1, x_1 y_2 - x_2 y_1)$
- 两条直线$l_1 = (a_1, b_1, c_1)^T, l_2 = (a_2, b_2, c_2)^T$的交点$X$:
  - $X = l_1 \times l_2 = \left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right) = (b_1 c_2 - b_2 c_1, a_2 c_1 - a_1 c_2, a_1 b_2 - a_2 b_1)$
  - $l_1 \parallel l_2 \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} \Rightarrow X = l_1 \times l_2 = (b_1 c_2 - b_2 c_1, a_2 c_1 - a_1 c_2, 0) \leftarrow$ 无穷远点


(419,341)
(624,564)
(625,779)
(772,778)
(461,1041)
(1108,1402)


$l_2 = \begin{pmatrix} -0.0095 \\ 0.0087 \\ 1.0000 \end{pmatrix}$
$l_1 = \begin{pmatrix} -0.0009 \\ -0.0006 \\ 1.0000 \end{pmatrix}$
$l_2 = \begin{pmatrix} -0.0028 \\ 0.0015 \\ 1.0000 \end{pmatrix}$

$$P_1 = l_1 \times l_2 := \begin{pmatrix} 704.3122 \\ 652.2940 \\ 1.0000 \end{pmatrix}$$

$$P_2 = l_1 \times l_3 := \begin{pmatrix} 704.6585 \\ 651.7407 \\ 1.0000 \end{pmatrix}$$

$$P_3 = l_2 \times l_3 := \begin{pmatrix} 703.1033 \\ 650.0489 \\ 1.0000 \end{pmatrix}$$

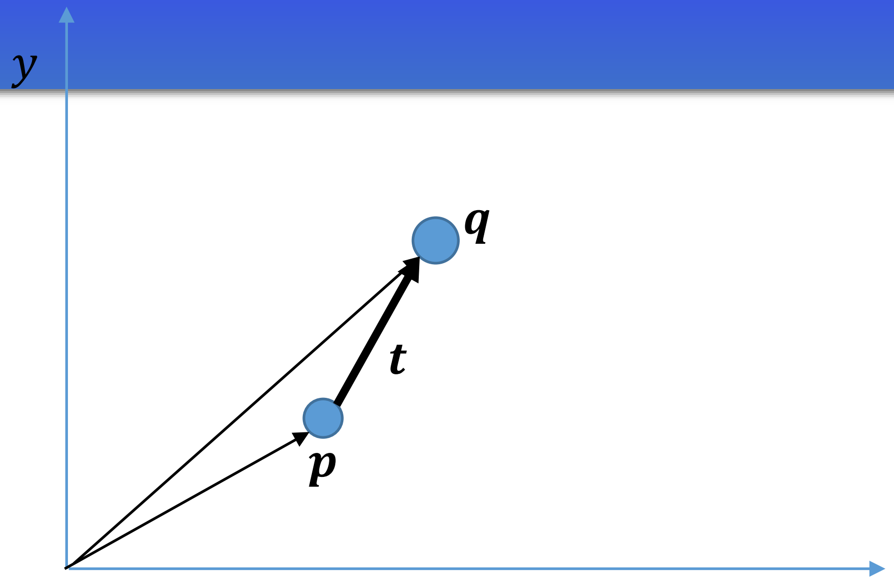
$$P_1 = P_2 = P_3?$$

空间点

- $\boldsymbol{x}=(x \quad y \quad z)^T \in \mathcal{R}^3$， $\mathcal{R}^3$：3D欧氏空间
- 用齐次坐标(homogeneous coordinates)表示$\boldsymbol{x}=(x \quad y \quad z)^T$：
  - $\tilde{\boldsymbol{x}}=(\tilde{x} \quad \tilde{y} \quad \tilde{z} \quad \widetilde{w})^T \in \mathcal{P}^3$
  - $x=\frac{\tilde{x}}{\widetilde{w}}, y=\frac{\tilde{y}}{\widetilde{w}}, z=\frac{\tilde{z}}{\widetilde{w}},(\widetilde{w} \neq 0)$
  - $\mathcal{P}^3=\mathcal{R}^4-(0 \quad 0 \quad 0 \quad 0)^T$:3D投影空间(3D Projective Space)
  - $\overline{\boldsymbol{x}}=(x \quad y \quad z \quad 1)^T$: $\boldsymbol{x}$的增广向量
- 无穷远点（理想点）：
  - $\tilde{\boldsymbol{x}}=(\tilde{x} \quad \tilde{y} \quad \tilde{z} \quad 0)^T$
- 例：
  - $\boldsymbol{x}=(1 \quad 2 \quad 3)^T \rightarrow \overline{\boldsymbol{x}}=(1,2,3,1)^T, \tilde{\boldsymbol{x}}=(2,4,6,2)^T, \ldots \tilde{\boldsymbol{x}}=\alpha(2,4,6,2)^T, \alpha \neq 0$
  - $\tilde{\boldsymbol{x}}=(3 \quad 4.5 \quad 1.5 \quad 3)^T \rightarrow \boldsymbol{x}=(1,1.5,0.5)^T$

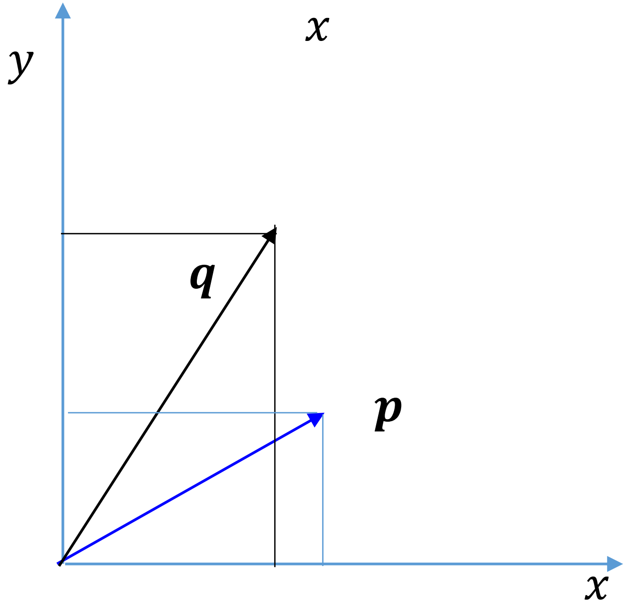
平面几何变换

- 平移：
  - 平移向量$\boldsymbol{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$
  - $\boldsymbol{q} = \boldsymbol{p} + \boldsymbol{t}$



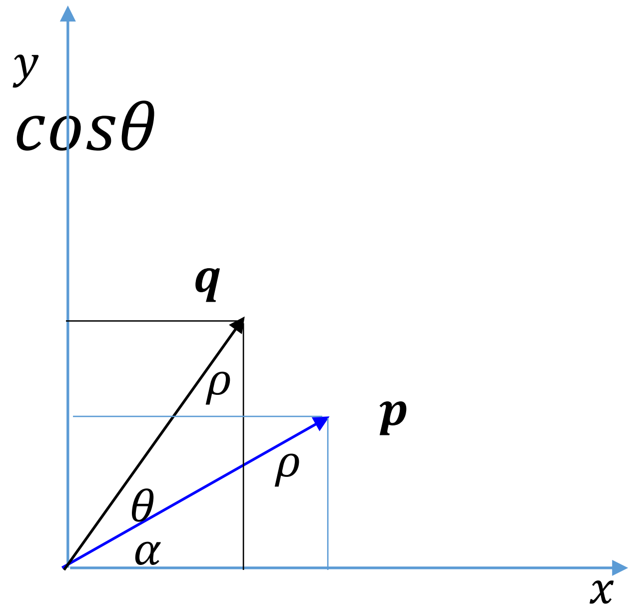
- 缩放：
  - $\begin{cases} q_x = s_x p_x \\ q_y = s_y p_y \end{cases}$
  - $\boldsymbol{q} = S\boldsymbol{p}, S = \boxed{\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}}$

2D缩放矩阵



平面几何变换

- 旋转：
  - $\begin{cases} p_x = \rho \cdot cos\alpha \\ p_y = \rho \cdot sin\alpha \end{cases}$
  - $\begin{cases} q_x = \rho \cdot cos(\alpha+\theta) = \rho \cdot cos\alpha \cdot cos\theta - \rho \cdot sin\alpha \cdot sin\theta \\ \qquad = p_x \cdot cos\theta - p_y \cdot sin\theta = (cos\theta \quad -sin\theta)\boldsymbol{p} \\ q_y = \rho \cdot sin(\alpha+\theta) = \rho \cdot cos\alpha \cdot sin\theta + \rho \cdot sin\alpha \cdot cos\theta \\ \qquad = p_x \cdot sin\theta + p_y \cdot cos\theta = (sin\theta \quad cos\theta)\boldsymbol{p} \end{cases}$
  - $\boldsymbol{q} = R\boldsymbol{p}, R = \boxed{\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}}$ 2D旋转矩阵
  - $R^T R = I_{2\times2}$ ➔旋转阵R是正交规范阵

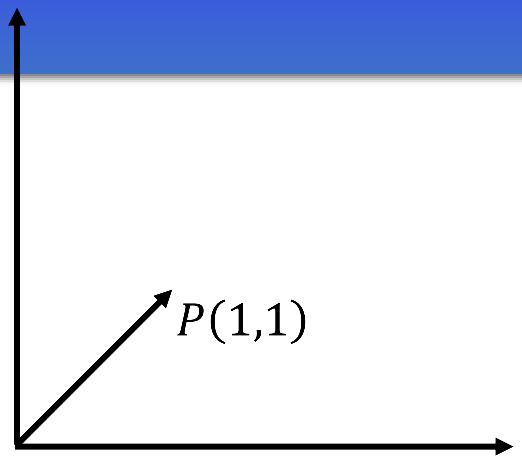


用矩阵表示平面几何变换

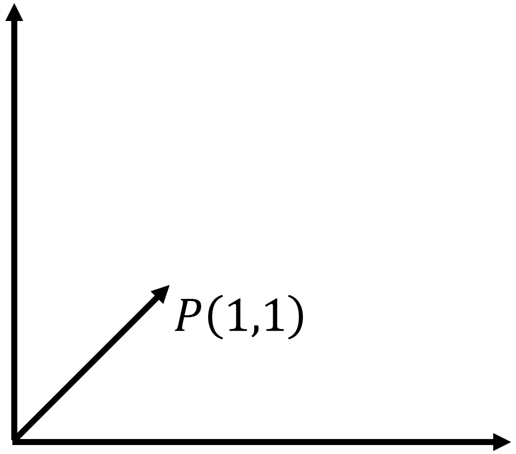
- 增广向量： $\boldsymbol{p} = (p_x \quad p_y)^{\boldsymbol{T}} \rightarrow \overline{\boldsymbol{p}} = (p_x \quad p_y \quad \boldsymbol{1})^{\boldsymbol{T}}$
- $\overline{\boldsymbol{q}} = T\overline{\boldsymbol{p}}$
- 平移变换: $T(t_x, t_y) = \begin{pmatrix} I_{2\times 2} & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$
- 缩放变换: $T(s_x, s_y) = \begin{pmatrix} S & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$
  - $s_x = s_y = s$: $T(s)$:等比例缩放
- 旋转变换: $T(\theta) = \begin{pmatrix} R & \boldsymbol{0} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$

用矩阵表示平面几何变换

- 欧氏变换(刚体变换,Rigid):
  - $T(\theta,t_x,t_y)=\begin{pmatrix} R & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix}=\begin{pmatrix} cos\theta & -sin\theta & t_x \\ sin\theta & cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}$
  - 仅有旋转和平移变换：$T(\theta,t_x,t_y)=T(t_x,t_y)T(\theta)$
  - 注意： $T(t_x,t_y)T(\theta)\neq T(\theta)T(t_x,t_y)$
  - 不改变长度、面积、垂直与平行关系

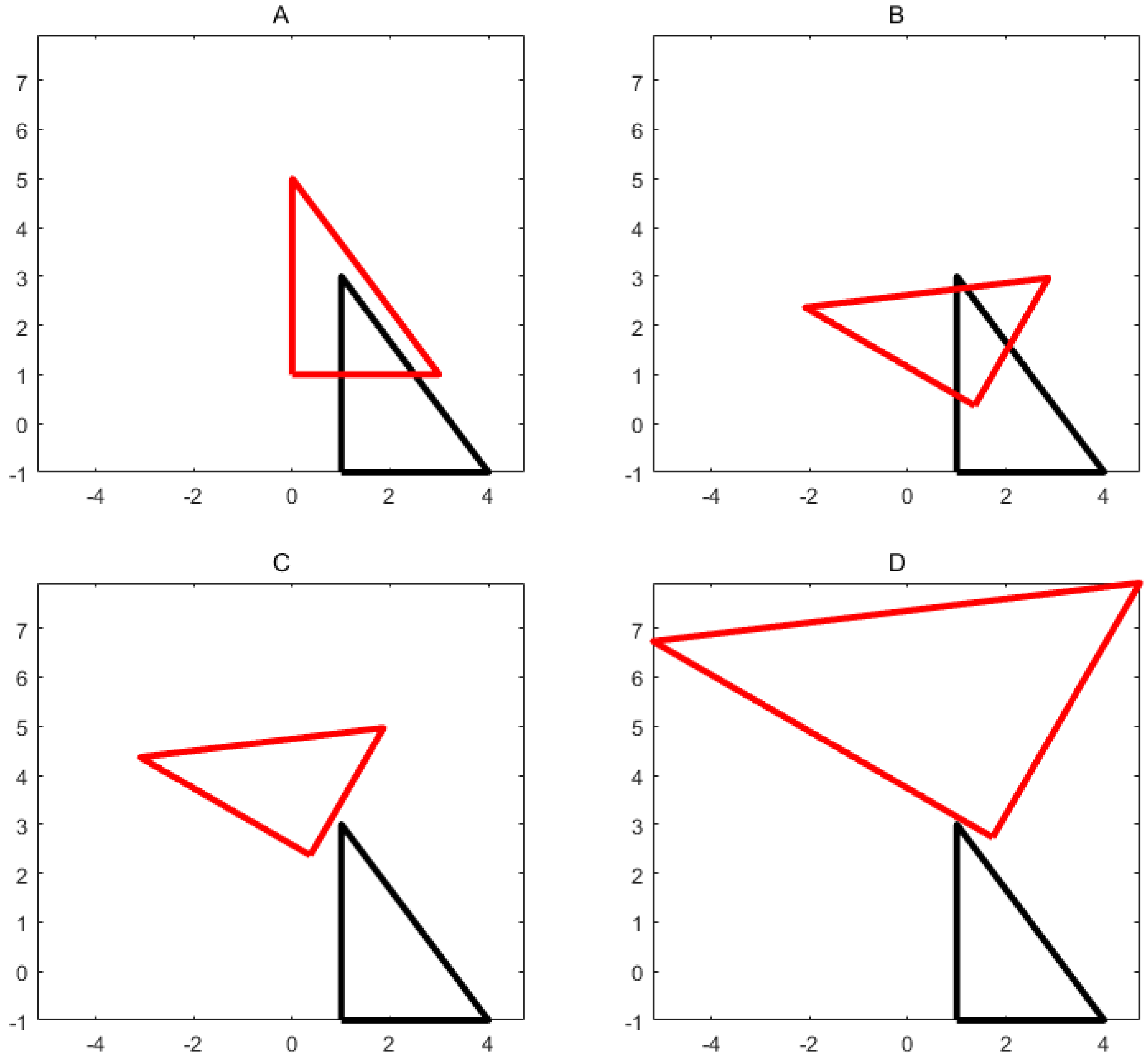


$T(1,1)T\left(\frac{\pi}{4}\right)$



$T\left(\frac{\pi}{4}\right)T(1,1)$

A,B,C,D所示的由黑色图形到红色图形的变换分别对应右边哪一个矩阵？哪些是欧氏变换？



$$T_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 2 \end{pmatrix}$$

$$T_4 = \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) & 0 \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) & 0 \end{pmatrix}$$

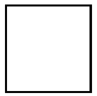
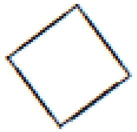
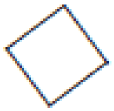
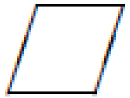
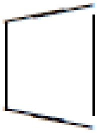
$$T_5 = \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) & -1 \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) & 2 \end{pmatrix}$$

$$T_6 = \begin{pmatrix} 2 \cdot \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) & -1 \\ \sin\left(\frac{\pi}{3}\right) & 2 \cdot \cos\left(\frac{\pi}{3}\right) & 2 \end{pmatrix}$$

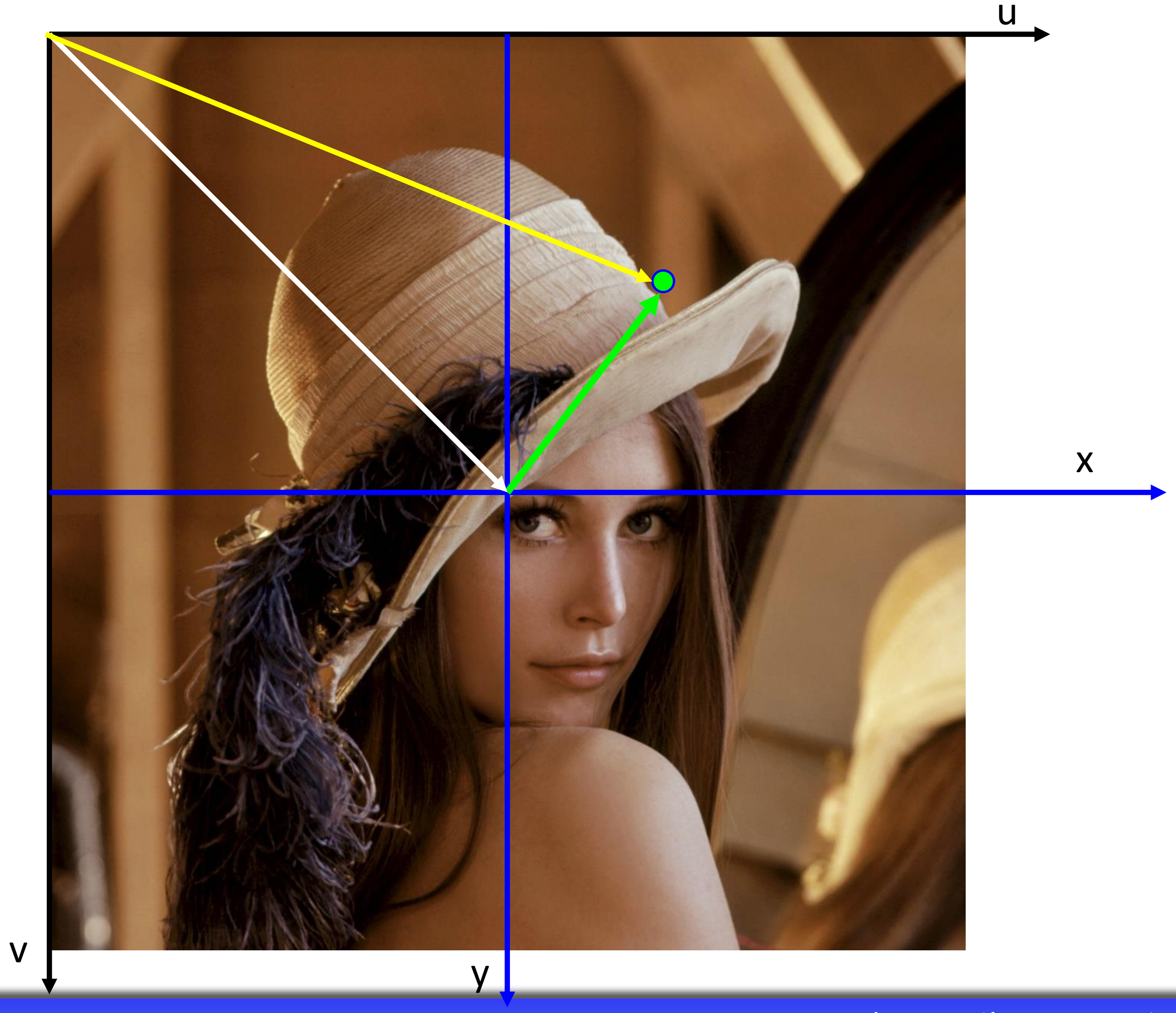
用矩阵表示平面几何变换

- 相似变换(Similarity): $T(s,\theta,t_x,t_y)=\begin{pmatrix} sR & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{pmatrix}=\begin{pmatrix} s\cdot cos\theta & -s\cdot sin\theta & t_x \\ s\cdot sin\theta & s\cdot cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}$
  - 旋转、平移、等比例缩放、反射
  - 变换前后的图形是相似形
  - 保持角度不变
- 仿射(Affine)变换： $q=Ap, A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$任意2*3矩阵
- 投影(Projective)变换： $\widetilde{\boldsymbol{q}}=H\widetilde{\boldsymbol{p}}$
  - $H=\begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix},\begin{cases} q_x=\dfrac{h_{11}p_x+h_{12}p_y+h_{13}}{h_{31}p_x+h_{32}p_y+h_{33}} \\ q_y=\dfrac{h_{21}p_x+h_{22}p_y+h_{23}}{h_{31}p_x+h_{32}p_y+h_{33}} \end{cases}$

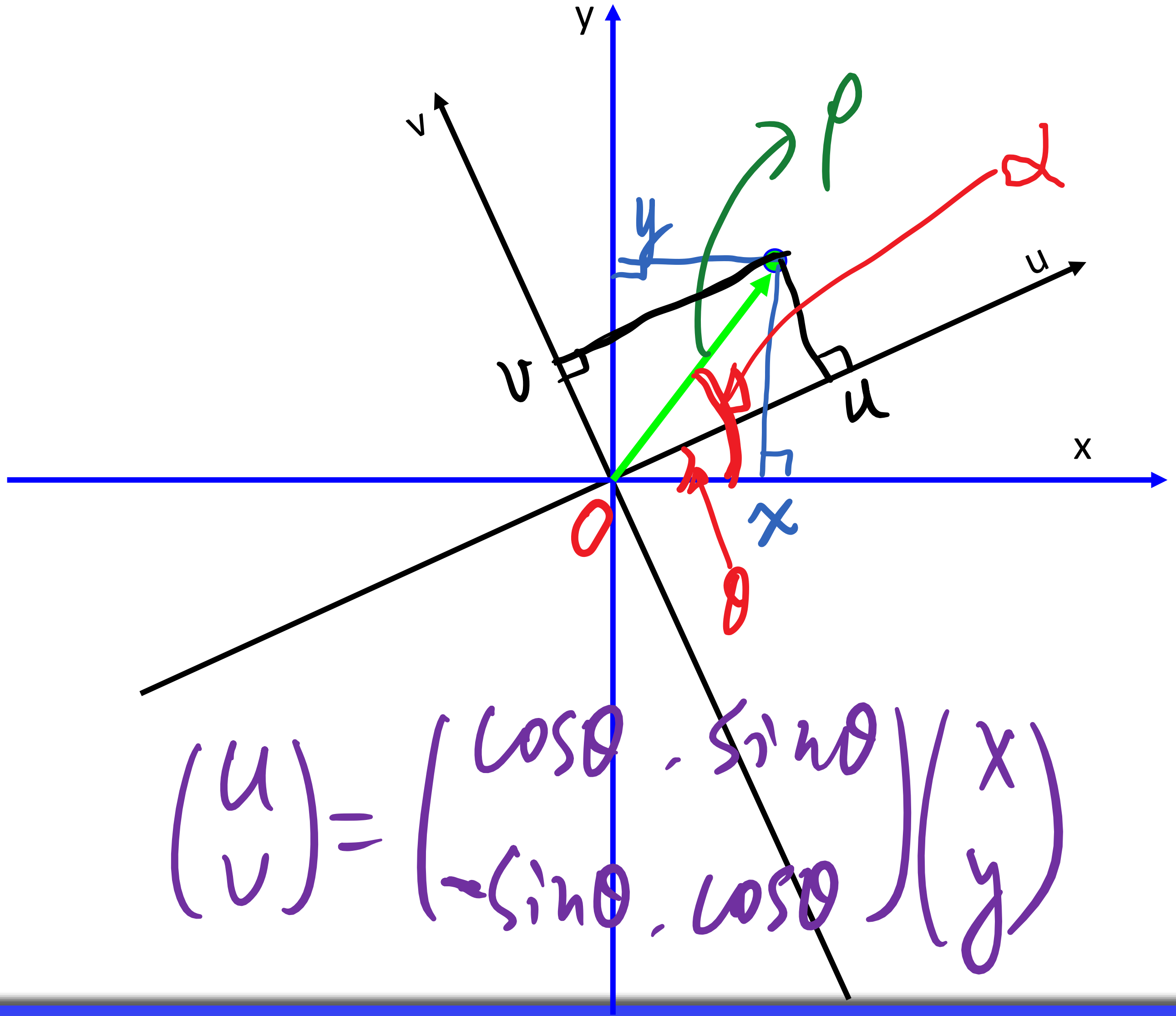
平面几何变换

Transformation	**Matrix**	**# DoF**	**Preserves**	**Icon**
translation	$\left[\begin{array}{c|c} \boldsymbol{I} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\left[\begin{array}{c|c} \boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	3	lengths	
similarity	$\left[\begin{array}{c|c} s\boldsymbol{R} & \boldsymbol{t} \end{array}\right]_{2\times 3}$	4	angles	
affine	$\left[\begin{array}{c} \boldsymbol{A} \end{array}\right]_{2\times 3}$	6	parallelism	
projective	$\left[\begin{array}{c} \tilde{\boldsymbol{H}} \end{array}\right]_{3\times 3}$	8	straight lines	

同一个目标在不同坐标系中的坐标之间的变换



同一个目标在不同坐标系中的坐标之间的变换



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

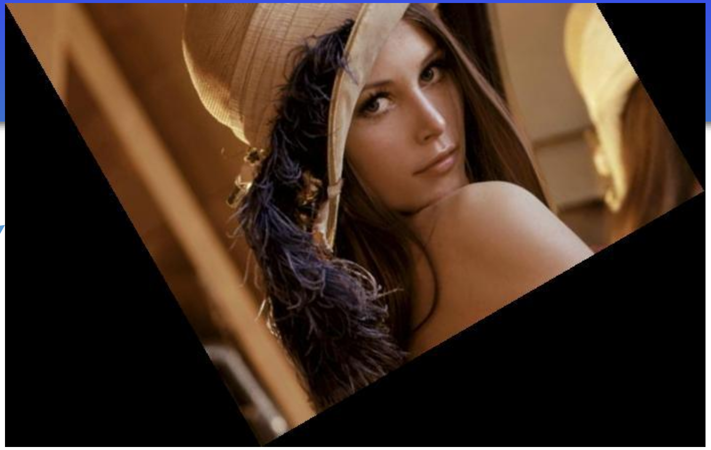
$x = \rho\cos\alpha,\ y = \rho\sin\alpha$

$$\begin{aligned} u &= \rho\cos(\alpha-\theta) \\ &= \rho\cos\alpha\cos\theta + \rho\sin\alpha\sin\theta \\ &= \underline{x\cos\theta + y\sin\theta} \end{aligned}$$

$$\begin{aligned} v &= \rho\sin(\alpha-\theta) \\ &= \rho\sin\alpha\cos\theta - \rho\cos\alpha\sin\theta \\ &= \underline{y\cos\alpha - x\sin\theta} \end{aligned}$$

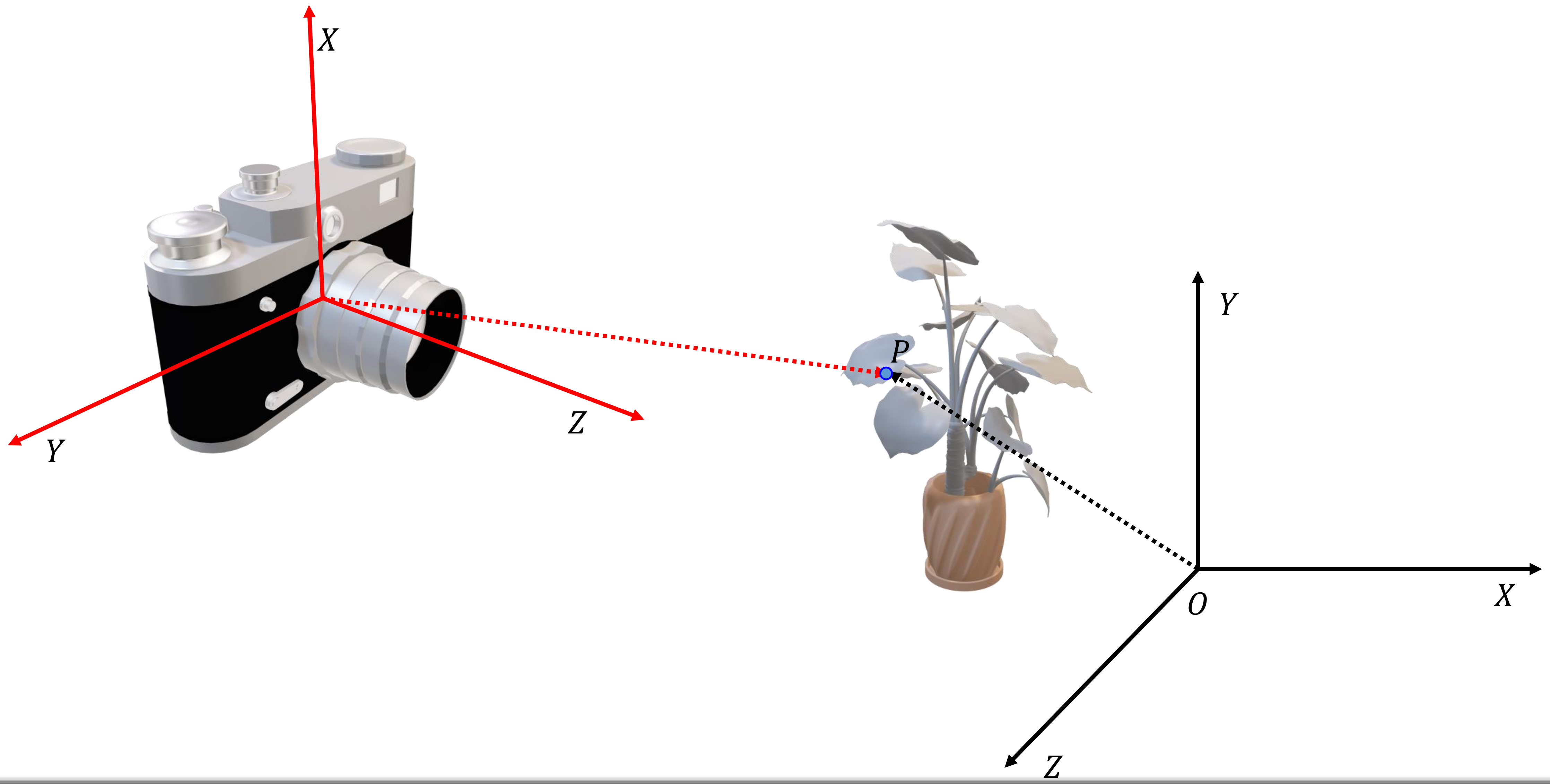
图像旋转

旋转中心：左上角

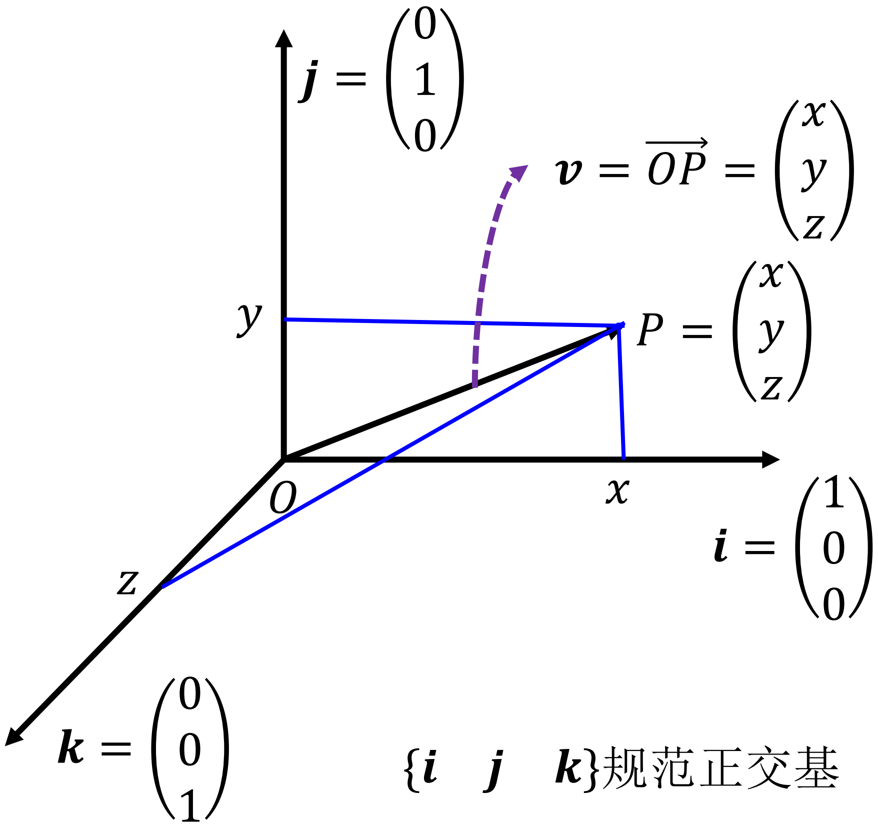


旋转中心：图像中心

空间坐标变换



空间坐标变换



$\{\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k}\}$规范正交基

$$\begin{cases} x = \boldsymbol{v}^T\boldsymbol{i} \\ y = \boldsymbol{v}^T\boldsymbol{j} \\ z = \boldsymbol{v}^T\boldsymbol{k} \end{cases}$$

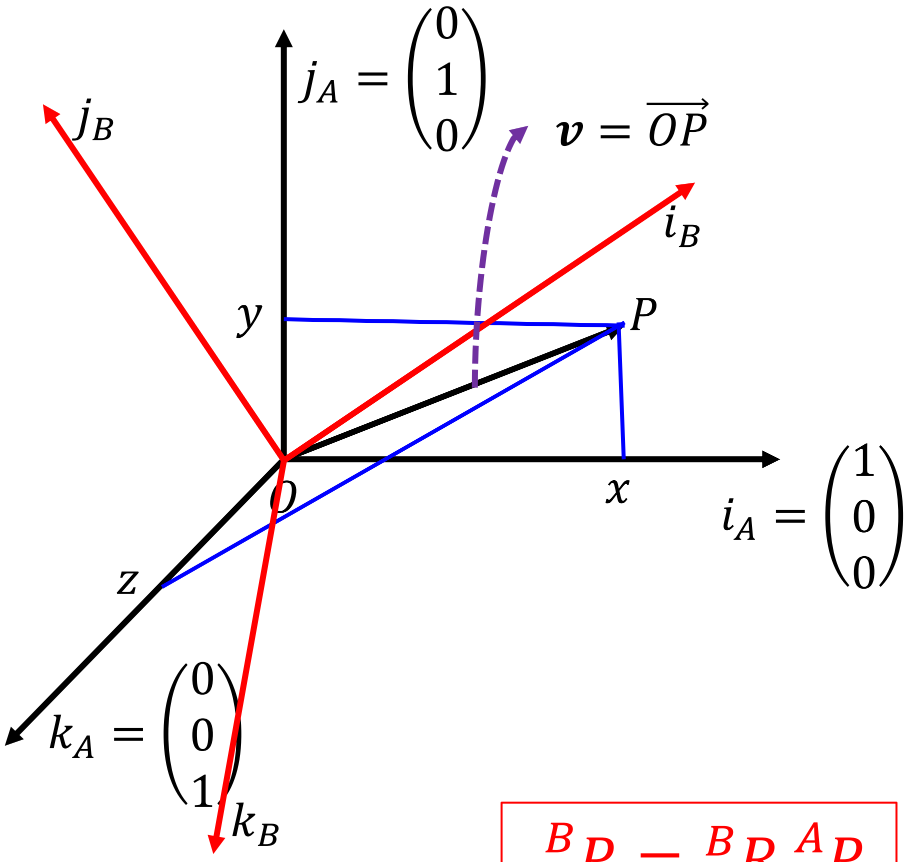
$$P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \boldsymbol{v}^T\boldsymbol{i} \\ \boldsymbol{v}^T\boldsymbol{j} \\ \boldsymbol{v}^T\boldsymbol{k} \end{pmatrix} = (\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k})^T\boldsymbol{v} = \boldsymbol{R}^T\boldsymbol{v}$$

$\boldsymbol{R} = (\boldsymbol{i} \quad \boldsymbol{j} \quad \boldsymbol{k})$：旋转矩阵

$$\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}_{3\times3}$$

$$\boldsymbol{det}(\boldsymbol{R}) = \boldsymbol{1}$$

空间坐标变换



$$\boxed{{}^{B}P = {}^{B}_{A}R\,{}^{A}P}$$

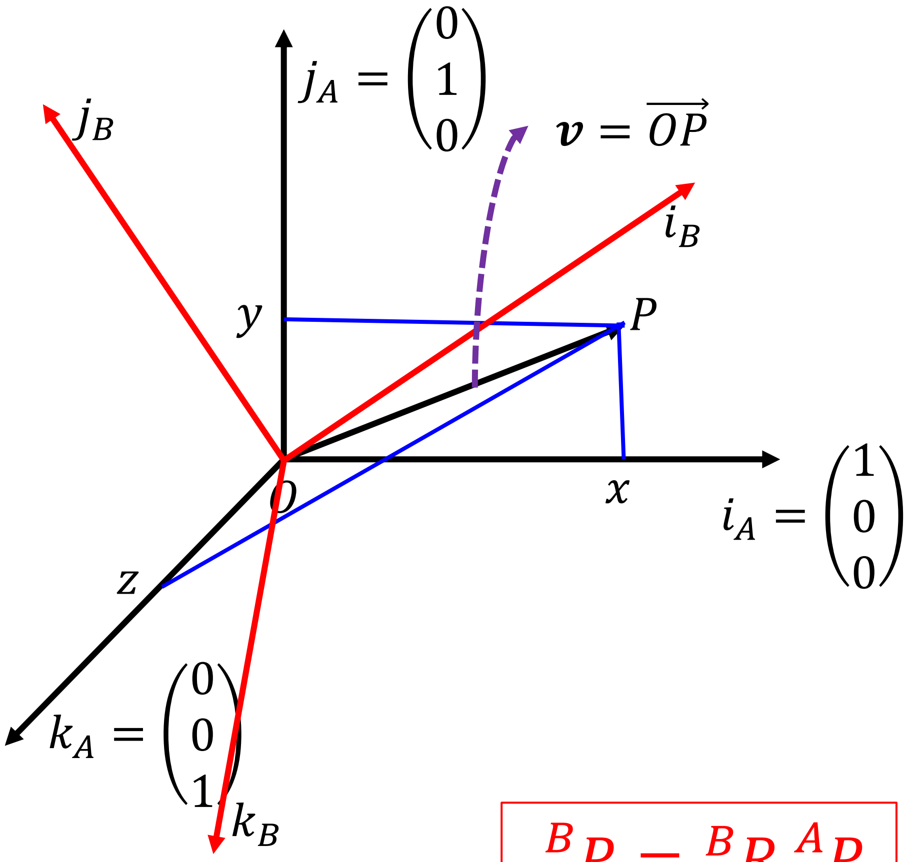
$${}^{A}P = \begin{pmatrix}x\\y\\z\end{pmatrix} \qquad {}^{B}P = f({}^{A}P)?$$

$$\begin{cases} {}^{A}P = R_A^T\boldsymbol{v}, R_A = (\boldsymbol{i}_A \quad \boldsymbol{j}_A \quad \boldsymbol{k}_A) \\ {}^{B}P = R_B^T\boldsymbol{v}, R_B = (\boldsymbol{i}_B \quad \boldsymbol{j}_B \quad \boldsymbol{k}_B) \end{cases}$$

$${}^{A}P = R_A^T\boldsymbol{v} \Rightarrow \boldsymbol{v} = R_A\,{}^{A}P$$

$${}^{B}P = R_B^T\boldsymbol{v} = R_B^T R_A\,{}^{A}P = {}^{B}_{A}R\,{}^{A}P$$

空间坐标变换



$$\boxed{{}^{B}P = {}_{A}^{B}R\,{}^{A}P}$$

$$\begin{cases} R_A = (\boldsymbol{i}_A \quad \boldsymbol{j}_A \quad \boldsymbol{k}_A) \\ R_B = (\boldsymbol{i}_B \quad \boldsymbol{j}_B \quad \boldsymbol{k}_B) \end{cases}$$

$${}^{B}P = {}_{A}^{B}R\,{}^{A}P$$

$${}_{A}^{B}R = R_B^T R_A = \begin{pmatrix} \boldsymbol{i}_B^T\boldsymbol{i}_A & \boldsymbol{i}_B^T\boldsymbol{j}_A & \boldsymbol{i}_B^T\boldsymbol{k}_A \\ \boldsymbol{j}_B^T\boldsymbol{i}_A & \boldsymbol{j}_B^T\boldsymbol{j}_A & \boldsymbol{j}_B^T\boldsymbol{k}_A \\ \boldsymbol{k}_B^T\boldsymbol{i}_A & \boldsymbol{k}_B^T\boldsymbol{j}_A & \boldsymbol{k}_B^T\boldsymbol{k}_A \end{pmatrix}$$

$$= ({}^{B}\boldsymbol{i}_A \quad {}^{B}\boldsymbol{j}_A \quad {}^{B}\boldsymbol{k}_A) = \begin{pmatrix} {}^{A}\boldsymbol{i}_B^T \\ {}^{A}\boldsymbol{j}_B^T \\ {}^{A}\boldsymbol{k}_B^T \end{pmatrix}$$

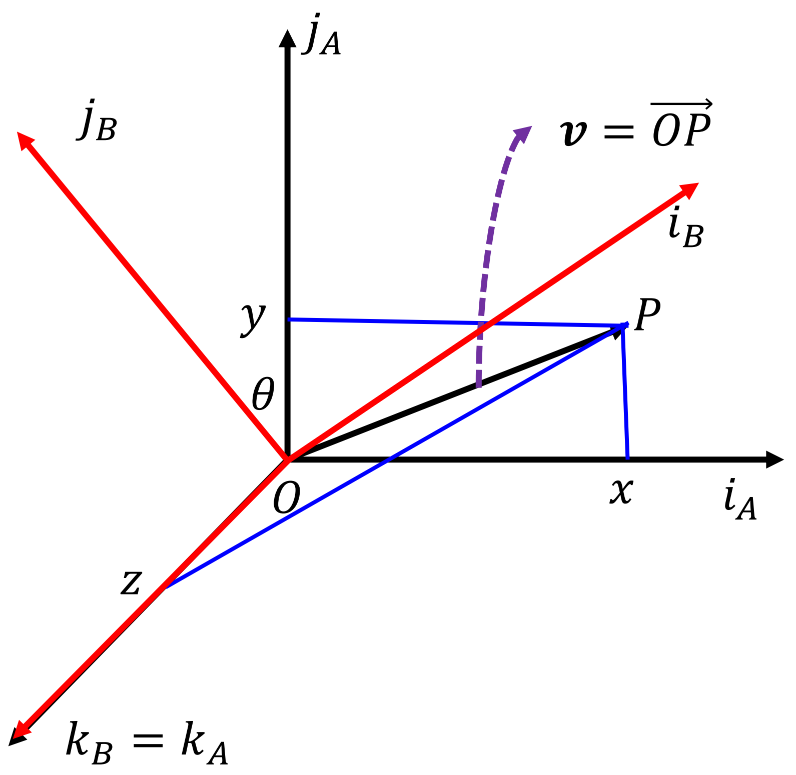
$${}_{A}^{B}R^{-1} = (R_B^T R_A)^{-1} = R_A^{-1} R_B^{-T} = R_A^T R_B = {}_{A}^{B}R^T$$

$${}_{A}^{B}R\,{}_{A}^{B}R^T = {}_{A}^{B}R^T\,{}_{A}^{B}R = I_{3\times 3}$$

$$det({}_{A}^{B}R) = det(R_B^T)det(R_A) = 1$$

空间坐标变换

例：绕z轴旋转θ



$$^{A}P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad ^{B}P = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, z' = z$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$^{B}P = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = {}^{B}_{A}R\, {}^{A}P$$

$${}^{B}_{A}R = R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

空间坐标变换

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

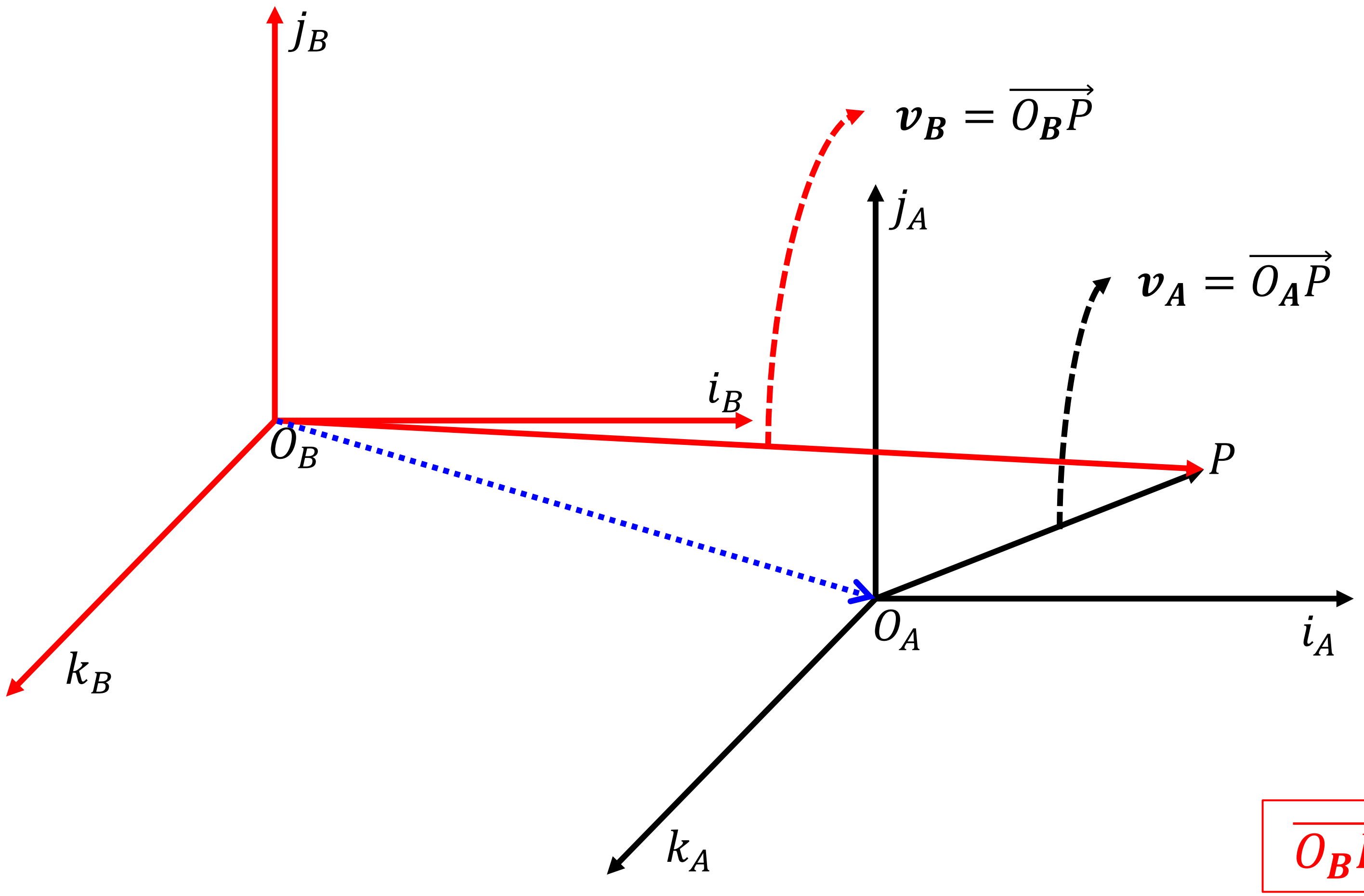
$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_z(\gamma)\, R_y(\beta)\, R_x(\alpha)$$

三维旋转的三个自由度：
α, β, γ：欧拉角(Euler Angles)

空间坐标变换



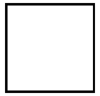
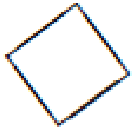
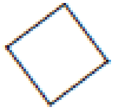
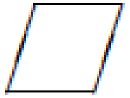
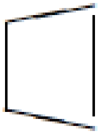
$$\overrightarrow{O_{\boldsymbol{B}}P} = \overrightarrow{O_{\boldsymbol{A}}P} + \overrightarrow{O_{\boldsymbol{B}}O_{\boldsymbol{A}}} \Longrightarrow {}^{B}P = {}^{A}P + {}^{B}O_A$$

空间坐标变换

- 平移：
  - 3×1平移向量$\boldsymbol{t} = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$
  - $\boldsymbol{q} = \boldsymbol{p} + \boldsymbol{t}$
- 旋转：
  - 3×3旋转矩阵$\boldsymbol{R}$(规范正交阵)
  - $\boldsymbol{q} = \boldsymbol{R}\boldsymbol{p}$

空间坐标变换

- 欧氏变换(刚体变换):
  - 3×1平移向量$\boldsymbol{t} = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$
  - 3×3旋转矩阵$\boldsymbol{R}$(规范正交阵)
  - $\boldsymbol{q} = \boldsymbol{R}\boldsymbol{p} + \boldsymbol{t}$
- 用齐次坐标表示欧氏变换:
  - $\bar{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} \\ 0^T & 1 \end{pmatrix} \bar{\boldsymbol{p}}$

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{3\times 4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{3\times 4}$	6	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{3\times 4}$	7	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{3\times 4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{4\times 4}$	15	straight lines	

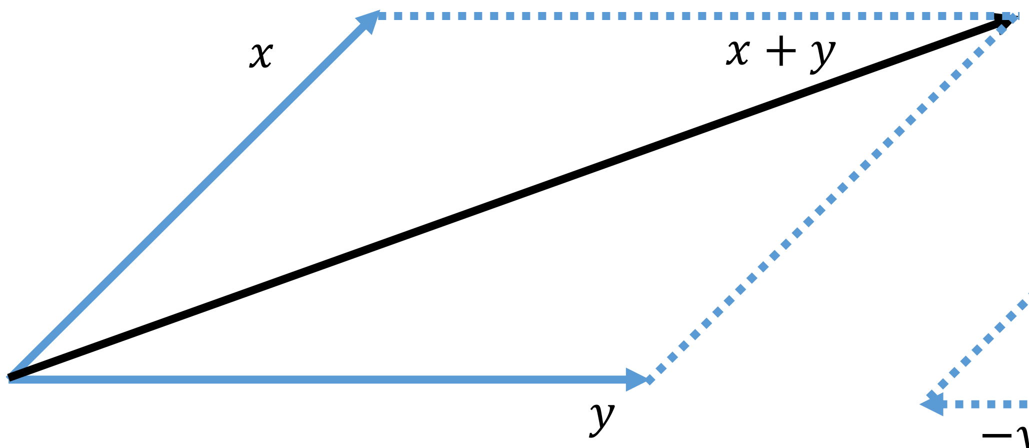
线性代数

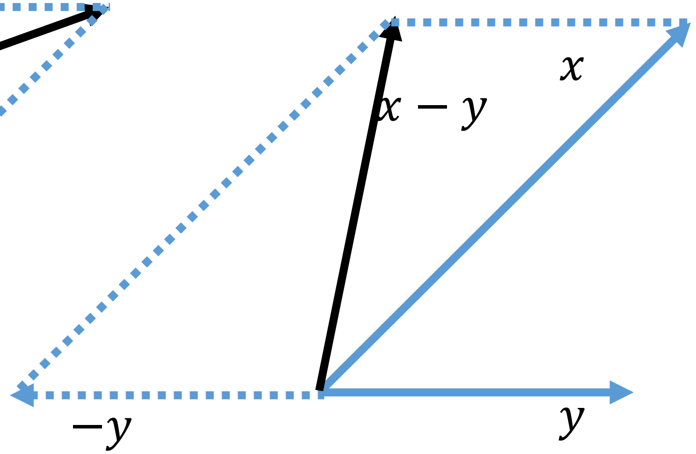
向量

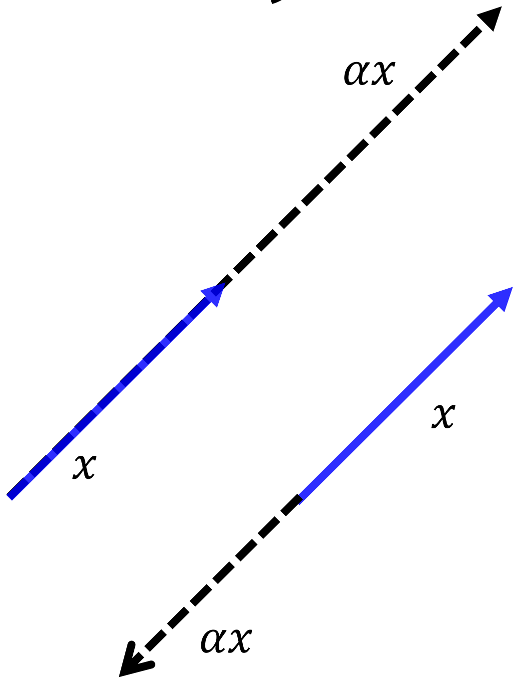
- 列向量$\boldsymbol{v} \in \mathcal{R}^{n\times 1}$: $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$
- 行向量$\boldsymbol{v}^T \in \mathcal{R}^{1\times n}$: $\boldsymbol{v}^T = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$
- 零向量$\mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \end{pmatrix}^T$
- 本课程用向量表示：
  - 几何元素（点、直线、平面、方向）：$\boldsymbol{p} = \begin{pmatrix} x & y \end{pmatrix}^T, \boldsymbol{q} = \begin{pmatrix} x & y & z \end{pmatrix}^T$
  - 数据（图像、特征描述、直方图……）（不完全符合向量的数学定义）

向量的运算

- $\boldsymbol{x} = (x_1 \quad x_2 \quad \cdots \quad x_n)^T, \boldsymbol{y} = (y_1 \quad y_2 \quad \cdots \quad y_n)^T$
- $\alpha\boldsymbol{x} = (\alpha x_1 \quad \alpha x_2 \quad \cdots \quad \alpha x_n)^T, \alpha \in \mathcal{R}$
- $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1 \quad x_2 + y_2 \quad \cdots \quad x_n + y_n)^T$





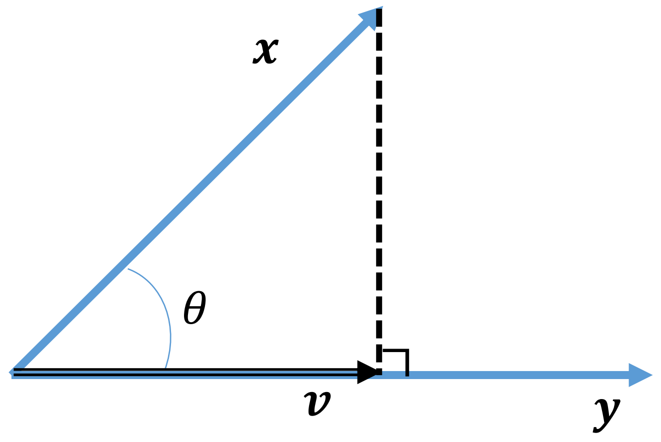


向量的运算

- 向量内积：
  - $\boldsymbol{x} = (x_1 \quad x_2 \quad \cdots \quad x_n)^T, \boldsymbol{y} = (y_1 \quad y_2 \quad \cdots \quad y_n)^T$
  - $\boldsymbol{x}^T\boldsymbol{y} = \boldsymbol{y}^T\boldsymbol{x} = \sum_{i=1}^{n} x_i y_i$
- 向量模长(2-范数,Norm)
  - $\|\boldsymbol{x}\|_2 = \sqrt[2]{\boldsymbol{x}^T\boldsymbol{x}} = \sqrt[2]{\sum_{i=1}^{n} x_i^2}$
  - $\|\boldsymbol{x}\|_2 \geq 0$
  - $\|\boldsymbol{x}\|_2 = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{0}$
  - $\|\alpha\boldsymbol{x}\|_2 = |\alpha|\|\boldsymbol{x}\|_2, \alpha \in \mathcal{R}$
  - $\|\boldsymbol{x}+\boldsymbol{y}\|_2 \leq \|\boldsymbol{x}\|_2 + \|\boldsymbol{y}\|_2$
  - $\boldsymbol{x}$与$\boldsymbol{y}$之间的欧氏距离$d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-y\|_2 = \sqrt[2]{(\boldsymbol{x}-\boldsymbol{y})^T(\boldsymbol{x}-\boldsymbol{y})} = \sqrt[2]{\boldsymbol{x}^T\boldsymbol{x}+\boldsymbol{y}^T\boldsymbol{y}-2\boldsymbol{x}^T\boldsymbol{y}}$
- 单位向量: x: $\|x\|_2 = 1$

向量的运算

- 两个向量的夹角
  - $\boldsymbol{x}^T\boldsymbol{y} = \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2 cos\theta$
  - $\theta = 0: \boldsymbol{x} \parallel \boldsymbol{y}, \boldsymbol{x}^T\boldsymbol{y} = \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2$
  - $\theta = \pm\frac{\pi}{2}: \boldsymbol{x} \perp \boldsymbol{y}, \boldsymbol{x}^T\boldsymbol{y} = 0$
  - $0 \le \boldsymbol{x}^T\boldsymbol{y} \le \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2$
- x在y上的投影向量v：
  - $\boldsymbol{v} = \boldsymbol{x}^T\boldsymbol{y}\frac{\boldsymbol{y}}{\|\boldsymbol{y}\|_2}$
  - $\boldsymbol{v} \parallel \boldsymbol{y}$
  - $\|\boldsymbol{v}\|_2 = \boldsymbol{x}^T\boldsymbol{y}$
  - $\boldsymbol{x} \perp \boldsymbol{y} \Rightarrow \boldsymbol{v} = \boldsymbol{0}$



矩阵

- $A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \in \mathcal{R}^{m\times n}, A = [a_{i,j}]_{i=1,j=1}^{m,n}$
- $A = \begin{bmatrix} — & \boldsymbol{a}_1^T & — \\ — & \boldsymbol{a}_2^T & — \\ & \vdots & \\ — & \boldsymbol{a}_m^T & — \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \tilde{\boldsymbol{a}}_1 & \tilde{\boldsymbol{a}}_2 & \cdots & \tilde{\boldsymbol{a}}_n \\ | & | & & | \end{bmatrix}$
  - $\boldsymbol{a}_i^T = (a_{i,1} \quad a_{i,2} \quad \cdots \quad a_{i,n}), i = 1 \cdots m, \ \ \tilde{\boldsymbol{a}}_j^T = (a_{1,j} \quad a_{2,j} \quad \cdots \quad a_{m,j}), j = 1 \cdots n$
- 单位阵$I_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{R}^{n\times n}$，对角阵$A_{n\times n} = \begin{bmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{bmatrix} \in \mathcal{R}^{n\times n}$

矩阵的运算

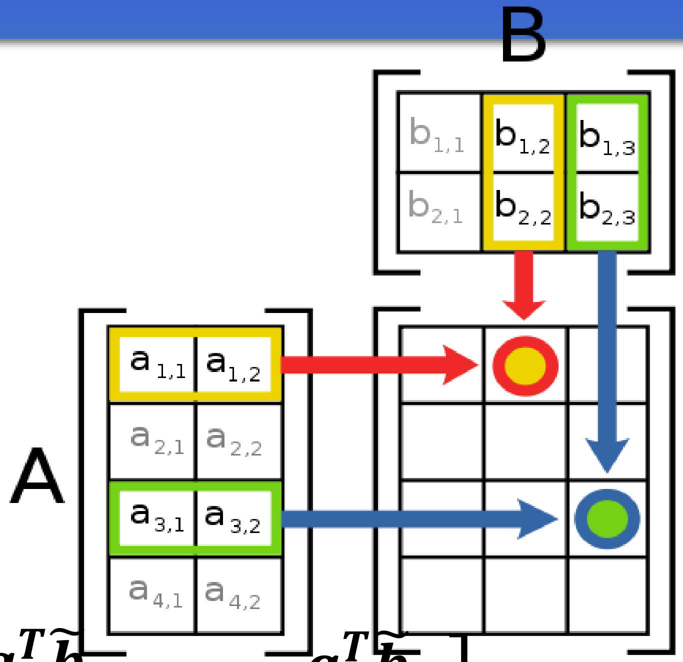
- $A = [a_{i,j}]_{i=1,j=1}^{m,n}, B = [b_{i,j}]_{i=1,j=1}^{m,n} \Rightarrow C = A + B = [a_{i,j} + b_{i,j}]_{i=1,j=1}^{m,n}$

- $\alpha A = [\alpha a_{i,j}]_{i=1,j=1}^{m,n}, \alpha \in \mathcal{R}$

- $A^T = [a_{j,i}]_{i=1,j=1}^{m,n} = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,m} & a_{2,m} & \cdots & a_{n,m} \end{bmatrix}$
- 对称阵： $A^T = A$

矩阵的运算

- $A \in \mathcal{R}^{m\times p}, B \in \mathcal{R}^{p\times n} \Longrightarrow C = AB \in \mathcal{R}^{m\times n}$
- $c_{i,j} = \sum_{k=1}^{p} a_{i,k} b_{k,j}$



- $C = AB = \begin{bmatrix} — & \boldsymbol{a}_1^T & — \\ — & \boldsymbol{a}_2^T & — \\ & \vdots & \\ — & \boldsymbol{a}_m^T & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ \tilde{\boldsymbol{b}}_1 & \tilde{\boldsymbol{b}}_2 & \cdots & \tilde{\boldsymbol{b}}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1^T\tilde{\boldsymbol{b}}_1 & \boldsymbol{a}_1^T\tilde{\boldsymbol{b}}_2 & \cdots & \boldsymbol{a}_1^T\tilde{\boldsymbol{b}}_n \\ \boldsymbol{a}_2^T\tilde{\boldsymbol{b}}_1 & \boldsymbol{a}_2^T\tilde{\boldsymbol{b}}_2 & \cdots & \boldsymbol{a}_2^T\tilde{\boldsymbol{b}}_n \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{a}_m^T\tilde{\boldsymbol{b}}_1 & \boldsymbol{a}_m^T\tilde{\boldsymbol{b}}_2 & \cdots & \boldsymbol{a}_m^T\tilde{\boldsymbol{b}}_n \end{bmatrix}$

矩阵的运算

- $A = \begin{bmatrix} — & \boldsymbol{a}_1^T & — \\ — & \boldsymbol{a}_2^T & — \\ & \vdots & \\ — & \boldsymbol{a}_m^T & — \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \tilde{\boldsymbol{a}}_1 & \tilde{\boldsymbol{a}}_2 & \cdots & \tilde{\boldsymbol{a}}_n \\ | & | & & | \end{bmatrix} \in \mathcal{R}^{m\times n}$

- $I_{m\times m}A = AI_{n\times n} = A$

- $B_{n\times n} = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_n \end{bmatrix} \rightarrow AB_{n\times n} = \begin{bmatrix} | & | & & | \\ b_1\tilde{\boldsymbol{a}}_1 & b_2\tilde{\boldsymbol{a}}_2 & \cdots & b_n\tilde{\boldsymbol{a}}_n \\ | & | & & | \end{bmatrix}$

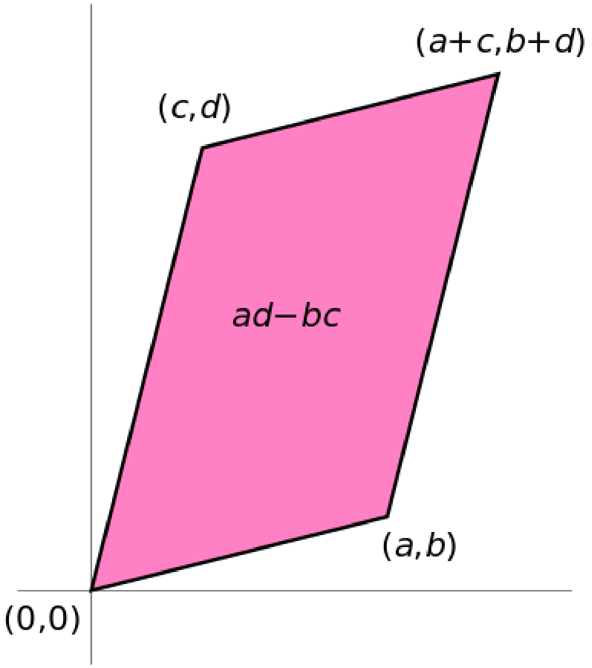
- $C_{m\times m} = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_m \end{bmatrix} \rightarrow C_{m\times m}A = \begin{bmatrix} — & c_1\boldsymbol{a}_1^T & — \\ — & c_2\boldsymbol{a}_2^T & — \\ & \vdots & \\ — & c_m\boldsymbol{a}_m^T & — \end{bmatrix}$

矩阵的运算

- 矩阵乘法满足结合律：
  - $A(BC) = (AB)C$
- 矩阵乘法对加法的分布律
  - $A(B + C) = AB + AC$
- 矩阵乘法不满足交换律
  - 一般情况：$AB \neq BA$
- $(AB)^T = B^T A^T$
- 矩阵的逆 A^{-1}: $A^{-1}A = AA^{-1} = I$
  - 如果A^{-1}存在，A称为可逆阵，或非奇异阵(non-singular)，否则A是奇异阵
  - $(AB)^{-1} = B^{-1}A^{-1}, A^{-T} \triangleq (A^T)^{-1} = (A^{-1})^T$
- $IA = AI = A$

矩阵的运算

- 矩阵的行列式
  - $det(A) \in \mathcal{R}$
  - $det(A) = 0 \Leftrightarrow A$是奇异阵
  - $det(AB) = det(BA)$
  - $det(A^{-1}) = \frac{1}{det(A)}$
  - $det(A^T) = det(A)$
- 矩阵的迹(trace)
  - $tr(A) = \sum_{i=1}^{n} a_{i,i}$
  - $tr(AB) = tr(BA)$
  - $tr(A + B) = tr(B) + tr(A)$
  - $tr(\alpha A) = \alpha tr(A), \alpha \in R$



$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : det(A) = ad - bc$$

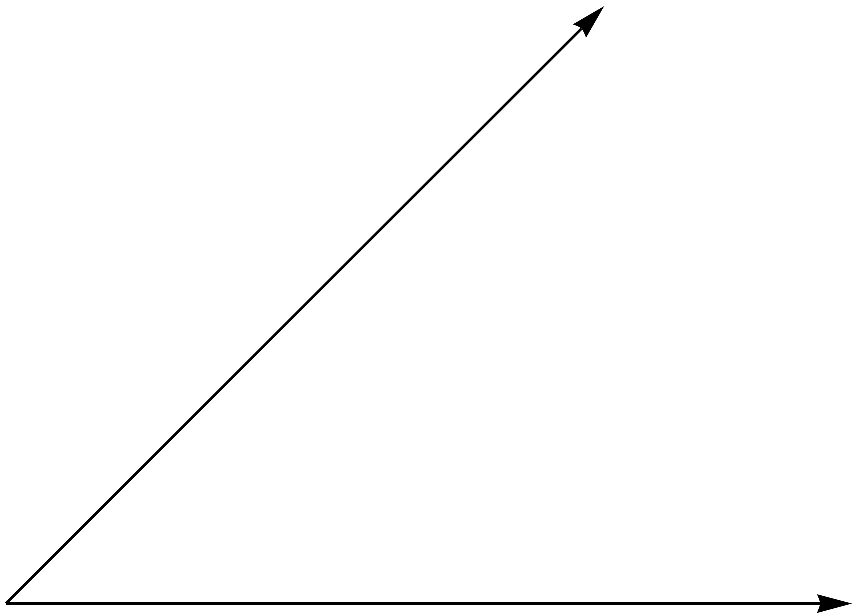
矩阵的运算

- $A = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \tilde{a}_1 & \tilde{a}_2 & \cdots & \tilde{a}_n \\ | & | & & | \end{bmatrix} \in \mathcal{R}^{m\times n}$
- $\boldsymbol{x} = (x_1 \quad x_2 \quad \cdots \quad x_n)^T, \boldsymbol{y} = (y_1 \quad y_2 \quad \cdots \quad y_m)^T$
- $\boldsymbol{u} = A\boldsymbol{x} = \sum_{j=1}^{n} x_j \tilde{a}_j \in \mathcal{R}^{m\times 1} \longleftarrow A$的各列的线性组合
- $\boldsymbol{u} = A\boldsymbol{x} = (a_1^T x \quad a_2^T x \qquad a_m^T x)^T \in \mathcal{R}^{m\times 1}$
  - u_i: $\boldsymbol{x}$在向量a_i上的投影
- $\boldsymbol{v} = \boldsymbol{y}^T A = \sum_{i=1}^{m} y_i a_i^T \in \mathcal{R}^{1\times n} \longleftarrow A$的各行的线性组合
- $\boldsymbol{v} = \boldsymbol{y}^T A = (y^T \tilde{a}_1 \quad y^T \tilde{a}_2 \qquad y^T \tilde{a}_n)^T \in \mathcal{R}^{1\times n}$
  - v_j: $\boldsymbol{y}$在向量$\tilde{a}_j$上的投影

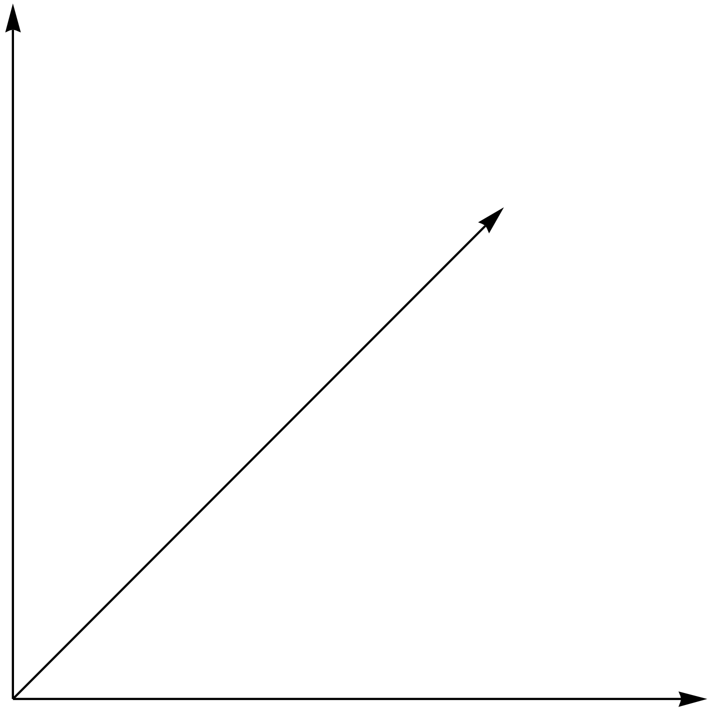
线性无关(Linear Independence)

- $V = \{\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \qquad\qquad \boldsymbol{v}_n\}$
- $\boldsymbol{v}_1$可以由$\{\boldsymbol{v}_2 \quad \boldsymbol{v}_3 \qquad\qquad \boldsymbol{v}_n\}$线性表示
  - $\boldsymbol{v}_1 = \alpha_{1,2}\boldsymbol{v}_2 + \alpha_{1,3}\boldsymbol{v}_3 + \cdots \alpha_{1,n}\boldsymbol{v}_n$， $\alpha_{1,j} \in \mathcal{R}$
  - $\boldsymbol{v}_1 - \alpha_{1,2}\boldsymbol{v}_2 - \alpha_{1,3}\boldsymbol{v}_3 - \cdots - \alpha_{1,n}\,\boldsymbol{v}_n = 0$
- $\{\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \qquad\qquad \boldsymbol{v}_n\}$线性相关
  - $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 + \cdots \alpha_n\boldsymbol{v}_n = 0$， $\exists\alpha_j \in \mathcal{R}, \alpha_j$不全为0
- $\{\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \qquad\qquad \boldsymbol{v}_n\}$线性无关
  - $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 + \cdots \alpha_n\boldsymbol{v}_n = 0 \rightarrow \alpha_j = 0, \forall j$
- $V\boldsymbol{\alpha} = \boldsymbol{0}, V = (\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \qquad\qquad \boldsymbol{v}_n), \boldsymbol{\alpha} = (\alpha_1 \quad \alpha_2 \qquad\qquad \alpha_n)^T$
  - 有非零解：线性相关
  - 只有零解：线性无关

线性无关(Linear Independence)



线性无关



线性相关

线性无关(Linear Independence)

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

线性无关

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}2\\3\\4\end{pmatrix}\right\}$$

$$\begin{pmatrix}2\\3\\4\end{pmatrix}=4\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}-\begin{pmatrix}1\\0\\0\end{pmatrix}$$

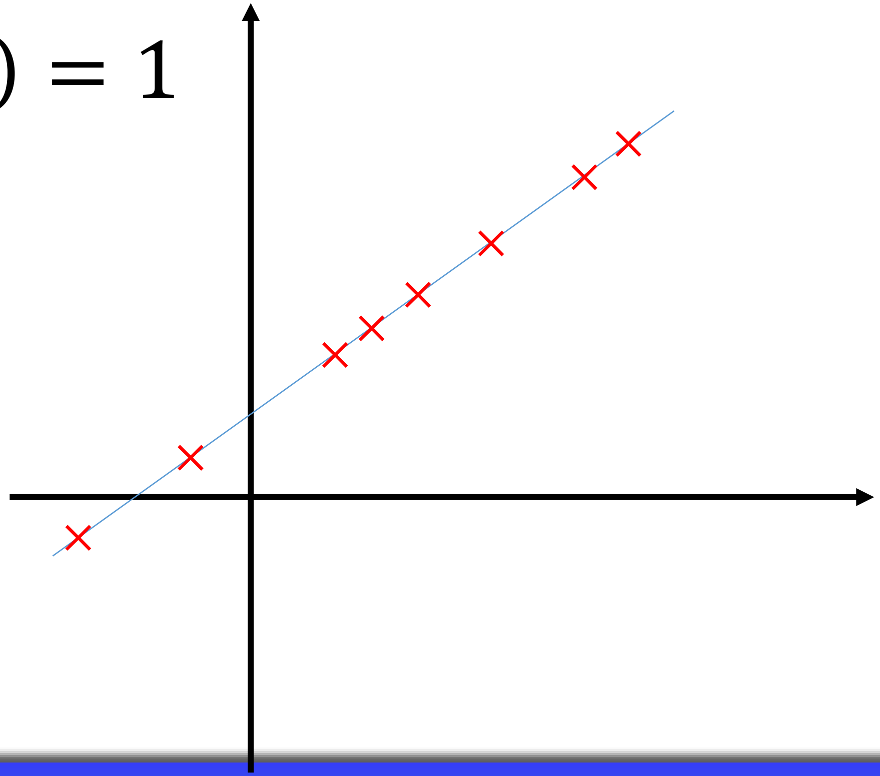
线性相关

矩阵的秩

- 列秩：矩阵A的线性无关的列向量的极大数目
- 行秩：矩阵A的线性无关的行向量的极大数目
- 行秩=列秩：$\mathrm{rank}(A)$
- $A \in \mathcal{R}^{m\times n}$:
  - $rank(A) \le min(m,n)$
  - $rank(AA^T) = rank(A^TA) = rank(A) = rank(A^T)$
- $A \in \mathcal{R}^{n\times n}$:
  - $rank(A) < n \rightarrow \det(A) = 0, A$是奇异阵、不可逆
  - $rank(A) = n \rightarrow \det(A) \neq 0, A$是奇异阵、可逆

矩阵的秩

- 例：假设平面上有n个点：$\left\{p_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, i = 1..n, n \geq 3\right\}$，如果这n个点共线:$l: ax + by + c = 0, a \neq 0, b \neq 0$，那么矩阵 $A = \begin{pmatrix} p_i & p_i & \cdots & p_n \end{pmatrix}$
- 如果$c = 0$，那么$y = \alpha x$, A的秩$rank(A) = 1$
  - A的第二行可以由第一行线性表示
- 如果$c \neq 0$，那么$y = \alpha x + \beta, \beta \neq 0$
  - A的第二行无法由第一行线性表示, $rank(A) = 2$



特征值与特征向量

- $\boldsymbol{Av = \lambda v, v \neq 0}$
- 矩阵A作用于$\boldsymbol{v}$，等价于对$\boldsymbol{v}$做了一个缩放操作：$\boldsymbol{\lambda v}$
- $tr(A) = \sum_{i=1}^{n} \lambda_i, det(A) = \prod_{i=1}^{n} \lambda_i$
- 不同特征值对应的特征向量是线性无关的
- $AV = VD$
  - $V = (\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \qquad \boldsymbol{v}_n), D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = diag(\lambda_1, \lambda_2, \ldots \lambda_n)$
  - 如果$\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \qquad \boldsymbol{v}_n$线性无关➔ V可逆➔$A = VDV^{-1}, V^{-1}AV = D$

实对称阵

- $A = A^T \in \mathcal{R}^{n\times n}$
- 实对称阵的特征值都是实数
- 不同特征值对应的特征向量是正交的
- $A = VDV^{-1} = A^T = V^{-T}DV^T \Longrightarrow V^{-1} = V^T$
- $V^{-1} = V^T \Longrightarrow V^TV = I_{n\times n} \Longrightarrow \boldsymbol{v}_i^T\boldsymbol{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$
- $V^TV = I_{n\times n}$: V 是规范正交阵

奇异值分解(Singular Value Decomposition, SVD)

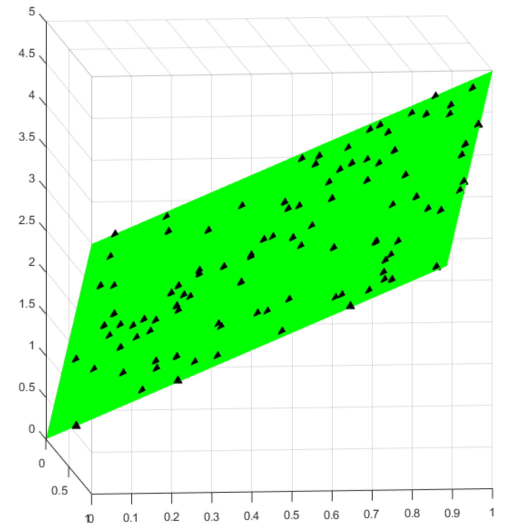
- 任何一个$m \times n$的矩阵$A \in \mathcal{R}^{m\times n}$都可以分解为$A = U\Sigma V^T$
  - $U \in \mathcal{R}^{m\times m}$， U的每一列$\boldsymbol{u}_i$是AA^T的1个特征向量， $U^TU = I_{m\times m}$
  - $V \in \mathcal{R}^{n\times n}$， V的每一列$\boldsymbol{v}_i$是A^TA的1个特征向量， $V^TV = I_{n\times n}$
  - $\Sigma \in \mathcal{R}^{m\times n}$， 是一个对角阵，包含$r$个奇异值，$\{\sigma_1 \quad \sigma_2 \quad \cdots \quad \sigma_r\}$，奇异值$\sigma_i$是$AA^T$与$A^TA$的非零特征值的平方根， $r = rank(A)$
  - $A = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$
- 实对称阵$AA^T = U\Sigma V^T V\Sigma U^T = U\Sigma^2 U^T$，对$AA^T$做特征分解得到： $AA^T = \Phi \boxed{D} \Phi^T$
  - $\Phi = U$： U是AA^T的特征向量构成的矩阵， AA^T的特征值$\lambda_i = \sigma_i^2$
- 实对称阵$A^TA = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T$，对$A^TA$做特征分解得到：$A^TA = \Psi \boxed{D} \Psi^T$
  - $\Psi = V$： V是A^TA的特征向量构成的矩阵，A^TA的特征值$\lambda_i = \sigma_i^2$

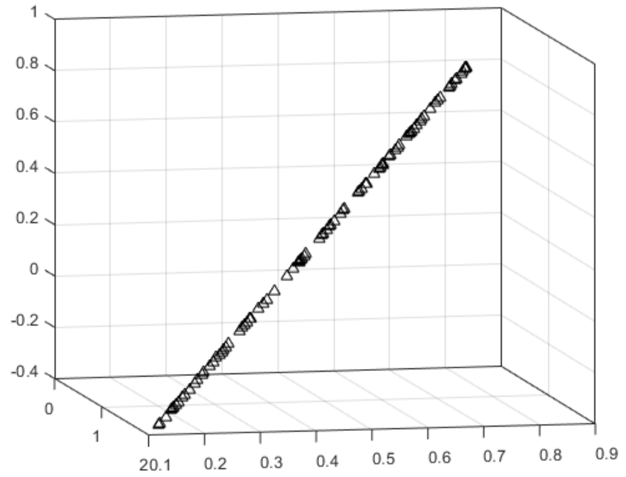
注意特征值与奇异值的区别

为什么AA^T与A^TA的特征值相同？

示例：3D空间的平面与直线

- 例：假设空间中有n个点$\left\{p_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}, i = 1..n, n \geq 3\right\}$，
  - 如果矩阵$A = (p_i \quad p_i \quad \cdots \quad p_n)$的秩$rank(A) = 2$，那么这些点共面或者共线。如果$rank(A) = 1$，那么这些点一定共线，为什么？
- 如何判断任意n个点共面或共线？
  - 对A做奇异值分解，检查三个奇异值中最小的是否接近0
  - 对实对称阵AA^T做特征值分解，检查三个特征值中最小的是否接近0
- 如果这n个点共面，平面点只需要两个坐标就可以表示，如何得到这个n个3D点在平面上的2D平面坐标？



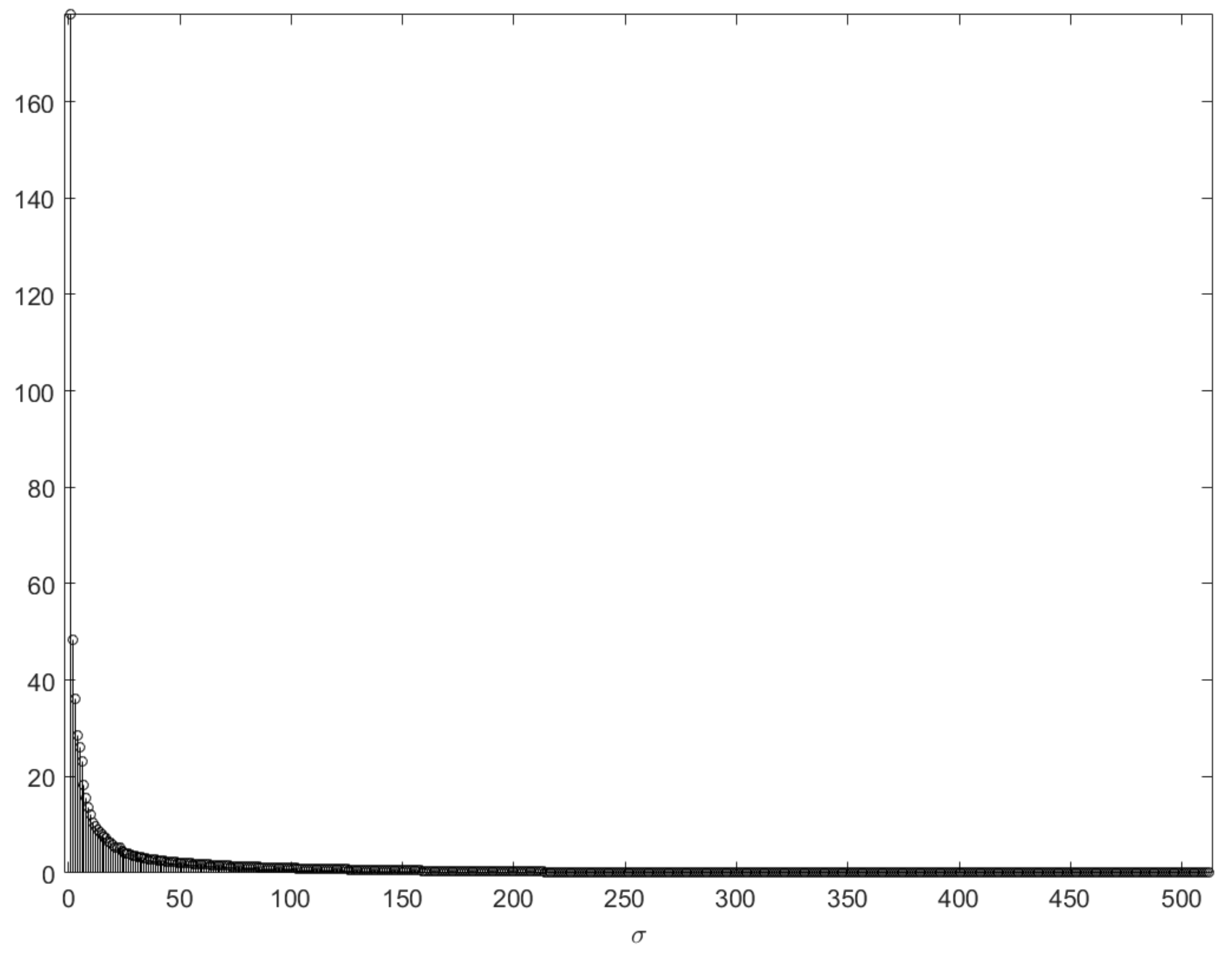


用SVD压缩图像

$A \in \mathcal{R}^{512\times512}$ 2^{18}个值

$$A = U\Sigma V^T = \sum_{i=1}^{512} \sigma_i \, \boldsymbol{u}_i \boldsymbol{v}_i^T$$

用SVD压缩图像



$A \in \mathcal{R}^{512\times512}$ 2^{18}个值

用SVD压缩图像

$A \in \mathcal{R}^{512\times512}$　　2^{18}个值

$$A \approx \sum_{i=1}^{k} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$$

$$\{\sigma_i \quad \boldsymbol{u}_i \quad \boldsymbol{v}_i\}_{i=1}^{k}$$

$1025 \times k \approx 2^{10} \times k$个值

用SVD压缩图像

$A \in \mathcal{R}^{512\times512}$ 2^{18}个值

$k = 32$

用SVD压缩图像

$A \in \mathcal{R}^{512\times512}$ 2^{18}个值

$k = 64$

用SVD压缩图像

$A \in \mathcal{R}^{512\times512}$　　2^{18}个值

$k = 128$