LAB100

Week 14: Further Matrices

Your aim for this workshop is to learn how to do matrix computations using R and how to use matrices within functions.

- Perform element-wise calculations
- Perform matrix multiplication
- Solve a system of simultaneous equations.
- Write functions involving matrices

In this workshop you will be using the following matrices.

```
A <- matrix( c(1,-3,2,-5,1,2,2,3,-4), ncol=3, nrow=3, byrow=FALSE )
B <- matrix( c(10,6,8,16,8,12,17,9,14), ncol=3, nrow=3, byrow=FALSE )
C <- matrix( c(1,-4,5,2,-3,-6), ncol=3, nrow=2, byrow=FALSE )
D <- matrix( c(8,7,8,14,1,-4), ncol=2, nrow=3, byrow=FALSE )
E <- matrix( c(2,2,2,2), ncol=2, nrow=2, byrow=FALSE )
```

Remember that you need to change the working directory and open a new R script at the start of each workshop.

1 Element-wise Calculation

In RStudio matrices can be thought of as a structured collection of information. In the first workshop you were introduced to vectors as a collection of data that enabled you to perform the same calculation on each value. Likewise, we can treat matrices in a similar manner. Using the matrices defined in the introduction, type and execute the following commands that perform element-wise commands:

```
E+1
B-A
5*C
3 - D/2
A*B
```

```
exp( D )
sqrt( B )
cos( pi*E )
asin( 1/C )
A^2
```

Note that the last command in each column are performed element-wise and are not matrix multiplication. Therefore, RStudio executes these commands like this:

$$\texttt{A*B} = \begin{bmatrix} \texttt{1*10} & \texttt{-5*16} & \texttt{2*17} \\ \texttt{-3*6} & \texttt{1*8} & \texttt{3*9} \\ \texttt{2*8} & \texttt{2*12} & \texttt{-4*14} \end{bmatrix} \qquad \texttt{A\^{}2} = \begin{bmatrix} \texttt{1\^{}2} & \texttt{-5\^{}2} & \texttt{2\^{}2} \\ \texttt{-3\^{}2} & \texttt{1\^{}2} & \texttt{3\^{}2} \\ \texttt{2\^{}2} & \texttt{2\^{}2} & \texttt{-4\^{}2} \end{bmatrix}$$

The element-wise calculations can only be performed when the dimensions of two matrices are identical. Performing the calculation `C*D` will produced the error message `Error in C * D : non-conformable arrays`.

2 Matrix Multiplication

Matrix multiplication is performed by the operator `%*%`; for example, execute the command `A%*%B`. However, this will only work if the two matrices have matching dimensions. The `dim` command calculates the dimensions of a matrix, where the first number indicates the number of rows and the second indicates the number of columns:

```
dim( A )
dim( B )
dim( C )
```

```
dim( D )

dim( E )
```

Here, we see that matrix C and matrix D have a matching dimension and so we can execute the following matrix multiplication:

```
F <- C%*%D
F
dim( F )
```

When the dimensions of the matrices do not match, for example `B%*%C`, RStudio reports the message `Error in B%*%C: non-conformable arguments`.

3 Simultaneous Equations

Consider the following system of linear equations:

$$\begin{array}{rcrcrcr} x & - & 5y & + & 2z & = & 0 \\ -3x & + & y & + & 3z & = & -8 \\ 2x & + & 2y & - & 4z & = & 4 \end{array}$$

Recall from MATH103 that this system can be represented by the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -5 & 2 & 0 \\ -1 & 1 & 3 & -8 \\ 2 & 2 & -4 & 4 \end{array}\right]$$

The square matrix on the left-hand-side is equal to matrix A from the introduction to this workshop and the right-hand-side vector is created in RStudio by:

```
b <- c(0, -8, 4)
```

One method that calculates the solution to this system of linear equations is to pre-multiply this vector by the inverse of matrix A. We can find the inverse by using the function `solve` introduced in last week's workshop.

```
Ainv <- solve(A)
Ainv %*% b
```

Alternatively, we can use the `solve` command by supplying two arguments; the first being the square matrix and the second being the vector:

```
solve( A, b )
```

4 Functions involving matrices

We have already seen several functions that take, or can take, matrices as inputs (e.g. `det`, `t()`, `solve`, `diag`). However, we can also write our functions involving matrices.

As a simple first example, suppose we want to write a function that returns the *trace* (i.e. the sum of the diagonal elements) of a square matrix. Note that we already know how to extract the diagonal elements of a matrix using the function `diag`.

```
Trace <- function(A) {
     diag_e <- diag(A)
     trace <- sum(diag_e)
     return(trace)
}
```

As a more complicated example, we can consider writing a function to determine whether a supplied square matrix is *nilpotent* and if so, to what degree.

A matrix $\mathbf{A}$ is nilpotent with degree n, where n is a positive integer, if $\mathbf{A}^n = \mathbf{0}$ and $\mathbf{A}^m \neq \mathbf{0}$ for all positive integers $m < n$.

To help us develop the function, we can note that if a matrix is nilpotent it must be nilpotent with degree less than or equal to the dimension of the matrix. We can check whether a particular matrix is equal to the zero matrix by obtaining the maximum absolute entry of the matrix. We can also use the fact that nilpotent matrices must have determinant equal to zero. In each case, a suitable tolerance limit is required, because R will introduce some small rounding error. Here we take 1×10^{-8}.

```
nilpotent <- function(A) {
      dm <- dim(A)[1]
      if (abs(det(A))> 1e-8) return("Matrix is not nilpotent")
      if (max(abs(A))< 1e-8) return(1)
      An <- A
      for (n in 2:dm) {
           An <- An%*%A
           if (max(abs(An))< 1e-8) return(n)
      }
      return("Matrix is not nilpotent")
}
```

Quiz 1: Matrix multiplication

For $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ and $\mathbf{E}$ as defined by `A`,`B`,`C`,`D` and `E` at the start of this worksheet, find

$$\mathbf{C}\mathbf{A}\mathbf{B}^T\mathbf{D} + \mathbf{E}\mathbf{C}\mathbf{D}$$

The result is

(**A**) $\begin{bmatrix} 2456 & 1237 \\ 5412 & 3112 \end{bmatrix}$

(**B**) $\begin{bmatrix} 2456 & 5412 \\ 1237 & 3112 \end{bmatrix}$

(**C**) $\begin{bmatrix} 3525 & 2157 \\ 5456 & 3536 \end{bmatrix}$

(**D**) $\begin{bmatrix} 3525 & 5456 \\ 2157 & 3112 \end{bmatrix}$

(**E**) $\begin{bmatrix} 5321 & 2654 \\ 3345 & 3411 \end{bmatrix}$

Quiz 2: System of linear equations

Find the value for x_1, x_2, x_3, x_4 and x_5 that satisfies these linear equations:

$$\begin{array}{rcrcrcrcrl} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 = & 2.0 \\ 2x_1 & - & x_2 & + & x_3 & + & x_4 & - & x_5 = & -1.5 \\ x_1 & + & 3x_2 & & & - & x_4 & + & 2x_5 = & 5.5 \\ 2x_1 & - & x_2 & + & x_3 & + & x_4 & + & x_5 = & 0.5 \\ & & x_2 & - & x_3 & - & x_4 & + & x_5 = & 2.5 \end{array}$$

Quiz 3: Nilpotent matrices

Use the `nilpotent` function defined in Section 4 to determine whether the following matrices are nilpotent and if so the degree.

$$V = \begin{bmatrix} -4 & 0 & -6 & -4 & -2 & 0 \\ 10 & 0 & 14 & 10 & 4 & 2 \\ 7 & 1 & 7 & 6 & 2 & 1 \\ -2 & 0 & -4 & -4 & -2 & -2 \\ -3 & 1 & -5 & -2 & 0 & -1 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \quad W = \begin{bmatrix} 4 & 6 & -6 & 6 & 4 \\ 2 & 2 & -2 & 4 & 2 \\ 4 & 6 & -4 & 8 & 4 \\ 0 & -2 & 0 & 0 & 0 \\ -2 & 2 & 4 & -2 & -2 \end{bmatrix}$$

$$X = \begin{bmatrix} -\frac{1}{8} & \frac{5}{4} & -\frac{5}{8} & \frac{11}{8} \\ -\frac{9}{8} & \frac{1}{4} & \frac{3}{8} & \frac{11}{8} \\ -\frac{3}{4} & \frac{1}{2} & \frac{1}{4} & \frac{5}{4} \\ \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} & -\frac{3}{8} \end{bmatrix} \quad Y = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 2 \\ -6 & 12 & 6 & -2 & -4 & 2 \\ 6 & -12 & -7 & 2 & 5 & -1 \\ -2 & 4 & 1 & 0 & -1 & 1 \\ -4 & 10 & 5 & -2 & -3 & 1 \\ 2 & -2 & -2 & 0 & 2 & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0.3 & 0.0625 & -0.5 & 0.15 \\ -2.4 & -0.50 & 4.0 & -1.2 \\ 0.06 & 0.0125 & -0.1 & 0.03 \\ 0.60 & 0.125 & -1.0 & 0.3 \end{bmatrix}$$

Match the matrices to the following answers:

(**A**) Nilpotent, degree 2

(**B**) Nilpotent, degree 3

(**C**) Nilpotent, degree 4

(**D**) Nilpotent, degree 5

(**E**) Matrix is not nilpotent

Quiz 4: Raising a matrix to a power

Write a function in R that takes as its inputs a matrix $\mathbf{X}$ and a positive integer n and returns $\mathbf{X}^n$. Hence find the $(3,2)$ element (the entry of the third row and second column) of the 3×3 matrix $\mathbf{W}^{25}$ where

$$\mathbf{W} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Quiz 5: Approximating a matrix exponential

The exponential of a square matrix $\mathbf{A}$ is defined by the infinite series

$$\exp(\mathbf{A}) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n$$

where $\mathbf{A}^0 = \mathbf{I}$.
Using your function for finding matrix powers, combined with a `for` loop, approximate $\exp(\mathbf{W})$ by computing

$$\tilde{\mathbf{W}} = \sum_{n=0}^{10} \frac{1}{n!} \mathbf{W}^n$$

where W is as defined in the previous question. Give the $(3,2)$ entry of $\tilde{\mathbf{W}}$ to 3 decimal places. [Recall that $n!$ may be computed in R using `factorial(n)`.]