12. Write and test subroutines or procedures for the following:

    (i) STORE $(n.x, y)$, which replaces the n-vector y by the n-vector x.

    (ii) PROD (m, n, A, x, y), which multiplies the n-vector x by the $m \times n$ matrix A and stores the result in the m-vector y.

    (iii) MULT (k, m, n, A, B, C), which computes $C = AB$, where A is $k \times m$, B is $m \times n$, and C is $k \times n$.

    (iv) DOT (n, x, y, a), which computes (in double-precision arithmetic) the dot product $\sum_{i=1}^{n} x_i y_i$ and stores the answer as a single-precision real number, a. *Note:* x_i, y_i, a are single-precision numbers.

13. Let A be an $n \times n$ invertible matrix, and let u and v be two vectors in $\mathbb{R}^n$. Find the necessary and sufficient conditions on u and v in order that the matrix

$$\begin{bmatrix} A & u \\ v^T & 0 \end{bmatrix}$$

be invertible, and give a formula for the inverse when it exists.

14. Let D be a matrix in partitioned form:

$$D = \begin{bmatrix} A & B \\ C & I \end{bmatrix}$$

Prove that if $A - BC$ is nonsingular, then so is D.

15. (Continuation) Prove the stronger result that the dimension of the null space of D is no greater than the dimension of the null space of $A - BC$.

16. Are these matrices positive definite?

    (a) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix}$

17. For what values of a is this matrix positive definite?

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}$$

18. A square matrix A is said to be skew-symmetric if $A^T = -A$. Prove that if A is skew-symmetric, then $x^T Ax = 0$ for all x.

19. (Continuation) Prove that the diagonal elements and the determinant of a skew-symmetric matrix are 0.

20. (Continuation) Let A be any square matrix, and define

$$A_0 = \tfrac{1}{2}(A + A^T) \qquad A_1 = \tfrac{1}{2}(A - A^T)$$

Prove that A_0 is symmetric, that A_1 is skew-symmetric, that $A = A_0 + A_1$, and that for all x, $x^T Ax = x^T A_0 x$. This explains why, in discussing quadratic forms, we can confine our attention to symmetric matrices.

21. Give an example of a symmetric matrix A containing all positive elements such that $x^T Ax$ is sometimes negative.

22. Can a matrix have a right inverse and a left inverse that are not equal?

The algorithm for the Cholesky factorization will then be as follows

$$
\begin{array}{l}
\textbf{input } n, (a_{ij}) \\
\textbf{for } k = 1, 2, \ldots, n \textbf{ do} \\
\quad \ell_{kk} \leftarrow \left(a_{kk} - \displaystyle\sum_{s=1}^{k-1} \ell_{ks}^2 \right)^{1/2} \\
\quad \textbf{for } i = k+1, k+2, \ldots, n \textbf{ do} \\
\qquad \ell_{ik} \leftarrow \left(a_{ik} - \displaystyle\sum_{s=1}^{k-1} \ell_{is}\ell_{ks} \right) \Big/ \ell_{kk} \\
\quad \textbf{end} \\
\textbf{end} \\
\textbf{output } (\ell_{ij})
\end{array}
$$

Theorem 2 guarantees that $\ell_{kk} > 0$. Observe that Equation (13) gives us for $j \leqq k$

$$a_{kk} = \sum_{s=1}^{k} \ell_{ks}^2 \geqq \ell_{kj}^2$$

from which we conclude that

$$|\ell_{kj}| \leqq \sqrt{a_{kk}} \qquad (1 \leqq j \leqq k)$$

Hence, any element of L is bounded by the square root of a corresponding diagonal element in A. This implies that the elements of L do not become large relative to A even without any pivoting. (Pivoting is explained in the next section.)

In both the Cholesky and Doolittle algorithms, the dot products of vectors should be carried out in double precision in order to avoid a buildup of roundoff errors. (See Problem 6 in Section 4.1.)

PROBLEM SET 4.2

1. Prove these facts, needed in the proof of Theorem 2.

   (a) If U is upper triangular and invertible, then U^{-1} is upper triangular.

   (b) The inverse of a unit lower triangular matrix is unit lower triangular.

   (c) The product of two upper (lower) triangular matrices is upper (lower) triangular.

2. Prove that if a nonsingular matrix A has an LU-factorization in which L is a unit lower triangular matrix, then L and U are unique.

3. Prove that algorithms (2), (3), (6), and (7) always solve $Ax = b$ if A is nonsingular.

4. Prove that an upper or lower triangular matrix is nonsingular if and only if its diagonal elements are all different from zero.

5. Show that if all the principal minors of A are nonsingular and $\ell_{ii} \neq 0$ for each i, then $u_{kk} \neq 0$ for $1 \leqq k \leqq n$.

6. Prove that the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ does not have an LU-factorization. *Caution*: This is *not* a simple consequence of the theorem proved in this section.

7. (a) Write the *row version* of the Doolittle algorithm that computes the kth row of L and the kth row of U at the kth step. (Consequently at the kth step, the order of computing is $\ell_{k1}, \ell_{k2}, \dots, \ell_{k,k-1}, u_{kk}, \dots, u_{kn}$.)

   (b) Write the *column version* of the Doolittle algorithm, which computes the kth column of U and the kth column of L at the kth step. (Consequently, the order of computing is $u_{1k}, u_{2k}, \dots, u_{kk}, \ell_{k+1,k}, \dots, \ell_{nk}$ at the kth step.)

8. By use of the equation $UU^{-1} = I$, obtain an algorithm for finding the inverse of an upper triangular matrix. Assume that U^{-1} exists; that is, the diagonal elements of U are all nonzero.

9. Count the number of arithmetic operations involved in the algorithms (2), (3), (6), and (7).

10. A matrix $A = (a_{ij})$ in which $a_{ij} = 0$ when $j > i$ or $j < i - 1$ is called a **Stieltjes matrix**. Devise an efficient algorithm for inverting such a matrix.

11. Let A be an $n \times n$ matrix. Let $(p_1, p_2, \dots, p_n)$ be a permutation of $(1, 2, \dots, n)$ such that (for $i = 1, 2, \dots, n$) row i in A contains nonzero elements only in columns $p_1, p_2, \dots, p_i$. Write an algorithm to solve $Ax = b$.

12. Show that every matrix of the form $A = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ has an LU-factorization. Show that even if L is *unit* lower triangular the factorization is not unique. (This problem, as well as Problems 13 and 15, illustrate Taussky's Maxim: If a conjecture about matrices is false, it can usually be disproved with a 2×2 example.)

13. Show that every matrix of the form $A = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ has an LU-factorization. Does it have an LU-factorization in which L is a *unit* lower triangular?

^{c}14. Devise an efficient algorithm for inverting an $n \times n$ lower triangular matrix A. *Suggestion*: Utilize the fact that A^{-1} is also lower triangular. Code your algorithm and test it on the matrix whose elements are $a_{ij} = (i+j)^2$ when $i \geq j$. Use $n = 10$. Form the product AA^{-1} as a test of the computed inverse.

15. Find all the LU-factorizations of $A = \begin{bmatrix} 1 & 5 \\ 3 & 15 \end{bmatrix}$ in which L is unit lower triangular.

16. If A is invertible and has an LU decomposition then all principal minors of A are nonsingular.

17. Let the system $Ax = b$ have the following property: There are two permutations of $(1, 2, \dots, n)$ called $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ such that, for each i, equation number p_i contains only the variables $x_{q_1}, x_{q_2}, \dots, x_{q_i}$. Write an efficient algorithm to solve this system.

18. Count the number of multiplications and/or divisions needed to invert a unit lower triangular matrix.

19. Prove or disprove: If A has an LU-factorization in which L is *unit* lower triangular, then it has an LU-factorization in which U is *unit* upper triangular.

20. Assuming that its LU-factorization is known, give an algorithm for inverting A. (Use Problems 8 and 14.)

21. Develop an algorithm for inverting a matrix A that has the property $a_{ij} = 0$ if $i + j \leq n$.

22. Use the Cholesky Theorem to prove that these two properties of a symmetric matrix A are equivalent: (a) A is positive definite: (b) there exists a linearly independent set of vectors $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ in $\mathbb{R}^n$ such that $A_{ij} = (x^{(i)})^T (x^{(j)})$.

23. Establish the correctness of the following algorithm for solving $Ux = b$ in the case that U is upper triangular:

$$
\begin{array}{l}
\textbf{for } j = n, n-1, \ldots, 1 \textbf{ do} \\
\quad x_j \leftarrow b_j / u_{jj} \\
\quad \textbf{for } i = 1, 2, \ldots, j-1 \textbf{ do} \\
\quad\quad b_i \leftarrow b_i - u_{ij} x_j \\
\quad \textbf{end} \\
\textbf{end}
\end{array}
$$

24. Prove that if all the leading principal minors of A are nonsingular, then A has a factorization LDU in which L is unit lower triangular, U is unit upper triangular, and D is diagonal.

25. (Continuation) If A is a symmetric matrix whose leading principal minors are nonsingular, then A has a factorization LDL^T in which L is unit lower triangular and D is diagonal.

26. (Continuation) Write an algorithm to compute the LDL^T-factorization of a symmetric matrix A. Your algorithm should do approximately half as much work as the standard Gaussian algorithm. *Note:* This algorithm can fail if some principal minors of A are singular. (This modification of the Cholesky algorithm does not involve square root calculations.)

27. Prove: A is positive definite and B is nonsingular if and only if BAB^T is positive definite.

28. If A is positive definite, does it follow that A^{-1} is also positive definite?

29. Consider

$$
A = \begin{bmatrix} 2 & 6 & -4 \\ 6 & 17 & -17 \\ -4 & -17 & -20 \end{bmatrix}
$$

Determine *directly* the factorization $A = LDL^T$ where D is diagonal and L is unit lower triangular—that is, do *not* use Gaussian elimination.

30. Develop an algorithm for finding directly the UL-factorization of a matrix A where L is *unit* lower triangular and U is upper triangular. Give an algorithm for solving $ULx = b$.

31. Find the LU-factorization of the matrix

$$
A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}
$$

in which L is lower triangular and U is unit upper triangular.

32. Factor the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ so that $A = LL^T$, where L is lower triangular.

33. Determine directly the LL^T-factorization, in which L is a lower triangular matrix with positive diagonal elements, for the matrix

$$A = \begin{bmatrix} 4 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{17}{16} & \frac{1}{4} \\ 1 & \frac{1}{4} & \frac{33}{64} \end{bmatrix}$$

34. Suppose that the nonsingular matrix A has a Cholesky factorization. What can be said about the determinant of A?

35. Determine the LU-factorization of the matrix $A = \begin{bmatrix} 1 & 5 \\ 3 & 16 \end{bmatrix}$ in which *both* L and U have unit diagonal elements.

c36. Solve this system by the Cholesky method:

$$\begin{cases} 0.05x_1 + 0.07x_2 + 0.06x_3 + 0.05x_4 = 0.23 \\ 0.07x_1 + 0.10x_2 + 0.08x_3 + 0.07x_4 = 0.32 \\ 0.06x_1 + 0.08x_2 + 0.10x_3 + 0.09x_4 = 0.33 \\ 0.05x_1 + 0.07x_2 + 0.09x_3 + 0.10x_4 = 0.31 \end{cases}$$

37. Consider the symmetric tridiagonal positive definite matrix

$$A = \begin{bmatrix} 136.01 & 90.860 & 0.0 & 0.0 \\ 90.860 & 98.810 & -67.590 & 0.0 \\ 0.0 & -67.590 & 132.01 & 46.260 \\ 0.0 & 0.0 & 46.260 & 177.17 \end{bmatrix}$$

Using five significant figures, factor A in the following ways.

(a) $A = LU$, where L is unit lower triangular and U is upper triangular.

(b) $A = LDU$, where L is unit lower triangular, D is diagonal, and U is unit upper triangular.

(c) $A = LU$, where L is lower triangular and U is unit upper triangular.

(d) $A = LL^T$, where L is lower triangular.

38. Determine the LU-factorization of the matrix

$$A = \begin{bmatrix} 6 & 10 & 0 \\ 12 & 26 & 4 \\ 0 & 9 & 12 \end{bmatrix}$$

in which L is a lower triangular matrix with twos on its main diagonal.

39. Prove that if a singular matrix has a Doolittle factorization, then that factorization is not unique.

40. Prove the uniqueness of the factorization $A = LL^T$, where L is lower triangular and has positive diagonal.

41. A matrix A that is symmetric and positive definite (SPD) has a square root X that is SPD. Thus $X^2 = A$. Find X if $A = \begin{bmatrix} 13 & 10 \\ 10 & 17 \end{bmatrix}$.

42. Develop algorithms for solving the linear system $Ax = b$ in two special cases: (a) $a_{ij} = 0$ when $j \leqq n - i$; (b) $a_{ij} = 0$ when $j > n + 1 - i$.

43. Using Equations (10), (11), and (12), find all the Doolittle factorizations of the matrix

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & 11 \end{bmatrix}$$

In this example, the algorithm works, although $u_{22} = 0$.

44. Prove that if A is symmetric, then in its LU-factorization the columns of L are multiples of the rows of U.

45. Find all UL-factorizations and all UL-factorizations in which L is *unit* lower triangular for the matrix $A = \begin{bmatrix} 1 & 5 \\ 3 & 17 \end{bmatrix}$.

46. Define a *P-matrix* to be one in which $a_{ij} = 0$ if $j \leqq n - i$ and a *Q-matrix* to be a P-matrix in which $a_{i,n-i+1} = 1$ for $i = 1, 2, \ldots, n$. Find the *PQ-factorization* of the matrix $A = \begin{bmatrix} 3 & 15 \\ -1 & -1 \end{bmatrix}$.

47. (Continuation) Devise an algorithm to obtain the PQ-factorization of a given matrix. Similarly, devise an algorithm for solving a system of equations of the form $PQx = b$.

48. Assuming that the LU-factorization of A is available, write an algorithm to solve the equation $x^T A = b^T$.

^{c}49. Write a subprogram or procedure that implements the general LU-factorization algorithm. The diagonal elements that are prescribed can be stored in an array D. An associated logical array can be used to indicate whether an element of D belongs to the diagonal of L or U. Test the routine on some Hilbert matrices, whose elements are $a_{ij} = (i + j - 1)^{-1}$. For each matrix, produce the Doolittle, Crout, and Cholesky factorizations plus one or more others with specified diagonal entries.

50. If A has a Doolittle factorization what is a simple formula for the determinant of A?

51. Let

$$A = \begin{bmatrix} 25 & 0 & 0 & 0 & 1 \\ 0 & 27 & 4 & 3 & 2 \\ 0 & 54 & 58 & 0 & 0 \\ 0 & 108 & 116 & 0 & 0 \\ 100 & 0 & 0 & 0 & 24 \end{bmatrix}$$

Determine the most general LU-factorization of A in which L is unit lower triangular. Show that the Doolittle algorithm produces one of these LU-factorizations.

52. Let A be a symmetric matrix whose leading principal minors are nonnegative. Does the matrix $A + \varepsilon I$ have the same properties for $\varepsilon > 0$?

53. Consider the LU-factorization of a 2×2 matrix A. Show that if ℓ_{22} and u_{22} are specified, then the equations that determine the remaining elements of L and U are nonlinear.

54. Prove: If A is symmetric and nonnegative definite, then $A = LL^T$ for some lower triangular matrix L. The terminology **nonnegative definite** means that $x^T A x \geqq 0$ for all x.

55. Find the precise conditions on a, b, c in order that the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ will be nonnegative definite.

56. Prove that if the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is nonnegative definite, then it has a factorization LL^T in which L is lower triangular.

57. Prove or disprove: A symmetric matrix is nonnegative definite if and only if all of its leading principal minors are nonnegative.

58. Find necessary and sufficient conditions on a, b, c in order that the matrix $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ has a factorization LL^T in which L is lower triangular.

59. In this problem, use the notation $X_{i:j,k}$ to denote the part of the kth column of the matrix X consisting of entries i to j. Similarly, let $X_{k,i:j}$ be the part of row k in X consisting of entries i to j.
(a) Refer to Equation (11) and show that it can be written as

$$U_{k,k+1:n} = (A_{k,k+1:n} - L_{k,1:k-1}M)/\ell_{kk}$$

in which M is the matrix whose rows are $U_{i,k+1:n}$, for $1 \leqq i \leqq k-1$.
(b) Carry out the analogous transformation of Equation (12). The computations discussed in this problem have the form $y \leftarrow y - Mx$. They can be carried out very efficiently on a vector supercomputer. (See Kincaid and Oppe [1988] and Oppe and Kincaid [1988] for further details.)

4.3 Pivoting and Constructing an Algorithm

In the previous section, an abstract version of Gaussian elimination was presented in the guise of the LU-factorization of a matrix. In this section, the traditional form of Gaussian elimination will be described and related to the abstract form. Then we shall take up the modifications of the process necessary to produce a satisfactory computer realization of it. Throughout this discussion, we shall use the words *equation* and *row* of a matrix system interchangeably.

Basic Gaussian Elimination

Here is a simple system of four equations in four unknowns that will be used to illustrate the Gaussian algorithm:

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 34 \\ 27 \\ -38 \end{bmatrix} \tag{1}$$

In the first step of the process, we subtract 2 times the first equation from the second. Then we subtract $\frac{1}{2}$ times the first equation from the third. Finally, we subtract -1 times the first equation from the fourth. The numbers 2, $\frac{1}{2}$, and -1 are called the **multipliers** for the first step in the elimination process. The number 6 used as the divisor in forming each of these multipliers is called the **pivot element** for this step.

and all the rest are similar. Here is the algorithm:

input $n, (a_i), (b_i), (c_i), (d_i)$
for $i = 2, 3, \ldots, n$ **do**
  $d_i \leftarrow d_i - (a_{i-1}/d_{i-1})c_{i-1}$
  $b_i \leftarrow b_i - (a_{i-1}/d_{i-1})b_{i-1}$
end
$x_n \leftarrow b_n/d_n$
for $i = n-1, n-2, \ldots, 1$ **do**
  $x_i \leftarrow (b_i - c_i x_{i+1})/d_i$
end
output (x_i)

PROBLEM SET 4.3

1. Solve the following linear systems twice. First, use Gaussian elimination and give the factorization $A = LU$. Second, use Gaussian elimination with scaled row pivoting and determine the factorization of the form $PA = LU$.

(a)

$$\begin{bmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 6 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} -1 & 1 & 0 & -3 \\ 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 3 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 4 & 10 \\ 3 & -13 & 3 & 3 \\ -6 & 4 & 2 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ -39 \\ -16 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & -9 & 2 & 1 \\ 8 & 16 & 6 & 5 \\ 2 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 14 \\ -3 \\ 0 \end{bmatrix}$$

2. Show that Equation (8) defining the Gaussian elimination algorithm can also be written in the form

$$a_{ij}^{(k+1)} = \begin{cases} a_{ij}^{(k)} & \text{if } i \leqq k \text{ or } j < k \\ a_{ij}^{(k)} - \left(a_{ik}^{(k)}/a_{kk}^{(k)}\right) a_{kj}^{(k)} & \text{if } i > k \text{ and } j \geqq k \end{cases}$$

3. Let $(p_1, p_2, \ldots, p_n)$ be a permutation of $(1, 2, \ldots, n)$ and define the matrix P by $P_{ij} = \delta_{p_i j}$. Let A be an arbitrary $n \times n$ matrix. Describe PA, AP, P^{-1}, and PAP^{-1}.

^{c}4. Gaussian elimination with *full* pivoting treats both rows and columns in an order different from the natural order. Thus, in the first step, the pivot element a_{ij} is chosen so that $|a_{ij}|$ is the largest in the entire matrix. This determines that row i will be the pivot row and column j will be the pivot column. Zeros are created in column j by subtracting multiples of row i from the other rows. Write the algorithm to carry out this process. Two permutation vectors will be required.

5. Let A be an $n \times n$ matrix with scale factors $s_i = \max_{1 \leqq j \leqq n} |a_{ij}|$. Assume that all s_i are positive, and let B be the matrix whose elements are (a_{ij}/s_i). Prove that if forward elimination is applied to A and to B, then the two final L-arrays are the same. Find the formulæthat relate the final A and B matrices (after processing).

6. It is sometimes advisable to modify a system of equations $Ax = b$ by introducing new variables $y_i = d_i x_i$, where d_i are positive numbers. If the x_i correspond to physical quantities, then this change of variables corresponds to a change in the units by which x_i is measured. Thus, if we decide to change x_1 from centimeters to meters, then $y_1 = 10^{-2} x_1$. In matrix terms, we define a diagonal matrix D with d_i as diagonal entries, and put $y = Dx$. The new system of equations is $AD^{-1}y = b$. If d_j is chosen as $\max_{1 \leqq i \leqq n} |a_{ij}|$, we call this **column equilibration**. Modify the factorization and solution algorithms to incorporate column equilibration. (The two algorithms together still will solve $Ax = b$.)

7. Show that the multipliers in the Gaussian algorithm with *full* pivoting (both row and column pivoting) lie in the interval $[-1, 1]$. (See Problem 4.)

8. Let the $n \times n$ matrix A be processed by forward elimination, with the resulting matrix called B, and permutation vector $p = (p_1, p_2, \ldots, p_n)$. Let P be the matrix that results from the identity matrix by writing its rows in the order $p_1, p_2, \ldots, p_n$. Prove that the LU-decomposition of PA is obtained as follows: Put $C = PB, L_{ij} = C_{ij}$ for $j < i$, and $U_{ij} = C_{ij}$ for $i \leqq j$. (Of course, $U_{ij} = 0$ if $i > j, L_{ij} = 0$ if $j > i$, and $L_{ii} = 1$.)

9. If the factor U in the LU-decomposition of A is known, what is the algorithm for calculating L?

10. Show how Gaussian elimination with scaled row pivoting works on this example (forward phase only).

$$\begin{bmatrix} 2 & -2 & -4 \\ 1 & 1 & -1 \\ 3 & 7 & 5 \end{bmatrix}$$

Display the scale array (s_1, s_2, s_3) and the *final* permutation array (p_1, p_2, p_3). Show the *final* A-array, with the multipliers stored in the correct locations.

11. Carry out the instructions in Problem 10 on the matrix

$$\begin{bmatrix} 3 & 7 & 3 \\ 1 & \frac{7}{3} & 4 \\ 4 & \frac{4}{3} & 0 \end{bmatrix}$$

12. Assume that A is tridiagonal. Define $c_0 = 0$ and $a_n = 0$. Show that if A is *columnwise* diagonally dominant

$$|d_i| > |a_i| + |c_{i-1}| \qquad (1 \leqq i \leqq n)$$

then the algorithm for tridiagonal systems will, in theory, be successful since no zero pivot entries will be encountered. Refer to Equation (19) for the notation.

13. Write a special Gaussian elimination algorithm to solve linear equations when A has the property $a_{ij} = 0$ for $i > j+1$. Do not use pivoting. Include the processing of the right-hand side in the algorithm. Count the operations needed to solve $Ax = b$.

14. Count the operations in the algorithm in the text for tridiagonal systems.

15. Rewrite the algorithm for tridiagonal systems so that the order of processing the equations and variables is reversed.

16. Prove the theorem concerning the number of long operations in Gaussian elimination.

17. Show how Gaussian elimination with scaled row pivoting works on this example:

$$A = \begin{bmatrix} -9 & 1 & 17 \\ 3 & 2 & -1 \\ 6 & 8 & 1 \end{bmatrix}$$

Show the scale array. The final A-array should contain the multipliers stored in the correct positions. Determine P, L, and U, and verify that $PA = LU$.

18. Show how the factorization phase of Gaussian elimination with scaled row pivoting works on the matrix

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 2 \\ 3 & -5 & -1 \end{bmatrix}$$

Show all intermediate steps—that is, multipliers, scale array s, and index array p—and the final array A as it would appear after the algorithm had finished working on it.

19. This problem shows how the solution to a system of equations can be *unstable* relative to perturbations in the data. Solve $Ax = b$ with $b = (100,\ 1)^T$, and with each of the following matrices. (See Stoer and Bulirsch [1980, p. 185].)

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0.01 \end{bmatrix}$$

20. Assume that $0 < \varepsilon < 2^{-22}$. If the Gaussian algorithm without pivoting is used to solve the system

$$\begin{cases} \varepsilon x_1 + 2x_2 = 4 \\ x_1 - x_2 = -1 \end{cases}$$

on the `MARC-32`, what will be the solution vector (x_1, x_2)?

21. Solve the following system by Gaussian elimination with full pivoting (as described in Problem 4):

$$\begin{bmatrix} -9 & 1 & 17 \\ 3 & 2 & -1 \\ 6 & 8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ -3 \end{bmatrix}$$

22. Solve the system

$$\begin{cases} 0.2641x_1 + 0.1735x_2 + 0.8642x_3 = -0.7521 \\ 0.9411x_1 + 0.0175x_2 + 0.1463x_3 = 0.6310 \\ -0.8641x_1 - 0.4243x_2 + 0.0711x_3 = 0.2501 \end{cases}$$

using Gaussian elimination with: (a) no pivoting, and (b) scaled row pivoting.

23. Write an algorithm to solve the system $Ax = b$ under the following conditions: There is a permutation $(p_1, p_2, \ldots, p_n)$ of $(1, 2, \ldots, n)$ such that for each i, equation p_i contains only the variable x_i.

24. Repeat the preceding problem assuming that for each i, equation i contains only the variable x_{p_i}.

25. Repeat the preceding problem assuming that for each i, equation p_i contains only the variable x_{p_i}. In this case, give the *simplest* algorithm.

26. (a) Show that if we apply Gaussian elimination without pivoting to a symmetric matrix A, then $\ell_{i1} = a_{1i}/a_{11}$.
(b) From this, show that if the first row and column of $A^{(2)}$ are removed, the remaining $(n-1) \times (n-1)$ matrix is symmetric. Conclude then that elements below the diagonal in this smaller matrix need not be computed. Use induction to infer that this simplification will occur in each succeeding step of the factorization phase.
(c) Show that the computation required is almost halved compared to the nonsymmetric case.
(d) Use this simplification to solve the system

$$\begin{cases} 0.6428x_1 + 0.3475x_2 - 0.8468x_3 = 0.4127 \\ 0.3475x_1 + 1.8423x_2 + 0.4759x_3 = 1.7321 \\ -0.8468x_1 + 0.4759x_2 + 1.2147x_3 = -0.8621 \end{cases}$$

27. Consider the matrix

$$\begin{bmatrix} 0 & 4 & 25 & 79 \\ 9 & 7 & 39 & 89 \\ 0 & 16 & 2 & 99 \\ 0 & 6 & 6 & 49 \end{bmatrix}$$

Circle the entry that will be used as the next pivot element in Gaussian elimination with scaled row pivoting. The scale array is $s = (80, 89, 160, 30)$.

28. Show the resulting matrix after the forward phase of Gaussian elimination with scaled row pivoting is applied to the matrix

$$\begin{bmatrix} 2 & -2 & -4 \\ 1 & 1 & -1 \\ 3 & 7 & 5 \end{bmatrix}$$

In the final matrix, write the multipliers in the appropriate locations.

29. Determine $\det(A)$ where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

without computing the determinant by expansion by minors.

30. Use Gaussian elimination with scaled row pivoting to find the determinant of

$$A = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix}$$

31. Consider the system

$$\begin{cases} x_2 + 2x_3 = 1 \\ 2x_1 - x_2 = 2 \\ 2x_2 + x_3 = 3 \end{cases}$$

Determine the factorization $PA = LU$ where P is a permutation matrix. Use this factorization to obtain $\det(A)$.

32. Consider

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 6 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

Use Gaussian elimination with scaled row pivoting to obtain the factorization

$$PA = LDU$$

where L is a unit lower triangular matrix, U is a unit upper triangular matrix, D is a diagonal matrix, and P is a permutation matrix.

33. In the next few problems, we fix n and denote by J the set $\{1, 2, \ldots, n\}$. A **permutation** of J is a map $p : J \twoheadrightarrow J$, where the double arrow indicates that p is *surjective*. Thus each element of J is the image, $p(i)$, of some element i in J. The **identity** permutation is defined by $u(i) = i$ for all $i \in J$. Show that if p and q are permutations of J, then so is $p \circ q$, which is defined as usual by the equation $(p \circ q)(i) = p(q(i))$. Prove that $p \circ (q \circ r) = (p \circ q) \circ r$ and that $p \circ u = u \circ p = p$.

34. (Continuation) Prove that each permutation p has an inverse p^{-1}, which satisfies $p \circ p^{-1} = u = p^{-1} \circ p$. The set of all permutations of J is called the **symmetric group** on J.

35. (Continuation) Give an algorithm for determining the inverse of any given permutation. (A permutation of J can be represented as a vector $(p(1), p(2), \ldots, p(n))$ in a computer.)

36. Let a system of equations, $Ax = b$, be given, in which A is $n \times n$. Let p and q be permutations of $\{1, 2, \ldots, n\}$. Write an algorithm for solving the system under the assumption that for $i = 1, 2, \ldots, n$, the equation numbered p_i contains only the variable x_{q_i}.

37. (Continuation) Repeat the preceding problem under the assumption that for each i, the variables $x_{q_1}, x_{q_2}, \ldots, x_{q_{i-1}}$ do not appear in the equation numbered p_i.
38. (Continuation) Repeat Problem 36 under the assumption that for each i, the variable x_{q_i} occurs only in the equations numbered $p_1, p_2, \ldots, p_i$.
39. (Difficult) Find an algorithm to solve $Ax = b$ under the assumption that all elements a_{ij} are zero unless $|i - j| \leqq 1$ or $(i,j) = (1,n)$ or $(i,j) = (n,1)$. Use Gaussian elimination without pivoting.
40. Count the number of long operations involved in the LU-factorization of an $n \times n$ matrix, assuming that no pivoting is employed.
41. Let A be an $n \times n$ matrix that is diagonally dominant in its columns. Thus

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} |a_{ij}| < |a_{jj}| \qquad (1 \leqq j \leqq n)$$

Determine whether Gaussian elimination without pivoting preserves this diagonal dominance.

42. In a diagonally dominant matrix A, define the *excess in row* i by the equation

$$e_i = |a_{ii}| - \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

Show that in the proof of Theorem 5 the following is true:

$$|a_{ii} - a_{i1}a_{1i}/a_{11}| \geqq \sum_{\substack{j=2 \\ j \neq i}}^{n} |a_{ij} - a_{i1}a_{1j}/a_{11}| + e_i$$

Thus, the excess in row i is not diminished in Gaussian elimination.

[c]43. Refer to Problem 26, and write a program to carry out the factorization phase of Gaussian elimination on a symmetric matrix. Assume that pivoting is not required.
44. Refer to Problems 33–35 if necessary. Let p be a permutation of $\{1, 2, \ldots, n\}$, and let P be the corresponding permutation matrix. (Thus $P_{ij} = \delta_{p_i j}$.) Let q be the inverse of p and Q the permutation matrix corresponding to q. Prove that $P^{-1} = Q$.
45. (Continuation) Prove that if P is a permutation matrix, then $P^{-1} = P^T$.
46. If A is $n \times n$ and B is $n \times m$, how many multiplications and divisions are required to solve $AX = B$ by Gaussian elimination with scaled row pivoting? What if $B = I$?
47. Prove or disprove: If A is tridiagonal and P is a permutation matrix, then PAP^{-1} is tridiagonal.
48. Suppose that the scale array is recomputed in each major step of Gaussian elimination with scaled row pivoting. Prove that for a symmetric and diagonally dominant matrix, Gaussian elimination without pivoting is the same as with scaled row pivoting.
49. Prove Theorem 3.

^{c}50. Write and test programs to carry out Gaussian elimination with scaled row pivoting. Suitable test cases are in Problems 19, 21, 22, and 26.

51. In the scaled row pivoting algorithm for Gaussian elimination, suppose that the scale numbers s_i are redefined by

$$s_i = |a_{i1}| + |a_{i2}| + \cdots + |a_{in}|$$

Prove that if the resulting algorithm is applied to a diagonally dominant matrix then the natural pivot order $(1, 2, \ldots, n)$ will be chosen. Write and test programs to carry out the ideas in Problem 26. This is Gaussian elimination without pivoting on a symmetric system.

^{c}52. Write and test programs that include column equilibration (Problem 6) in the Gaussian algorithms.

^{c}53. Write and test programs to solve $Ax = b$ and $y^T A = c^T$ using only one factorization of A (with scaled row pivoting) and two other subprograms to solve for x and y.

^{c}54. Write and test programs to solve $Ax = b$ using column equilibration, row equilibration, and full pivoting. (Refer to Problems 4 and 6 for the terminology.)

^{c}55. Write a subroutine GAUSSJ (N, A, B, X, P, S, D) that solves an $n \times n$ system $Ax = b$ by the **Gauss-Jordan** method, with column equilibration and scaled row pivoting. In the Gauss-Jordan algorithm (without pivoting) at the kth major step, multiples of row k are subtracted from *all* the other rows so that the coefficient of x_k is 0 in all rows except the kth row. At the end, the matrix will be a diagonal matrix (rather than an upper triangular matrix, as in the Gaussian elimination method). With scaled row pivoting, row p_k is used as pivot row to produce 0 coefficients of x_k in all rows except row p_k. Column equilibration, as discussed in Problem 6, should be carried out at the beginning. The divisors needed in this process should be stored in array D, since they will be needed at the end in obtaining x.

56. Prove or disprove the natural conjecture that if a matrix has the property

$$0 \neq |a_{ii}| \geqq \sum_{\substack{j=1 \\ j \neq i}} |a_{ij}| \qquad (1 \leqq i \leqq n)$$

then the Gaussian elimination without pivoting will preserve this property.

57. (a) Prove that computing the determinant of a matrix by expansion by minors involves $(n-1)(n!)$ ops.

(b) Prove that Cramer's rule requires $(n^2 - 1)(n!)$ ops.

(c) Prove that the Gauss-Jordan method involves $\frac{1}{2}n(n+1)^2 \approx \frac{1}{2}n^3$ ops and therefore is 50 percent more expensive than Gaussian elimination.

4.4 Norms and the Analysis of Errors

To discuss the errors in numerical problems involving vectors, it is useful to employ *norms*. Our vectors are usually in one of the spaces $\mathbb{R}^n$, but a norm can be defined on any vector space.

31. Find the explicit form for the iteration matrix $I - Q^{-1}A$ in the Gauss-Seidel method when

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

32. Characterize the family of all $n \times n$ nonsingular matrices A for which *one step* of the Gauss-Seidel algorithm solves $Ax = b$, starting at the vector $x = 0$.

33. Give an example of a matrix A that is not diagonally dominant, yet the Gauss-Seidel method applied to $Ax = b$ converges.

34. How does the Chebyshev acceleration method simplify if the basic method is the Jacobi method?

35. Prove that if the number $\delta = \|I - Q^{-1}A\|$ is less than 1, then

$$\|x^{(k)} - x\| \leqq \frac{\delta}{1-\delta}\|x^{(k)} - x^{(k-1)}\|$$

36. Prove that

$$T_n(t) = \tfrac{1}{2}(b^n + b^{-n}) \qquad b = t + \sqrt{t^2 - 1}$$

37. Prove Theorem 9 in the case $a > 1$.

38. Prove that a positive definite matrix A is Hermitian using the definition; namely, A is positive definite if $x^*Ax > 0$ for all nonzero $x \in \mathbb{C}^n$. Thus, in particular, a real positive definite matrix A must be symmetric if this definition is used. However, if the alternative definition $x^TAx > 0$ for all nonzero $x \in \mathbb{R}^n$ is used, then a real positive definite matrix need not be symmetric.

*4.7 Steepest Descent and Conjugate Gradient Methods

In this section, some special methods will be developed for solving the system

$$Ax = b$$

for the case when A is a real $n \times n$ *symmetric* and *positive definite* matrix. These hypotheses mean that

$$A^T = A$$

and

$$x^TAx > 0 \quad \text{for} \quad x \neq 0$$

Throughout we use the inner-product notation for real vectors x and y

$$\langle x, y\rangle = x^Ty = \sum_{i=1}^{n} x_iy_i$$

Summary

We summarize these results in the following table.

Method	**Equation**	**Computes**
power	$x^{(k+1)} = Ax^{(k)}$	largest eigenvalue λ_1
inverse power	$Ax^{(k+1)} = x^{(k)}$	smallest eigenvalue λ_n
shifted power	$x^{(k+1)} = (A - \mu I)x^{(k)}$	eigenvalue farthest from μ
shifted inverse power	$(A - \mu I)x^{(k+1)} = x^{(k)}$	eigenvalue closest to μ

PROBLEM SET 5.1

1. Let A be an $n \times n$ matrix that has a linearly independent set of n eigenvectors, $\{u^{(1)}, u^{(2)}, \ldots, u^{(n)}\}$. Let $Au^{(i)} = \lambda_i u^{(i)}$, and let P be the matrix whose columns are the vectors $u^{(1)}, u^{(2)}, \ldots, u^{(n)}$. What is $P^{-1}AP$?
2. Show that if the normalized and unnormalized versions of the power method are started at the same initial vector, then the values of r in the two algorithms will be the same.
3. In the power method, let $r_k = \phi(x^{(k+1)})/\phi(x^{(k)})$. We know that $\lim_{k\to\infty} r_k = \lambda_1$. Show that the relative errors obey

$$\frac{r_k - \lambda_1}{\lambda_1} = \left(\frac{\lambda_2}{\lambda_1}\right)^k c_k$$

where the numbers c_k form a convergent (and hence bounded) sequence.

4. (Continuation) Show that $r_{k+1} - \lambda_1 = (c + \delta_k)(r_k - \lambda_1)$ where $|c| < 1$ and $\lim_{n\to\infty} \delta_k = 0$, so that Aitken acceleration is applicable.
5. Prove that in Aitken acceleration, $(s_n - r)/(r_{n+2} - r) \to 0$ as $n \to \infty$, provided that $c \neq 0$.
6. Count the number of multiplications and/or divisions involved in carrying out m steps of the basic (unnormalized) power method.
7. In the normalized power method, show that if $\lambda_1 > 0$ then the vectors $x^{(k)}$ converge to an eigenvector.
8. Devise a simple modification of the power method to handle the following case: $\lambda_1 = -\lambda_2 > |\lambda_3| \geqq |\lambda_4| \geqq \cdots \geqq |\lambda_n|$.
9. What can you prove about Aitken acceleration if the sequence $[r_n]$ satisfies only the hypothesis $|r_{n+1} - r| \leqq c|r_n - r|$ with $0 < c < 0.2$?
10. Let the eigenvalues of A satisfy $\lambda_1 > \lambda_2 \cdots > \lambda_n$ (all real, but not necessarily positive). What value of the parameter β should be used in order for the power method to converge most rapidly to λ_1 when applied to $A + \beta I$?
11. Prove that $I - AB$ has the same eigenvalues as $I - BA$, if either A or B is nonsingular.
12. If the power method is applied to a real matrix with a real starting vector, what will happen if a dominant eigenvalue is complex? Does the theory outlined in the text apply?

^{c}13. Employ the power method using the matrix in Example 1 with starting vector $(1, 2, 3)^T$. Take 100 steps and explain the apparent convergence at an early stage that is followed by later convergence to a different value.

14. What is the characteristic polynomial of this matrix?

$$\begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

15. Prove that for any complex number λ

$$\dim\{x : Ax = \lambda x\} = n - \operatorname{rank}(A - \lambda I)$$

16. Let $A = LU$ where L is unit lower triangular and U is upper triangular. Put $B = UL$ and show that B and A have the same eigenvalues.

^{c}17. Write a subroutine or procedure to apply M steps of the normalized power method on a given $n \times n$ matrix A, starting with a given vector x. Incorporate Aitken acceleration. In each step, print the current vector $x^{(k)}$, the current ratio r_k, and the current *accelerated* value s_k as defined in the text. Test the procedure on $A - \mu I$, where

(i)

$$A = \begin{bmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad \mu = 0, 3, 6, 11$$

(ii)

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 7 & -1 & 3 \\ 1 & -1 & 5 \end{bmatrix} \quad \text{and} \quad \mu = 0, 5, 10$$

18. Suppose that A has a row, say row k, such that $\sum_{j=1,\, j\neq k}^{n} |a_{kj}| = 0$. Let B denote the matrix obtained by removing row k and column k from A. Show that a_{kk} is an eigenvalue of A and that the remaining eigenvalues of A are eigenvalues of B.

19. An $n \times n$ matrix is said to be **defective** if its eigenvectors do not span $\mathbb{R}^n$. Show that the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is defective.

20. Prove that if an $n \times n$ matrix has n distinct eigenvalues, then it is not defective.

21. Prove that the converse of the theorem in Problem 20 is not true.

22. In some experiments to compute eigenvalues of real matrices, it was observed that real eigenvalues occurred frequently. Prove that a real $n \times n$ matrix must have at least one real eigenvalue if n is odd.

23. (Continuation) A real polynomial can be factored into real quadratic and linear factors. The quadratic factors may have complex roots. Compute the probability that a real quadratic, $x^2 + ax + b$, will have real roots, assuming that the coefficients a and b are uniformly distributed random variables chosen from the square $|a| \leqq \rho$, $|b| \leqq \rho$. Show that this probability converges to 0 as $\rho \to \infty$, and converges to $1/2$ as $\rho \to 1$. What does this suggest about the eigenvalues of real matrices?

24. To compute $x^{(k)} = A^k x$ for high values of k, one can perform repeated squarings of matrices:

$$A \to A^2 \to A^4 \to A^8 \to \cdots \to A^{2^m}$$

Then a single matrix-vector product produces $x^{(2^m)} = A^{2^m} x$. Compare this procedure to the ordinary power method. Count the number of multiplications required by each method in arriving at $x^{(2^m)}$. Prove that for large m, the squaring method is always more economical. Devise a means of avoiding overflow by scaling the successive powers of A.

^{c}25. Write a computer program for the inverse power method and test it on some matrices of your own choosing.

^{c}26. (a) Write a computer program to reproduce the results given in Example 1 on your computer system. Modularize your program into a number of subroutines or procedures to compute each major segment of the algorithm. (For example, you may wish to construct routines for (i) matrix times a vector, (ii) dot product, (iii) replace one vector by another, (iv) norm of a vector, (v) normalize a vector, and so on.)

(b) Add Aitken acceleration to your code and rerun it. Write up your conclusions from this project.

^{c}27. (a) Repeat 26 (a) for Example 2.

(b) Repeat 26 (b) for Example 2.

^{c}28. Write and test a computer program to compute the eigenvalue farthest from a given complex number. Test the routine on the example matrix in this section.

29. Determine an approximate value for the spectral radius, $\rho(A)$, of the following matrix by taking two iterations of the power method using the infinity norm for ϕ. Use the initial vector $(1, 1, 1)^T$

$$A = \begin{bmatrix} 2 & 0 & -1 \\ -2 & -10 & 0 \\ -1 & -1 & 4 \end{bmatrix}$$

Prove that when ϕ is a norm, the ratios r_k in the power method converge to $|\lambda_1|$.

30. Show that a preferable form of the Aitken acceleration for computation is

$$s_k = r_k - \frac{(\Delta r_k)^2}{\Delta^2 r_k}$$

where

$$\Delta r_k = r_{k+1} - r_k \qquad \Delta^2 r_k = \Delta r_{k+1} - \Delta r_k$$

These are *forward differences*, which are discussed in Chapter 6.

^{c}31. Construct an example to show that Aitken acceleration will produce meaningless results if it is not stopped at an appropriate stage.

Conjunto de problemas 6

Ejercicio 1

Dado el sistema de ecuaciones lineales:

$$2x_1 - 6\alpha x_2 = 3$$
$$3\alpha x_1 - x_2 = 3/2$$

a. Obtenga el valor o los valores de α para el(los) cual(es) el sistema no tiene solución.

b. Obtenga el valor o los valores de α para el(los) cual(es) el sistema tiene infinitas soluciones.

c. Suponiendo que hay una solución única para determinar α, obtenga la solución.

Ejercicio 2

Cuente el número de operaciones requeridas durante el proceso de eliminación Gaussiana aplicado a la siguiente matriz pentadiagonal

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$