Mathematical Analysis

Lecture 1:Real numbers

Outline

-The set of real numbers . $\mathbb{R}$

-Review of subsets of the real number system
-Subsets.
-Properties of real numbers.
-Absolute value and properties.

Objectives (real numbers 2 sessions)

- Identify the Subsets of the Set of Real Numbers
- Recognize subsets of the real numbers
- Use Inequality symbols
- Evaluate absolute value
- Use absolute value to express distance
- Identify properties of the real numbers
- Simplify algebraic expressions

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rev

Notation

- Set - A collection of objects
- { } - the set of….
- Examples of set:

  A = { 2, 4, 6, 8}

  B = {x| x is an odd number}

rev

Examples of Real Numbers

- 9 9.0000
- 1/8 0.125
- -2/3 - 0.6
- $\sqrt{2}$ 1.14213562...

rev

Reviewing subsets of $\mathbb{R}$

Natural numbers $\mathbb{N}$

- counting numbers
- positive integers
- {1, 2, 3, 4, ….}

Whole numbers

- nonnegative integers
- { 0 } $\cup$ {1, 2, 3, 4, ….}
- {0, 1, 2, 3, 4, …}

2 + x = 1 hasn't solution in N

rev

Reviewing subsets of $\mathbb{R}$

Integers $\mathbb{Z}$

- numbers that consist of positive integers, negative integers, and zero,

$$\mathbb{Z} = \{\ldots -n, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, n, \ldots\} = (-\mathbb{N}) \cup \{0\} \cup \mathbb{N}$$

Reviewing subsets of $\mathbb{R}$

Rational Numbers $\mathbb{Q}$

- Numbers that can be expressed as a quotient a/b, where *a* and *b* are integers.
- Include fractions and decimals.
- Fractions when turned into decimals

  Decimals either terminate or repeat

Examples:

- 1/2 = .5
- 1/3 = .3
- 4/11 = .363636…
- 7 = 7/1

Reviewing subsets of $\mathbb{R}$

Irrational Numbers $\mathbb{I}$

- Decimals that do not end or repeat
- Transcendent numbers (π, e)
- Examples:
  - $\sqrt{2}$ = 1.414213562...
  - π = 3.141592654…
  - .01011011101111…
  - And all square roots of non-perfect squares

Reviewing subsets of $\mathbb{R}$

Irrational Numbers

Don't worry about irrational numbers.
The combination of the rational numbers along with the irrational numbers make up the real number system.

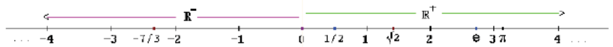
Real Numbers $\mathbb{R}$

Real numbers $\mathbb{R}$

The set of all rational and irrational numbers.

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$$

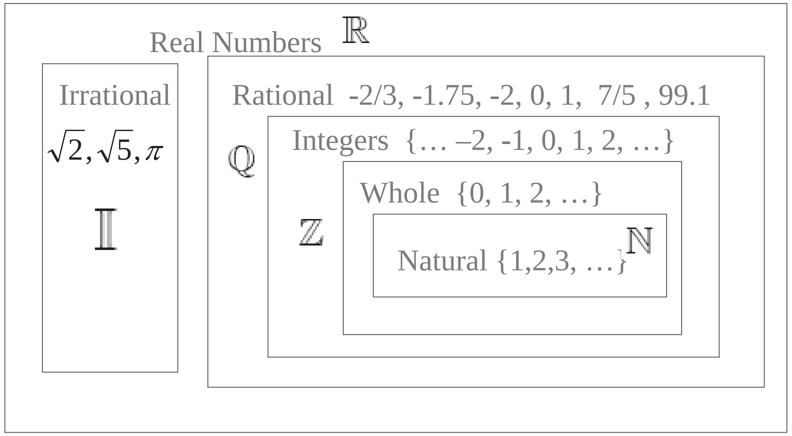
Real numbers are identified with the real line (there is a one-to-one correspondence)



$$\mathbb{R} = \mathbb{R}^- \cup \{0\} \cup \mathbb{R}^+$$

Nonnegative, zero and positive numbers

Structure of Real Number System



What are the Real Numbers?

- **Some common definitions**
  - Extension of the rational numbers to include the irrational numbers
  - Converging sequence of rational numbers, the limit of which is a real number
  - A point on the number line
- **Microscope analogy: If you magnify the number line at a very high power,**
  - Would the real numbers look the same?
  - Would the rational numbers look the same or be a row of dots separated by spaces?

Properties of real Numbers

The properties of the real number system fall into three categories: algebraic properties, order properties and completeness property

- **Algebraic** properties say how can be added, subtracted and multiplied real numbers
- **Order** properties of real numbers give useful rules to use with inequalities
- **Completeness** property of real number, it says that there are no holes or gaps in it.

Algebraic properties

- Commutative (addition & multiplication)
- Associative (addition & multiplication)
- Identity (additive = 0 & multiplicative = 1)
- Inverse (additive = -x & multiplicative = 1/x) except zero
- Distributive (multiplication over addition)

- ALL these properties are useful when manipulating algebraic expressions & equations rev

Real numbers are ordered

$(\mathbb{R}, \leq)$ is a total order compatible with + and ×

- $x \leq y \leftrightarrow y\text{-}x \in \mathbb{R}^{+} \cup \{0\}$ (≤ reflexive,antisimetric and transitive)
- If $x, y \in \mathbb{R}$ then $x < y \vee x = y \vee x > y$
- If $x, y \in \mathbb{R}^{+}$ then $x+y \ \mathbb{R}^{+} \wedge x \bullet y \in \mathbb{R}^{+}$

The extended real line is defined as:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$$

We extend the real number system by adjoining two "ideal points" denoted by the symbols +∞,-∞ ("plus infinity" and "minus infinity").

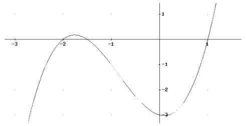
∧ AND
∨ OR

Inequalities

$$a \leq b \ \Rightarrow \ a+c \leq b+c$$

$$a \leq b \ \Rightarrow \begin{cases} a \cdot c \leq b \cdot c & \text{si } c > 0 \\ a \cdot c \geq b \cdot c & \text{si } c < 0 \end{cases}$$

$4x+13 \leq 2x+7$	$4x+13 \leq 6x+7$	$2x^3+5x^2-x-6 \leq 0$
$4x-2x \leq 7-13$	$4x-6x \leq 7-13$	$2(x-1)(x+2)\left(x+\frac{3}{2}\right) \leq 0$
$2x \leq -6$	$-2x \leq -6$	$x \in \left]-\infty,-2\right] \cup \left[\frac{-3}{2},1\right]$
$x \leq \frac{-6}{2} = -3$	$x \geq \frac{-6}{-2} = 3$	
$x \in \left]-\infty,-3\right]$	$x \in \left[3,+\infty\right[$	



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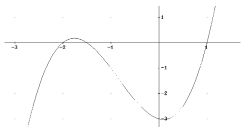

Absolute Value

- Formal definition

$$|x| = |-x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- Properties

$$|x| \geq 0$$

$$|x| \leq a \ (>0) \ \Leftrightarrow \ -a \leq x \leq a \ \Leftrightarrow \ -a \leq x \ \wedge \ x \leq a \ \Leftrightarrow \ x \in [-a,a]$$

$$|x| \geq b \ (>0) \ \Leftrightarrow \ b \leq x \ \vee \ x \leq -b \ \Leftrightarrow \ x \in \left]-\infty,-b\right] \cup \left[b,+\infty\right[$$

$$|x \cdot y| = |x| \cdot |y|$$

$$|x+y| \leq |x|+|y| \quad \text{(Minkowski inequality)}$$

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$$|x \cdot y| = |x| \cdot |y|$$

$|x + y| \leq |x| + |y|$ (Minkowski inequality)

Exercise: Find $x \in \mathbb{R}$ such as $||x| - 2| \leq 1$

Absolute Value: distance

- If *x* and *y* are real number then the distance between both is *d(x,y)=|x-y|*
- Open interval with center in "a" and radius "r"

$$I = \{x \in \mathbb{R} \, / \, d(x,a) < \delta\} =]a-\delta, a+\delta[$$

Exercise: Find $x \in \mathbb{R}$ such as $||x| - 2| \leq 1$

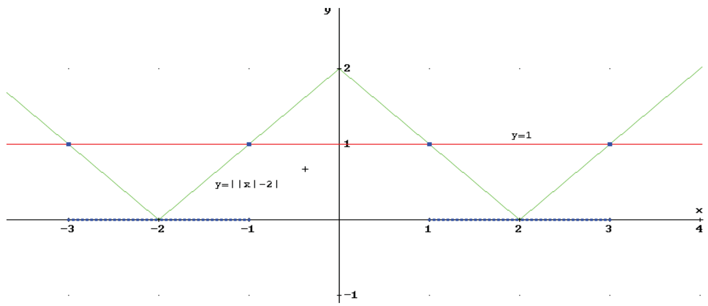
- by second property of absolute value

$$\boxed{||x| - 2| \leq 1 \ \Leftrightarrow \ 1 \leq |x| \leq 3 \ \Leftrightarrow \ |x| \leq 3 \ \wedge \ |x| \geq 1}$$

$$|x| \leq 3 \ \Leftrightarrow \ x \in [-3, 3]$$

- third property of absolute value $|x| \geq 1 \ \Leftrightarrow \ x \in]-\infty, -1] \cup [1, +\infty[$

$$[-3,3] \cap (]-\infty,-1] \cup [1,+\infty[) = [-3,-1] \cup [1,3]$$



Completeness property

The completeness property says that "there are enogh real number to complete the real number line, in the sense that there are no holes or gaps in it"

- This property is essential to the idea of a limit
- Many theorems of calculus would fail it the real number systems were not complete

Real

Don't leave common sense at the door!

- Remember to use logic!
- Can an absolute value ever be less than or equal to a negative value?? NO! (therefore if such an inequality were presented, the solution would be the empty set)
- Can an absolute value ever be more than or equal to a negative value?? YES! ALWAYS! (therefore if such an inequality were given, the solution would be all reals)