Mathematical Analysis

Real valued functions

Outline

- Definitions & terminology
  - function, domain, image, range, image of a set, strictly increasing, strictly decreasing, monotonic
- Properties
  - One-to-one (injective). Inverse
  - Increasing and decreasing functions
  - Even and odd functions
- Elementary functions
  - Polynomial, rational, and irrational functions
  - identity, absolute value.
  - Exponential, logarithmic
  - Trigonometric, inverse
- Derivability

Introduction

A function from a set *D* to a set *R* is a **rule** that assigns to every element in *D* a unique element in *R*. The set *D* of all input values is the **domain** of the function, and the set *R* of all output values is the **range** of the function.

$f: x \to y$

$x \in$ **Domain**

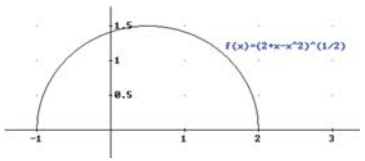
$y \in$ **Range**

Exercise

Obtain the domain of $f(x) = +\sqrt{2 + x - x^2}$

It's necessary that $2 + x - x^2 \geq 0 \Leftrightarrow x^2 - x - 2 \leq 0$ The function $y = x^2 - x - 2$ is a parabola with solutions $x_1=-1$ and $x_2=2$. Then for $x \in [-1, 2]$

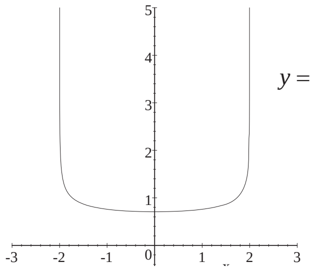
$$D(f) = [-1, 2]$$



Exercise

Obtain the domain of $f(x)=\dfrac{1}{\sqrt[4]{(2+x)(2-x)}}=\dfrac{1}{\sqrt[4]{4-x^2}}$

$$x\in D(f)\iff \left\{\begin{matrix}\overbrace{(2+x)(2-x)}^{4-x^2}\geq 0\\ \sqrt[4]{(2+x)(2-x)}\neq 0\end{matrix}\right\}\iff\left\{\begin{matrix}x\in[-2,2]\\ x\neq 2\,,\,x\neq -2\end{matrix}\right\}\iff x\in\,]-2,2[$$

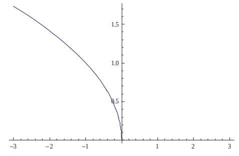


$y=\dfrac{1}{\sqrt[4]{4-x^2}}$

Exercise

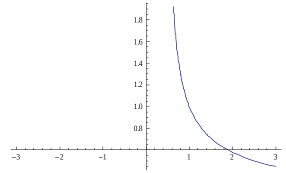
Obtain the domain of $f(x)=\sqrt{-x}+\dfrac{1}{\sqrt{2x-1}}$

It's necessary that $\underbrace{-x\geq 0}_{x\leq 0}$ and $\underbrace{2x-1>0}_{x>\frac{1}{2}}$ both simultaneously



$$D(f)=\,]-\infty,0]\cap\left]\frac{1}{2},+\infty\right[=\varnothing$$

This function doesn't exist



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Definition: Injection

- **Definition**: A function f is said to be **one-to-one** or **injective** (or an injection) if

  $\forall$ x and y in the domain of f, $f(x)=f(y)\Rightarrow x=y$
- A monotonic function is injective
- Intuitively, an injection simply means that each element in the range has at most one preimage (antecedent)
- It is useful to think of the contrapositive of this definition

  $x\neq y \;\Rightarrow\; f(x)\neq f(y)$

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Inverse Functions (1)

•**Definition**: Let f: X→Y be injective. The **inverse** function of f is the function that assigns to an element $y\in$Y the unique element $x\in$X such that f(x)=y

•The inverse function is denote f^{-1}.

•When f is **injective**, its inverse exists and

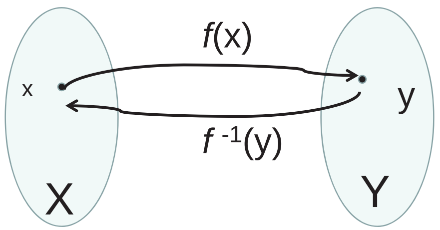
$f(x)=y \iff f^{-1}(y)=x$

•If inverse exists then:

$f: x\to y$ and

$f^{-1}: y\to x$

Inverse Functions: Representation



Domain Range

A function and its inverse

Domain and range are changed between a function and its inverse

rev

Inverse Functions (2)

Inverse function:

$$\boxed{f : \underset{x}{\mathbb{R}} \longrightarrow \underset{y}{\mathbb{R}}} \Rightarrow \boxed{f^{-1} : \underset{y}{\mathbb{R}} \longrightarrow \underset{x}{\mathbb{R}}}$$

$$f(x) = y \quad \Leftrightarrow \quad x = f^{-1}(y)$$

$$D(f^{-1}) = R(f) \;,\; R(f^{-1}) = D(f)$$

Domain and range are changed between a function and its inverse

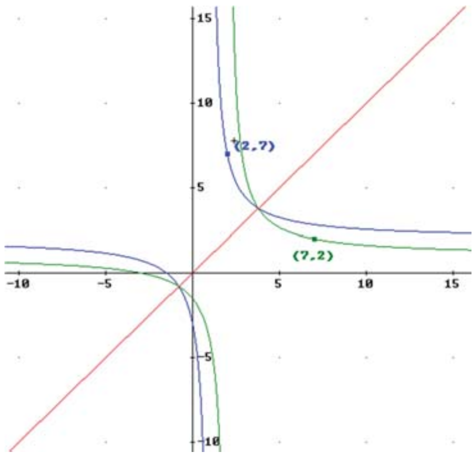
Exercise: $f(x) = \dfrac{2x+3}{x-1}$ has a inverse and $f^{-1}(x) = \dfrac{x+3}{x-2}$

$$D(f) = \mathbb{R} - \{1\} = R(f^{-1}) \quad ; \quad D(f^{-1}) = \mathbb{R} - \{2\} = R(f)$$

$$f(x) = \frac{2x+3}{x-1} = y$$
$$\Downarrow$$
$$y \cdot (x-1) = 2x+3$$
$$xy - y = 2x+3$$
$$xy - 2x = y+3$$
$$x(y-2) = y+3$$
$$x = \frac{y+3}{y-2}$$
$$\Downarrow$$

f inyective and $f^{-1}(y) = \dfrac{y+3}{y-2}$

$$f^{-1}(x) = \frac{x+3}{x-2}$$



Graphs of f and f-1 are symmetric to the bisector of the the first-third quadrant.

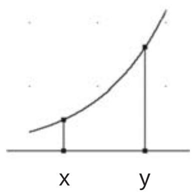
Review of definitions:

- Increasing// decreasing
- Bounded //Unbounded or not bounded
- Even// Odd
- Periodic

rev

More Definitions

- **Definition**: A function *f* is
  - **increasing** if *f*(x)≤*f*(y) whenever x<y and x and y are in the domain of *f*.
  - **strictly increasing** if *f*(x)<*f*(y) whenever x<y and x and y are in the domain of *f*.



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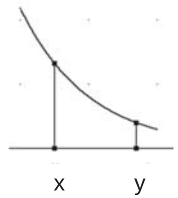
More Definitions

- **Definition**: A function *f* is
  - **decreasing** if *f*(x) ≥ *f*(y) whenever x<y and x and y are in the domain of *f*.
  - **strictly decreasing** if *f*(x)>*f*(y) whenever x<y and x and y are in the domain of *f*.



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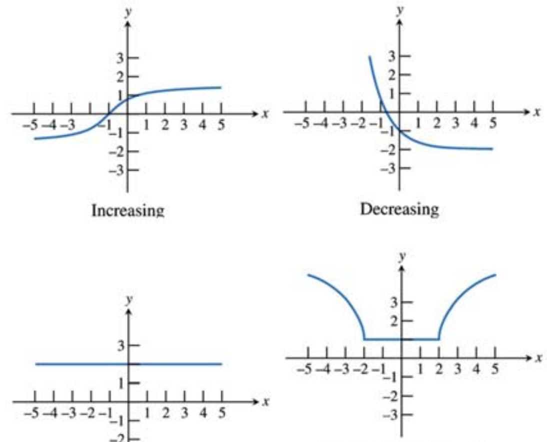
More Definitions

- A function that is increasing or decreasing is said to be **monotonic**.
- Constant functions are increasing and decreasing at the same time.

Increasing and Decreasing Functions



Increasing

Decreasing

Constant

Decreasing on (−∞, −2]
Constant on [−2, 2]
Increasing on [2, ∞)

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Boundedness

- A function f is **bounded below** if there is some number L that is less than or equal to every number in the range I of f. Any such number L is called a **lower** bound of f in I .

$$L \Leftrightarrow [\ f(x) \geq L \,, \forall x \in I\ ]$$

- A function f is **bounded above** if there is some number K that is greater than or equal to every number in the range I I of f. Any such number K is called an **upper** bound of f in I

$$K \Leftrightarrow [\ f(x) \leq K \,, \forall x \in I\ ]$$

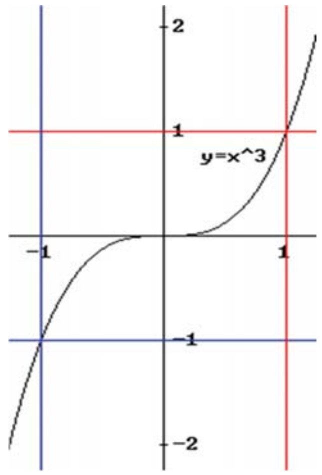
- A function f is **bounded** if it is bounded in both above and below. That means:

$$[\ |f(x)| \leq K \,, \forall x \in I\ ] \Leftrightarrow [-K \leq f(x) \leq K]$$

rev

Domain and boundedness

- Upper bounded in
  [-1,1] and]-∞ ,-1]
- Not upper bounded in
  [1,+ ∞[
- Lower bounded in
  [-1,1] and [1,+ ∞[
- No lower bounded in
  ]- ∞,-1]
- Bounded in
  [-1,1]
- No bounded in
  ]- ∞,-1] and [1,+ ∞[



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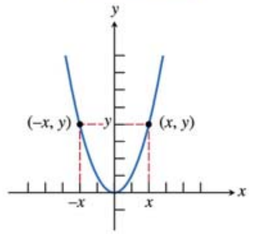
Symmetry

- With respect to the y-axis
  - *f(x) = f(-x)*
  - *(x, y) →(-x, y)*
  - The function is **even**
- With respect to the origin
  - *f(-x) = -f(x)*
  - *(x, y) →(-x,-y)*
  - The function is **odd**

Symmetry with respect to the *y*-axis

Example: $f(x) = x^2$

Graphically



Numerically

x	$f(x)$
−3	9
−2	4
−1	1
1	1
2	4
3	9

Algebraically

For all x in the domain of f,

$$f(-x) = f(x)$$

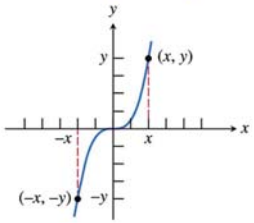
Functions with this property (for example, x^n, n even) are **even** functions.

Even functions: $x^6 - x^2 \,,\ \dfrac{x^3 + x}{x^5 + x} \,,\ |x|$

Symmetry with respect to the origin

Example: $f(x) = x^3$

Graphically



Numerically

x	y
−3	−27
−2	−8
−1	−1
1	1
2	8
3	27

Algebraically

For all x in the domain of f,

$$f(-x) = -f(x).$$

Functions with this property (for example, x^n, n odd) are **odd** functions.

Odd functions: x , $x^3 - x$, $\dfrac{x^4+1}{x^3+x}$, $\log\left(1+\dfrac{2x}{1-x}\right)$

Periodic functions

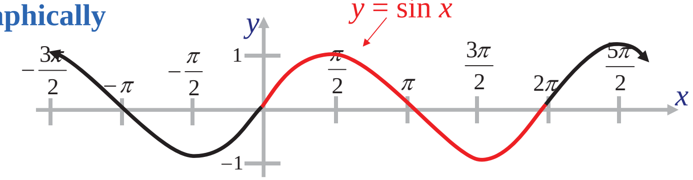
A periodic function is a function f such that $f(x) = f(x + p)$, for every real number x in the domain where p is a constant.

The smallest positive number p, if there is one, for which $f(x + p) = f(x)$ for all x, is the period of the function.

Periodic functions

Example: $f(x)=\sin x$

Graphically

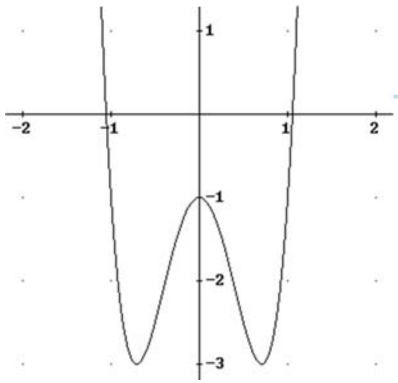


Numerically

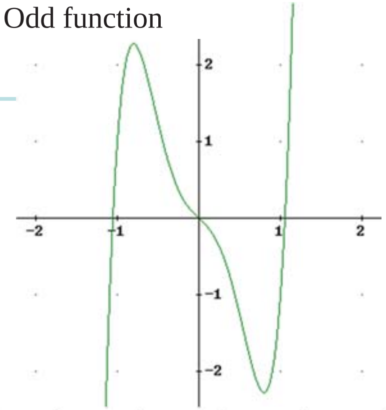
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y= sin x	0	1	0	-1	0

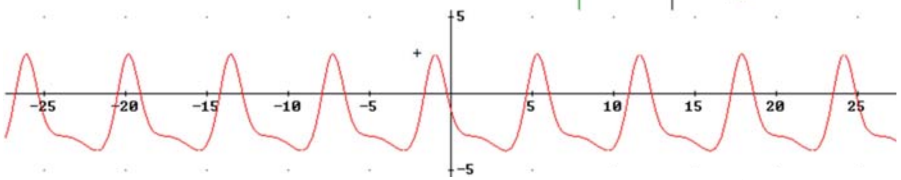
Algebraically f periodic, T period $\Leftrightarrow f(x) = f(x+T)$

Even function



Odd function





Periodic function

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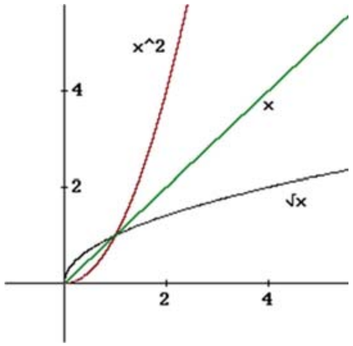
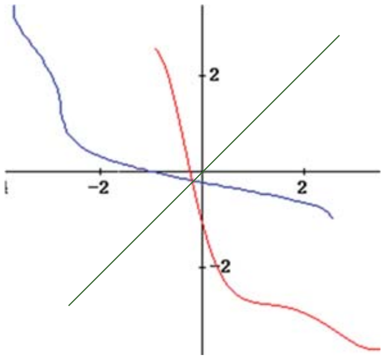
More about symmetries





A function and its inverse are symmetric respect the bisector y=x

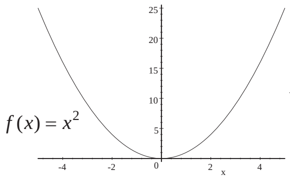
Elementary functions

- A elementary function is a function of one variable built from a finite number of **exponentials**, **logarithms**, constants, and **nth roots** through composition and combinations using the four elementary operations (+ – × ÷).

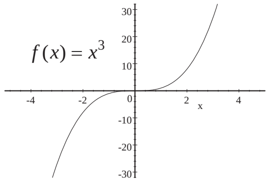
Review of elementary functions

- Polynomial:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \;,\; a_i \in \mathbb{R} \;,\; i: 0,1,\ldots,n$$



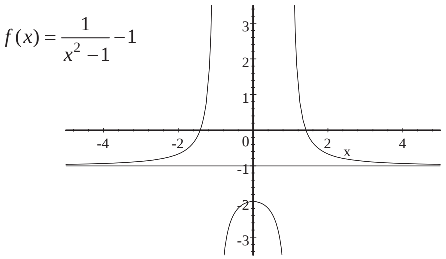
$D(f) = \mathrm{R}$



Review of elementary functions

- Rational

Domain of $f(x) = \dfrac{P(x)}{Q(x)}$? $\quad Q(x) \neq 0$



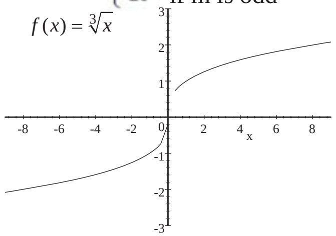
The domain are all the real numbers except those numbers that cancel out the denominator

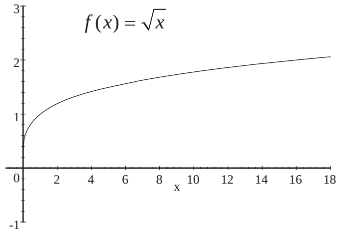
Review of elementary functions

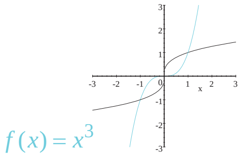
- Irrational $f(x)=\sqrt[m]{x}$, $m \in \mathbb{N}$

$$D(f)=\begin{cases} \mathbb{R}^+ \cup \{0\} & \text{if m is even} \\ \mathbb{R} & \text{if m is odd} \end{cases}$$

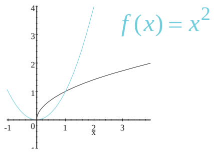
Domain of *f(x)*= $\sqrt[m]{f(x)}$?





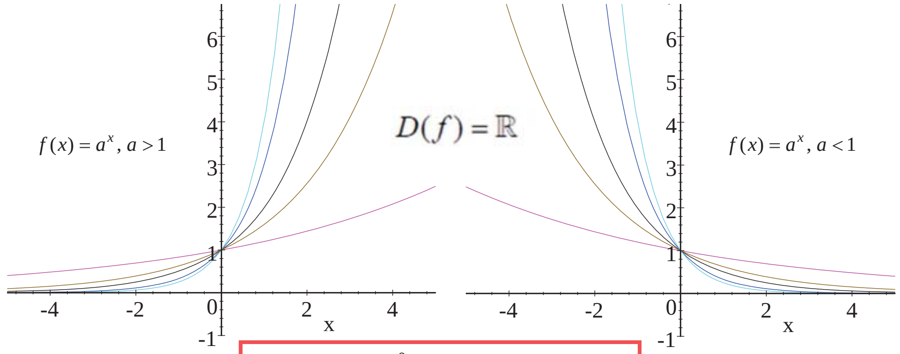


inverses



Review of elementary functions

- Exponential $f(x)=a^x$, $a>0$



$a^x > 0$, $a^0 = 1$

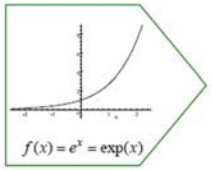
$a^x \cdot a^y = a^{x+y}$, $a^x / a^y = a^{x-y}$

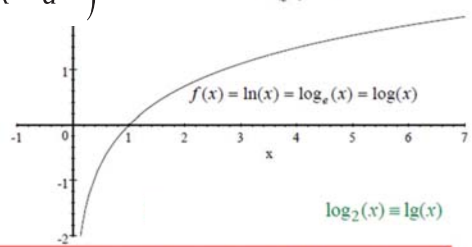
$(a^x)^y = a^{x \cdot y}$

Review of elementary functions

Logaritmic (inverse of exponential)**:** $f(x)=\log_a(x)$, $a>0$

$\left(y=\log_a(x) \;\Leftrightarrow\; x=a^y\right)$ $\quad D(f)=\mathbb{R}^+$





$f(x)=x^\alpha \equiv e^{\alpha \log(x)}$; $\alpha \in \mathbb{R}$, $x>0$

$\log_a(1)=0$, $\log(e)=1$

$\log_a(x \cdot y)=\log_a(x)+\log_a(y)$, $\log_a(x/y)=\log_a(x)-\log_a(y)$

$\log_a(x^y)=y\log_a(x)$, $\log(e^k)=k$

$x^{\log_a(y)}=y^{\log_a(x)}$, $\log_a(x)=\dfrac{\log_b(x)}{\log_b(a)}=\dfrac{1}{\log_b(a)}\cdot\log_b(x)=k\cdot\log_b(x)$

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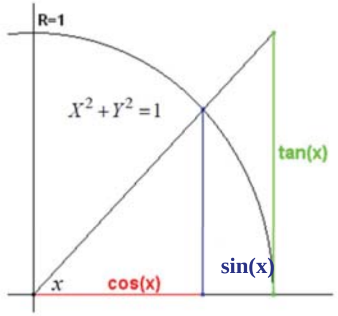
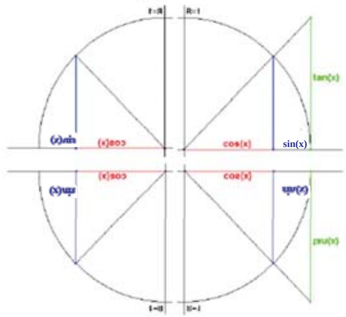
Review of elementary functions

Trigonometric: sin (*x*), cos(*x*), tan(*x*)





$\cos^2(x)+\quad{}^2(x)=1$

x	$\sin(x)$	$\cos(x)$	$\tan(x)$
0	0	1	0
$\pi/2$	1	0	?

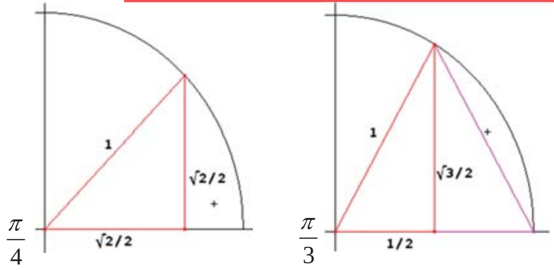
$\underbrace{\sin(-x)=-\sin(x)}_{\text{even}}$, $\underbrace{\cos(-x)=\cos(x)}_{\text{odd}}$

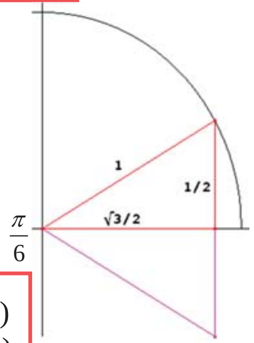
$\sin\left(x+\frac{\pi}{2}\right)=\cos(x)$, $\cos\left(x+\frac{\pi}{2}\right)=-\sin(x)$

$\sin(x+\pi)=-\sin(x)$, $\cos(x+\pi)=-\cos(x)$

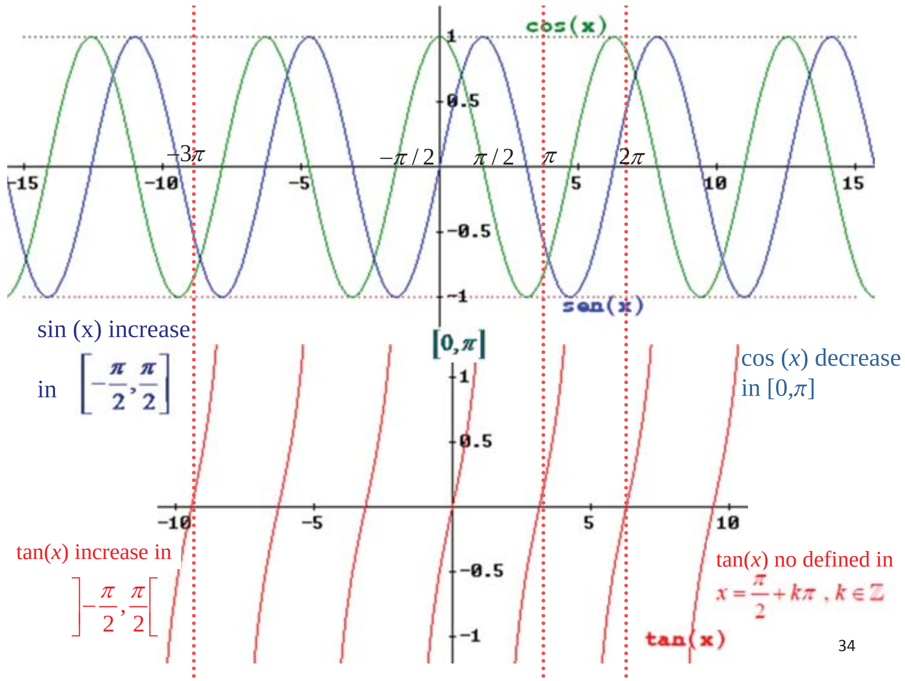
rev

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	$3\pi/2$
$\sin(x)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0	-1
$\cos(x)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1	0
$\tan(x)$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	?	0	?

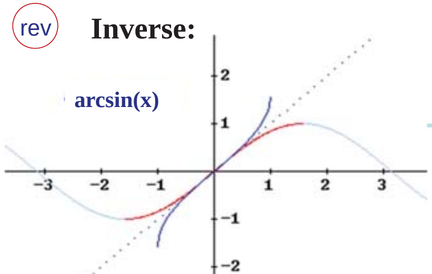


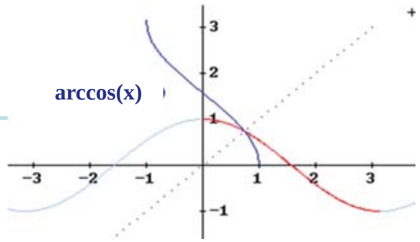


$\sin(x \pm y)=\sin(x)\cos(y) \pm \cos(x)\sin(y)$
$\cos(x \pm y)=\cos(x)\cos(y) \mp \sin(x)\sin(y)$
$\cos(2x)=\cos^2(x)-\sin^2(x)$
$\sin(2x)=2\sin(x)\cos(x)$

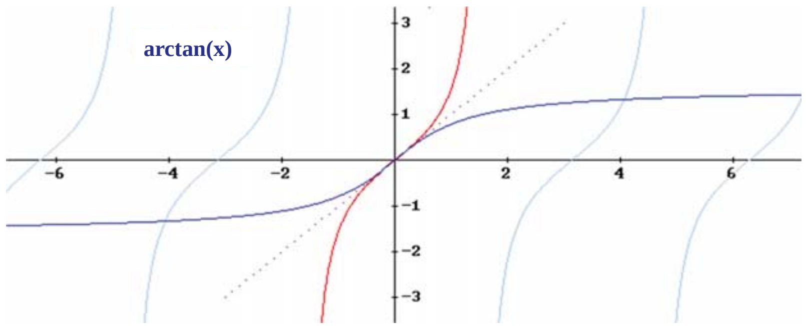


rev **Inverse:**





$\arcsin:[-1,1]\to[-\frac{\pi}{2},\frac{\pi}{2}]$, $\arccos:[-1,1]\to[0,\pi]$, $\arctan:\mathbb{R}\to]-\frac{\pi}{2},\frac{\pi}{2}[$

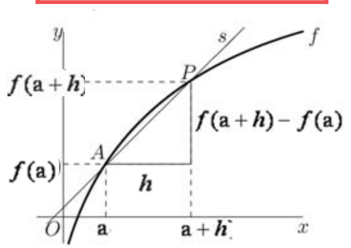


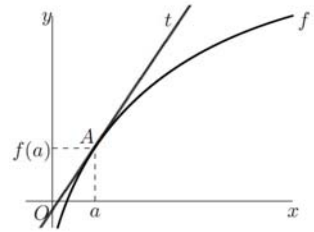
Derivative of a function

The derivative of a function f at a point a denoted $f'(a)$ is provided this limit exists

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

when $h\to 0$ the secant between two points tends to the derivative





f `(a) gives the slope of the curve *y =f(x)* at the point *P(a,f(a))*
The tangent line to the curve at P is the line through P with this slope

Differentiable functions

•If *f ’(a)* exits, we say that *f* is differentiable (has derivative) at *a*.
•If *f ’(x)* exits at every point in the domain of *f(x)*, we call *f(x)* differentiable
•A function is continuous at every point where it has derivative
•If *f*, *g* are differentiable at a point *a*, then

$$(f \pm g),\ (fg),\ \left(\frac{f}{g}\right) \text{ and } (f \circ g) \text{ are differentiable at } a$$

$$\boxed{\begin{array}{l} (f \pm g)'(a) = f'(a) \pm g'(a) \quad \text{and} \quad (\alpha f)'(a) = \alpha f'(a)\ ,\ \ \alpha \in \mathbb{R} \\ (fg)'(a) = f'(a)g(a) + f(a)g'(a) \\ \left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2} \\ (f \circ g)'(a) = f'(g(a))g'(a) \qquad \text{(chain rule)} \end{array}}$$

Derivatives

$f(x)=k \quad\Rightarrow\quad f'(x)=0$	
$f(x)=x^n \quad\Rightarrow\quad f'(x)=n\cdot x^{n-1}$	$f(x)=\sqrt{x} \quad\Rightarrow\quad f'(x)=\frac{1}{2\sqrt{x}}$
$f(x)=\log(x) \quad\Rightarrow\quad f'(x)=\frac{1}{x}$	$f(x)=\log_a(x) \quad\Rightarrow\quad f'(x)=\frac{1}{x\cdot\log(a)}$
$f(x)=e^x \quad\Rightarrow\quad f'(x)=e^x$	$f(x)=a^x \quad\Rightarrow\quad f'(x)=a^x\log(a)$
$f(x)=\mathrm{sen}(x) \quad\Rightarrow\quad f'(x)=\cos(x)$	$f(x)=\tan(x) \quad\Rightarrow\quad f'(x)=\frac{1}{\cos^2(x)}$
$f(x)=\cos(x) \quad\Rightarrow\quad f'(x)=-\sin(x)$	
$f(x)=\arctan(x) \quad\Rightarrow\quad f'(x)=\frac{1}{1+x^2}$	$f(x)=\mathrm{arcsen}(x) \quad\Rightarrow\quad f'(x)=\frac{1}{\sqrt{1-x^2}}$

Derivatives

$$f(x)=\log(x)+3\arctan(x) \quad\Rightarrow\quad f'(x)=\frac{1}{x}+\frac{3}{1+x^2}$$

$$g(x)=\frac{x^3-5x}{x^2+8} \quad\Rightarrow\quad g'(x)=\frac{(3x^2-5)(x^2+8)-(x^3-5x)(2x)}{(x^2+8)^2}$$

$$h(x)=x^3\cdot\sqrt{\mathrm{sen}(x)} \quad\Rightarrow\quad h'(x)=3x^2\cdot\sqrt{\mathrm{sen}(x)}+\frac{x^3}{2\sqrt{\mathrm{sen}(x)}}\cdot\cos(x)$$

Applications of derivatives

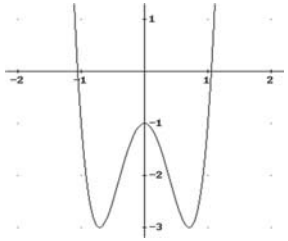
Increasing and decreasing

•If *f ’(x)>0* then *f* is strictly increasing for all values near enough to *x*
•If *f ’(x)<0* then *f* is strictly decreasing for all values near enough to *x*

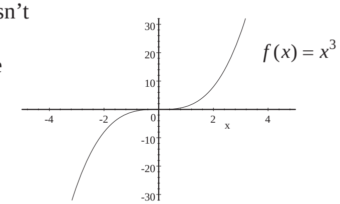
Extreme values

•If *f* has a local maximum or minimum value at an interior point *c* of its domain, and if *f ’* is defined at *c*, then

f ´(c) = 0



f ´(c) = 0 doesn’t imply a local extreme value



Applications of derivatives

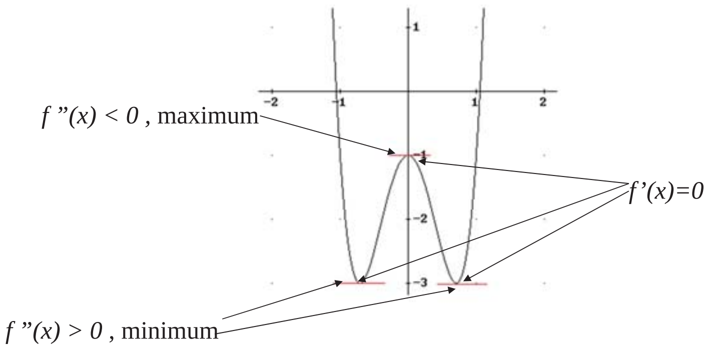
The guy gets on a bus and starts threatening everybody:
"I'll integrate you! I'll differentiate you!!!"

So everybody gets scared and runs away.
Only one person stays.
The guy comes up to him and says: "Aren't you scared, I'll integrate you, I'll differentiate you!!!"
And the other guy says; "No, I am not scared, I am e^x."

Concavity and curve sketching

Extreme values:

Let y=f(x) be twice-differentiable on an interval I and $f'(c)=0$
•if $f''(x) > 0$ on I, f has a minimum in c
•if $f''(x) < 0$ on I, f has maximum in c



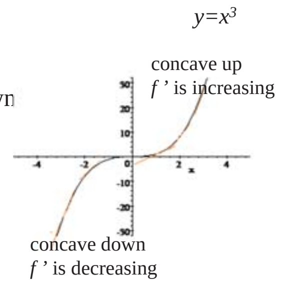
Concavity and curve sketching

Concave up and down:

The graph of a differentiable function $y = f(x)$ is
•concave up on an open interval I if f' is increasing on I; The graph is above the tangent line
•concave down on an open interval I if f' is decreasing on I; the graph is below the tangent line

The second derivative test for concavity:

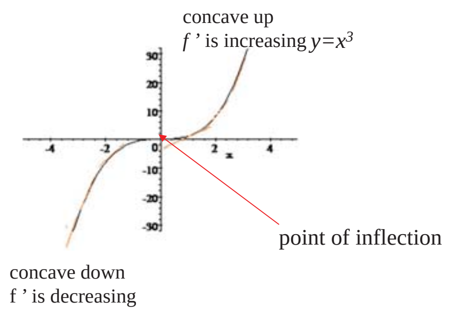
Let $y=f(x)$ be twice-differentiable on an interval I
•if $f''(x) > 0$ on I, the graph of f over I is concave up
•if $f''(x) < 0$ on I, the graph of f over I is concave down



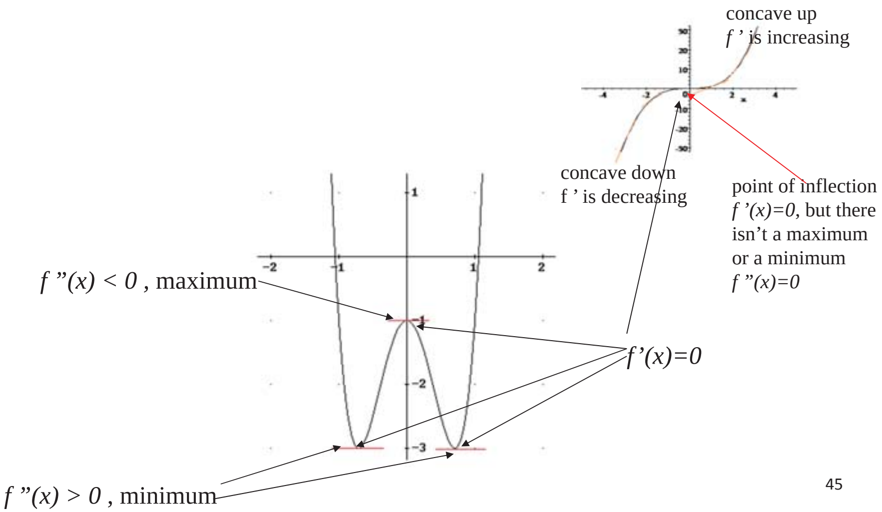
Concavity and curve sketching

Points of inflection:

A point where the graph of a function has a tangent line and where the concavity changes there is a point of inflection
At a point of inflection $(c, f(c))$, either $f''(c)=0$ or fails to exits.

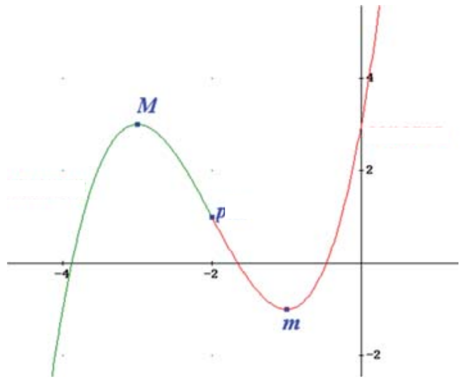


Concavity and curve sketching



Example

$$f(x) = x^3 + 6x^2 + 9x + 3$$
$$f'(x) = 3x^2 + 12x + 9$$
$$f''(x) = 6x + 12$$



$f'(x) = 0 \Leftrightarrow x \in \{-3,-1\}$

$f'(x) < 0 \Leftrightarrow x \in]-3,-1[$

$f'(x) > 0 \Leftrightarrow x \in]-\infty,-3[\cup]-1,+\infty[$

f increasing in $-\infty,-3[\cup]-1,+\infty[$

f decreasing in $]-3,-1[$

relative maximum at $(-3,3)$
relative minimum at $(-1,-1)$

$f''(x) = 0 \Leftrightarrow x = -2$

$f''(x) < 0 \Leftrightarrow x \in]-\infty,-2[$

$f''(x) = 0 \Leftrightarrow x \in]-2,+\infty[$

f is concave down in $]-\infty,-2[$

f is concave up in $]-2,+\infty[$

point of inflection at $(-2,1)$

rev

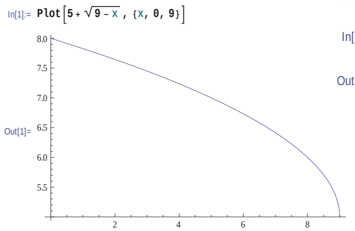
Exercise: Is the function $\boldsymbol{f(x) = 5 + \sqrt{9-x}}$ strictly decreasing in [0,9]?

$D(f) =]-\infty, 9]$

$$x_1, x_2 \in D(f), x_1 < x_2 \Rightarrow -x_1 > -x_2 \Rightarrow 9 - x_1 > 9 - x_2 \Rightarrow \sqrt{9-x_1} > \sqrt{9-x_2}$$
$$\Downarrow$$
$$f(x_1) > f(x_2)$$

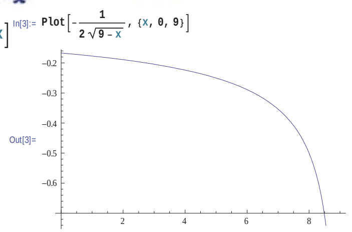
and f is strictly decreasing in the domain, [0,9]

We can observe: $\boldsymbol{f'(x) = -\dfrac{1}{2\sqrt{9-x}} < 0, \text{ if } x \in]0,9[}$



In[2]:= D[5 + √(9 - x), x]

Out[2]= $-\dfrac{1}{2\sqrt{9-x}}$



rev

Exercise: Is the function $\boldsymbol{f(x) = 5 + \sqrt{9-x}}$ strictly decreasing in [0,9]?. Found the inverse function f^{-1} in that interval.

$$x_1, x_2 \in D(f), x_1 < x_2 \Rightarrow -x_1 > -x_2 \Rightarrow 9 - x_1 > 9 - x_2 \Rightarrow \sqrt{9-x_1} > \sqrt{9-x_2}$$
$$\Downarrow$$

$D(f) =]-\infty, 9]$ $\qquad f(x_1) > f(x_2)$

and f is strictly decreasing in the domain, [0,9]

We can observe: $\boldsymbol{f'(x) = -\dfrac{1}{2\sqrt{9-x}} < 0, \text{ if } x \in]0,9[}$

$$f : \underbrace{[0,9]}_{x} \to \underbrace{[5,8]}_{5+\sqrt{9-x}} \Rightarrow f^{-1} : \underbrace{[5,8]}_{x} \to \underbrace{[0,9]}_{-x^2+10x-16}$$

$$f(x) = 5 + \sqrt{9-x} = y \Rightarrow x = 9 - (y-5)^2 = f^{-1}(y)$$