Mathematical Analysis

Integration

Contents

Introduction

In this chapter we’re going to develop a method to calculate the areas and volumes of very general shapes. This method is called integration, is a tool for calculating much more than areas and volumes. The integral is of fundamental importance in statistics, sciences, and engineering. In this chapter we’re going to focus on thee integral concept and in its use in computing areas of various regions with curves boundaries.

Introduction

- First we choose a unit of measurement

Unit

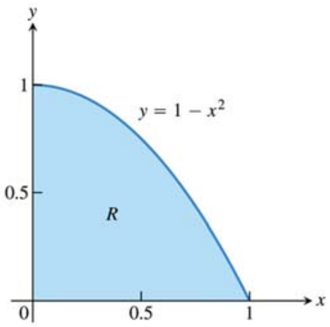
- Second we count the area

Rectangles /Triangles

Problem (more general areas):

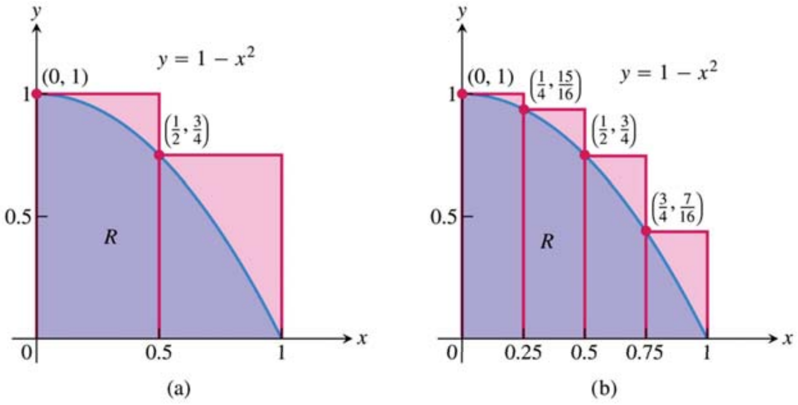
Introduction

How can I find the area of the shaded region R that lies above the x-axis, below the graph $y = 1 - x^2$?



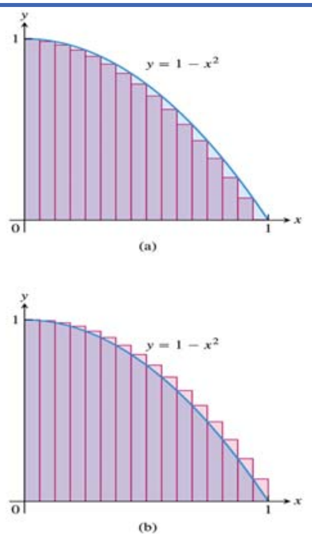
The area of the region R cannot be found by a simple formula

Introduction



a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area by the amount shaded in light red.

Introduction



(a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

Introduction

Finite approximations for the area of R

Number of subintervals	Lower sum	Midpoint rule	Upper sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

Contents

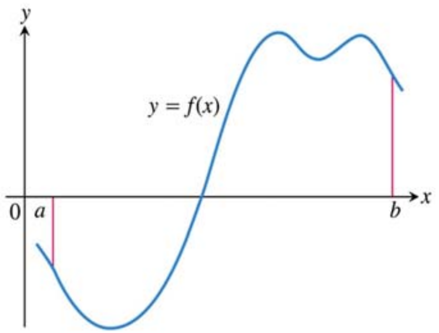
Riemann's integral

- We're going to begin with an arbitrary function f definite on an closed interval $[a,b]$. We subdivide the interval $[a,b]$ into subintervals, not necessarily of equals widths (or lengths), and form sums in the same way as for the finite approximations. To do so, we choose n-1 points between a and b and satisfying

$$P=\{a=x_0, x_1,x_2,\ldots,x_{k-1},x_k,x_{k+1},\ldots,x_n=b\}$$

P is a partition of $[a,b]$ into subintervals

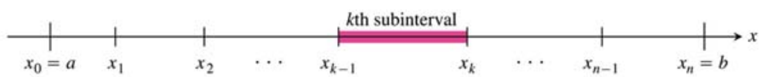
Riemann's integral



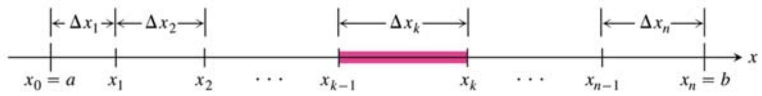
A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

Subintervals

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the **kth subinterval of P** is $[x_{k-1}, x_k]$, for k an integer between 1 and n.



The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1, the width of the second $[x_1, x_2]$ is denoted Δx_2, and the width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$. If all n subintervals have equal width, then the common width Δx is equal to $(b - a)/n$.



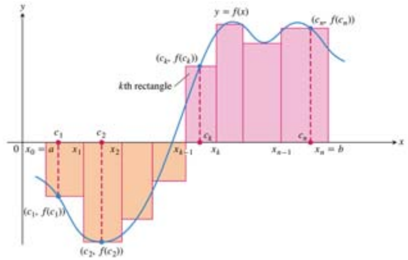
Subintervals

In each subinterval we select some point. The point chosen in the kth subinterval $[x_{k-1}, x_k]$ is called c_k. Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$. These rectangles can be above or below the x-axis, depending on whether $f(c_k)$ is positive or negative, or on the x-axis if $f(c_k) = 0$

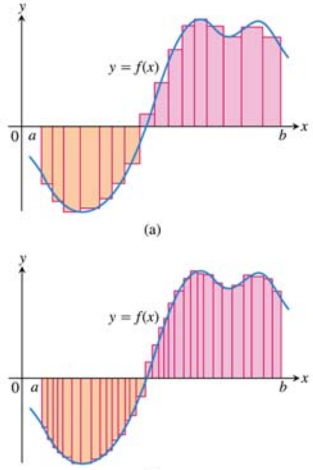
On each subinterval we form the product $f(c_k) \cdot \Delta x_k$. This product is positive, negative, or zero, depending on the sign of $f(c_k)$. When $f(c_k) > 0$, the product $f(c_k) \cdot \Delta x_k$ is the area of a rectangle with height $f(c_k)$ and width Δx_k. When $f(c_k) < 0$, the product $f(c_k) \cdot \Delta x_k$ is a negative number, the negative of the area of a rectangle of width Δx_k that drops from the x-axis to the negative number $f(c_k)$.

Riemann's integral

Subintervals



Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of *f* and the *x-axis* with increasing accuracy



(a)

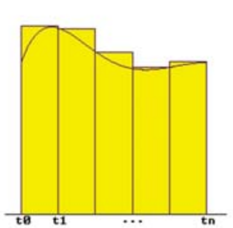
(b)

Riemann's integral

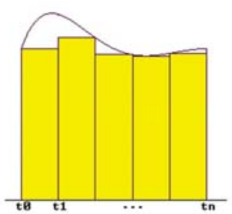
Definitions:Upper and lower sums

Let f:[a,b] $\to \mathbb{R}$ bounded and $P=\{a=x_0, x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, xn=b\}$ a partition of *[a,b]*,

The upper and the lower sums associated to f and P are:



$U(f,P)$, (*upper*)



$L(f,P)$, (*lower*)

Riemann's integral

Integrable function

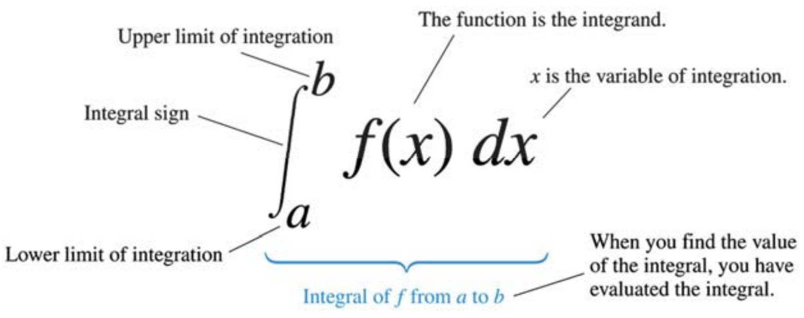
- A function f is called integrable Riemann in $[a,b]$ if and only if exists a sequence of partitions $\{P_n\}$ of [a,b] such as

$$\lim_n (U(f,P_n) - L(f,P_n)) = 0$$

- In this case,

$$\int_a^b f = \lim_n (U(f,P_n) - L(f,P_n)) = 0$$

Integral symbol



Example

- **Show that *f(x)=x* is integrable over [0,1]**

We consider the sequence which divides into n equal parts of length $h=1/n$ $P_n=\left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right\}$

In each subinterval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$ we consider the rectangles

$\frac{1}{n}\cdot f\left(\frac{k-1}{n}\right)$ and $\frac{1}{n}\cdot f\left(\frac{k}{n}\right)$, (area=base × high), which are

respectively a lower and a upper bound

Example (cont')

The lower and thee upper sums are thee limits of the series

$$L(f,P_n)=\sum_{k=1}^{n}\frac{1}{n}\cdot f\left(\frac{k-1}{n}\right)=\sum_{k=1}^{n}\frac{1}{n}\cdot\left(\frac{k-1}{n}\right)=\frac{0+1+\cdots+(n-1)}{n^2}=\frac{(n-1)n}{2n^2}\xrightarrow[n\to+\infty]{}\frac{1}{2}$$

$$U(f,P_n)=\sum_{k=1}^{n}\frac{1}{n}\cdot f\left(\frac{k}{n}\right)=\sum_{k=1}^{n}\frac{1}{n}\cdot\left(\frac{k}{n}\right)=\frac{1+2+\cdots+n}{n^2}=\frac{n(n+1)}{2n^2}\xrightarrow[n\to+\infty]{}\frac{1}{2}$$

They exits and are equal, so f is integrable over [0,1] and

$$\int_0^1 f=\lim_n(U(f,P_n)-L(f,P_n))=\frac{1}{2}$$

Integrable functions

- f monotonic in $[a,b] \Longrightarrow f$ integrable over $[a,b]$
- f continuous in $[a,b] \Longrightarrow f$ integrable over $[a,b]$
- Continuous functions in an $[a,b]$ interval, except in a finite number of points are integrable
- f integrable in $[a,b] \Longrightarrow |f|$ integrable over $[a,b]$ and

$$\left|\int_a^b f\right|\leq\int_a^b|f|$$

Examples

$f(x)=k\ (cte)$

$f(x)=x$

$f(x)=x^2$

$\left.\begin{matrix} f(x)=k\ (cte) \\ f(x)=x \\ f(x)=x^2 \end{matrix}\right\}$ f is integrable $\forall\,[a,b]$

$f(x)=\begin{cases} 0 & \text{if } x\in[0,2]-\{1\} \\ 2 & \text{if } x=1 \end{cases}$ $\Big\}$ f is continuous in $x=1$ and integrable over $[0,2]$

$f(x)=\begin{cases} 0, & \text{if } x\in\mathbb{Q} \\ 1, & \text{if } x\notin\mathbb{Q} \end{cases}$ $\Big\}$ f is discontinuous in each point of $[0,1]$ and no integrable

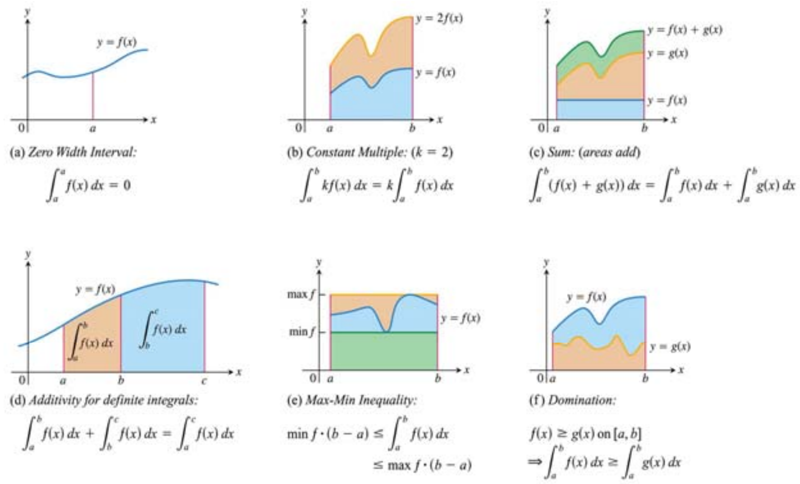
Properties

Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A Definition when $f(a)$ exists
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k
4. *Sum and Difference:* $\int_a^b (f(x)\pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. *Max-Min Inequality:* If f has maximum value $\max f$ and minimum value $\min f$ on $[a,b]$, then
$$\min f\cdot(b-a) \le \int_a^b f(x)\,dx \le \max f\cdot(b-a).$$
7. *Domination:* $f(x)\ge g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

   $f(x)\ge 0$ on $[a,b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)

Properties



Geometric interpretations of Rules :

Examples

Example: f is integrable over $[1,2]$ $\quad x\le f(x)\le x^2,\qquad x\in[1,2],$

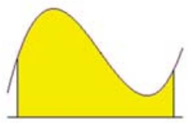
$$\int_1^2 x\,dx \le \int_1^2 f(x)\,dx \le \int_1^2 x^2\,dx \;\Rightarrow\; \frac{3}{2}\le \int_1^2 f(x)\,dx \le \frac{7}{3}$$

Example: $\left|\int_0^1 \frac{\cos(nx)}{x+1}\,dx\right| \le \int_0^1 \left|\frac{\cos(nx)}{x+1}\right| dx \le \int_0^1 \frac{dx}{1+x} \le \int_0^1 dx = 1$

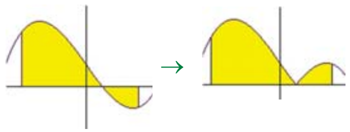
Plane area

If f is integrable over $[a,b]$, the area of the region between the x-axis, the graph $y = f(x)$, and thee vertical lines $x=a$, $x=b$ is defined by

$$A=\int_a^b |f|$$



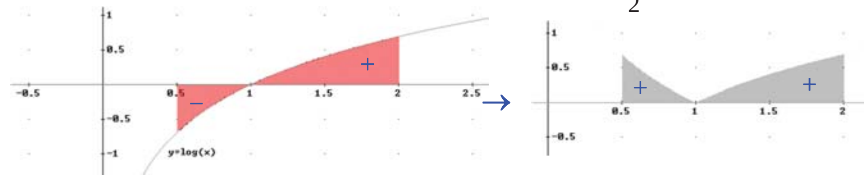
f is positive in $[a,b]$



f has a change of sign in $[a,b]$

Example

Example: The area limited by $y = \log(x)$ and $x = \frac{1}{2}, x = 2$



$$A = \int_{1/2}^{2} |\log(x)|\, dx = -\int_{1/2}^{1} \log(x)dx + \int_{1}^{2} \log(x)dx \overset{?}{=} \frac{3\log(2)-1}{2}$$

Summary

Summary:

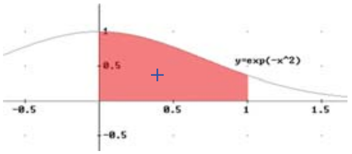
To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Contents

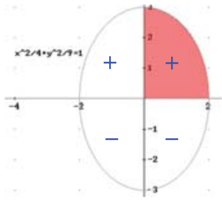
Exact calculus of a integral

Example: The area limited by $y = e^{-x^2}$, $x = 0$, $x = 1$ and *OX* axis



$$A = \int_0^1 e^{-x^2} dx \overset{?}{\approx} 0.7468$$

Example: The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$A = 4b \int_0^a \sqrt{1 - \frac{x^2}{a^2}} dx \overset{?}{=} \pi ab$$

Definition

- A **primitive** of *f(x)* is a function *g(x)* such that as derivative.

$$\frac{d}{dx} g(x) = f(x)$$

Often a primitive of f is denoted by symbol $\int f(t)dt$ (without the extremes of integration). Since two functions that have the same derivative can at most differ by a constant and since we have seen that we can characterize all primitives of f, denoted by Φ, with the following equation

$$\Phi(x) = \int_a^x f(t)dt + c \quad \text{c constant}$$

Definition (cont')

Letting $x = a$ we find $c = \Phi(a)$ and letting $x=b$ we find

$$\int_a^b f(t)dt = \Phi(b) - \Phi(a)$$

Sometimes we write the previous expression as

$$\Phi(x)\,\Big|_a^b$$

so the key object we need to evaluate $\int_a^b f(t)dt$ is a primitive of *f(x)*

Barrow's rule

If *h(x)* is a primitive of *f(x)*, definite in [*a*,*b*] and integrable, then

$$\int_a^b f = \int_a^b f(x)\, dx = h(x)\Big]_a^b \equiv h(b) - h(a) \qquad (h'(x) = f(x))$$

Examples:

$$\int_a^b x^p\, dx \underset{(p\neq -1)}{=} \frac{x^{p+1}}{p+1}\Bigg]_a^b = \frac{b^{p+1} - a^{p+1}}{p+1} \qquad p\neq -1$$

$$\int_0^\pi \operatorname{sen}(x)\, dx = -\cos(x)\Big]_0^\pi = -\cos(\pi) + \cos(0) = 1 + 1 = 2$$

$$\int_{-1}^1 e^x\, dx = e^x\Big]_{-1}^1 = e - \frac{1}{e} = \frac{e^2 - 1}{e}$$

Method for finding primitives

While computing derivatives is a straightforward technique for every function finding the primitive of a function can be very hard. Here we report some integration methods that work with relatively simple functions

Integration by parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

Example

$$\left(\int_a^b u\cdot dv = [u\cdot v]_a^b - \int_a^b v\cdot du\right)$$

$$\int_0^1 x\cdot\cos(x)\,dx = \begin{pmatrix} u = x & du = dx \\ dv = \cos(x)dx & v = \mathrm{sen}(x)\end{pmatrix}$$

$$= x\cdot\mathrm{sen}(x)\Big]_0^1 - \int_0^1 \mathrm{sen}(x)\,dx$$

$$= x\cdot\mathrm{sen}(x)\Big]_0^1 - \left(-\cos(x)\Big]_0^1\right)$$

$$= \mathrm{sen}(1) + \cos(1) - 1$$

Example

$$\int_{1/2}^2 \log(x)dx = \int_{1/2}^2 \log(x)\cdot 1dx = \begin{pmatrix} u = \log(x) & du = \dfrac{dx}{x} \\ dv = dx & v = x\end{pmatrix}$$

$$= x\log(x)\Big]_{1/2}^2 - \int_{1/2}^2 x\cdot\frac{1}{x}dx = x\log(x)\Big]_{1/2}^2 - \int_{1/2}^2 dx$$

$$= 2\log(2) - \frac{1}{2}\log\left(\frac{1}{2}\right) - \left(x\Big]_{1/2}^2\right) = \frac{5\log(2) - 3}{2}$$

Example

$$\int_0^1 \arctan(x)dx = \int_0^1 \arctan(x)\cdot 1dx = \begin{pmatrix} u=\arctan(x) & du=\dfrac{dx}{1+x^2} \\ dv=dx & v=x \end{pmatrix}$$

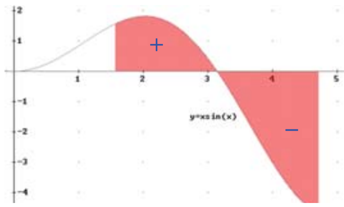
$$= x\arctan(x)\Big]_0^1 - \int_0^1 x\cdot\frac{1}{1+x^2}dx = x\arctan(x)\Big]_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}dx$$

$$=1\cdot\arctan(1)-0\cdot\arctan(0) \; -\frac{1}{2}\log(1+x^2)\Big]_0^1 = \frac{\pi}{4}-\frac{1}{2}\log(2)$$

Example

Find the plane area limited by $y=sin(x)$, the OX axis and the vertical lines $x=\frac{\pi}{2}$, $x=\frac{3\pi}{2}$

$$A=\int_{\pi/2}^{3\pi/2}\left|x\,\mathrm{sen}(x)\right|dx; \longrightarrow \; A=\int_{\pi/2}^{\pi}x\,\mathrm{sen}(x)\,dx-\int_{\pi}^{3\pi/2}x\,\mathrm{sen}(x)\,dx$$



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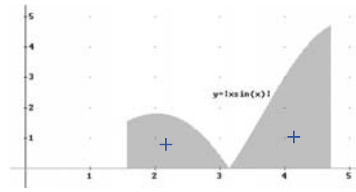


Integrating by part $\begin{pmatrix} u=x\;,\; du=dx \\ dv=\mathrm{sen}(x)\;,\; v=-\cos(x)\end{pmatrix}$,

$$A=\big(\mathrm{sen}(x)-x\cos(x)\big)\Big]_{\pi/2}^{\pi}+\big(\mathrm{sen}(x)-x\cos(x)\big)\Big]_{\pi}^{3\pi/2}=2\pi$$

Substitution or change of variable

$$\int_{g(a)}^{g(b)} f(x)\,dx = \left(\underbrace{x=g(t)}_{\leftarrow\ \mathrm{cdv}\ \rightarrow}\right) = \int_a^b f(g(t))g'(t)\,dt$$

Procedure:

- Find a suitable $x= g(t)$
- Compute $dx= g'(t)\,dt$
- Substitute for x and dx in the original integral

When you use the substitution method with a definite integral always remember to change the integration bounds

Example

Example: $\int_0^1 x\sqrt{1+x^2}\,dx=\frac{1}{2}\int_0^1 2x\sqrt{1+x^2}\,dx \overset{\leftarrow}{=} \begin{pmatrix} 1+x^2=t \\ 2xdx=dt\end{pmatrix}$

$$=\frac{1}{2}\int_1^2\sqrt{t}\,dt=\frac{1}{3}\left(t^{3/2}\Big]_1^2\right)=\frac{2\sqrt{2}-1}{3}$$

Calculate $\int_1^4 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx$ using substitutions in both cases

↓

$\left(\text{Sol: } 4\sqrt{3}-\frac{8\sqrt{2}}{3}\right)$

Integration bounds!

$$\int_0^1 x\cdot\sqrt[3]{x-1}\,dx \overset{\rightarrow}{=} \begin{pmatrix} x-1=t^3 \\ x=t^3+1 \\ dx=3t^2dt\end{pmatrix} = \int_{-1}^{0}(t^3+1)\cdot t\cdot 3t^2\,dt=$$

$$=3\int_{-1}^{0}(t^6+t^3)\,dt=3\left(\frac{t^7}{7}+\frac{t^4}{4}\Big]_{-1}^{0}\right)=-\frac{9}{28}$$

Example

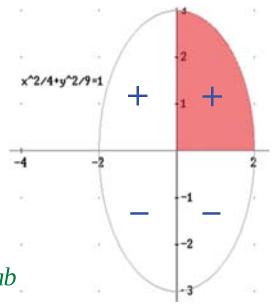
Example: Find the inner area of the ellipse: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

We know that: $A=4b\int_0^a\sqrt{1-\dfrac{x^2}{a^2}}\,dx$

To solve this integral we can do a trigonometric substitution

$$\begin{pmatrix} x=a\cos(t) \\ dx=-a\sin(t)dt \end{pmatrix}$$

$$A \overset{?}{=} 4ab\int_0^{\pi/2}\sin^2(t)dt \overset{?}{=} 2ab\int_0^{\pi/2}(1-\cos(2t))\,dt=\pi ab$$



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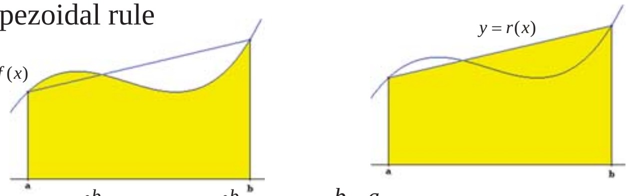
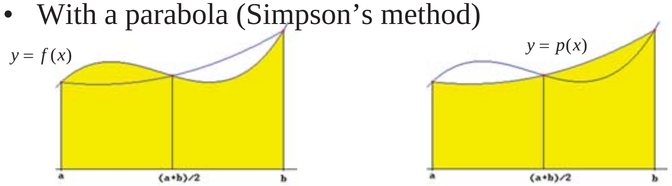
Approximate calculus of Riemann's integral

- Trapezoidal rule



$$\int_a^b f(x)\,dx \cong \int_a^b r(x)\,dx=\frac{b-a}{2}\big(f(a)+f(b)\big)$$

- With a parabola (Simpson's method)

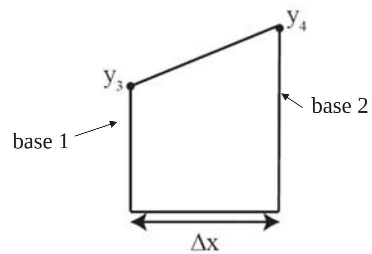


$$\int_a^b f(x)\,dx \cong \int_a^b p(x)\,dx \overset{?}{=} \frac{b-a}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

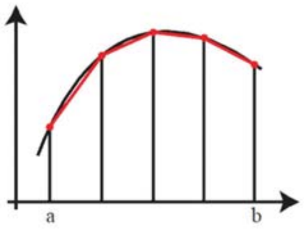
Trapezoidal rule

- The trapezoidal rule divides up the area under the function into trapezoids, rather than rectangles. The area of a trapezoid is the height times the average of the parallel bases

$$\text{Area}=\text{height}\left(\frac{\text{base 1}+\text{base 2}}{2}\right)=\left(\frac{y_3+y_4}{2}\right)\Delta x$$



Trapezoidal rule



Trapezoidal rule = sum of areas of trapezoids.

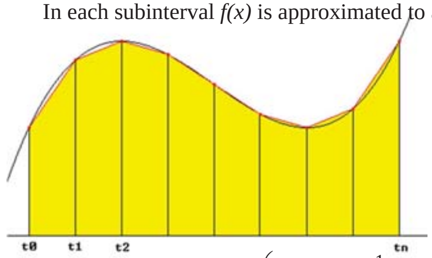
$$\text{Total Trapezoidal Area} = \Delta x\left(\frac{y_0+y_1}{2}+\frac{y_1+y_2}{2}+\frac{y_2+y_3}{2}+\ldots+\frac{y_{n-1}+y_n}{2}\right)$$
$$= \Delta x\left(\frac{y_0}{2}+y_1+y_2+\ldots+y_{n-1}+\frac{y_n}{2}\right)$$

Error estimation for trapezoidal rule

- We've divided up $[a,b]$ into n equal parts of length

$P=\{a=t_0,t_1,\ldots,t_n=b\}=\{a,a+h,a+2h,\ldots,a+nh=b\}$ partition of $[a,b]$

In each subinterval *f(x)* is approximated to a line



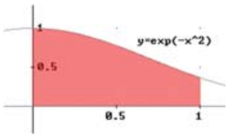
$$\int_a^b f(x)\,dx \cong T_n f=\frac{h}{2}\left(f(a)+2\sum_{k=1}^{n-1} f(a+kh)+f(b)\right)$$

$$E_n=\left|\int_a^b f(x)\,dx - T_n f\right| \le \frac{n\cdot h^3}{12}M_2=\frac{(b-a)^3}{12n^2}M_2 \ , \qquad M_2 \ge \max_{x\in[a,b]}\left|f''(x)\right|$$

Example

Example: Find $\int_0^1 e^{-x^2}dx$ using the trapezoidal rule

Exact value using *Mathematica* : 0.7468241328…



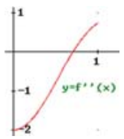
***If we make 10 subdivisions of the interval*:**

With $n=10,\ h=\frac{1}{10}$ we find $[0,1]:\ P=\left\{0,\frac{1}{10},\frac{2}{10},\ldots,\frac{9}{10},1\right\}$

$$\int_0^1 e^{-x^2}dx \cong T_{10} f=\frac{1}{20}\left(1+2\left(e^{-1/100}+e^{-4/100}+\cdots+e^{-81/100}\right)+\frac{1}{e}\right)=0.7462107961\ldots$$

To calculate the error estimation we use the second derivative of e^{-x^2} in [0,1]

$$f'(x)=-2xe^{-x^2} \Rightarrow f''(x)=2e^{-x^2}\left(2x^2-1\right)\Rightarrow\left|f''(x)\right|\le 6 \overset{?}{\underset{\text{improve it}}{\overset{\text{we can}}{=}}} M_2$$



$$E_{10}\le\frac{6}{12\cdot 10^2}=\frac{1}{200}=0.005 \quad \text{(practically two exact decimals)}$$

Example (cont')

- **Five exact decimals (at least)**

We need to calculate $E_n<10^{-6}$ and calculate (n), the number of subdivisions. Using $M_2=6$,

$$E_n\le\frac{6}{12n^2}=\frac{1}{2n^2}<10^{-6} \iff n^2>5\cdot 10^5 \Rightarrow n\ge 708$$

We'll obtain the desired result with 708 subdivisions of [0,1] , $h=\frac{1}{708}$

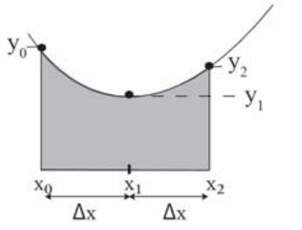
$$\int_0^1 e^{-x^2}dx \cong T_{708} f=\frac{1}{1416}\left(f(0)+2\sum_{k=1}^{707} f\left(\frac{k}{708}\right)+f(1)\right) \overset{?}{=} 0.7468240104\ldots$$

Mathematica

The final result $\int_0^1 e^{-x^2}dx=0.7468241328\ldots$, is better than the expected one

Simpson's rule

- This approach often yields more accurate results than the trapezoidal rule does. Here, we match quadraties (i.e. parabolas), instead of straight or slanted lines, to the graph. This approach requires an even number of intervals.



Simpson's rule

$$\text{Area under parabola} = (\text{base})(\text{weighted average height}) = (2\Delta x)\left(\frac{y_0 + 4y_1 + y_2}{6}\right)$$

Simpson's rule for n intervals (n must be even!)

$$\text{Area} = (2\Delta x)\left(\frac{1}{6}\right)[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + (y_4 + 4y_5 + y_6) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)]$$

Notice the following pattern in the coefficients:

1	4	1				
		1	4	1		
				1	4	1
1	4	2	4	2	4	1

Simpson's rule

Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n)$$

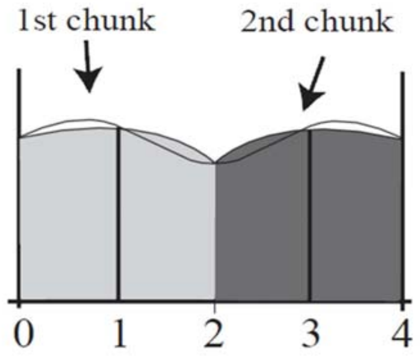
The pattern of coefficients in parentheses is:

		1	4	1			=	sum 6
	1	4	2	4	1		=	sum 12
1	4	2	4	2	4	1	=	sum 18

To double check – plug in $f(x) = 1$ (n even!).

$$\frac{\Delta x}{3}(1 + 4 + 2 + 4 + 2 + \cdots + 2 + 4 + 1) = \frac{\Delta x}{3}\left(1 + 1 + 4\left(\frac{n}{2}\right) + 2\left(\frac{n}{2} - 1\right)\right) = n\Delta x \quad (n \text{ even})$$

Simpson's rule

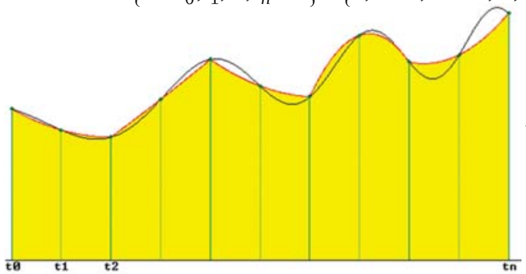


$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n)$$

Error estimation for Simpson's rule

We've divided up $[a,b]$ in n (even) equal parts of length $h=\dfrac{b-a}{n}$

$$P=\{a=t_0,t_1,\dots,t_n=b\}=\{a,a+h,a+2h,\dots,a+nh=b\}$$



$$\frac{h}{3}\underbrace{(1+4+2+4+2+\cdots+2+4+1)}_{\text{scheme of coefficients}}$$

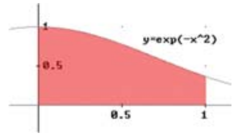
$$\int_a^b f(x)\,dx \cong S_n f=\frac{h}{3}\left(f(a)+4\sum_{k=0}^{n/2-1} f(a+(2k+1)h)+2\sum_{k=1}^{n/2-1} f(a+2kh)+f(b)\right)$$

$$E_n=\left|\int_a^b f(x)\,dx-S_n f\right|\le\frac{n\cdot h^5}{180}M_4=\frac{(b-a)^5}{180n^4}M_4\ ,\qquad M_4\ge \max_{x\in[a,b]}\left|f^{(iv)}(x)\right|$$

Example

Example: Find $\int_0^1 e^{-x^2}dx$ using Simpson's rule

Exact value using *Mathematica* : 0.7468241328…



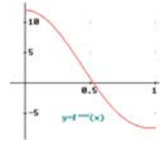
***If we make 10 subdivisions of the interval*:**

With $n=10,\ h=\dfrac{1}{10}$ we find $[0,1]:\ P=\left\{0,\dfrac{1}{10},\dfrac{2}{10},\dots,\dfrac{9}{10},1\right\}$

$$\int_0^1 e^{-x^2}dx\cong S_{10}f=\frac{1}{30}\left(\begin{array}{l}1+4\left(e^{-1/100}+e^{-9/100}+\cdots+e^{-81/100}\right)\\ +2\left(e^{-4/100}+e^{-16/100}+\cdots+e^{-64/100}\right)+\dfrac{1}{e}\end{array}\right)=0.\underline{746824}9482\dots$$

To calculate the error estimation we use the forth derivative of e^{-x^2} in $[0,1]$

$$f^{(iv)}(x)=4e^{-x^2}\left(4x^4-12x^2+3\right)\Rightarrow\left|f^{(iv)}(x)\right|\overset{?}{\le}76\longrightarrow M_4$$



$$E_{10}\le\frac{76}{180\cdot 10^4}<0.000043$$

Example (cont')

We want to calculate the number of subdivisions (n) such as $E_n<10^{-9}$. Using that $M_4=76$,

$$E_n\le\frac{76}{180n^4}<10^{-9}\ \Leftrightarrow\ n^4>\frac{38}{9}\cdot 10^8\ \Rightarrow\ n\ge 144$$

and consequently, doing 144 subdivisions of [0,1], h=1/144 and

$$\int_0^1 e^{-x^2}dx\cong S_{144}f=\frac{1}{432}\left(f(0)+4\sum_{k=0}^{71}f\left(\frac{2k+1}{144}\right)+2\sum_{k=1}^{71}f\left(\frac{2k}{144}\right)+f(1)\right)$$

$$\overset{?}{=}\ 0.\underline{7468241328}31439\dots$$

Mathematica

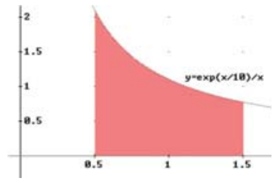
$$\int_0^1 e^{-x^2}dx=0.746824132812427\dots$$

this method gives us ten exacts decimals (better than expected)

Example

- **Example:** Approximate $\int_{0.5}^{1.5}\frac{\exp(x/10)}{x}dx$ using trapezoidal and Simpson's rules with 10 subdivision (you must obtain 6 exact decimals).

Real value of the integral *Mathematica* : 1.203798181…



***Trapezoid* (n=10):**

$h=0.1$ y $P=\{0.5,0.6,0.7,\dots,1.4,1.5\}$

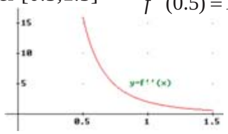
$$\int_{0.5}^{1.5}\frac{\exp(x/10)}{x}dx\cong T_{10}f=\frac{1}{20}\Big(f(0.5)+2\big(f(0.6)+f(0.7)+\cdots+f(1.4)\big)+f(1.5)\Big)$$

$$=1.\underline{20}6748518\dots$$

we can bound the error (with a graph) f'' over [0.5,1.5]

$$f''(0.5)=16.000346\ldots<17=M_2$$

$$f''(x)=\frac{\exp(x/10)}{x}\left(x^2-20x+200\right)$$



$$M_2=22$$

$$E_{10}\le\frac{17}{12\cdot 10^2}<0.02$$

Example (cont')

- ***Trapezoid (six exact decimals)***

$$M_2 = 17,\quad E_n \le \frac{17}{12n^2} < 10^{-7} \iff n^2 > \frac{17}{12}\cdot 10^7 \Rightarrow n \ge 3764$$

with 3764 subdivisions of [0.5,1.5], h=1 / 3764 and

$$\int_{0.5}^{1.5} \frac{\exp(x/10)}{x}\,dx \cong T_{3764} f = \frac{1}{7528}\left(f(0.5) + 2\sum_{k=1}^{3763} f\left(0.5 + \frac{k}{3764}\right) + f(1.5) \right)$$

$$\overset{?}{=} \underline{1.203798}202$$

Mathematica

$$\int_{0.5}^{1.5} \frac{\exp(x/10)}{x}\,dx = 1.203798181\ldots$$

Approximate calculus

Example (cont')

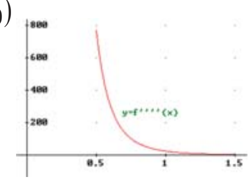
We are going to bound the error (with the graph) $f^{(iv)}$ over [0.5,1.5]

$$f^{(iv)}(x) = \frac{\exp(x/10)}{10^4 x^5}\left(x^4 - 40x^3 + 1200x^2 - 24000x + 240000\right)$$

$$f^{(iv)}(0.5) = 768.000002\ldots < 769 = M_4$$

hand made bound; $M_4 = 1037$

$$E_{10} \le \frac{769}{180 \cdot 10^4} < 0.00045 \quad \text{(three exact decimals)}$$



- ***Simpson (6 exact decimals)***

$$M_4 = 769,\quad E_n \le \frac{769}{180n^4} < 10^{-7} \iff n^4 > \frac{769}{18}\cdot 10^6 \Rightarrow n \ge 82$$

Now h=1/ 82 and

$$\int_{0.5}^{1.5} \frac{\exp(x/10)}{x}\,dx \cong S_{82} f = \frac{1}{246}\left(f(0.5) + 4\sum_{k=0}^{40} f\left(0.5 + \frac{2k+1}{82}\right) + 2\sum_{k=1}^{40} f\left(0.5 + \frac{2k}{82}\right) + f(1.5) \right)$$

$$\underset{\text{DERIVE}}{\overset{?}{=}} \underline{1.2037981}92\ldots \quad \text{(seven exact decimals)}$$

Approximate calculus

Example (cont')

- ***Simpson (n=10)***

Now h=0.1 and P={0.5, 0.6, 0.7,…, 1.4,1.5} and

$$\int_{0.5}^{1.5} \frac{\exp(x/10)}{x}\,dx \cong S_{10} f = \frac{1}{30}\left(\begin{array}{l} f(0.5) + 4\big(f(0.6) + f(0.8) + \cdots + f(1.4)\big) \\ \quad + 2\big(f(0.7) + f(0.9) + \cdots + f(1.3)\big) + f(1.5) \end{array} \right)$$

$$= \underline{1.203}846491\ldots$$

(three exact decimals)