Session 3: Implications

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Implications

An *implication* is a propositional form of the type $A \to B$ that is a tautology. If $A \to B$ is an implication, we will denote it by $A \Rightarrow B$ or by $A \vdash B$, and we will say that A *implies* B. A is called *antecedent* and B is called *consequent*.

We saw in classroom exercise 3 of Session 2 that there exists an implication $A \Rightarrow B$ if and only if "whenever there is a 1 at a position of the truth table of A, there is also a 1 at the same position of the truth table of B". In other words, "whenever A is true, B is also true". That is, if "B is deduced from A".

Remark. Compare the definitions of equivalence and implication:

- An *equivalence* is a biconditional $A \leftrightarrow B$ that is a tautology.
- An *implication* is a conditional $A \to B$ that is a tautology.

For example, in the classroom exercise 2 of Session 2, it is proved that the following conditional is, in fact, an implication: $(P \to Q) \wedge P \to Q$. Then,

- we can write $(P \to Q) \wedge P \Rightarrow Q$ or $(P \to Q) \wedge P \vdash Q$
- and we can say that $(P \to Q) \wedge P$ implies Q.

When the antecedent of an implication $A \Rightarrow B$ consists of the conjunction of several propositional forms, that is, $A = P_1 \wedge P_2 \wedge \cdots \wedge P_n$, we may denote the implication in this way:

$$\{P_1, P_2, \ldots, P_n\} \vdash B$$

(this is the notation that is used in the list of implications of the file Table.pdf).

So, we can write the previous example as $\{P \to Q, P\} \vdash Q$. This implication has the name *modus ponens*.

2 Usual implications

Now, we introduce a list of usual implications (with their corresponding names) that can be proved using either truth tables or equivalences (as in the classroom exercise 2 of Session 2). We also include "short names" (that you can use in exercises).

1. **Simplification** (Simp)

   If we have the conjunction of two propositional forms, then we can take out one of them.

   - $\{P \wedge Q\} \vdash P$
   - $\{P \wedge Q\} \vdash Q$

2. **Addition** (Add)

   If we have a propositional form, then we can add (using disjunction) another propositional form.

   - $\{P\} \vdash P \vee Q$
   - $\{Q\} \vdash P \vee Q$

3. **Modus ponendo ponens (or modus ponens)** (MP). This means *"the way that affirms by affirming"*.

   If we have a conditional and the antecedent is true, then we can get the consequent .

   - $\{P \to Q, P\} \vdash Q$

4. **Modus tollendo tollens (or modus tollens)** (MT). This means *"the way that denies by denying"*.

   If we have a conditional and we have the negation of the consequent, then we can get the negation of the antecedent.

   - $\{P \to Q, \neg Q\} \vdash \neg P$

5. **Disjunctive syllogism (or modus tollendo ponens)** (DS or MTP).

   If we have a disjunction and the negation of one of the involved propositional forms, then we can affirm the other propositional form.

   - $\{P \vee Q, \neg P\} \vdash Q$

6. **Hypothetical syllogism** (HS)

   If the consequent of a conditional is the antecedent of another one, then we can "connect" them.

   - $\{P \to Q, Q \to R\} \vdash (P \to R)$

3 A "deduction" is a sequence of implications

Notice that, when we logically "deduce things from other things", we are essentially considering sequences of implications. More specifically, our starting point is a set of hypotheses (or *premises*) that we assume that are true and, then, we "concatenate" implications until deducing a "conclusion". This is the *natural* way of the human thought. Let's see an example.

Assume that *John* is a certain student the subject Discrete Mathematics of Engineering in Computer Science. Assume that John has two ways to pass the subject: either passing the written exam or passing an oral exam. Then the following rule (or hypothesis) is naturally assumed:

$H_1 =$"If John passes the written exam or passes the oral exam, then he will pass Discrete Mathematics"

Suppose that John realizes the written exam and he passes it. Then everybody will naturally deduce that John will pass Discrete Mathematics, isn't it? But, what is the process? It is nothing but a concatenation of some implications that we have seen in the previous section. To see it we need to formalize the involved propositions (as we did in Section 1):

$W =$"John passes the written exam"

$O =$"John passes the oral exam"

$D =$"John passes Discrete Mathematics"

With these "small" propositions we can formalize H_1 as

$$H_1 = W \vee O \to D.$$

The hypotheses of our reasoning are:

- $H_1 = W \vee O \to D$ (that is, "If John passes the written exam or passes the oral exam, then he will pass Discrete Mathematics")
- $H_2 = W$ (that is, "John passes the written exam")

The *addition rule* says that $\{P\} \vdash P \vee Q$. That is, if we have a propositional form, then we can add (using disjunction) another propositional form. In our case we have W (that is, it is true). Then we can apply this rule replacing P by W and Q by O:

$$\{W\} \vdash W \vee O.$$

Then, we "have deduced" $W \vee O$.

But the *modus ponens rule* says that $\{P \to Q, P\} \vdash Q$, that is, if we have a conditional and the antecedent is true, then we can get the consequent. But... notice that we can apply this rule taking $P = W \vee O$ and $Q = D$! Indeed, the conditional $W \vee O \to D$ is true because it is H_1; moreover

$W \vee O$ (the antecedent) is true because we have just deduced it. Therefore we conclude D ("John passes Discrete Mathematics").

You can see, then, that the *natural* reasoning, in this case, consists of applying, first, the addition rule and, finally, the modus ponens rule.

The scheme that we will use to formalize the reasoning is the following one:

H1:	$W \vee O \to D$	
H2:	W	
3:	$W \vee O$	Addition (2)
4:	D	Modus Ponens (1,3)

The propositional forms which appear in the lines of the scheme are called *hypotheses* or *premises*. The first two lines correspond with our *initial* hypotheses (or premises): H_1 and H_2. In the third line we write the result of applying the addition rule to H_2 (notice that, from H_2, we have deduced $W \vee O$). The forth line shows the result of applying modus ponens to lines 1 and 3 (H_1 is a conditional and 3 is the antecedent of this conditional).

4 Inference rules

Although the essence of "deducing things from others" is the *implication*, we can use other laws in the process. For example, we can replace a hypothesis (or premise) by an equivalent propositional form.

The process that allows us to deduce a conclusion from some hypotheses (or premises) is called *inference*. It is a finite sequence of steps that are governed by the following *inference rules*:

1. **Use of the hypotheses**

   Every hypothesis (or premise) that is obtained during the inference can be used at any step of the process.

2. **Use of equivalences**

   Every propositional form can be replaced by an equivalent one.

3. **Use of tautologies**

   At every step of the process we can introduce a tautology as a new hypothesis (or premise).

4. **Use of implications**

   At every step of the process we can apply an implication rule (of those listed in Section 2) to any set of deduced hypotheses.

5. **Conjunction** (Conj).

   If we have a hypothesis P and another hypothesis Q, then we can introduce $P \wedge Q$ as new hypothesis. In the file Table.pdf we have represented this rule by

- $\{P, Q\} \vdash P \wedge Q$

You can use the short name *Conj* for it.

Example 1. Let us prove that, from the hypotheses $P \to Q$, $P \leftrightarrow \neg R$ and $\neg Q$, we can deduce, using inference rules, the conclusion R:

H1:	$P \to Q$	
H2:	$P \leftrightarrow \neg R$	
H3:	$\neg Q$	
4:	$\neg P$	Modus tollens (1,3)
5:	$(P \to \neg R) \wedge (\neg R \to P)$	Conditional-biconditional (2)
6:	$\neg R \to P$	Simplification (5)
7:	$\neg\neg R$	Modus tollens (4,6)
8:	R	Double negation (7)

Notice that, in this inference process, we have used implications, but also equivalences.

Remark. Notice that, in virtue of the Hypothetical Syllogism, when we perform an inference process such that $H_1, H_2, \ldots, H_n$ are the initial hypotheses and C is the conclusion, we are, actually, proving the implication

$$H_1 \wedge H_2 \wedge \cdots \wedge H_n \Rightarrow C.$$

Then, in the above example, we have proved the implication

$$(P \to Q) \wedge (P \leftrightarrow \neg R) \wedge \neg Q \Rightarrow R,$$

that is, that $(P \to Q) \wedge (P \leftrightarrow \neg R) \wedge \neg Q$ implies R.

Although it is interesting to see a first overview of the notion of inference, its accurate learning needs some guided practice, which we will start in the classroom exercises.