Session 4: Conditional and biconditional inference

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

In the previous session we defined the *inference* as the process that allows us to deduce a conclusion from some hypotheses (or premises) using *inference rules*. But we should name this process, more accurately, *direct inference*. We saw several examples of *direct inference* in the classroom exercises.

In the next two sessions we are going to enlarge the concept of *inference* allowing certain *new rules* and giving rise to 3 new types of inference (in addition to the *direct* one): *conditional inference*, *biconditional inference* and *inference by contradiction*.

1 Conditional inference

The *conditional inference* can be used to prove a conditional $Q \to R$ from several hypotheses $H_1, H_2, \ldots, H_n$. That is, it can be used when *the conclusion of the inference is a conditional*.

It is based on the equivalence given by the *exportation law*:

$$H \to (Q \to R) \equiv H \wedge Q \to R.$$

This shows that

$H \to (Q \to R)$ is an implication if and only if $H \wedge Q \to R$ is an implication.

In other words:

$\{H\} \vdash (Q \to R)$ if and only if $\{H, Q\} \vdash R$.

Using this fact, when the conclusion of an inference is a conditional $Q \to R$, we can proceed as follows:

1. Add Q to the set of hypotheses as an additional hypothesis, and
2. deduce R from Q and the other hypotheses by direct inference.

Example 1. We will prove that the hypotheses

H1: $U \to R$
H2: $R \wedge S \to P \vee T$
H3: $Q \to U \wedge S$
H4: $\neg T$

let us conclude $Q \to P$.

H1:	$U \to R$		
H2:	$R \wedge S \to P \vee T$		
H3:	$Q \to U \wedge S$		
H4:	$\neg T$		
5:		Q	Additional hypotesis (conditional inference)
6:		$U \wedge S$	Modus ponens (3,5)
7:		U	Simplification (6)
8:		R	Modus ponens (1,7)
9:		S	Simplification (6)
10:		$R \wedge S$	Conjunction (8,9)
11:		$P \vee T$	Modus ponens (2,10)
12:		P	Disjunctive syllogism (4,11)
13:	$Q \to P$		Conclusion (conditional inference)

Observe the above scheme: We have added Q as a new hypothesis and we have proved P using direct inference; this process is written indented to the right of the previous hypotheses. Finally we have added, in the las line (and with no indentation) the conditional conclusion we wanted to prove ($Q \to P$).

Notice that, when the conclusion is a conditional, it is not compulsory to use *conditional inference*. It is an available option that you may use or not (perhaps you are able to reach the conclusion using direct inference).

2 Biconditional inference

Biconditional inference can be used when the conclusion is of the type $P \leftrightarrow Q$ (a biconditional). It is based on the equivalence *conditional-biconditional*:

$$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P).$$

When we have to prove a biconditional $P \leftrightarrow Q$ we will proceed as follows:

1. We prove that the hypotheses let us deduce the conditional $P \to Q$ (called *direct conditional*), and
2. we also prove that the hypotheses let us deduce $Q \to P$ (called *converse conditional*).

In other words, the proof of $P \leftrightarrow Q$ is decomposed into two inferences: the one whose conclusion is the *direct conditional* $(P \to Q)$ and the one whose conclusion is the *converse conditional* $(Q \to P)$. Of course, in each one of both inferences, we can apply *conditional inference*.

Example 2. We will prove that the hypotheses

H1: $\neg(\neg P \wedge \neg Q)$
H2: $S \to \neg Q$
H3: $\neg P \vee S$

let us conclude $\neg P \leftrightarrow Q$.

H1:	$\neg(\neg P \wedge \neg Q)$		
H2:	$S \to \neg Q$		
H3:	$\neg P \vee S$		
	Direct conditional:		
4:		$\neg P$	Additional hypotesis (conditional inference)
5:		$P \vee Q$	De Morgan and Double Negation (1)
6:		Q	Modus tollendo ponens (5,4)
7:	$\neg P \to Q$		Conclusion (conditional inference)
	Converse conditional:		
8:		Q	Additional hypotesis (conditional inference)
9:		$\neg S$	Modus tollens (2,8)
10:		$\neg P$	Modus tollendo ponens (3,9)
11:	$Q \to \neg P$		Conclusion (conditional inference)
12:	$(\neg P \to Q) \wedge (Q \to \neg P)$		Conjunction (7,11)
13:	$Q \leftrightarrow \neg P$		Conditional-biconditional (12)

In lines 4,5,6 and 7 we have proved that the hypotheses H_1, H_2 and H_3 imply the direct conditional $\neg P \to Q$. Inside this process we have used *conditional inference* (because the conclusion is a conditional).

Similarly, in lines 8,9,10 and 11 we have proved that the hypotheses H_1, H_2 and H_3 imply the converse conditional $Q \to \neg P$ (using, again, *conditional inference*).

Finally, in lines 12 and 13, we have applied Conjunction and Conditional-biconditional in order to deduce the desired conclusion $Q \leftrightarrow \neg P$.