Sessions 15 and 16: Binary relations and properties.

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Introduction

The mathematical notion of a relation, like that of a set, is a very general one. It is one of the key concepts of mathematics and examples of relations occur throughout the subject. Although one can define the concept of n-ary relation (for arbitrary $n \geq 2$) we will focus our attention on 2-ary relations (or binary relations). Two special types of (binary) relation are particularly important: equivalence relations and order relations.

2 Definition of binary relation and examples

A **binary relation** R between two sets A and B is a subset of the cartesian product $A \times B$.

The first thing to notice is that a relation as we have defined is a set; namely a set of ordered pairs. If R is a relation between A and B, we will say that $a \in A$ **is related to** $b \in B$ if $(a, b) \in R$. Thus the relation R itself is simply the set of all related pairs of elements. For the most part we will adopt the commonly used notation and write $a \, R \, b$ to denote that a **is related to** b, and $a \not R \, b$ to denote $(a, b) \notin R$ or a **is not related to** b. If $A = B$ it is also common to refer to R as a **relation on** A.

Example 1. (a) Let $A = \{\text{Cities of the world}\}$, $B = \{\text{Countries of the world}\}$ and the relation between A and B given by

$$R = \{(a, b) \in A \times B \mid a \text{ is the capital city of } b\}.$$

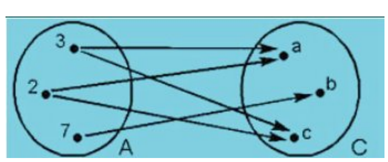
For example we have: (Paris) R (France), (Moscow) R (Russia), (Tirana) R (Albania), etc.

Also we have: (London) $\not R$ (Zimbabwe), (Naples) $\not R$ (Italy), etc.

(b) Let $A = \{2, 3, 7\}$ and $C = \{a, b, c\}$. Consider the following binary relation between A and C:

$$R = \{(3, a), (3, c), (2, a), (2, c), (7, b)\}.$$

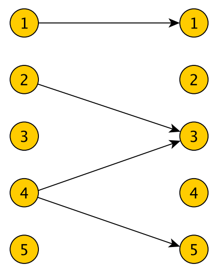
Since A and C are small finite sets, we can represent this relation using Venn diagrams:



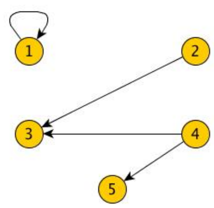
(c) Let $A = \{1, 2, 3, 4, 5\}$ and R the relation on A defined by

$$R = \{(1, 1), (2, 3), (4, 3), (4, 5)\}.$$

As in the previous example, we can represent this relation by means of Venn diagrams:



However, taking advantage of the fact that the relation is defined on a single set A, it is better in this case to represent it by means of a directed graph whose vertices are the elements of A and whose edges represent the pairs of the relation:



Notice that a binary relation between two sets A and B **can be thought as the graph of a correspondence from** A **to** B. Then... what is the difference between a binary relation and a correspondence? Technically, there is no difference except the **way of thinking**: a correspondence is thought as an **assignment** of elements and a relation is thought as a **criterium that relates** elements.

3 Operations with relations

Binary relations are **sets** and, therefore, we can consider **union**, **intersection**, **complement** and **difference** of relations between two sets A and B.

Binary relations are **correspondences** and, therefore, we can give the following definitions:

- Given a relation R, the **inverse relation** of R, denoted R^{-1}, is the relation associated with the inverse correspondence of R (viewed as a correspondence). That is:

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}.$$

- Given a relation R between A and B, and a relation S between B and C, the **composition** $S \circ R$ is the relation between A and C given by the composition of S and R viewed as correspondences. That is:

$$S \circ R = \{(a,c) \in A \times C \mid \exists b \in B \text{ with } (a,b) \in R \ \wedge \ (b,c) \in S\}.$$

- The **domain** and **image** of a relation R between A and B are, respectively, the domain and image of R viewed as a correspondence. That is:

$$Dom(R) = \{a \in A \mid \exists b \in B, (a,b) \in R\},$$
$$Im(R) = \{b \in B \mid \exists a \in A, (a,b) \in R\}$$

4 Matrix representation of a binary relation

A binary relation can be represented by a matrix whenever the sets that are involved in the relation are finite. Suppose that $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_p\}$ and R is a binary relation between A and B. Then, the associated matrix to R is the boolean matrix (that only consists on ones and zeros) with m arrows and p columns

$$M_R = \begin{pmatrix} r_{11} & \cdots & r_{1p} \\ \vdots & \vdots & \vdots \\ r_{m1} & \cdots & r_{mp} \end{pmatrix} \text{ with } r_{ij} := \begin{cases} 1 & \text{if} \quad a_i R\, b_j \\ 0 & \text{if} \quad a_i \not{R}\, b_j \end{cases}$$

Example 2. Consider the sets $A = \{2, 3, 5\}$ and $B = \{4, 6, 9, 10\}$, and the relation

$$R := \{(2,4), (2,6), (2,10), (3,6), (3,9), (5,10)\} \subseteq A \times B$$

(that is, aRb if and only if a divides b). Then the associated matrix to R is

$$M_R = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

5 Properties of a binary relation on a set

Up to now we have not justified our assertion that relations are important in mathematics. Indeed, if all we were able to do with relations between sets were to draw diagrams to represent them, the concept of a relation would not be significant. Its importance is mainly due to special kinds of relations which satisfy additional properties. We look now at some of the properties which a relation on a set may have.

Some properties that a binary relation on a set may satisfy:

Reflexive: if $\forall x \in A$, xRx

Symmetric: if $\forall x, y \in A$, $xRy \longrightarrow yRx$.

Antisymmetric: if $\forall x, y \in A$, $(xRy) \wedge (yRx) \longrightarrow x = y$.

Transitive: if $\forall x, y, z \in A$, $(xRy) \wedge (yRz) \longrightarrow (xRz)$.

5.1 Reflexive relations

A binary relation on a set A is *reflexive* if **every** element of A is related to itself.

For example, the relation of equality, "$=$", is reflexive. Observe that, for any $a \in A$, a is equal to itself (that is, $a = a$). So "$=$" is reflexive.

If the set is finite, in the directed graph a reflexive relation **we have arrows for all values in the the set pointing back to themselves**:

Example 3. (a) On the set of real numbers $\mathbb{R}$ we consider the relation "$\leq$", that is, for every pair of real numbers a, b, we say that aRb if and only if $a \leq b$. (Usually we do not name "R" to this relation; instead, we write $a \leq b$ directly). Notice that this relation is reflexive because $a \leq a$ for every real number a (that is, every real number is related to itself).

(b) On the set of real numbers $\mathbb{R}$ we consider the relation "$\leq$", that is, for every pair of real numbers a, b, we say that aRb if and only if $a < b$. (As before, usually we do not name "R" to this relation; instead, we write $a < b$ directly). Notice that this relation is not reflexive because, for example, $1 \nless 1$ (that is, the number 1 is not related to itself); in fact, this happens for every real number.

(c) Consider the set $A = \{a, b, c\}$ and the relation

$$R = \{(a, a), (b, b), (a, b), (a, c)\}.$$

This relation is not reflexive because c is not related to itself (that is, $(c, c) \notin R$).

5.2 Symmetric relations

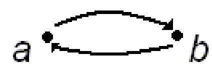
A binary relation R on a set A is *symmetric* if

$$\forall\, x, y \in A,\ xRy \longrightarrow yRx,$$

that is, **for every** couple of elements x, y of A, if x is related to y then y is related to x.

The relation of equality again is symmetric. If $x = y$, we can also write that $y = x$.

If the set A is finite, in the directed graph of a symmetric relation: for each arrow we have also an opposite arrow, i.e. either there is no arrow between x and y, or there are two arrows between them (one of them pointing from x to y and an arrow back from y to x).



Example 4.

(a) On the set of real numbers $\mathbb{R}$ we consider the relation "$\leq$". Notice that this relation is not symmetric because, for example, $2 \leq 3$ but $3 \not\leq 2$ (that is, 2 is related to 3 but 3 is not related to 2). The same happens for the relation "$<$".

(b) Consider the set $A = \{a, b, c\}$ and the relation

$$R = \{(a, a), (b, b), (a, b), (a, c)\}.$$

This relation is not symmetric because a is related to b but b is not related to a. (That is, in the directed graph, there is only one arrow between a and b).

(c) Consider the set $A = \{a, b, c\}$ and the relation

$$R = \{(a, a), (b, b), (a, b), (b, a), (c, c)\}.$$

This relation is symmetric because the a and b are the unique different elements which are related and aRb and bRa. In the directed graph, a and b are the unique different vertices with some arrow joining them and there is an arrow in each direction (from a to b and from b to a).

5.3 Antisymmetric relations

A binary relation on a set A is antisymmetric if, $\forall x, y \in A$, $(xRy) \wedge (yRx) \longrightarrow x = y$. Perhaps the application of the "transposition law" and the "De Morgan law" will help you to understand this property:

$$\forall\, x, y \in A, \;\; x \neq y \longrightarrow (x \not R\, y) \vee (y \not R\, x).$$

In other words, R is symmetric if, for every pair of different elements x, y of A, *at most one* of this conditions hold: xRy or yRx (maybe none of them, or one of them, but not both).

If A is finite it means that, in the directed graph of the relation, between two **different** vertices there is, at most, one arrow (*never* two arrows).

For example, if A is a set with two or more elements, the relation of equality, "=" is not anti-symmetric because, if we consider two different elements of A, x and y, we have that $x = y$ and $y = x$.

Example 5. (a) In the set of real numbers $\mathbb{R}$, the relation "$\leq$" is antisymmetric because, if x and y are two **different** real numbers, either $x \leq y$ or $y \leq x$, but **not both conditions simultaneously**. The same happens with the relation "$<$".

(b) Consider the set $A = \{a, b, c\}$ and the relation

$$R = \{(a, a), (b, b), (a, b), (a, c)\}.$$

This relation is antisymmetric.

(c) Consider the set $A = \{a, b, c\}$ and the relation

$$R = \{(a, a), (b, b), (a, b), (b, a), (c, c)\}.$$

This relation is not antisymmetric because aRb and bRa.

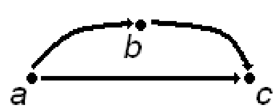
Notice that "antisymmetric" is not the same as "non-symmetric". Why?

5.4 Transitive relations

A binary relation R on a set A is transitive if, $\forall x, y, z \in A$, $(xRy) \wedge (yRz) \longrightarrow (xRz)$.

An informal simile that may help you to understand this property is the following one: the relation R is transitive if you may think that it "transmits something" from one element to another. If the relation behaves in this way, xRy means that R "transmits something" to y and yRz means that R "transmits something" to z; then it must occur that, if both conditions (xRy and yRz) are true, then "the thing has been transmitted from x to z", that is, xRz.

If the set A is finite, the relation R is transitive if, whenever we have an arrow from a to b and an arrow from b to c then there is an arrow from a to c:



(Notice that a, b, c are not necessarily different!!)

For example, the equality relation is transitive because, if $x = y$ and $y = z$ then $x = z$.

Example 6. (a) The relation $\leq$ on the set of real numbers is transitive because, if $x \leq y$ and $y \leq z$ then $x \leq z$. The same happens with "$<$".

(b) Consider the set $A = \{a, b\}$ and the relation

$$R = \{(b, b), (a, b), (b, a)\}.$$

This relation is not transitive because we have that aRb and bRa, but $a \not R a$. That is, the conditional

$$xRy \ \wedge \ yRz \longrightarrow xRz$$

is not true for all possible choices of elements x, y, z in A: for the specific choice $x = a$, $y = b$ and $z = a$, the conditional is false (the left-hand-side is true but the right-hand-side is false).

6 Characterization of the properties

The objective of this section consists of providing methods to check whether the above explained properties are satisfied or not. One of these methods uses "boolean matrices" and operations between them. Then, we include a first section concerning boolean matrices.

6.1 Boolean matrices

A boolean matrix is a matrix of ones and zeroes.

Notice that the matrix associated with a binary relation on a finite set is a boolean matrix.

We define the following operations in the set $\{1, 0\}$: $1 \wedge 1 = 1$, $1 \wedge 0 = 0$, $0 \wedge 1 = 0$, $0 \wedge 0 = 0$, $1 \vee 1 = 1$, $1 \vee 0 = 1$, $0 \vee 1 = 1$ and $0 \vee 0 = 0$. (That is, we consider the meaning of the known connectives $\wedge$ and $\vee$, interpreting 1 as "true" and 0 as "false"). Similarly we define $\neg 1 = 0$ and $\neg 0 = 1$. Using this, for each pair of boolean matrices M and N of the same dimensions, we define the following operations:

- $M \wedge N$ is the matrix whose dimensions coincide with the ones of M and N and whose elements are computed as $m_{ij} \wedge n_{ij}$, where m_{ij} and n_{ij} are the elements of M and N located at position (i, j).
- $M \vee N$ is the matrix whose dimensions coincide with the ones of M and N and whose elements are computed as $m_{ij} \vee n_{ij}$, where m_{ij} and n_{ij} are the elements of M and N located at position (i, j).
- $\neg M$ is the matrix whose dimensions coincide with those of M and whose elements are computed as $\neg m_{ij}$, where m_{ij} is the element of M located at position (i, j).

Example 7. If $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ then

$$M \wedge N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ M \vee N = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \ \neg M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Remark. Notice that, to operate two matrices with $\vee$, we can "sum" (usual sum of integers) element by element and then replace the non-zero elements by 1. Similarly, we can operate two matrices with $\wedge$ "multiplying" (usual product of integers) element by element.

Also we define the boolean product of two boolean matrices M and N of respective dimensions $m \times p$ and $p \times s$:

The **boolean product** of M and N, denoted by $M \odot N$, is the $m \times s$-matrix obtained performing the usual product MN but replacing the product by $\wedge$ and the sum by $\vee$. In other words, the element of $M \odot N$ located at position (i, j) is

$$(m_{i1} \wedge n_{1j}) \vee (m_{i2} \wedge n_{2j}) \vee \cdots \vee (m_{ip} \wedge n_{pj}).$$

Remark. The boolean product $M \odot N$ can also be calculated by computing the usual product MN and, after that, replacing all the non-zero entries by ones.

Example 8. If $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}$ then

$$MN = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow M \odot N = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Also we can *compare* two boolean matrices:

Given two matrices M and N of the same dimensions, we will write $M \leq N$ if every element of M is less than or equal to the corresponding element of N, that is, $m_{ij} \leq n_{ij}$ for all i, j.

Example 9.

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \leq \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

The following proposition shows the relationship between the operations with binary relations (as sets) and the operations with the boolean matrices obtained by their associated matrices (its proof is evident):

Proposition 1. Let R and S be two binary relations between finite sets A and B. Then:

(a) $M_{R^c} = \neg M_R$.

(b) $M_{R^{-1}} = M_R^t$.

(c) $M_{R \cup S} = M_R \vee M_S$.

(d) $M_{R \cap S} = M_R \wedge M_S$.

The following proposition shows that the composition of binary relations is associated with the boolean product of their respective matrices:

Proposition 2. Let A, B and C be finite sets and consider a relation R between A and B and a relation S between B and C. Then

$$M_{S \circ R} = M_R \odot M_S.$$

Proof. Assume that $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_p\}$. Using the definition of the composition of relations, for each pair of indices i, j such that $1 \leq i \leq m$ and $1 \leq j \leq p$ it holds that

$$a_i(S \circ R)c_j \Leftrightarrow \text{ there exists some } b_k \text{ such that } a_i R b_k \text{ and } b_k S c_j.$$

That is, $a_i(S \circ R)c_j$ if and only if there is a "1" at the position (i, k) of the matrix M_R and also at the position (k, j) of the matrix M_S. Using the definition of boolean product it is easy to deduce that this happens if and only there is a "1" at the position (i, j) of the boolean matrix $M_R \odot M_S$. □

Finally, we see that the inclusion of relations corresponds with the comparison of the corresponding matrices:

Proposition 3. Let R and S be two binary relation between finite sets A and B. Then

$$R \subseteq S \Leftrightarrow M_R \leq M_S.$$

The proof of this proposition is evident.

6.2 Characterization using operations of relations

For any set A, we will denote by Δ to the "equality relation" on A, that is, $a \Delta b$ if and only if $a = b$. In other words:

$$\Delta = \{(a, a) \in A \times A \mid a \in A\}.$$

The following theorem gives a characterization of all properties in terms of relation operations:

Theorem 1. Let R be a binary relation on a set A. Then

(a) R is reflexive if and only if $\Delta \subseteq R$.

(b) R is symmetric if and only if $R = R^{-1}$.

(c) R is antisymmetric if and only if $R \cap R^{-1} \subseteq \Delta$.

(d) R is transitive if and only if $R \circ R \subseteq R$.

Proof. (a) R is reflexive if and only if aRa for all $a \in A$, that is, if and only if $(a, a) \in R$ for all $a \in A$. This is equivalent to say that $\Delta \subseteq R$.

(b) R is symmetric if and only if, for all $a, b \in A$, it holds that $(a, b) \in R \leftrightarrow (b, a) \in R$. But $(b, a) \in R$ is equivalent to say that $(a, b) \in R^{-1}$. We deduce, then, that R is symmetric if and only if $R = R^{-1}$.

(c) R is antisymmetric if and only if, for all $a, b \in A$, the following conditional is true

$$aRb \ \wedge \ bRa \longrightarrow a = b.$$

The result follows by observing that the left-hand-side of the conditional is equivalent to $(a, b) \in R \cap R^{-1}$, and the right-hand-side is equivalent to $(a, b) \in \Delta$.

(d) Notice that R is transitive if and only if, for all $a, b, c \in A$, the following conditional is true:

$$aRb \ \wedge \ bRc \longrightarrow aRc.$$

$\Rightarrow$ Let us prove first the direct implication, that is, "if R is transitive then $R \circ R \subseteq R$":

Assume that R is transitive and take an arbitrary element (a, c) of $R \circ R$. This means that there exists $b \in A$ such that aRb and bRc. Since R is transitive we have that aRc, that is, $(a, c) \in R$.

$\Leftarrow$ Let us prove, now, the converse implication, that is, "if $R \circ R \subseteq R$ then R is transitive":

Assume that $R \circ R \subseteq R$ and suppose that a, b, c are three elements of A such that aRb and bRc. This means, by definition of composition of relations, that $(a, c) \in R \circ R$. But, this last set is contained into R, by assumption. Therefore $(a, c) \in R$, that is, aRc. This proves the transitivity of R.

□

Notice that, if A is a finite set, the matrix associated with the "equality relation" in A, Δ, is just the *identity matrix* I. Taking this fact and above propositions 1, 2 and 3 into account, it is easy to translate all the characterizations given in Theorem 1 in terms of matrices of relations:

Theorem 2. Let A be a finite set and let R be a binary relation on A. Then:

(a) R is reflexive if and only if all the elements on the main diagonal of M_R are ones (or, in other words, $I \leq M_R$).

(b) R is symmetric if and only if $M_R = M_R^t$ (that is, M_R is a symmetric matrix).

(c) R is antisymmetric if and only $M_R \wedge M_R^t \leq I$, that is, if there are not symmetric positions outside the main diagonal of M_R whose values are 1 simultaneously.

(d) R is transitive if and only if $M_R \odot M_R \leq M_R$.

This theorem is specially useful for those exercises in which we must check properties of binary relations defined on finite sets.