Session 2: Equivalences and simplification of propositional forms

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Boolean equivalences

In the previous section we defined the concept of “equivalence between two propositional forms”. Recall that two propositional forms (with the same variables) are equivalent if their truth tables are equal. The following equivalences are called *Boolean equivalences*[1]. You can prove them making the truth tables of each involved propositional form.

1. Associative laws (A)

$$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$$
$$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$$

2. Commutative laws (C)

$$P \vee Q \equiv Q \vee P$$
$$P \wedge Q \equiv Q \wedge P$$

3. Distributive laws (D)

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$
$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

4. Identity laws (Id)

$$P \vee \phi \equiv P$$
$$P \wedge \tau \equiv P$$

5. Inverse Element laws (Inv)

$$P \vee \neg P \equiv \tau$$
$$P \wedge \neg P \equiv \phi$$

We have included, between parentheses, a **short name** for each law. You can use this short names in the exercises.

[1] The term “Boolean” comes from the fact that these properties define the concept of “Boolean algebra”(we will study it in a forthcoming lesson).

2 More equivalences concerning disjunction and conjunction

Next, we introduce a list with more equivalences (including the short names):

- Absorption laws (Abs)

  $$\tau \vee P \equiv \tau, \quad \phi \wedge P \equiv \phi$$

- Simplification laws (Simp)

  $$P \vee (P \wedge Q) \equiv P, \quad P \wedge (P \vee Q) \equiv P$$

- Idempotent laws (Idemp)

  $$P \vee P \equiv P$$
  $$P \wedge P \equiv P$$

- De Morgan's laws (DM)

  $$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$
  $$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

- Double negation law (DN)

  $$\neg(\neg P) \equiv P$$

All these equivalences can be deduced from the Boolean laws. We omit the proof of this fact here, although there are some exercises about it.

3 Equivalences concerning conditional and biconditional

Finally, we give a list of equivalences concerning conditional and biconditional (an also conjunction and disjunction).

- Conditional-disjunction equivalence (CD)

  $$P \to Q \equiv \neg P \vee Q$$

- Conditional-biconditional equivalence (CB)

  $$(P \to Q) \wedge (Q \to P) \equiv P \leftrightarrow Q$$

- Transposition law (T)

$$P \to Q \equiv \neg Q \to \neg P$$

- Exportation law (E)

$$(P \land Q) \to R \equiv P \to (Q \to R)$$

As before, these equivalences can be deduced from the Boolean ones.

4 Exercises

Try to solve the following exercises **before** the next class. Check the solutions only if, after a reasonable period of time, you are not able to find a solution.

Exercise 1. Prove that the Idempotent law $P \lor P \equiv P$ can be deduced from the Boolean laws.

Exercise 2. Prove that the Absorption law $\tau \lor P \equiv \tau$ can be deduced from the Boolean laws.

Remark: Next class you will learn how to simplify propositional forms using the above equivalences. In the file **Table.pdf** you can find the list of equivalences (among other things). Print this file for the next class.

5 Solutions

(1)

$$
\begin{aligned}
P &\equiv P \vee \phi && \text{by Identity Elements law (Id)}\\
&\equiv P \vee (P \wedge \neg P) && \text{by Inverse Element law (Inv)}\\
&\equiv (P \vee P) \wedge (P \vee \neg P) && \text{by Distributive law (D)}\\
&\equiv (P \vee P) \wedge \tau && \text{by Inverse Element law (Inv)}\\
&\equiv P \vee P && \text{by Identity Elements law (Id)}
\end{aligned}
$$

(2)

$$
\begin{aligned}
\tau \vee P &\equiv (P \vee \neg P) \vee P && \text{by Inv}\\
&\equiv (\neg P \vee P) \vee P && \text{by C}\\
&\equiv \neg P \vee (P \vee P) && \text{by A}\\
&\equiv \neg P \vee P && \text{by Exercise 1}\\
&\equiv P \vee \neg P && \text{by C}\\
&\equiv \tau && \text{by Inv}
\end{aligned}
$$