Practices of discrete mathematics

Session 8 (directed graphs)

1. Directed graphs: basic concepts
2. Paths and connectivity
3. Eulerian directed graphs

Definition

A directed graph (or digraph) is a triple (V, A, φ) where:

1. V is a non-empty set whose elements are called vértices.
2. A is a finite set whose elements are called edges (or arcs).
3. $\varphi : A \to V \times V$ is a map (called incidence map) that assigns, to each edge A, an element of $V \times V$, that is, an ordered pair of vertices.

Example

Consider the graph $G = (V, A, \varphi)$, where

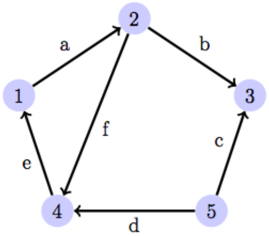
$$V = \{v_1, v_2, v_3, v_4, v_5\}, \quad A = \{a, b, c, d, e, f\}$$

and the incidence map φ is defined as follows:

$$\varphi(a) = (1, 2), \ \ \varphi(b) = (2, 3), \ \ \varphi(c) = (5, 3),$$

$$\varphi(d) = (5, 4), \ \ \varphi(e) = (4, 1), \ \ \varphi(f) = (2, 4)$$

This graph has the following diagram:



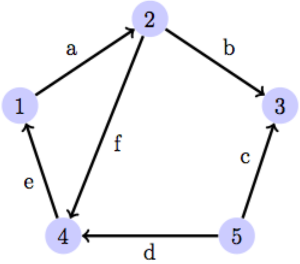
Basic concepts

- If an edge corresponds with a pair of vertices (u, v), we will say that u is the *initial vertex* and that v is the *final vertex* of this edge.
- The *underlying graph* of a directed graph is the non-dicrected graph obtained disregarding the orientations of the edges.

Basis concepts

- We define the out-degree (denoted by $deg^{+}(v)$) of a vertex v to be the number of edges with initial vertex v.
- We define the in-degree (denoted by $deg^{-}(v)$) of a vertex v to be the number of edges with final vertex v
- The degree of a vertex v, $deg(v)$, is the sum $deg^{+}(v) + deg^{-}(v)$.
- Un sinkis a vertex with out-degree 0.
- A source is a vertex with in-degree 0.

Example



- The out-degrees are:
  $deg^+(v_1) = 1$, $deg^+(v_2) = 2$, $deg^+(v_3) = 0$,
  $deg^+(v_4) = 1$, $deg^+(v_5) = 2$
- The in-degrees are:
  $deg^-(v_1) = 1$, $deg^-(v_2) = 1$, $deg^-(v_3) = 2$,
  $deg^-(v_4) = 2$, $deg^-(v_5) = 0$
- The degrees are:
  $deg(v_1) = 2$, $deg(v_2) = 3$, $deg(v_3) = 2$,
  $deg(v_4) = 3$, $deg(v_5) = 2$
- The vertex 3 is a sink.
- The vertex 5 is a source.

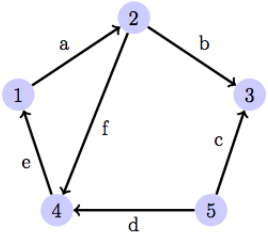
Degrees' formula

Propiedad

If $G = (V, A, \varphi)$ is a directed graph then

$$\sum_{v \in V} deg^{+}(v) = \sum_{v \in V} deg^{-}(v) = \text{number of edges}$$

Ejemplo:



$\sum_{v \in V} deg^{+}(v) = 1 + 2 + 0 + 1 + 2 = 6$

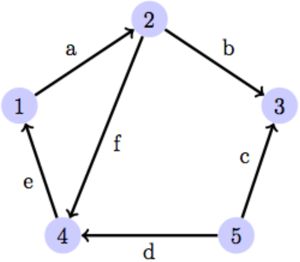
$\sum_{v \in V} deg^{-}(v) = 1 + 1 + 2 + 2 + 0 = 6$

nº de aristas $= 6$

Adjancenty matrix

Let $G = (V, A, \varphi)$ be a directed graph whose set of vertices is $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of G is the square matrix $M_A = (m_{ij})$ of size $n \times n$ such that m_{ij} is the number of edges with initial vertex v_i and final vertex v_j.

Example:



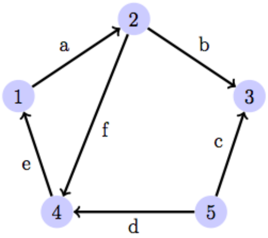
$$M_A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Incidency matrix

Let G be a directed graph without loops with set of vertices $V = \{v_1, v_2, \ldots, v_m\}$ and set of edges $A = \{e_1, e_2, \ldots, e_n\}$. The incicency matrix of G is defined to be the matrix $M_I = (m_{ij})$, of size $m \times n$, given by:

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the initial vertex of } e_j \\ -1 & \text{if } v_i \text{ is the final vertex of } e_j, \\ 0 & \text{otherwise} \end{cases}$$

Example:



$$M_I = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

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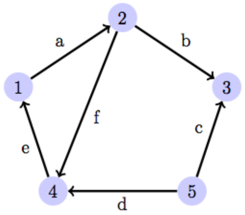
Paths and accesibility

- A directed path in a directed graph is a finite sequence of vertices and edges

$$v_0 \ e_1 \ v_1 \ e_2 \ \dots \ e_n \ v_n$$

  such that each edge e_i has initial vertex v_{i-1} and final vertex v_i.
- A vertex v is accesible from a vertex u if there exists a directed graph with initial vertex u and final vertex v.

Example:



In this graph the vertex 3 is accessible from 4 because there exists a directed path from v_4 to v_3:

$$v_4 \ e \ v_1 \ a \ v_2 \ b \ v_3.$$

However 4 is not accesible from 3 because there is no directed path from 3 to 4.

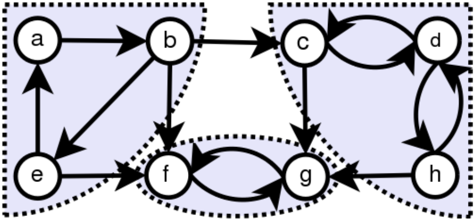
Weak and strong connectivity

- A directed graph is weakly connected if its underlying graph is connected.
- The weakly connected components are the connected components of the underlying graph.
- A vertex u is strongly connected with a vertex v if u is accessible from v and v is accessible from u.
- A directed graph is strongly connected if every pair of vertices are strongly connected.
- Given a vertex v of a directed graph, the vertices that are strongly connected with v determine a strongly connected subgraph called strongly connected component.

Examples

- The graph of the previous example is weakly connected (because its underlying graph is connected) but it is not strongly connected (because the vertices 3 and 4 are not strongly connected). It has 3 strongly connected components. Describe them.

- The following graph is weakly connected but it is not strongly connected. It has 3 strongly connected components (the shadowed subgraphs):



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Eulerian directed graphs

A directed graph is Eulerian if it contains a closed directed path, that is, a closed path that contains every edge once.

Euler Theorem (part 1)

Let G be a weakly connected graph. Then G is Eulerian if and only if, for any vertex u, the in-degree and the out-degree of u coincide.

Euler Theorem (part 2)

Let G be a weakly connected graph that is not Eulerian. Then there exists an Eulerian non-closed path joining u and v if and only if

- For every vertex w different from u and v, it holds that $\deg^+(w) = \deg^-(w)$.
- For the vertices u and v it holds that

$$deg^+(u) = deg^-(u) + 1 \;\text{ y }\; deg^+(v) = deg^-(v) - 1.$$