Stochastic matrices. Markov Chains.

- A probability vector is a vector with non-negative components, whose sum is 1.
- A stochastic matrix (Or Markov matrix) is a square matrix whose columns are probability vectors.
- Given a square matrix P, we will say that a non-zero vector $\vec{v}$ is stationary for P if $P\vec{v} = \vec{v}$.
- A Markov chain is a sequence of probability vectors $\vec{x}_0$, $\vec{x}_1, \ldots$ such that there exists a stochastic matrix P (transition matrix) that verifies

$$\vec{x}_1 = P\vec{x}_0, \quad \vec{x}_2 = P\vec{x}_1, \quad \vec{x}_k = P\vec{x}_{k-1}, \ldots$$

- A Markov chain is convergent to $\vec{v}$ if

$$\vec{v} = \lim_{k \to \infty} x_k$$

In this case $\vec{v}$ is a stationary probability vector for P.

Calculation of stationary vectors of a stochastic matrix

Assume P is a stochastic matrix. The stationary vectors of P are the non-zero vectors, $\vec{x}$, satisfying that

$$\mathrm{P}\vec{x} = \vec{x},$$

It is, the non-zero solutions of the homogeneous system:

$$(\mathrm{P} - \mathrm{I})\vec{x} = \vec{0}$$

$$\mathit{Scilab}: \quad \mathit{kernel}(\mathrm{P} - \mathrm{I})$$

Regular matrices

- A stochastic matrix P is regular if there exists a natural number n such that all the entries of the matrix P^n are strictly positive (P^n does not have zero entries).
- **Theorem.** If P is a regular stochastic matrix, there exists a **unique** stationary probability vector [1] for P. Besides, if $\vec{x}_0$ is any probability vector and $\vec{x}_{k+1} = P\vec{x}_k$ for each $k \geq 0$, then the Markov chain $\{\vec{x}_k\}$ converges to the mentioned stationary probability vector,

[1] Note that to obtain a probability stationary vector from a stationary vector is enough if we divide the vector by the sum of its entries.