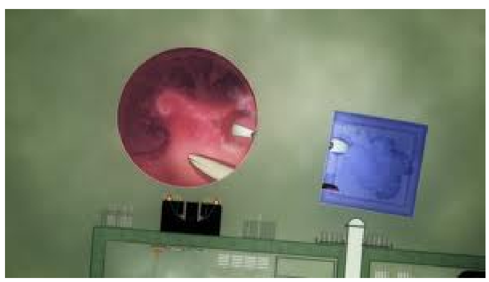
Lesson 1

Resolution of systems of linear equations using elementary operations. Echelon matrices



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I Rudiments on matrices and vectors

Throughout these course (unless otherwise be stated) a *scalar* will be a real number, although we must bear in mind all the time that (except in a few number of situations) there is no algebraic obstruction to consider elements of other infinite fields instead of the field of real numbers $\mathbb{R}$ (for example the one of complex numbers $\mathbb{C}$, or of rational numbers $\mathbb{Q}$). We can also consider finite fields, as $\mathbb{Z}_p$ (the field of classes of integers modulo a prime number p); in this case almost all remains valid but there are some exceptions (for example the solutions set of a system of linear equations cannot be infinite).

I.1 Basic definitions and notation

Definition I.1. An $m \times n$ *matrix* is a rectangular array of scalars with m *rows* and n *columns*. Each scalar in the matrix is an *entry* or *element*.

Matrices are usually named by upper case roman letters such as A. For instance,

$$A = \begin{bmatrix} 1 & 2.2 & 5 \\ 3 & 4 & -7 \end{bmatrix}$$

has 2 rows and 3 columns and so is a 2×3 matrix. Read that aloud as “two-by-three”; the number of rows is always first. We denote entries with the corresponding lower-case letter so that $a_{i,j}$ is the number in row i and column j of the array. The entry in the second row and first column is $a_{2,1} = 3$. Note that the order of the subscripts matters: $a_{1,2} \neq a_{2,1}$ since $a_{1,2} = 2.2$.

We shall use $\mathcal{M}_{m \times n}$ to denote the collection of $m \times n$ matrices.

Definition I.2. A *vector* (or *column vector*) is a matrix with a single column. A matrix with a single row is a *row vector*. The entries of a vector are its *components*. A column vector whose components are all zeros is a *zero vector*.

Vectors are an exception to the convention of representing matrices with capital roman letters. We use lower-case roman or greek letters overlined with an arrow: $\vec{a}$, $\vec{b}$, ... or $\vec{\alpha}$, $\vec{\beta}$, ... (boldface is also common: $\boldsymbol{a}$ or $\boldsymbol{\alpha}$). For instance, this is a column vector with a third component of 7.

$$\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$$

A zero vector is denoted $\vec{0}$. There are many different zero vectors, e.g., the one-tall zero vector, the two-tall zero vector, etc. Nonetheless we will usually say “the” zero vector, expecting that the size will be clear from the context.

We will allow also to denote a vector by an ordered list of its components, separated by commas, and between parentheses. That is:

$$(u_1, \ldots, u_n) := \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

Sometimes we will use this notation to save space.

Definition I.3. An $n \times n$ matrix (that is, with the same number of rows and columns, n) will be called a *square* matrix of order n.

Example I.4.

- $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix}$ is a 2×3 matrix.
- $\begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$ is a square matrix of order 2.
- $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is row vector.
- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column vector or <u>vector</u>).
- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the 3×2 zero matrix.

The set of all vectors of n components will be denoted by $\mathbb{R}^n$.

Definition I.5. The *vector sum* of $\vec{u}$ and $\vec{v}$ is the vector of the sums.

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

Note that the vectors must have the same number of entries for the addition to be defined. This entry-by-entry addition works for any pair of matrices, not just vectors, provided that they have the same number of rows and columns:

Example I.6.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -2 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1+2 & -1+3 & 0-2 \\ 0+0 & 2-1 & -1+0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix}$$

Definition I.7. The *scalar multiplication* of the real number r and the vector $\vec{v}$ is the vector of the multiples.

$$r \cdot \vec{v} = r \cdot \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} rv_1 \\ \vdots \\ rv_n \end{bmatrix}$$

We write scalar multiplication in either order, as $r \cdot \vec{v}$ or $\vec{v} \cdot r$, or without the '$\cdot$' symbol: $r\vec{v}$. (Do not refer to scalar multiplication as 'scalar product' because we use that name for a different operation.)

Example I.8.

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 3-1 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix} \qquad 7 \cdot \begin{bmatrix} 1 \\ 4 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ 28 \\ -7 \\ -21 \end{bmatrix}$$

Notice that the definitions of vector addition and scalar multiplication agree where they overlap, for instance, $\vec{v} + \vec{v} = 2\vec{v}$.

As with the addition operation, the entry-by-entry scalar multiplication operation extends beyond just vectors to any matrix:

Example I.9.

$$3 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 & 3(-1) & 3 \cdot 0 \\ 3 \cdot 0 & 3 \cdot 2 & 3(-1) \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \\ 0 & 6 & -3 \end{bmatrix}$$

Definition I.10. A vector obtained from other vectors combining the two operations (sum of vectors and scalar multiplication) is a *linear combination.*

Example I.11. For example:

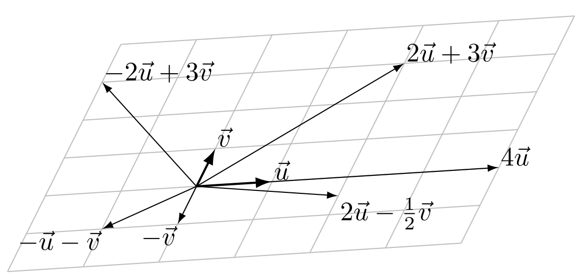
$$2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

is a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

with coefficients 2, 1 and -4 .

Example I.12.



Definition I.13. Given some vectors $\vec{v}_1, \ldots, \vec{v_m} \in \mathbb{R}^n$, we will call the *span* of $\vec{v}_1, \ldots, \vec{v_m}$ to the set of all linear combinations of those vectors. We will denote it by $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m)$.

Remark I.14. Notice that the zero vector $\vec{0} \in \mathbb{R}^n$ is a linear combination of any set of vectors $\{\vec{v}_1, \ldots, \vec{v}_m\} \subseteq \mathbb{R}^n$ because

$$\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \ldots + 0\vec{v}_m.$$

Therefore **$\vec{0}$ belongs to the span of any set of vectors**.

We can extend to matrices (of the same size), in an obvious way, the definition of linear combination of vectors:

Example I.15. This is an example of linear combination of two matrices:

$$\begin{aligned} 3\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} + 2\begin{bmatrix} 2 & 3 & -2 \\ 0 & -1 & 0 \end{bmatrix} &= \begin{bmatrix} 3\cdot 1 + 2\cdot 2 & 3(-1) + 2\cdot 3 & 3\cdot 0 + 2(-2) \\ 3\cdot 0 + 2\cdot 0 & 3\cdot 2 + 2(-1) & 3(-1) + 2\cdot 0 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 3 & -4 \\ 0 & 4 & -3 \end{bmatrix} \end{aligned}$$

The following theorem, whose trivial proof we leave as an exercise, provides several properties involving the two operations among matrices and vectors that we have seen:

Theorem I.16. *Let A and B be $m \times n$ matrices and let $\lambda, \lambda_1, \lambda_2$ be scalars.*

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + O = A$
- $A + (-A) = O$
- $\lambda_1(\lambda_2 A) = (\lambda_1\lambda_2)A$
- $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$
- $\lambda(A + B) = \lambda A + \lambda B$
- $1A = A$

UTILITARIAN SUMMARY I.17.

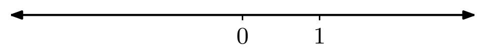
- An $m\times n$ *matrix* is a rectangular array of scalars with m *rows* and n *columns.* Each scalar in the matrix is an *entry* or *element.*
- An $m \times 1$ matrix (for some m) is a *column vector* or *vector*.
- Two matrices (and vectors) of the same size can be added. A matrix can be multiplied by a scalar. Both operations have the usual properties "that one expects".
- **This is one of the most important concepts in this course:** A **linear combination** of some vectors (or matrices) of the same size is any vector (or matrix) that can be obtained from them using the above considered two operations (sum of matrices and product by a scalar).
- The **span** of a set of vectors is the set of all their linear combinations.
- The zero vector $\vec{0}$ belongs to any span.

I.2 Vectors from a geometric point of view

"Higher-dimensional geometry" sounds exotic. It is exotic — interesting and eye-opening. But it isn't distant or unreachable.

We begin by defining one-dimensional space to be $\mathbb{R}^1$. To see that the definition is reasonable, we picture a one-dimensional space

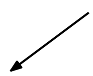
and make a correspondence with $\mathbb{R}$ by picking a point to label 0 and another to label 1.



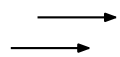
Now, with a scale and a direction, finding the point corresponding to, say, $+2.17$, is easy — start at 0 and head in the direction of 1, but don't stop there, go 2.17 times as far.

The basic idea here, combining magnitude with direction, is the key to extending to higher dimensions.

An object comprised of a magnitude and a direction is a *vector* (we use the same word as in the prior section because we shall show below how to describe such an object with a column vector). We can draw a vector as having some length, and pointing somewhere.

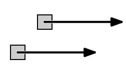


There is a subtlety here — these vectors

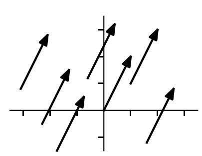


are equal, even though they start in different places, because they have equal lengths and equal directions. Again: those vectors are not just alike, they are equal.

How can things that are in different places be equal? Think of a vector as representing a displacement (the word vector is Latin for "carrier" or "traveler"). These two squares undergo equal displacements, despite that those displacements start in different places.



Sometimes, to emphasize this property vectors have of not being anchored, we can refer to them as *free* vectors. Thus, these free vectors are equal as each is a displacement of one over and two up.



More generally, vectors in the plane are the same if and only if they have the same change in first components and the same change in second components: the vector extending from the point (a_1, a_2) to the point (b_1, b_2) equals the vector from (c_1, c_2) to (d_1, d_2) if and only if $b_1 - a_1 = d_1 - c_1$ and $b_2 - a_2 = d_2 - c_2$.

Saying 'the vector that, were it to start at (a_1, a_2), would extend to (b_1, b_2)' would be unwieldy. We instead describe that vector as

$$\begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \end{bmatrix}$$

so that the ‘one over and two up’ arrows shown above picture this vector.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We often draw the arrow as starting at the origin, and we then say it is in the *canonical position* (or *natural position* or *standard position*). When the vector

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

is in canonical position then it extends to the endpoint (v_1, v_2).

We typically just refer to “the point

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}\text{”}$$

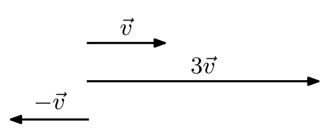
rather than “the endpoint of the canonical position of” that vector. Thus, we will call each of these $\mathbb{R}^2$.

$$\mathbb{R}^2 = \{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \}$$

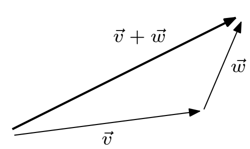
In the prior section we defined vectors and vector operations with an algebraic motivation;

$$r \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} rv_1 \\ rv_2 \end{bmatrix} \qquad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$$

we can now understand those operations geometrically. For instance, if $\vec{v}$ represents a displacement then $3\vec{v}$ represents a displacement in the same direction but three times as far, and $-1\vec{v}$ represents a displacement of the same distance as $\vec{v}$ but in the opposite direction.

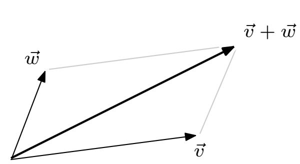


And, where $\vec{v}$ and $\vec{w}$ represent displacements, $\vec{v} + \vec{w}$ represents those displacements combined.



The long arrow is the combined displacement in this sense: if, in one minute, a ship’s motion gives it the displacement relative to the earth of $\vec{v}$ and a passenger’s motion gives a displacement relative to the ship’s deck of $\vec{w}$, then $\vec{v} + \vec{w}$ is the displacement of the passenger relative to the earth.

Another way to understand the vector sum is with the *parallelogram rule*. Draw the parallelogram formed by the vectors $\vec{v}$ and $\vec{w}$. Then the sum $\vec{v} + \vec{w}$ extends along the diagonal to the far corner.



The above drawings show how vectors and vector operations behave in $\mathbb{R}^2$. We can extend to $\mathbb{R}^3$, or to even higher-dimensional spaces where we have no pictures, with the obvious generalization: the free vector that, if it starts at $(a_1, \ldots, a_n)$, ends at $(b_1, \ldots, b_n)$, is represented by this column.

$$\begin{bmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{bmatrix}$$

Vectors are equal if they have the same representation. **We aren't too careful about distinguishing between a point and the vector whose canonical representation ends at that point**.

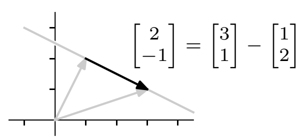
$$\mathbb{R}^n = \{ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \mid v_1, \ldots, v_n \in \mathbb{R} \}$$

And we do addition and scalar multiplication component-wise (as defined in the preceding section).

Having considered points, we now turn to the lines. In $\mathbb{R}^2$, the line through $(1, 2)$ and $(3, 1)$ is comprised of (the endpoints of) the vectors in this set.

$$\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \end{bmatrix} \mid t \in \mathbb{R} \} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \text{span}(\begin{bmatrix} 2 \\ -1 \end{bmatrix})$$

The last expression of the above equality denotes the set of all vectors that can be written as the sum of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and a linear combination of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. That description expresses this picture.

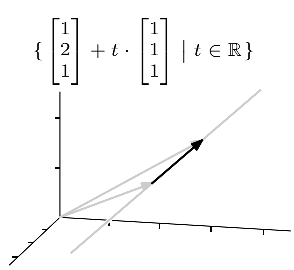


The vector associated with the parameter t

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

has its whole body in the line — it is a *direction vector* for the line. Note that points on the line to the left of $x = 1$ are described using negative values of t.

In $\mathbb{R}^3$, the line through $(1, 2, 1)$ and $(2, 3, 2)$ is the set of (endpoints of) vectors of this form



and lines in even higher-dimensional spaces work in the same way.

In $\mathbb{R}^3$, a line uses one parameter so that a particle on that line is free to move back and forth in one dimension, and a plane involves two parameters. For example, the plane through the points $(1,0,5)$, $(2,1,-3)$, and $(-2,4,0.5)$ consists of (endpoints of) the vectors in

$$\{\begin{bmatrix}1\\0\\5\end{bmatrix}+t\begin{bmatrix}1\\1\\-8\end{bmatrix}+s\begin{bmatrix}-3\\4\\-4.5\end{bmatrix}\mid t,s\in\mathbb{R}\}=\begin{bmatrix}1\\0\\5\end{bmatrix}+\operatorname{span}(\begin{bmatrix}1\\1\\-8\end{bmatrix},\begin{bmatrix}-3\\4\\-4.5\end{bmatrix})$$

(the column vectors associated with the parameters

$$\begin{bmatrix}1\\1\\-8\end{bmatrix}=\begin{bmatrix}2\\1\\-3\end{bmatrix}-\begin{bmatrix}1\\0\\5\end{bmatrix}\qquad\begin{bmatrix}-3\\4\\-4.5\end{bmatrix}=\begin{bmatrix}-2\\4\\0.5\end{bmatrix}-\begin{bmatrix}1\\0\\5\end{bmatrix}$$

are two vectors whose whole bodies lie in the plane). As with the line, note that we describe some points in this plane with negative t's or negative s's or both.

Generalizing,

Definition I.18. A set of the form $\vec{p}+\operatorname{span}(\vec{v}_1,\dots,\vec{v}_k)$, where $\vec{p},\vec{v}_1,\dots,\vec{v}_k\in\mathbb{R}^n$, is an *affine subspace*.

For example, in $\mathbb{R}^4$

$$\{\begin{bmatrix}2\\\pi\\3\\-0.5\end{bmatrix}+t\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\mid t\in\mathbb{R}\}=\begin{bmatrix}2\\\pi\\3\\-0.5\end{bmatrix}+\operatorname{span}(\begin{bmatrix}1\\0\\0\\0\end{bmatrix})$$

is a line,

$$\{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+t\begin{bmatrix}1\\1\\0\\-1\end{bmatrix}+s\begin{bmatrix}2\\0\\1\\0\end{bmatrix}\mid t,s\in\mathbb{R}\}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}+\operatorname{span}(\begin{bmatrix}1\\1\\0\\-1\end{bmatrix},\begin{bmatrix}2\\0\\1\\0\end{bmatrix})$$

is a plane, and

$$\{\begin{bmatrix}3\\1\\-2\\0.5\end{bmatrix}+r\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}+s\begin{bmatrix}1\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}2\\0\\1\\0\end{bmatrix}\mid r,s,t\in\mathbb{R}\}=\begin{bmatrix}3\\1\\-2\\0.5\end{bmatrix}+\operatorname{span}(\begin{bmatrix}0\\0\\0\\-1\end{bmatrix}+\begin{bmatrix}1\\0\\1\\0\end{bmatrix}+\begin{bmatrix}2\\0\\1\\0\end{bmatrix})$$

is a *three-dimensional* affine subspace (in Lesson 4 we will see the concrete notion of dimension). Again, the intuition is that a line permits motion in one direction, a plane permits motion in combinations of two directions, etc.

Definition I.19. Any affine subspace of $\mathbb{R}^n$ that contains $\vec{0}$ is called a **vector** *subspace* of $\mathbb{R}^n$.

For example:

- The vector subspaces of $\mathbb{R}^2$ are, geometrically: The "origin" $\{\vec{0}\}$ (that is the span of the zero vector), the lines passing the origin (that are the spans of single non-zero vectors) and the whole plane $\mathbb{R}^2$ (that is the span of the vectors $(1,0)$ and $(0,1)$, for example).

- Analogously, the vector subspaces of $\mathbb{R}^3$ are, geometrically: $\{\vec{0}\}$, the lines passing through the origin, the planes passing through the origin and the whole $\mathbb{R}^3$.

Notice that any vector subspace of $\mathbb{R}^n$ has the form $\operatorname{span}(\vec{v}_1, \ldots, \vec{v_k})$, for $\vec{v}_1, \ldots, \vec{v_k} \in \mathbb{R}^n$. That is, the vector subspaces of $\mathbb{R}^n$ are the spans of vectors of $\mathbb{R}^n$. In Lesson 4 we will generalize the notion of vector subspace and we will go deeper into this concept.

UTILITARIAN SUMMARY I.20.

- The geometric idea of vector of $\mathbb{R}^2$ or $\mathbb{R}^3$ (commonly represented by an arrow) can be generalized to $\mathbb{R}^n$, identifying it with a *column vector* of n components (what was already called a *vector* in the preceding section).
- Given some vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, the set of all their linear combinations is called the **span** of $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, and it is denoted by $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_k)$.
- Any subset of $\mathbb{R}^n$ of the form $\vec{p} + \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_k)$ (that is, the sum of a vector (or a "point") $\vec{p}$ and a linear combination of $\vec{v}_1, \ldots, \vec{v}_k$) is called an **affine subspace** of $\mathbb{R}^n$. When the vector $\vec{p}$ can be taken to be $\vec{0}$ (that is, when the origin belongs to the affine subspace) then it is called **vector subspace**.
- In Lesson 4 we will generalize more and more the notions of vector and vector (sub)space and we will go deeper into their properties.

I.3 Length, angles and orthogonality

The notions of length (or norm) of a vector, angle between vectors and orthogonality have a clear geometrical meaning in the plane and in the three-dimensional space. The objective of this section is to extend these notions to $\mathbb{R}^n$.

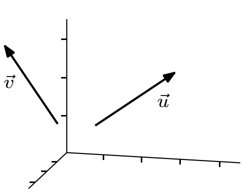
Definition I.21. The *length* (or *norm*) of a vector $\vec{v} \in \mathbb{R}^n$ is the square root of the sum of the squares of its components.

$$\|\vec{v}\| = \sqrt{v_1^2 + \cdots + v_n^2}$$

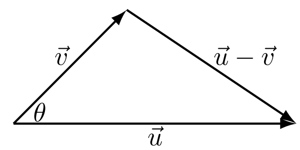
Remark I.22. This is a natural generalization of the Pythagorean Theorem.

Note that for any nonzero $\vec{v}$, the vector $\vec{v}/\|\vec{v}\|$ has length one. We say that the second vector *normalizes* $\vec{v}$ to length one.

We can use that to get a formula for the angle between two vectors. Consider two vectors in $\mathbb{R}^3$ where neither is a multiple of the other



(the special case of multiples will prove below not to be an exception). They determine a two-dimensional plane — for instance, put them in canonical poistion and take the plane formed by the origin and the endpoints. In that plane consider the triangle with sides $\vec{u}$, $\vec{v}$, and $\vec{u}-\vec{v}$. Apply the Law of Cosines:



$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\,\|\vec{u}\|\,\|\vec{v}\|\cos\theta$ where θ is the angle between the vectors. The left side gives

$$(u_1-v_1)^2+(u_2-v_2)^2+(u_3-v_3)^2 \\ = (u_1^2-2u_1v_1+v_1^2)+(u_2^2-2u_2v_2+v_2^2)+(u_3^2-2u_3v_3+v_3^2)$$

while the right side gives this.

$$(u_1^2+u_2^2+u_3^2)+(v_1^2+v_2^2+v_3^2)-2\,\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

Canceling squares $u_1^2, \ldots, v_3^2$ and dividing by 2 gives the formula.

$$\theta = \arccos\left(\frac{u_1v_1+u_2v_2+u_3v_3}{\|\vec{u}\|\,\|\vec{v}\|}\right)$$

To give a definition of angle that works in higher dimensions we cannot draw pictures but we can make the argument analytically.

Definition I.23. The *dot product* (or *inner product* or *scalar product*) of two n-components **real** vectors $\vec{u}$ and $\vec{v}$ is following linear combination of their components:

$$\vec{u}\bullet\vec{v} = u_1v_1+u_2v_2+\cdots+u_nv_n$$

Note that the dot product of two vectors is a real number, not a vector, and that the dot product of a vector from $\mathbb{R}^n$ with a vector from $\mathbb{R}^m$ is not defined unless n equals m. Note also this relationship between dot product and length: $\vec{u}\bullet\vec{u} = u_1u_1+\cdots+u_nu_n = \|\vec{u}\|^2$.

Remark I.24. The notion of dot product has geometric sense when the set of scalars is the set of real numbers $\mathbb{R}$. In the case of complex numbers, instead of the dot product, it is used a different "product" with more geometric meaning.

The following proposition, whose proof we leave as an exercise, provides some basic properties of the dot product.

Proposition I.25. *Let $\vec{u}, \vec{v}, w$ be vectors of $\mathbb{R}^n$. Their dot products satisfy the following properties:*

(a) $\vec{u}\bullet\vec{v} = \vec{v}\bullet\vec{u}$ *(commutative law)*

(b) $(\vec{u}+\vec{v})\bullet\vec{w} = \vec{u}\bullet\vec{w}+\vec{v}\bullet\vec{w}$ *(distributive law)*

(c) $\lambda(\vec{u}\bullet\vec{w}) = (\lambda\vec{u})\bullet\vec{v} = \vec{u}\bullet(\lambda\vec{v})$ *for any scalar* λ.

(d) $\vec{0}\bullet\vec{u} = 0$.

(e) $\vec{u}\bullet\vec{u} \geq 0$*, and the equality is true if and only if* $\vec{u}=0$.

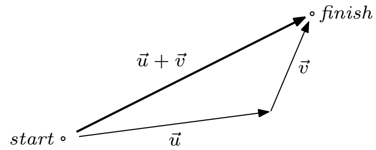
Still reasoning with letters but guided by the pictures, we use the next theorem to argue that the triangle formed by $\vec{u}$, $\vec{v}$, and $\vec{u}-\vec{v}$ in $\mathbb{R}^n$ lies in the planar subset of $\mathbb{R}^n$ generated by $\vec{u}$ and $\vec{v}$.

Theorem I.26 (Triangle Inequality)**.** *For any $\vec{u},\vec{v}\in\mathbb{R}^n$,*

$$\|\vec{u}+\vec{v}\,\| \leq \|\vec{u}\,\| + \|\vec{v}\,\|$$

with equality if and only if one of the vectors is a nonnegative scalar multiple of the other one.

This is the source of the familiar saying, "The shortest distance between two points is in a straight line."



PROOF:

Since all the numbers are positive, the inequality holds if and only if its square holds.

$$\begin{aligned}
\|\vec{u}+\vec{v}\,\|^2 &\leq (\,\|\vec{u}\,\| + \|\vec{v}\,\|\,)^2 \\
(\,\vec{u}+\vec{v}\,)\cdot(\,\vec{u}+\vec{v}\,) &\leq \|\vec{u}\,\|^2 + 2\,\|\vec{u}\,\|\,\|\vec{v}\,\| + \|\vec{v}\,\|^2 \\
\vec{u}\cdot\vec{u}+\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v} &\leq \vec{u}\cdot\vec{u}+2\,\|\vec{u}\,\|\,\|\vec{v}\,\|+\vec{v}\cdot\vec{v} \\
2\,\vec{u}\cdot\vec{v} &\leq 2\,\|\vec{u}\,\|\,\|\vec{v}\,\|
\end{aligned}$$

This holds if and only if the relationship obtained by multiplying both sides by the nonnegative numbers $\|\vec{u}\,\|$ and $\|\vec{v}\,\|$

$$2\,(\,\|\vec{v}\,\|\,\vec{u}\,)\cdot(\,\|\vec{u}\,\|\,\vec{v}\,) \leq 2\,\|\vec{u}\,\|^2\,\|\vec{v}\,\|^2$$

holds. And this inequality can be rewritten as

$$0 \leq \|\vec{u}\,\|^2\,\|\vec{v}\,\|^2 - 2\,(\,\|\vec{v}\,\|\,\vec{u}\,)\cdot(\,\|\vec{u}\,\|\,\vec{v}\,) + \|\vec{u}\,\|^2\,\|\vec{v}\,\|^2.$$

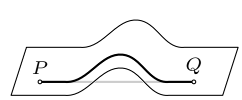
But factoring shows that it is true

$$0 \leq (\,\|\vec{u}\,\|\,\vec{v} - \|\vec{v}\,\|\,\vec{u}\,)\cdot(\,\|\vec{u}\,\|\,\vec{v} - \|\vec{v}\,\|\,\vec{u}\,)$$

since it only says that the square of the length of the vector $\|\vec{u}\,\|\,\vec{v} - \|\vec{v}\,\|\,\vec{u}$ is not negative.

Finally, equality holds when, and only when, $\|\vec{u}\,\|\,\vec{v} - \|\vec{v}\,\|\,\vec{u}$ is $\vec{0}$. The check that $\|\vec{u}\,\|\,\vec{v} = \|\vec{v}\,\|\,\vec{u}$ if and only if one vector is a nonnegative real scalar multiple of the other is easy. QED

This result supports the intuition that even in higher-dimensional spaces, lines are straight and planes are flat. We can easily check from the definition that linear surfaces have the property that for any two points in that surface, the line segment between them is contained in that surface. But if the linear surface were not flat then that would allow for a shortcut.



Because the Triangle Inequality says that in any $\mathbb{R}^n$ the shortest cut between two endpoints is simply the line segment connecting them, linear surfaces have no bends.

Back to the definition of angle measure. The heart of the Triangle Inequality's proof is the $\vec{u} \cdot \vec{v} \leq \|\vec{u}\| \, \|\vec{v}\|$ line. We might wonder if some pairs of vectors satisfy the inequality in this way: while $\vec{u} \cdot \vec{v}$ is a large number, with absolute value bigger than the right-hand side, it is a negative large number. The next result says that does not happen.

Corollary I.27 (Cauchy-Schwartz Inequality)**.** *For any $\vec{u}, \vec{v} \in \mathbb{R}^n$,*

$$|\,\vec{u} \cdot \vec{v}\,| \leq \|\,\vec{u}\,\| \, \|\vec{v}\,\|$$

with equality if and only if one vector is a scalar multiple of the other.

PROOF:

The Triangle Inequality's proof shows that $\vec{u} \cdot \vec{v} \leq \|\vec{u}\| \, \|\vec{v}\|$. So, if $\vec{u} \cdot \vec{v}$ is positive or zero then we are done. If $\vec{u} \cdot \vec{v}$ is negative then this holds:

$$|\,\vec{u} \cdot \vec{v}\,| = -(\,\vec{u} \cdot \vec{v}\,) = (-\vec{u}\,) \cdot \vec{v} \leq \|-\vec{u}\,\| \, \|\vec{v}\,\| = \|\vec{u}\,\| \, \|\vec{v}\,\|.$$

We leave the equality condition as an exercise. QED

The Cauchy-Schwartz inequality assures us that the next definition makes sense because the fraction has absolute value less than or equal to one.

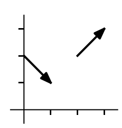
Definition I.28. The *angle* between two nonzero vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ is

$$\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\,\| \, \|\vec{v}\,\|}\right)$$

(by definition, the angle between the zero vector and any other vector is right).

Definition I.29. Two vectors from $\mathbb{R}^n$ are *orthogonal*, if their dot product is zero.

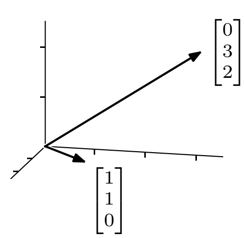
Example I.30. These vectors are orthogonal.



$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

We've drawn the arrows away from canonical position but nevertheless the vectors are orthogonal.

Example I.31. The angle between these angles of $\mathbb{R}^3$



is

$$\arccos(\frac{(1)(0)+(1)(3)+(0)(2)}{\sqrt{1^2+1^2+0^2}\sqrt{0^2+3^2+2^2}}) = \arccos(\frac{3}{\sqrt{2}\sqrt{13}})$$

approximately 0.94 radians. Notice that these vectors are not orthogonal.

UTILITARIAN SUMMARY I.32.

- The *length* or *norm* of a vector is the square root of the sum of the squares of its components. Dividing the vector by its norm, it can be *normalized* (that is, it can be transformed into a vector of norm 1).
- The *dot product* of two real vectors (of the same length) is a real number.
- The angle of two vectors of $\mathbb{R}^2$ or $\mathbb{R}^3$ can be computed from the dot product and the norms of the vectors. Using this:
- The *angle* of two vectors is defined in general, for vectors in $\mathbb{R}^n$.
- Two vectors are *orthogonal* (perpendicular) if their dot product is zero.

I.4 Multiplication of matrices

We begin with the definition of the product of a row vector and a column vector:

Definition I.33. Let us consider a row vector and a column vector of the same size:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}, \qquad \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

We define their *product* as the dot product of the transposed row vector (that is, with its components arranged as a column instead of as a row) and the column vector:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \bullet \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots a_nb_n.$$

Now we will use the above definition to define the matrix multiplication:

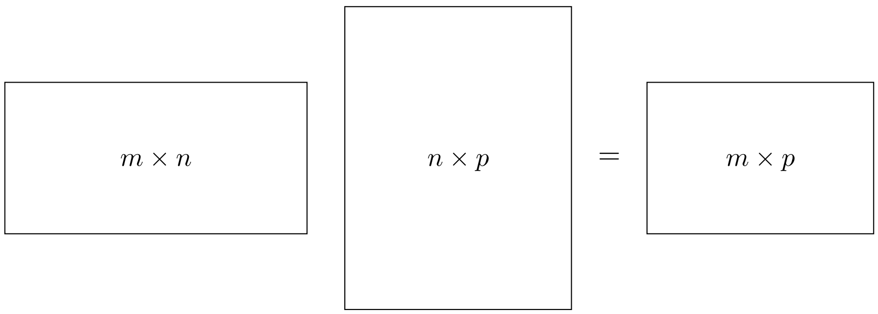
Definition I.34. Let A be an $m \times n$ matrix and B an $n \times p$ matrix. The *product matrix* is defined as the $m \times p$ matrix C whose $c_{i,j}$ entry is the product of the ith row of A and the jth column of B. That is:

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j} = \sum_{s=1}^{n} a_{i,s}b_{s,j}$$

Example I.35.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ -1 & 3 \end{bmatrix}$$

The purpose of the following figure is to emphasize that two matrices can be multiplied only if their dimensions satisfy a certain relation:



Observe now the following example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 2 + 4 \cdot 3 \\ 1 \cdot 5 + 2 \cdot 0 + 4 \cdot 4 \\ 1 \cdot 3 + 2 \cdot 2 + 4 \cdot 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

We have multiplied a matrix by a column vector and the result is a linear combination of the columns of the matrix. It is evident that this is true in general, that is,

Proposition I.36. *The product of a matrix by a vector $\vec{u}$ is a linear combination of the columns of the matrix (whose coefficients are the components of $\vec{u}$):*

$$\begin{bmatrix} Col.1 & Col.2 & \cdots & Col.n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = u_1(Col.1) + u_2(Col.2) + \cdots + u_n(Col.n).$$

The next result (that we assume without proof) provides some properties of the multiplication of matrices:

Proposition I.37. *Let A, B and C be matrices. Whenever the products have sense, the following properties hold:*

- $(AB)C = A(BC)$ *(associative)*
- $(A + B)C = AC + BC$ *(distributive)*
- $A(B + C) = AB + AC$ *(distributive)*
- $\lambda(AB) = (\lambda A)B = A(\lambda B)$

Notice that **the multiplication of matrices is not commutative**:

Example I.38. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}$$

Notice also that **the product of two non-zero matrices can be a zero matrix**:

Example I.39.

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 1 & -5/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

UTILITARIAN SUMMARY I.40.

- The matrix multiplication is associative and distributive with respect to the sum (both sides).
- The matrix multiplication **is not commutative**.
- **The multiplication of a matrix A by a vector $\vec{x}$ (that is, $A\vec{x}$) is a linear combination of the columns of A** (whose coefficients are the components of $\vec{x}$).

I.5 Some types of matrices

Definition I.41. The *main diagonal* (or *principle diagonal* or *diagonal*) of a square matrix goes from the upper left to the lower right.

Definition I.42. An *identity matrix* is square and every entry is 0 except for 1's in the main diagonal.

$$I_{n\times n} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

If the size n of the identity matrix is understood, we write I instead of $I_{n\times n}$, for short.

Example I.43. Here is the 2×2 identity matrix leaving its multiplicand unchanged when it acts from the right.

$$\begin{bmatrix} 1 & -2 \\ 0 & -2 \\ 1 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & -2 \\ 1 & -1 \\ 4 & 3 \end{bmatrix}$$

Example I.44. Here the 3×3 identity leaves its multiplicand unchanged both from the left

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&3&6\\1&3&8\\-7&1&0\end{bmatrix}=\begin{bmatrix}2&3&6\\1&3&8\\-7&1&0\end{bmatrix}$$

and from the right.

$$\begin{bmatrix}2&3&6\\1&3&8\\-7&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}2&3&6\\1&3&8\\-7&1&0\end{bmatrix}$$

In short, **the $n\times n$ identity matrix is the identity element of the set of $n\times n$ matrices with respect to the operation of matrix multiplication**.

Definition I.45. A *diagonal matrix* is square and has 0's off the main diagonal.

$$\begin{bmatrix}a_{1,1}&0&\dots&0\\0&a_{2,2}&\dots&0\\&\vdots&&\\0&0&\dots&a_{n,n}\end{bmatrix}$$

Example I.46. From the left, the action of multiplication by a diagonal matrix is to rescale the rows.

$$\begin{bmatrix}2&0\\0&-1\end{bmatrix}\begin{bmatrix}2&1&4&-1\\-1&3&4&4\end{bmatrix}=\begin{bmatrix}4&2&8&-2\\1&-3&-4&-4\end{bmatrix}$$

From the right such a matrix rescales the columns.

$$\begin{bmatrix}1&2&1\\2&2&2\end{bmatrix}\begin{bmatrix}3&0&0\\0&2&0\\0&0&-2\end{bmatrix}=\begin{bmatrix}3&4&-2\\6&4&-4\end{bmatrix}$$

Definition I.47. A *scalar* matrix is diagonal and all the entries in the main diagonal are equal:

$$\begin{bmatrix}\lambda&0&\dots&0\\0&\lambda&\dots&0\\&\vdots&&\\0&0&\dots&\lambda\end{bmatrix}$$

Definition I.48. An *upper triangular* matrix is square and all its entries *below* the main diagonal are 0's, that is, it has the form

$$\begin{bmatrix}a_{1,1}&a_{1,2}&\dots&a_{1,n}\\0&a_{2,2}&\dots&a_{2,n}\\&&\ddots&\\0&0&\dots&a_{n,n}\end{bmatrix}$$

Example I.49. This is an upper triangular matrix

$$\begin{bmatrix}0&7&-1\\0&1&2\\0&0&9\end{bmatrix}$$

Definition I.50. A *lower triangular* matrix is square and all its entries *above* the main diagonal are 0's, that is, it has the form

$$\begin{bmatrix} a_{1,1} & 0 & \dots & 0 \\ a_{2,1} & a_{2,2} & \dots & 0 \\ & & \ddots & \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{bmatrix}$$

Example I.51. This is a lower triangular matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 6 & 1 & 0 \\ 8 & -2 & 9 \end{bmatrix}$$

Definition I.52. Given a $m \times n$ matrix A, its *transpose*, denoted by A^t is the $n \times m$ matrix whose rows are the columns of A.

Example I.53. Consider the matrix

$$D = \begin{bmatrix} 3 & 7 & 2 & -3 \\ -1 & 4 & 2 & 8 \\ 0 & 3 & -2 & 5 \end{bmatrix}.$$

. Its transpose is

$$D^t = \begin{bmatrix} 3 & -1 & 0 \\ 7 & 4 & 3 \\ 2 & 2 & -2 \\ -3 & 8 & 5 \end{bmatrix}$$

.

The next result is an evident property of the transpose:

Proposition I.54. *If A is any matrix then $(A^t)^t = A$.*

The following proposition, whose proof is trivial, shows that the operation "taking transpose" behaves well with respect to the sum of matrices and the scalar multiplication:

Proposition I.55. *If A and B are $m \times n$ matrices and $\alpha \in \mathbb{R}$ then $(A+B)^t = A^t + B^t$ and $(\alpha A)^t = \alpha A^t$.*

Now we will show that the transpose of the product of two matrices is the product of the trasposes, but **the order of the factors reverses**.

Proposition I.56. *If A is an $m \times k$ matrix and B is a $k \times n$ matrix then $(AB)^t = B^t A^t$.*

PROOF:

Notice that $(AB)^t$ is an $n \times m$ matrix. The element located at the position (j, i) of this matrix (that is, at row j and column i) is the same as the element (i, j) of AB, by the definition of transpose matrix. But this is the **product of the ith row of A by the jth column of** B, that coincides with the **product of the jth row of B^t by the ith column of** A^t; and this is the element of B^tA^t located at the position (j, i). QED

Using induction, this result generalizes to any number of factors:

Proposition I.57. *If $A_1, \dots, A_s$ are $m \times k$ matrices then $(A_1 \cdots A_s)^t = A_s^t \cdots A_1^t$.*

Definition I.58. A square matrix A is *symmetric* if $A^t = A$.

Example I.59. The following matrix is symmetric:

$$E = \begin{bmatrix} 2 & 3 & -9 & 5 & 7 \\ 3 & 1 & 6 & -2 & -3 \\ -9 & 6 & 0 & -1 & 9 \\ 5 & -2 & -1 & 4 & -8 \\ 7 & -3 & 9 & -8 & -3 \end{bmatrix}$$

Definition I.60. A square matrix A is *antisymmetric* if $A^t = -A$.

Example I.61. The following matrix is antisymmetric:

$$E = \begin{bmatrix} 0 & -3 & 9 & -5 & -7 \\ 3 & 0 & -6 & 2 & 3 \\ -9 & 6 & 0 & 1 & -9 \\ 5 & -2 & -1 & 0 & 8 \\ 7 & -3 & 9 & -8 & 0 \end{bmatrix}$$

Notice that the diagonal elements of an antisymmetric matrix are equal to 0 (why?).

UTILITARIAN SUMMARY I.62.

- The identity matrix of order n is the identity element of the matrix multiplication in the set of $\mathcal{M}_{n\times n}$.
- The operation "taking transpose" behaves well with respect to sum and scalar multiplication.
- The transpose of the product of matrices is equal to the product of transposes **but with the order of the factors reversed**.

II Resolution of systems of linear equations

II.1 The Gauss' Method

Definition II.1. A *linear combination* of $x_1, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n$$

where the numbers $a_1, \ldots, a_n \in \mathbb{R}$ are the combination's *coefficients*. A *linear equation* in the *variables* (or *unknowns*) $x_1, \ldots, x_n$ has the form $a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = d$ where $d \in \mathbb{R}$ is called *independent term.*

A vector $(s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$ is a *solution* of, or *satisfies*, that equation if substituting the numbers $s_1, \ldots, s_n$ for the variables gives a true statement: $a_1s_1 + a_2s_2 + \cdots + a_ns_n = d$. A *system of linear equations*

$$\begin{array}{rcl} a_{1,1}x_1 + \ a_{1,2}x_2 + \cdots + \ a_{1,n}x_n & = & d_1 \\ a_{2,1}x_1 + \ a_{2,2}x_2 + \cdots + \ a_{2,n}x_n & = & d_2 \\ & \vdots & \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n & = & d_m \end{array}$$

has the solution $(s_1, s_2, \ldots, s_n)$ if that vector is a solution of all of the equations in the system.

Example II.2. The combination $3x_1+2x_2$ of x_1 and x_2 is linear. The combination $3x_1^2+2\sin(x_2)$ is not linear, nor is $3x_1^2 + 2x_2$.

Example II.3. The ordered pair $(-1, 5)$ is a solution of this system.

$$\begin{aligned} 3x_1 + 2x_2 &= 7 \\ -x_1 + \ \ x_2 &= 6 \end{aligned}$$

In contrast, $(5, -1)$ is not a solution.

Finding the set of all solutions is *solving* the system. We don't need guesswork or good luck; there is an algorithm that always works. This algorithm is *Gauss' Method* (or *Gaussian elimination* or *linear elimination*).

Example II.4. To solve this system

$$\begin{aligned} 3x_3 &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ \tfrac{1}{3}x_1 + 2x_2 \qquad\quad &= 3 \end{aligned}$$

we transform it, step by step, until it is in a form that we can easily solve.

The first transformation rewrites the system by interchanging the first and third row.

$$\xrightarrow{\text{swap row 1 with row 3}} \quad \begin{aligned} \tfrac{1}{3}x_1 + 2x_2 \qquad\quad &= 3 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_3 &= 9 \end{aligned}$$

The second transformation rescales the first row by multiplying both sides of the equation by 3.

$$\xrightarrow{\text{multiply row 1 by 3}} \quad \begin{aligned} x_1 + 6x_2 \qquad\quad &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ 3x_3 &= 9 \end{aligned}$$

The third transformation is the only nontrivial one in this example. We mentally multiply both sides of the first row by -1, mentally add that to the second row, and write the result in as the new second row.

$$\xrightarrow{\text{add } -1 \text{ times row 1 to row 2}} \begin{aligned} x_1 + 6x_2 \qquad &= 9 \\ -x_2 - 2x_3 &= -7 \\ 3x_3 &= 9 \end{aligned}$$

The point of these steps is that we've brought the system to a form where we can easily find the value of each variable. The bottom equation shows that $x_3 = 3$. Substituting 3 for x_3 in the middle equation shows that $x_2 = 1$. Substituting those two into the top equation gives that $x_1 = 3$. Thus the system has a unique solution; the solution set is $\{ (3, 1, 3) \}$.

Most of this subsection and the next one consists of examples of solving linear systems by Gauss' Method. We will use it throughout the course. It is fast and easy. The following result (whose proof is very easy and we do not reproduce here) states that this method is "safe" in the sense that it never loses solutions or picks up extraneous solutions.

Theorem II.5 (Gauss' Method)**.** *If a linear system is changed to another by one of these operations*

(1) an equation is swapped with another

(2) an equation has both sides multiplied by a nonzero constant

(3) an equation is replaced by the sum of itself and a multiple of another

then the two systems have the same set of solutions.

Each of the three Gauss' Method operations has a restriction. Multiplying a row by 0 is not allowed because obviously that can change the solution set of the system. Similarly, adding a multiple of a row to itself is not allowed because adding -1 times the row to itself has the effect of multiplying the row by 0. Finally, we disallow swapping a row with itself because it's pointless.

Definition II.6. The three operations from Theorem II.5 are the *elementary operations.* They are:

- Type 1: *swapping* two equations,
- Type 2: *multiplying an equation by a (nonzero) scalar* (or *rescaling*), and
- Type 3: *row combination (adding, to a row, a multiple of another one).*

When writing out the calculations, we will abbreviate 'row i' by 'ρ_i'. For instance, we will denote a row combination operation (or elementary operation of type 3) by $\rho_i + k\rho_j$, with the row that changes written first. To save writing we will often combine addition steps when they use the same ρ_i; see the next example.

Example II.7. Gauss' Method systematically applies the row operations to solve a system. Here is a typical case.

$$\begin{aligned} x + y \phantom{{}+3z} &= 0 \\ 2x - y + 3z &= 3 \\ x - 2y - z &= 3 \end{aligned}$$

We begin by using the first row to eliminate the $2x$ in the second row and the x in the third. To get rid of the $2x$, we mentally multiply the entire first row by -2, add that to the second row, and write the result in as the new second row. To eliminate the x leading the third row, we multiply the first row by -1, add that to the third row, and write the result in as the new third row.

$$\xrightarrow[\rho_3-\rho_1]{\rho_2-2\rho_1} \begin{aligned} x + \quad y \qquad &= 0 \\ -3y + 3z &= 3 \\ -3y - z &= 3 \end{aligned}$$

To finish we transform the second system into a third system, where the last equation involves only one unknown. We use the second row to eliminate y from the third row.

$$\xrightarrow{\rho_3-\rho_2} \begin{aligned} x + \quad y \qquad &= 0 \\ -3y + \quad 3z &= 3 \\ -4z &= 0 \end{aligned}$$

Now the system's solution is easy to find. The third row shows that $z = 0$. Substitute that back into the second row to get $y = -1$ and then substitute back into the first row to get $x = 1$. Therefore this system has a unique solution, that is the vector

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Example II.8. The reduction

$$\begin{aligned} x + y + z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned} \quad \xrightarrow[\rho_3-3\rho_1]{\rho_3-2\rho_1} \quad \begin{aligned} x + y + z &= 9 \\ 2y - 5z &= -17 \\ 3y - 8z &= -27 \end{aligned}$$

$$\xrightarrow{\rho_3-(3/2)\rho_2} \quad \begin{aligned} x + y + \qquad z &= 9 \\ 2y - \qquad 5z &= -17 \\ -(1/2)z &= -(3/2) \end{aligned}$$

shows that $z = 3$, $y = -1$, and $x = 7$. Therefore the unique solution is the vector $(7, -1, 3)$.

As illustrated above, the point of Gauss' Method is to use the elementary reduction operations to set up back-substitution.

Definition II.9. In each row of a system, the first variable with a nonzero coefficient is the row's *leading variable* (or *pivot variable*). A system is in *echelon form* if each leading variable is to the right of the leading variable in the row above it (except for the leading variable in the first row).

Example II.10. The prior two examples only used the operation of row combination. This linear system requires the swap operation to get it into echelon form because after the first combination

$$\begin{aligned} x - y \qquad\qquad &= 0 \\ 2x - 2y + z + 2w &= 4 \\ y \qquad + w &= 0 \\ 2z + w &= 5 \end{aligned} \quad \xrightarrow{\rho_2-2\rho_1} \quad \begin{aligned} x - y \qquad\qquad &= 0 \\ z + 2w &= 4 \\ y \qquad + w &= 0 \\ 2z + w &= 5 \end{aligned}$$

the second equation has no leading y. To get one, we put in place a lower-down row that has a leading y.

$$\xrightarrow{\rho_2\leftrightarrow\rho_3}\begin{array}{rcrcrcrcl} x & - & y & & & & & = & 0 \\ & & y & & & + & w & = & 0 \\ & & & & z & + & 2w & = & 4 \\ & & & & 2z & + & w & = & 5 \end{array}$$

(Had there been more than one suitable row below the second then we could have swapped in any one.) With that, Gauss' Method proceeds as before.

$$\xrightarrow{\rho_4-2\rho_3}\begin{array}{rcrcrcrcr} x & - & y & & & & & = & 0 \\ & & y & & & + & w & = & 0 \\ & & & & z & + & 2w & = & 4 \\ & & & & & & -3w & = & -3 \end{array}$$

Back-substitution gives $w = 1$, $z = 2$, $y = -1$, and $x = -1$. Then, the unique solution is the vector $(-1, -1, 2, 1)$.

All of the systems seen so far have the same number of equations as unknowns. All of them have a solution, and for all of them there is only one solution. We finish this subsection by seeing some other things that can happen.

Example II.11. This system has more equations than variables.

$$\begin{array}{rcrcr} x & + & 3y & = & 1 \\ 2x & + & y & = & -3 \\ 2x & + & 2y & = & -2 \end{array}$$

Gauss' Method helps us understand this system also, since this

$$\xrightarrow[\rho_3-2\rho_1]{\rho_2-2\rho_1}\begin{array}{rcrcr} x & + & 3y & = & 1 \\ & & -5y & = & -5 \\ & & -4y & = & -4 \end{array}$$

shows that one of the equations is redundant. Echelon form

$$\xrightarrow{\rho_3-(4/5)\rho_2}\begin{array}{rcrcr} x & + & 3y & = & 1 \\ & & -5y & = & -5 \\ & & 0 & = & 0 \end{array}$$

gives that $y = 1$ and $x = -2$. The '$0 = 0$' reflects the redundancy. The system has a unique solution: $(-2, 1)$.

Another way that linear systems can differ from the above examples is that some linear systems do not have a unique solution. This can happen in two ways.

The first is that a system can fail to have any solution at all.

Example II.12. Contrast the system in the last example with this one.

$$\begin{array}{rcrcr} x & + & 3y & = & 1 \\ 2x & + & y & = & -3 \\ 2x & + & 2y & = & 0 \end{array}\xrightarrow[\rho_3-2\rho_1]{\rho_2-2\rho_1}\begin{array}{rcrcr} x & + & 3y & = & 1 \\ & & -5y & = & -5 \\ & & -4y & = & -2 \end{array}$$

Here the system is inconsistent: no pair of numbers satisfies all of the equations simultaneously. Echelon form makes this inconsistency obvious.

$$\xrightarrow{\rho_3-(4/5)\rho_2}\quad\begin{aligned} x+\ \ 3y&=\ \ 1\\ -5y&=-5\\ 0&=\ \ 2\end{aligned}$$

The solution set is empty.

Example II.13. The prior system has more equations than unknowns, but that is not what causes the inconsistency — Example II.11 has more equations than unknowns and yet is consistent. Nor is having more equations than unknowns necessary for inconsistency, as we see with this inconsistent system that has the same number of equations as unknowns.

$$\begin{aligned} x+2y&=8\\ 2x+4y&=8\end{aligned}\quad\xrightarrow{\rho_2-2\rho_1}\quad\begin{aligned} x+2y&=\ \ 8\\ 0&=-8\end{aligned}$$

The other way that a linear system can fail to have a unique solution, besides having no solutions, is to have many solutions.

Example II.14. In this system

$$\begin{aligned} x+\ \ y&=4\\ 2x+2y&=8\end{aligned}$$

any pair of real numbers (s_1, s_2) satisfying the first equation also satisfies the second. The solution set $\{(x,y) \mid x+y=4\}$ is infinite; some of its members are $(0,4)$, $(-1,5)$, and $(2.5,1.5)$.

The result of applying Gauss' Method here contrasts with the prior example because we do not get a contradictory equation.

$$\xrightarrow{\rho_2-2\rho_1}\quad\begin{aligned} x+y&=4\\ 0&=0\end{aligned}$$

Don't be fooled by the final system in that example. A '$0=0$' equation is not the signal that a system has many solutions.

Example II.15. The absence of a '$0=0$' does not keep a system from having many different solutions. This system is in echelon form has no '$0=0$', but has infinitely many solutions.

$$\begin{aligned} x+y+z&=0\\ y+z&=0\end{aligned}$$

Some solutions are: $(0,1,-1)$, $(0,1/2,-1/2)$, $(0,0,0)$, and $(0,-\pi,\pi)$. There are infinitely many solutions because any triple whose first component is 0 and whose second component is the negative of the third is a solution.

Nor does the presence of a '$0=0$' mean that the system must have many solutions. Example II.11 shows that. So does this system, which does not have any solutions at all despite that in echelon form it has a '$0=0$' row.

$$\begin{aligned} 2x\qquad\ -2z&=6\\ y+\ \ z&=1\\ 2x+\ y-\ \ z&=7\\ 3y+3z&=0\end{aligned}\quad\xrightarrow{\rho_3-\rho_1}\quad\begin{aligned} 2x\qquad -2z&=6\\ y+\ \ z&=1\\ y+\ \ z&=1\\ 3y+3z&=0\end{aligned}$$

$$\xrightarrow[\rho_4-3\rho_2]{\rho_3-\rho_2}\quad\begin{aligned} 2x\quad -2z&=\ \ 6\\ y+\ \ z&=\ \ 1\\ 0&=\ \ 0\\ 0&=-3\end{aligned}$$

We will finish this subsection with a summary of what we've seen so far about Gauss' Method.

UTILITARIAN SUMMARY II.16.

Gauss' Method uses the three elementary row operations to set a system up for back substitution. If any step shows a contradictory equation then we can stop with the conclusion that the system has no solutions. If we reach echelon form without a contradictory equation, and each variable is a leading variable in its row, then the system has a unique solution and we find it by back substitution. Finally, if we reach echelon form without a contradictory equation and at least one variable is not a leading variable, then the system has many solutions.

II.2 Describing the solution set

A linear system with a unique solution has a solution set with one element. A linear system with no solution has a solution set that is empty. In these cases the solution set is easy to describe. Solution sets are a challenge to describe only when they contain many elements.

Example II.17. This system has many solutions because in echelon form

$$\begin{array}{rcl} \begin{aligned} 2x \quad\;\; + z &= 3 \\ x - y - z &= 1 \\ 3x - y \quad\;\; &= 4 \end{aligned} & \xrightarrow[\rho_3-(3/2)\rho_1]{\rho_2-(1/2)\rho_1} & \begin{aligned} 2x \quad + \quad\quad z &= 3 \\ -y - (3/2)z &= -1/2 \\ -y - (3/2)z &= -1/2 \end{aligned} \\ & \xrightarrow{\rho_3-\rho_2} & \begin{aligned} 2x \quad + \quad\quad z &= 3 \\ -y - (3/2)z &= -1/2 \\ 0 &= 0 \end{aligned} \end{array}$$

not all of the variables are leading variables. Theorem II.5 shows that an (x, y, z) satisfies the first system if and only if it satisfies the third. So we can describe the solution set $\{(x, y, z) \mid 2x + z = 3 \text{ and } x - y - z = 1 \text{ and } 3x - y = 4\}$ in this way:

$$\{(x, y, z) \mid 2x + z = 3 \text{ and } -y - 3z/2 = -1/2\} \tag{$*$}$$

This description is better because it has two equations instead of three but it is not optimal because it still has some hard to understand interactions among the variables.

To improve it, use the variable that does not lead any equation, z, to describe the variables that do lead, x and y. The second equation gives $y = (1/2) - (3/2)z$ and the first equation gives $x = (3/2) - (1/2)z$. Thus we can describe the solution set in this way:

$$\{(x, y, z) = ((3/2) - (1/2)z, (1/2) - (3/2)z, z) \mid z \in \mathbb{R}\} \tag{$**$}$$

Compared with $(*)$, the advantage of $(**)$ is that z can be any real number. This makes the job of deciding which tuples are in the solution set much easier. For instance, taking $z = 2$ shows that $(1/2, -5/2, 2)$ is a solution. Giving to z all real numbers as values and replacing, we obtain the infinitely many solutions of the system. The variable z (whose values determine the solutions) is a *parameter*, and $(**)$ shows that the set of solutions is *parametrized* by z.

The expression $(**)$ is called a **parametric expression of the solution set**.

If we give an *arbitrary* value to z, say μ, then one obtains the solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} - \frac{1}{2}\mu \\ \frac{1}{2} - \frac{3}{2}\mu \\ 0 + \mu \end{bmatrix}.$$

Splitting the right hand side into two summands we obtain that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}.$$

This means that the set of solutions is

$$\{ \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix} \quad \text{such that } \mu \in \mathbb{R}\},$$

that is,

$$\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + \operatorname{span}(\begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}).$$

This is called the **vectorial expression of the solution set**. It means that the solutions are, exactly, those vectors obtained by adding, to $(3/2, 1/2, 0)$, a linear combination (multiple) of $(-1/2, -3/2, 1)$.

Definition II.18. In an echelon form linear system the variables that are not leading are called *free.*

Example II.19. Reduction of a linear system can end with more than one variable free. On this system Gauss' Method

$$\begin{array}{rcrcrcrcr} x & + & y & + & z & - & w & = & 1 \\ & & y & - & z & + & w & = & -1 \\ 3x & & & + & 6z & - & 6w & = & 6 \\ & & -y & + & z & - & w & = & 1 \end{array} \xrightarrow{\rho_3 - 3\rho_1} \begin{array}{rcrcrcrcr} x & + & y & + & z & - & w & = & 1 \\ & & y & - & z & + & w & = & -1 \\ & & -3y & + & 3z & - & 3w & = & 3 \\ & & -y & + & z & - & w & = & 1 \end{array}$$

$$\xrightarrow[\rho_2 + \rho_4]{\rho_3 + 3\rho_2} \begin{array}{rcrcrcrcr} x & + & y & + & z & - & w & = & 1 \\ & & y & - & z & + & w & = & -1 \\ & & & & & & 0 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

leaves x and y leading, and both z and w free. To get the parametric expression of the solution set, we work from the bottom. We first express the leading variable y in terms of z and w, with $y = -1 + z - w$. Moving up to the top equation, substituting for y gives $x + (-1 + z - w) + z - w = 1$ and solving for x leaves $x = 2 - 2z + 2w$. The solution set

$$\{ (2 - 2z + 2w, -1 + z - w, z, w) \mid z, w \in \mathbb{R} \} \tag{$**$}$$

has the leading variables in terms of the free variables z and w (which are the parameters here). This is the parametric expression and it gives the solution set *parametrized by* z *and* w. The vectorial expression is

$$\{\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}+\alpha\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}+\beta\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}\ \text{ such that }\alpha,\beta\in\mathbb{R}\}=\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}+\operatorname{span}(\begin{bmatrix}-2\\1\\1\\0\end{bmatrix},\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}).$$

It shows that any solution is the result of adding, to the vector $(2,-1,0,0)$, a linear combination of the vectors $(-2,1,1,0)$ and $(2,-1,0,1)$.

Notice that, when we have a system with an infinite solution set $\mathcal{S}$, the vectorial expression of the solutions shows that $\mathcal{S}$ **is an affine subspace**. That is:

Theorem II.20. Consider a linear system. If, after the Gaussian reduction, there is no inconsistency and there are free variables, then its solution set is infinite and it has the form

$$\vec{p}+\operatorname{span}(\vec{\beta}_1,\cdots,\vec{\beta}_k)$$

where $\vec{p}$ is any particular solution and where the number of vectors $\vec{\beta}_1,\ldots,\vec{\beta}_k$ equals the number of free variables.

Definition II.21. A linear system is *homogeneous* if the independent term of any equation is equal to 0.

Example II.22. The following linear system is homogeneous. We show its resolution using the Gauss' Method:

$$\begin{aligned} x\phantom{{}-y}+z+w&=0\\ 2x-y\phantom{{}+3z}+w&=0\\ x+y+3z+2w&=0\end{aligned}\ \xrightarrow[\rho_3-\rho_1]{\rho_2-2\rho_1}\ \xrightarrow{\rho_3+\rho_2}\ \begin{aligned} x\phantom{{}-y}+z+w&=0\\ -y-2z-w&=0\\ 0&=0\end{aligned}$$

In fact the solution set of this homogeneous system is infinite.

$$\{\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix}\alpha+\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix}\beta\ \big|\ \alpha,\beta\in\mathbb{R}\}=\operatorname{span}(\begin{bmatrix}-1\\-2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix})$$

Observe that the unique case in which a linear system has no solution is when, after applying Gauss' Method, we obtain a contradictory equation (of the type $0=c$ with $c\neq 0$). Notice that if **the system is homogeneous this is impossible** because, in any step of the reduction process, the independent terms of the equations are always equal to 0 (see the above example). Therefore:

Theorem II.23. *Any homogeneous linear system has, at least, one solution.*

UTILITARIAN SUMMARY II.24.

If we apply Gauss' Method to solve a linear system, three cases can occur with respect to the final obtained equations:

(1) There is a contradictory equation, that is, en equation of the form $0 = c$ with $c \neq 0$. In this case **the solution set is empty**.

(2) There is no contradictory equation and there are free variables. In this case, parametrizing the solution set shows that system **has infinitely many solutions**. Moreover the solution set is an affine subspace.

(3) There is no free variable. In this case the system **has a unique solution**.

Moreover, **if the system is homogeneous, case (1) cannot occur**.

II.3 Gauss' Method using matrices

Consider the following linear system:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 - x_2 &= -3 \end{aligned}.$$

It is very easy to check that this system can also be written using matrices as follows:

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

This is the *matrix form* of the system.

La matrix$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ is the *coefficient matrix*, the vector $\vec{b} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ is the *vector of independent terms* and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the *vector of unknowns*. With these notations, the system can be written for short as follows:

$$A\vec{x} = \vec{b}.$$

The system, then, can be regarded as a **unique equation**: we want to know the vectors $\vec{x}$ such that, if we multiply them by A (on the left), we obtain $\vec{b}$. This point of view will be very useful later.

We will call *augmented matrix* to the matrix obtained adjoining to A (on the right) the vector $\vec{b}$:

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & -1 & -3 \end{array}\right]$$

Obviously, these definitions can be reproduced for any linear system

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \cdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n = b_m \end{cases}$$

It is equivalent to the following equality:

$$\underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & & & \\ \cdots & & & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}}_{\text{Coefficient matrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{(*)} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{(**)}$$

(*) is the vector of unknowns and (**) is the vector of independent terms. The augmented matrix is

$$[A|b] := \left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \cdots & & & & \\ \cdots & & & & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{array}\right].$$

The augmented matrix is very useful to do the things more clear when one applies Gauss' Method, as it is shown in the following example.

Example II.25. We can abbreviate this linear system

$$\begin{aligned} x + 2y \phantom{{}-z} &= 4 \\ y - z &= 0 \\ x \phantom{{}+2y} + 2z &= 4 \end{aligned}$$

using the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right]$$

The vertical bar just reminds a reader of the difference between the coefficients on the system's left hand side and the constants on the right. In this notation, the processes of copying the variables and writing the +'s and ='s are lighter.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 4 \end{array}\right] \xrightarrow{\rho_3-\rho_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow{\rho_3+2\rho_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The second row stands for $y - z = 0$ and the first row stands for $x + 2y = 4$ so the solution set is $\{(4 - 2z, z, z) \mid z \in \mathbb{R}\}$.

Definition II.26. The first non-zero entry in each row (from left to right) is called the *pivot* of the row.

Definition II.27. A matrix is *row echelon* (or *echelon* for short) if

(1) its zero rows, if any, are at the bottom of the matrix and

(2) the pivot of any given row is always to the right of the pivot of the row above it.

Definition II.28. A matrix A is *row equivalent* to a matrix B if B can be obtained from A successively performing elementary row operations.

Definition II.29. An *row echelon form* (or *echelon form* for short) of a matrix A is a row echelon matrix B such that A is row equivalent to B.

Using this terminology, the "matrix version" of the Gauss' Method applied to a linear system $A\vec{x} = \vec{b}$ consists of the following:

(1) to obtain an echelon form of the augmented matrix $[A \mid \vec{b}]$ and

(2) to solve the system using backward substitution (if there are solutions).

Remark II.30. Notice that the relation of "being equivalent to" is an equivalence relation in the set $\mathcal{M}_{m\times n}$ of $m \times n$ matrices:

- It is reflexive because any matrix is row equivalent to itself.
- It is obviously transitive.
- It is symmetric because elementary row operations are reversible:

$$A \xrightarrow{\rho_i\leftrightarrow\rho_j} \xrightarrow{\rho_j\leftrightarrow\rho_i} A \qquad A \xrightarrow{k\rho_i} \xrightarrow{(1/k)\rho_i} A \qquad A \xrightarrow{\rho_i+k\rho_j} \xrightarrow{\rho_i-k\rho_j} A$$

Therefore, we have a *partition* of the set of $m \times n$ matrices $\mathcal{M}_{m\times n}$ into equivalence classes. In the next section we will see that each one of these equivalence classes has a *distinguished* element, called *reduced row echelon form.*

UTILITARIAN SUMMARY II.31.

- To solve a linear system is equivalent to solve a single equation of the type $A\vec{x} = \vec{b}$ where A is the coefficient matrix, $\vec{x}$ is the vector of unknowns and $\vec{b}$ is the vector of independent terms.
- The Gauss' Method can be applied with more clarity computing an echelon form of the augmented matrix $[A \mid \vec{b}]$.
- The relation "being row equivalent" is an equivalence relation. As a consequence the set of matrices is partitioned into equivalence classes.

II.4 Reduced Row Echelon Form: Gauss-Jordan Method

Given a matrix A there are many different matrices that are row equivalent to A, that is, there are many matrices in its *equivalence class.* Our objective now will be to find a "canonical" or "distinguished" one in this equivalence class. This "canonical" form will be called *Reduced Row Echelon Form*, or RREF for short.

Definition II.32. A matrix is *principal row echelon* (*principal echelon* for short) if it is echelon and, moreover, the pivot of any row is always 1.

Definition II.33. A matrix is *reduced row echelon* if it is principal echelon and, moreover, **any** pivot is the only non-zero entry in its column.

A reduced row echelon matrix looks like this in general:

$$\begin{bmatrix} 1 & * & 0 & * & 0 & \cdots & 0 & \\ 0 & & 1 & * & 0 & \cdots & 0 & \vdots \\ 0 & & 0 & & 1 & \cdots & 0 & \\ \vdots & & \vdots & & \vdots & & 0 & \vdots \\ & & & & & & 1 & \\ 0 & & 0 & & 0 & \cdots & 0 & \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ 0 & & 0 & & 0 & \cdots & 0 & \end{bmatrix}$$

The asterisks denote arbitrary content which could be several columns long.

Example II.34. The following matrix is *row echelon*, but it is not principal row echelon and, therefore, it is not reduced row echelon:

$$\begin{bmatrix} {\color{blue}2} & 3 & 0 & 5 \\ 0 & {\color{blue}1} & 0 & -2 \\ 0 & 0 & {\color{blue}1} & 4 \end{bmatrix}.$$

The blue entries are the *pivots*.

Example II.35. The following matrix is *principal row echelon*, but it is not *reduced row echelon*:

$$\begin{bmatrix} {\color{blue}1} & {\color{red}3} & 7 & 0 \\ 0 & {\color{blue}1} & 0 & {\color{red}-2} \\ 0 & 0 & 0 & {\color{blue}1} \end{bmatrix}.$$

The reason is that the pivots of the second and third rows are not the unique non-zero elements of their columns (see the red ones).

Example II.36. The following matrix is *reduced row echelon*:

$$\begin{bmatrix} {\color{blue}1} & {\color{red}0} & 1 & {\color{red}0} \\ 0 & {\color{blue}1} & 2 & {\color{red}0} \\ 0 & 0 & 0 & {\color{blue}1} \end{bmatrix}.$$

Notice that the pivots are the unique non-zero elements of their columns (see the red zeroes above them).

The following example shows a method to obtain, from a matrix A, a reduced row echelon matrix that is row equivalent to A:

Example II.37. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 1 & 0 & -1 & -1 \end{bmatrix}$$

The first stage to obtain a row equivalent reduced row echelon matrix is to apply Gauss' Method until obtaining an echelon form:

$$\xrightarrow{\rho_3-\rho_1} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{\rho_3+\rho_2} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 4 & 8 \end{bmatrix}$$

We can keep going to a **second stage** by making the pivots into 1's (to obtain a principal echelon matrix):

$$\xrightarrow{(1/4)\rho_3} \begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The **third stage** uses the pivots to eliminate all of the other entries in each column by combining upwards.

$$\xrightarrow[\rho_1+2\rho_3]{\rho_2-3\rho_3} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\rho_1-\rho_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Using one entry to clear out the rest of a column is **pivoting** on that entry.

Note that the row combination operations in the first stage move left to right, from column one to column three, while the combination operations in the third stage move right to left.

Definition II.38. The algorithm to obtain, from a matrix A, a row equivalent reduced row echelon matrix, and that is described in the preceding example, is called *Gauss-Jordan Method*.

It is clear, from the described example, that Gauss-Jordan Method can be applied to any matrix. Therefore we have the following theorem:

Theorem II.39. *Given any matrix A, there exists a reduced row echelon matrix such that it is row equivalent to A.*

In fact, it holds the stronger result:

Theorem II.40. *Given any matrix A, there exists a* **unique** *reduced row echelon matrix such that it is row equivalent to A.*

Although we include this theorem here without proof, if you are interested in you can find it either in Hefferon's book or in Lay's book, for example.

Remark II.41. This theorem says that, **in each equivalence class of the relation "being row equivalent to" there is one, and only one, reduced row echelon matrix**.

This remark gives rise to the following definition:

Definition II.42. Given a matrix A, we will call the *reduced row echelon form (RREF in short) of* A to the unique reduced row echelon matrix that is row equivalent to A.

Example II.43. Consider the following matrices:

$$A = \begin{bmatrix} 2 & -2 & 1 & 0 \\ 1 & 5 & 8 & -1 \\ 3 & 6 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -2 & 1 & 0 \\ 0 & 12 & 15 & -2 \\ -5 & 5 & 21 & -1 \end{bmatrix}.$$

Are both matrices row equivalent? Since each row equivalence class has only one reduced row echelon matrix, the most efficient way to answer this question is to compute the RREF of both matrices. Then, both matrices will be row equivalent if and only if both RREF coincide.

Using Gauss-Jordan Method you can check that A and B has the same RREF, that is

$$\begin{bmatrix} 1 & 0 & 0 & -13/141 \\ 0 & 1 & 0 & -16/141 \\ 0 & 0 & 1 & -2/47 \end{bmatrix}.$$

Therefore A and B are row equivalent.

We state the general fact involved in this example in the following proposition:

Proposition II.44. *Two matrices are row equivalent if and only if their RREFs are equal.*

UTILITARIAN SUMMARY II.45.

- Each matrix has a unique RREF, that can be computed using the Gauss-Jordan Method. Therefore, in each equivalence class of the relation "being row equivalent" there is **only one** reduced row echelon matrix. Therefore:
- Two matrices are row equivalent if and only if their RREFs are equal.

II.5 Solving linear systems using Gauss-Jordan Method

An alternative method to solve linear systems consists of computing the RREF of the augmented matrix applying Gauss-Jordan Method. We will explain this by means of several examples.

Example II.46. Consider the following linear system:

$$\begin{aligned} 3x_3 &= 9 \\ x_1 + 5x_2 - 2x_3 &= 2 \\ \tfrac{1}{3}x_1 + 2x_2 \qquad &= 3 \end{aligned}$$

First we write the system as an augmented matrix:

$$
\left[\begin{array}{ccc|c} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ \frac{1}{3} & 2 & 0 & 3 \end{array}\right]
\xrightarrow{\rho_1 \leftrightarrow \rho_3}
\left[\begin{array}{ccc|c} \frac{1}{3} & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right]
$$

$$
\xrightarrow{3\rho_1}
\left[\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{array}\right]
$$

$$
\xrightarrow{\rho_2 - \rho_1}
\left[\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{array}\right]
$$

$$
\xrightarrow{-\rho_2;\, (1/3)\rho_3}
\left[\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

$$
\xrightarrow{\rho_2 - 2\rho_3}
\left[\begin{array}{ccc|c} 1 & 6 & 0 & 9 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

$$
\xrightarrow{\rho_1 - 6\rho_2}
\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

Now we're in RREF. If we "translate" again from "augmented matrix" to "equations" we have the equivalent system

$$
\begin{aligned}
x_1 \qquad\quad &= 3 \\
x_2 \quad\; &= 1 \\
x_3 &= 3
\end{aligned}
$$

and, without doing anything more, we see that the solution to the system is given by $x_1 = 3$, $x_2 = 1$, and $x_3 = 3$; it happens to be a unique solution. Notice that we kept track of the steps we were taking; this is important for checking your work!

Example II.47. Consider now the linear system with variables x_1, x_2, x_3 and x_4 and with this augmented matrix:

$$
\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 1 & 1 & 1 & -1 & 2 \\ 0 & -1 & -2 & 3 & -3 \\ 5 & 2 & -1 & 4 & 1 \end{array}\right]
$$

$$
\xrightarrow{\rho_2 - \rho_1; \rho_4 - 5\rho_2}
\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & -1 & -2 & 3 & -3 \\ 0 & 2 & 4 & -6 & 6 \end{array}\right]
$$

$$
\xrightarrow{\rho_3 + \rho_2; \rho_4 - 2\rho_3}
\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

"Passing to equations":

$$\begin{array}{rrrrl} x_1 & & -x_3 & +2x_4 & = -1 \\ & +x_2 & +2x_3 & -3x_4 & = \ \ 3 \end{array}$$

Here the variables x_3 and x_4 are free; then the solution is not unique. Set $x_3 = \lambda$ and $x_4 = \mu$ where λ and μ are *arbitrary* real numbers. Then we can write x_1 and x_2 in terms of λ and μ as follows:

$$\{(x_1, x_2, x_3, x_4) = (-1 + \lambda - 2\mu, 3 - 2\lambda + 3\mu, \lambda, \mu) \mid \lambda, \mu \in \mathbb{R}\}.$$

The vector expression of the solution is:

$$\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \mid \lambda, \mu \in \mathbb{R}\} = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 0 \end{bmatrix} + \operatorname{span}(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix}).$$

Example II.48. Consider the linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 5 & 15 & 1 & 2 \\ 1 & 3 & 1 & 0 \\ 2 & 6 & 1 & 2 \end{array}\right].$$

Applying Gauss-Jordan Method you can deduce that its RREF is

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

The last obtained equation is $0 = 1$, which is contradictory. Therefore this system has no solution.

The three examples above are representative of the three types of linear system: with a unique solution, with infinitely many solutions, and without solutions. Let's analyze the reason of being of each type in terms of two things that can be seen in the RREF:

(1) the existence of a pivot in the column of independent terms and

(2) the number of pivots.

Case (a). If there exists a pivot in the column of independent terms (as in Example II.48) then we have an equation of the type $0 = 1$, that is, a contradictory equation. This means that the solution set is empty.

Case (b). If there is not a pivot in the column of independent terms then two situations can occur:

(b1) If the number of pivots is equal to the number of variables (that is, there are not free variables), as in Example II.46, then the system has a unique solution.

(b2) If the number of pivots is less than the number of variables (that is, there are free variables), as in Example II.47, then the system has infinitely many solutions.

In the next section we will re-write this case distinction in a clearer form using a new concept: the *rank* of a matrix.

UTILITARIAN SUMMARY II.49. Applying Gauss-Jordan Method to the augmented matrix of a linear system we can solve it without having to do back substitution. Moreover the number of solutions depends on: (1) the existence of a pivot in the last column of the RREF and (2) the number of pivots of the RREF:

(a) If there exists a pivot in the column of independent terms then the solution set is empty.

(b) If there is not a pivot in the last column:

  (b1) If the number of pivots is equal to the number of variables (that is, there are not free variables) then the system has a unique solution.

  (b2) If the number of pivots is less than the number of variables (that is, there are free variables) then the system has infinitely many solutions.

II.6 Rank of a matrix and Rouché-Fröbenius Theorem

Since the RREF of a matrix is unique, the following definition has sense:

Definition II.50. The *rank* of a matrix A, denoted by $\operatorname{rank}(A)$, is the number of pivots of its RREF.

When we apply Gauss-Jordan Method to obtain a RREF of a matrix A, in the first stage we must compute an echelon form of A. It seems evident that **the number of pivots of this echelon form and the number of pivots of the RREF (obtained at the end of the process) are equal!** This means that the rank of A coincides with the number of pivots of *any* row echelon form of A. Taking this observation into account, we can rewrite the above definition:

Definition II.51. (Rewritten Definition II.50) The *rank* of a matrix A, denoted by $\operatorname{rank}(A)$, is the number of pivots of *any* row echelon form of A.

Remark II.52. To compute the rank of a matrix we only need to compute an echelon form and then to count the number of pivots (**that coincides with the number of non-zero rows**).

Remark II.53. If $A\vec{x} = \vec{b}$ is a linear system and we compute an echelon form of the augmented matrix $[A \mid \vec{b}]$, we can read off at the same time both ranks, $\operatorname{rank}(A)$ and $\operatorname{rank}([A \mid \vec{b}])$. To read off the former, we only need to hide the last column of the echelon form and to count the pivots there.

Example II.54. Recall the RREF of the augmented matrix obtained in Example II.46:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

The system has, in this case, a unique solution.

The rank of this augmented matrix, $[A \mid \vec{b}]$ is 3 because it has 3 pivots. Hiding the last column, we see that the rank of the coefficient matrix A is also 3. Therefore

$$\operatorname{rank}(A) = \operatorname{rank}([A \mid \vec{b}]).$$

Observe that **this equality of ranks holds if and only if there is no pivot in the last column** (the 3 pivots are common pivots of both matrices, A and $[A \mid \vec{b}]$).

Also, this common rank (that is, the common number of pivots) is equal, in this case, to 3, that is the total number of unknowns. Therefore we are in the following situation:

$$\operatorname{rank}(A) = \operatorname{rank}([A \mid \vec{b}]) = \text{Number of unknowns}.$$

Example II.55. The RREF of the augmented matrix corresponding to Example II.47 is

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & -1 \\ 0 & 1 & 2 & -3 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Recall that, in this case, the system has infinitely many solutions. We have also here equality of ranks

$$\operatorname{rank}(A) = \operatorname{rank}([A \mid \vec{b}]),$$

what means that there is no pivot in the last column. Moreover, in this case, this common rank (the common number of pivots) is equal to 2, that is less than the number of unknowns.

Example II.56. The RREF of the augmented matrix in Example II.48 is

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Recall that the solution set is empty. In this case we have that

$$\operatorname{rank}(A) \neq \operatorname{rank}([A \mid \vec{b}]),$$

what means that there is a pivot in the last column of $[A \mid \vec{b}]$.

Thinking around these examples you will see evident that the case distinction given at the end of the preceding section can be written using ranks in the following form

Theorem II.57. (Rouché-Fröbenius Theorem) *Let $A\vec{x} = \vec{b}$ be a linear system. Then*

(a) If $\operatorname{rank}(A) \neq \operatorname{rank}([A \mid \vec{b}])$ *then the solution set is empty.*

(b) If $\operatorname{rank}(A) = \operatorname{rank}([A \mid \vec{b}])$ *then the system has solutions and*

  (b1) If $\operatorname{rank}(A) = \operatorname{rank}([A \mid \vec{b}]) = (\text{Number of unknowns})$ *then the system has a unique solution.*

  (b2) If $\operatorname{rank}(A) = \operatorname{rank}([A \mid \vec{b}]) < (\text{Number of unknowns})$ *then the system has infinitely many solutions.*

UTILITARIAN SUMMARY II.58.

- To compute the rank of a matrix, compute first any echelon form and count the number of pivots.
- To know the number of solutions of a linear system in terms of the ranks of the coefficient and augmented matrices, we can use the Rouché-Fröbenius Theorem.

II.7 Simultaneous resolution of linear systems

We can solve simultaneously several systems of linear equations **with the same coefficient matrix**:

$$Ax_1 = b_1 \qquad Ax_2 = b_2 \qquad \dots \qquad Ax_r = b_r.$$

To solve them we consider the *(super)augmented matrix*

$$\left[\begin{array}{c|ccc} A & \vec{b}_1 & \cdots & \vec{b}_r \end{array}\right]$$

and we apply Gauss-Jordan Method to the matrix A, but doing the elementary row operations we use in the process to the whole (super)augmented matrix. Let's see an example.

Example II.59. To solve simultaneously the systems:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 2x_1 + 5x_2 - x_3 &= 0 \\ -x_1 - 2x_2 + x_3 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + 2x_2 &= -1 \\ 2x_1 + 5x_2 - x_3 &= -5 \\ -x_1 - 2x_2 + x_3 &= 3 \end{aligned} \qquad \begin{aligned} x_1 + 2x_2 &= 4 \\ 2x_1 + 5x_2 - x_3 &= 9 \\ -x_1 - 2x_2 + x_3 &= -4 \end{aligned}$$

we consider the (super)augmented matrix

$$\left[\begin{array}{c|ccc} A & b_1 & b_2 & b_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & -1 & 4 \\ 2 & 5 & -1 & 0 & -5 & 9 \\ -1 & -2 & 1 & 0 & 3 & -4 \end{array}\right]$$

and we apply Gauss-Jordan to A, but applying the elementary row operations to the whole matrix, obtaining the following:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right]$$

To read off the solutions of each one of the 3 systems, we must hide the columns of independent terms of the others and get the information from there. Therefore, the solutions of the three systems are

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \qquad x_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

UTILITARIAN SUMMARY II.60.
We can solve simultaneously several linear systems **with the same coefficient matrix** performing Gauss-Jordan Method to the coefficient matrix, but applying the used elementary row operations to the whole (super)augmented matrix.

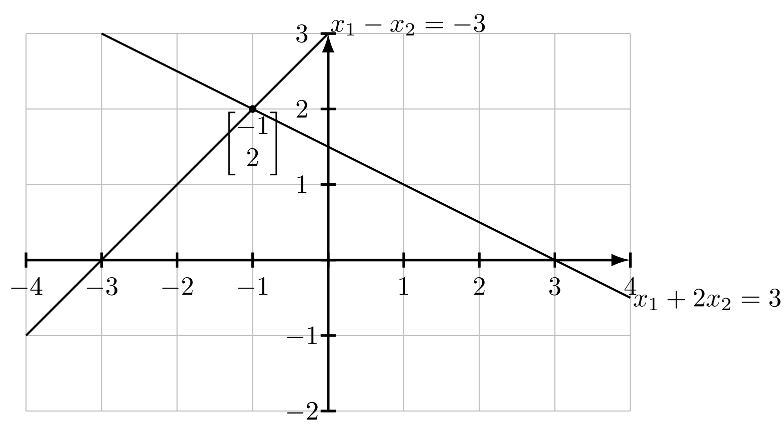
II.8 Two interpretations of the solutions of a linear system

Let us consider the following linear system of two equations and two unknowns:

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 - x_2 &= -3 \end{aligned}$$

It is easily shown that it has only one solution: $(x_1, x_2) = (-1, 2)$. The "classical" geometric interpretation of this is the following:

- Each equation represents a line on the plane.
- The solution is the point where both lines meet.



But there is a different (**and very useful**) geometric interpretation of the solution:

Consider now the matrix expression of the system:

$$\begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

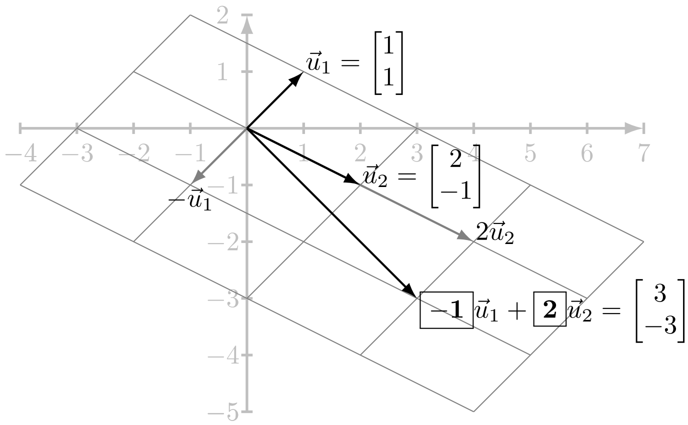
Taking into account that the product matrix-vector is a linear combination of the columns of the matrix (see Proposition I.36), the above equality is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

And since $(x_1, x_2) = ({\color{red}-1}, {\color{red}2})$ is the solution, one has that

$${\color{red}(-1)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + {\color{red}2} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}.$$

This means that **the vector of independent terms $\vec{b} = (3, -3)$ is a linear combination of the columns of the coefficient matrix** A; moreover the coefficients of this linear combination are just the components of the solution.



This argument is valid for *any* linear system

$$\underbrace{\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & & & \\ \cdots & & & \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}.$$

This equality is equivalent to

$$x_1 \underbrace{\begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix}}_{Col.1} + x_2 \underbrace{\begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{bmatrix}}_{Col.2} + \cdots + x_n \underbrace{\begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix}}_{Col.n} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}.$$

Then, the existence of a solution of the linear system is equivalent to the fact that $\vec{b}$ is a linear combination of the columns of A. That is:

Theorem II.61. *A linear system $A\vec{x} = \vec{b}$ has some solution if and only if the vector of independent terms $\vec{b}$ is a linear combination of the columns of A. In this case, each solution gives the coefficients of such a linear combination.*

UTILITARIAN SUMMARY II.62.
The solutions of a system of linear equations $A\vec{x} = \vec{b}$ can be interpreted in two ways:

- **Rows' point of view:** each equation represents an affine subspace and the solution set is the set of intersection points of all the affine subspaces.

- **Columns' point of view:** each solution gives the coefficients of the expression of $\vec{b}$ as a linear combination of the columns of A. Therefore **a linear system has solution if and only if $\vec{b}$ is a linear combination of the columns of A.**

II.9 Some computational considerations

Partial Pivoting Strategy

Consider the system below:

$$\begin{aligned} 0.001x + y &= 1 \\ x - y &= 0 \end{aligned} \tag{$*$}$$

The second equation gives $x = y$, so $x = y = 1/1.001$ and thus both variables have values that are just less than 1. A computer using a precision of two digits represents the system internally in this way:

$$\begin{aligned} (1.0 \times 10^{-2})x + (1.0 \times 10^{0})y &= 1.0 \times 10^{0} \\ (1.0 \times 10^{0})x - (1.0 \times 10^{0})y &= 0.0 \times 10^{0} \end{aligned}$$

The computer's row reduction step $\rho_2 - 1000\rho_1$ produces a second equation $-1001y = -999$, which the computer rounds to two places as $(-1.0 \times 10^3)y = -1.0 \times 10^3$. Then the computer decides from the second equation that $y = 1$ and from the first equation that $x = 0$. This y value is fairly good, but the x is very bad.

An experienced programmer may respond by going to *double precision* that retains sixteen significant digits. This will indeed solve many problems. However, double precision has twice the memory requirements and besides, **we can obviously tweak the system of the above example to give the same trouble in the seventeenth digit, so double precision isn't a panacea**. We need is a strategy to minimize the numerical trouble arising from solving systems on a computer.

A strategy that, many times (but not always) minimizes these errors due to rounding, is the *partial pivoting strategy*:

Strategy II.63. (Partial Pivoting)
At each step, search the positions on and below the pivotal position for the coefficient of maximum magnitude (absolute value). If necessary perform the appropriate row interchange to bring this maximal coefficient into the pivotal position.

For example, in the situation

$$\left[\begin{array}{ccccc|c} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & P & * & * & * \\ 0 & 0 & S & * & * & * \\ 0 & 0 & T & * & * & * \end{array}\right]$$

search, among P, S and T, the coefficient of maximum magnitude (absolute value) and, if necessary, interchange rows to bring this coefficient into the position of P.

Number of operations of Gauss and Gauss-Jordan Methods

To finish this section, we show the number of operations when Gauss and Gauss-Jordan Methods are applied to a linear system $A\vec{x} = \vec{b}$, where A is an $n \times n$ matrix.

GAUSS METHOD + BACK SUBSTITUTION

- Multiplications/divisions:
$$\frac{n^3}{3} + n^2 - \frac{n}{3}$$
- Additions/subtractions:
$$\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$$

Observe that the dominant term is $\frac{n^3}{3}$.

GAUSS-JORDAN METHOD

- Multiplications/divisions:
$$\frac{n^3}{2} + \frac{n^2}{2}$$
- Additions/subtractions:
$$\frac{n^3}{2} - \frac{n}{2}$$

Observe that the dominant term is $\frac{n^3}{2}$.

UTILITARIAN SUMMARY II.64. When Gauss' or Gauss-Jordan Methods are applied by a computer, the given "solution" may be very different from the correct one due to rounding errors. The **partial pivoting strategy** minimizes, many times, these errors.

III (General solution of $A\vec{x} = \vec{b}$) = (Particular solution) + (kernel of A)

Kernel of a matrix

Definition III.1. Given an $m \times n$ matrix A, its *kernel*, denoted by $\ker(A)$, is the set of solutions of the homogeneous linear system whose coefficient matrix is A, that is,

$$\ker(A) = \{\vec{x} \in \mathbb{R}^n \mid A \cdot \vec{x} = \vec{0}\}.$$

Example III.2. Consider the matrix

$$A = \begin{bmatrix} 7 & 0 & -7 & 0 \\ 8 & 1 & -5 & -2 \\ 0 & 1 & -3 & 0 \\ 0 & 3 & -6 & -1 \end{bmatrix}.$$

To compute the kernel of A we must solve the homogeneous system $A \cdot \vec{x} = \vec{0}$, that is,

$$\begin{array}{rcl} 7x \qquad - 7z \qquad &=& 0 \\ 8x + \ y - 5z - 2w &=& 0 \\ y - 3z \qquad &=& 0 \\ 3y - 6z - \ w &=& 0 \end{array} \xrightarrow{\rho_2 - (8/7)\rho_1} \begin{array}{rcl} 7x \quad - 7z \qquad &=& 0 \\ y + 3z - 2w &=& 0 \\ y - 3z \qquad &=& 0 \\ 3y - 6z - \ w &=& 0 \end{array}$$

$$\xrightarrow[\rho_4 - 3\rho_2]{\rho_3 - \rho_2} \begin{array}{rcl} 7x \quad - \quad 7z \qquad &=& 0 \\ y + \quad 3z - 2w &=& 0 \\ -6z + 2w &=& 0 \\ -15z + 5w &=& 0 \end{array}$$

$$\xrightarrow{\rho_4 - (5/2)\rho_3} \begin{array}{rcl} 7x \quad - \quad 7z \qquad &=& 0 \\ y + \ 3z - 2w &=& 0 \\ -6z + 2w &=& 0 \\ 0 &=& 0 \end{array}$$

The solution set is

$$\{ \begin{bmatrix} 1/3 \\ 1 \\ 1/3 \\ 1 \end{bmatrix} \alpha \mid \alpha \in \mathbb{R} \} = \operatorname{span}(\begin{bmatrix} 1/3 \\ 1 \\ 1/3 \\ 1 \end{bmatrix})$$

and therefore

$$\ker(A) = \operatorname{span}(\begin{bmatrix} 1/3 \\ 1 \\ 1/3 \\ 1 \end{bmatrix}).$$

Example III.3. Consider the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix}.$$

To compute its kernel we must solve the system $A \cdot \vec{x} = \vec{0}$, that is,

$$\begin{array}{rcl} 3x + 4y &=& 0 \\ 2x - \ y &=& 0 \end{array}$$

Aplying Gauss' Method:

$$\begin{array}{rl} 3x+4y=0 \\ 2x-\ \ y=0 \end{array} \xrightarrow{\rho_2-(2/3)\rho_1} \begin{array}{rl} 3x+ & 4y=0 \\ & -(11/3)y=0 \end{array}$$

Therefore this system has a unique solution: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence

$$\ker(A)=\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}=\{\vec{0}\}.$$

Remark III.4. Notice that the kernel of any matrix is always the span of a set of vectors, that is, a vector subspace of $\mathbb{R}^4$.

Expression of the general solution of a linear system in terms of a particular solution and the kernel of the coefficient matrix

Lemma III.5. *For a linear system $A\cdot\vec{x}=\vec{b}$, where $\vec{p}$ is any particular solution, the solution set equals the set*

$$\vec{p}+\ker(A),$$

that is, the solutions are exactly those vectors obtained by adding to $\vec{p}$ a vector in the kernel of A.

PROOF:

Firstly, let us prove that any vector in the set $\vec{p}+\ker(A)$ is a solution of $A\cdot\vec{x}=\vec{b}$. Take such a vector $\vec{s}=\vec{p}+\vec{v}$, with $\vec{v}\in\ker(A)$. On the one hand, since $\vec{v}$ is a solution of the homogeneous system $A\cdot\vec{x}=\vec{0}$ one has that $A\cdot\vec{v}=\vec{0}$. On the other hand, since $\vec{p}$ is a particular solution of the original system, one has that $A\cdot\vec{p}=\vec{b}$. Then

$$A\cdot\vec{s}=A\cdot(\vec{p}+\vec{v})=A\cdot\vec{p}+A\cdot\vec{v}=\vec{b}+\vec{0}=\vec{b}.$$

Therefore $\vec{s}$ is a solution of $A\cdot\vec{x}=\vec{b}$.

Secondly, let us prove that any solution of the original system belongs to the set $\vec{p}+\ker(A)$. Take, then, a solution $\vec{s}$ of $A\cdot\vec{x}=\vec{b}$ and define

$$\vec{v}:=\vec{s}-\vec{p}.$$

Notice that $\vec{s}=\vec{p}+(\vec{s}-\vec{p})=\vec{p}+\vec{v}$ and $\vec{v}$ belongs to $\ker(A)$ because

$$A\cdot\vec{v}=A\cdot(\vec{s}-\vec{p})=A\cdot\vec{s}-A\cdot\vec{p}=\vec{b}-\vec{b}=\vec{0}.$$

We conclude, then, that $\vec{s}\in\vec{p}+\ker(A)$ and we are done. QED

UTILITARIAN SUMMARY III.6.

- The kernel of a matrix A, $\ker(A)$, is the solution set of the homogeneous linear system whose coefficient matrix is A.
- The kernel of any matrix is always a vector subspace of $\mathbb{R}^n$, that is, the span of a set of vectors.
- If $A \cdot \vec{x} = \vec{b}$ is any system with non-empty solution set and $\vec{p}$ is any particular solution of this system, then its solution set (or *general solution*) is

$$\vec{p} + \ker(A).$$

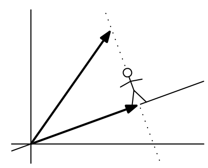
IV Linear systems, vectors and matrices in action: Orthogonal Projections and Least Squares Method

IV.1 Orthogonal projections

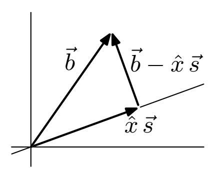
Orthogonal Projection onto a line

We first consider orthogonal projection of a vector $\vec{b}$ onto a line ℓ that passes through the origin. The problem we want to solve is the following: **Find the vector (or point, if you prefer) of ℓ that is closest to the vector (or point) $\vec{b}$.**

This picture shows someone walking out on the line until he/she is at a point $\vec{p}$ such that the tip of $\vec{b}$ is directly above him/her, where "above" does not mean parallel to the y-axis but instead means "perpendicular" (or "orthogonal") to the line. It is clear that this is the best choice!



We can describe the line as the span of some nonzerovector $\ell = \{x \cdot \vec{s} \mid x \in \mathbb{R}\} = \mathrm{span}(\vec{s})$. Since this chosen vector is on the line, it will be a multiple of the vector $\vec{s}$, that is, it will have the form $\hat{x} \cdot \vec{s}$ for some real number $\hat{x}$. This number $\hat{x}$ satisfies the property that the vector $\vec{b} - \hat{x} \cdot \vec{s}$ is orthogonal to $\vec{s}$.



Then $(\vec{b} - \hat{x} \cdot \vec{s}) \bullet \vec{s} = 0$ gives that $\hat{x} = \vec{b} \bullet \vec{s} / \vec{s} \bullet \vec{s}$. So, the vector of ℓ that is closest to $\vec{b}$ is

$$\hat{x} \cdot \vec{s} = \frac{\vec{b} \bullet \vec{s}}{\vec{s} \bullet \vec{s}} \cdot \vec{s}.$$

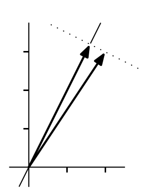
Definition IV.1. The *orthogonal projection* of $\vec{b}$ onto the line spanned by a nonzero vector $\vec{s}$ *is the following vector:*

$$\frac{\vec{b} \bullet \vec{s}}{\vec{s} \bullet \vec{s}} \cdot \vec{s}$$

Example IV.2. To orthogonally project the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ into the line $y = 2x$, we first pick a direction vector for the line.

$$\vec{s} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The calculation is easy.



$$\frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 2 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{8}{5} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8/5 \\ 16/5 \end{bmatrix}$$

Example IV.3. In $\mathbb{R}^3$, the orthogonal projection of a general vector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

into the y-axis is

$$\frac{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix}$$

which matches our intuitive expectation.

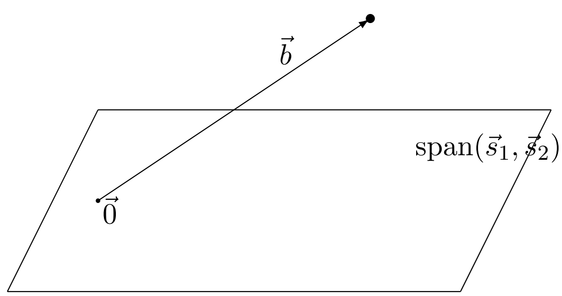
Orthogonal projection onto a span

Now we will solve a more general problem: **Given a vector (or point, if you prefer) $\vec{b}$ of $\mathbb{R}^n$, find the vector (or point) of a span of vectors of $\mathbb{R}^n$, $\mathrm{span}(\vec{s}_1, \dots, \vec{s}_k)$, that is closest to $\vec{b}$.** (In the preceding subsection we have solved the problem when $k = 1$).

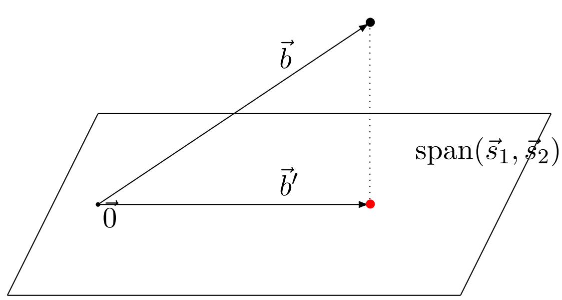
Let us consider an example. Let $\vec{b} = (6, 0, 0) \in \mathbb{R}^3$. Take the following vectors of $\mathbb{R}^3$,

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

and consider their span, $\mathrm{span}(\vec{s}_1, \vec{s}_2)$. We are going to compute the vector of $\mathrm{span}(\vec{s}_1, \vec{s}_2)$ that is closest to $\vec{b}$. Notice that it is a plane in $\mathbb{R}^3$ (passing through the origin) and, then, we have the following situation:



Geometrically is clear that the point of the plane that is closest to $\vec{b}$ can be obtained as in the case of a line: Imagine someone walking out on the plane until he/she is at a point $\vec{b}'$ such that the tip of $\vec{b}$ is directly above him/her, where "above" means "perpendicular" (or "orthogonal") to the plane. This vector $\vec{b}'$ is the best choice!



Since the vector $\vec{b}'$ is on the plane, it is a linear combination of $\vec{s}_1$ and $\vec{s}_2$. Therefore

$$\vec{b}' = \hat{x}_1\vec{s}_1 + \hat{x}_2\vec{s}_2$$

for some real coefficients $\hat{x}_1$ and $\hat{x}_2$. That is, if we define

$$\hat{\vec{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}$$

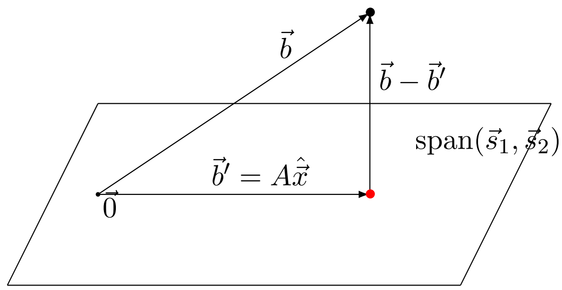
and A is that matrix whose columns are $\vec{s}_1$ and $\vec{s}_2$,

$$A = [\vec{s}_1 \;\; \vec{s}_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix},$$

then we have that

$$\vec{b}' = A\hat{\vec{x}}.$$

Remember that a matrix by a vector is a linear combination of the columns (see Proposition I.36)! This is the picture now:



The objective is, now, to compute the vector of coefficients $\hat{\vec{x}}$ and, then, to compute $\vec{b}'$ as the product $A\hat{\vec{x}}$. For this purpose, observe that the vector $\vec{b} - \vec{b}'$ must be orthogonal to the plane, that is, it must be orthogonal to the vectors $\vec{s}_1$ and $\vec{s}_2$. Then we can impose two conditions:

- $\vec{s}_1 \bullet (\vec{b} - \vec{b}') = 0$
- $\vec{s}_2 \bullet (\vec{b} - \vec{b}') = 0$

An equivalent form to express these 2 conditions in a more compact way is the following one (check it!):

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}}_{A^t}(\vec{b} - \vec{b}') = \vec{0}. \tag{0}$$

Replacing $\vec{b}'$ by $A\hat{\vec{x}}$ we have that (0) is equivalent to

$$A^t(\vec{b} - A\hat{\vec{x}}) = \vec{0} \;\;\Leftrightarrow\;\; A^t\vec{b} - A^tA\hat{\vec{x}} = \vec{0} \;\;\Leftrightarrow\;\; A^tA\hat{\vec{x}} = A^t\vec{b}.$$

This means that the vector $\hat{\vec{x}}$ we want to compute is a solution of the linear system

$$A^tA\vec{x} = A^t\vec{b}. \tag{1}$$

Let's compute the coefficient matrix and the vector of independent terms of this linear system:

$$A^tA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A^t\vec{b} = \begin{bmatrix}1 & 1 & 1\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}6\\0\\0\end{bmatrix} = \begin{bmatrix}6\\0\end{bmatrix}$$

So we must solve the system

$$\begin{bmatrix}3 & 3\\3 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}6\\0\end{bmatrix}.$$

Doing this we obtain the desired coefficients $\hat{x}_1$ and $\hat{x}_2$:

$$\hat{\vec{x}} = \begin{bmatrix}\hat{x}_1\\\hat{x}_2\end{bmatrix} = \begin{bmatrix}5\\-3\end{bmatrix}.$$

Now we can compute the desired vector $\vec{b}'$:

$$\vec{b}' = \hat{x}_1 \cdot \vec{s}_1 + \hat{x}_2 \cdot \vec{s}_2 = 5\begin{bmatrix}1\\1\\1\end{bmatrix} - 3\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}5\\2\\-1\end{bmatrix}.$$

This vector $\vec{b}'$ is called *orthogonal projection* of $\vec{b}$ onto the span $\mathrm{span}(\vec{s}_1, \vec{s}_2)$.

In this example we can "see" the geometry because we are "living" in the 3-dimensional space. However, using algebraic proofs, it is possible to show that **all we have done above works properly when we are "living" in** $\mathbb{R}^n$. The only differences are the number of components of the vectors and the size of the above matrix A.

Now, we formalize all these things in a definition and a strategy:

Definition IV.4. Let $\vec{b}, \vec{s}_1, \dots, \vec{s}_k$ be some vectors in $\mathbb{R}^n$. The vector $\vec{b}'$ of $\mathrm{span}(\vec{s}_1, \dots, \vec{s}_k)$ that is closest to $\vec{b}$ is called *orthogonal projection* of $\vec{b}$ onto $\mathrm{span}(\vec{s}_1, \dots, \vec{s}_k)$. It can be computed as follows:

Strategy IV.5. (Computing orthogonal projections)

1. Compute the matrix A whose **columns** are the vectors $\vec{s}_1, \dots, \vec{s}_k$.
2. Compute a solution $\hat{\vec{x}}$ of the linear system $A^tA \cdot \vec{x} = A^t\vec{b}$.
3. The orthogonal projection $\vec{b}'$ is $A \cdot \hat{\vec{x}}$.

Remark IV.6. It can be reasoned that there is always a **unique** orthogonal projection.

UTILITARIAN SUMMARY IV.7. To compute the orthogonal projection of a vector $\vec{b} \in \mathbb{R}^n$ onto the span of several vectors of $\mathbb{R}^n$:

(1) Compute the matrix A whose **columns** are the vectors that give the span.

(2) Compute a solution $\hat{\vec{x}}$ of the linear system $A^tA \cdot \vec{x} = A^t\vec{b}$.

(3) The orthogonal projection is $A \cdot \hat{\vec{x}}$.

When the span is spanned by only one vector $\vec{s} \neq \vec{0}$, then it is a line and the orthogonal projection is given directly by the following formula:

$$\frac{\vec{b} \bullet \vec{s}}{\vec{s} \bullet \vec{s}} \cdot \vec{s}$$

IV.2 Least Squares Method

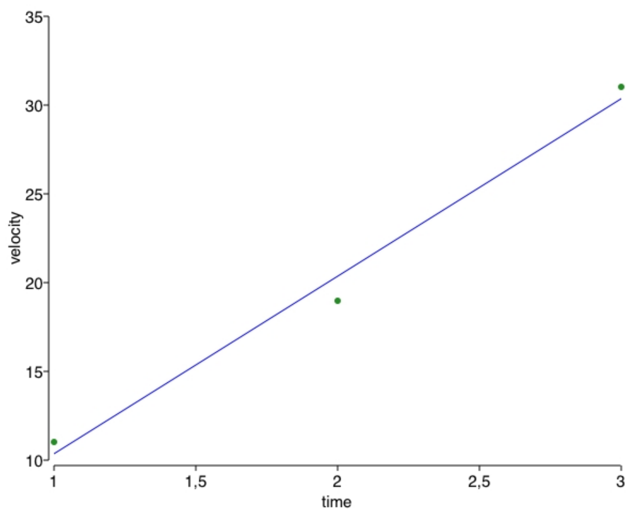
SpongeBob falls off of the leaning tower of Pisa and makes three (rather shaky) measurements of his velocity at three different times.

t (s)	v (m/s)
1	11
2	19
3	31

Having taken some calculus and physics, he believes that his data are best approximated by a straight line

$$v = at + b.$$

Then he should find a and b to best fit the data. In other words, he is looking for the equation of a line (more or less as the blue one in the next figure) that approximates our data the best possible.



Since he wants that the points given in the above table belong to the line, he should have that

$$\begin{aligned} 11 &= a \cdot 1 + b \\ 19 &= a \cdot 2 + b \\ 31 &= a \cdot 3 + b. \end{aligned}$$

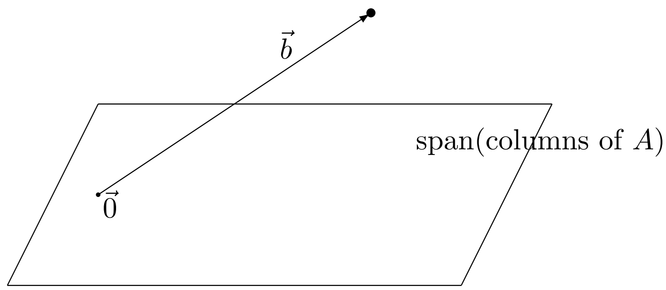
This means that the vector (a, b) should be a solution of the linear system

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 11 \\ 19 \\ 31 \end{bmatrix}}_{\vec{b}}. \tag{2}$$

But... notice that **this system $A\vec{x} = \vec{b}$ has clearly no solution!** (geometrically: it is clear from the above figure that the points are not aligned; algebraically: if we subtract the first equation from the second one we obtain $a = 8$; however, if we subtract the second equation from the third one we obtain $a = 12$). The reason is that **the vector $\vec{b}$ is not a linear combination of the columns of A** (see Section II.8), that is,

$$\vec{b} = \begin{bmatrix} 11 \\ 19 \\ 31 \end{bmatrix} \notin \operatorname{span}(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \operatorname{span}(\text{columns of } A),$$

where span(columns of A) is the span of the columns of A. It is, in this case, a plane in $\mathbb{R}^3$ passing through the origin. The situation, then, is as follows:

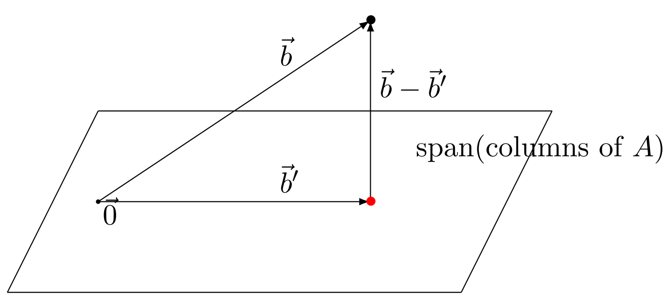


The vector $\vec{b}$ (or the point $\vec{b}$, if you prefer to identify $\vec{b}$ with its ending point) is **outside** the plane span(columns of A). This is the geometrical reason for which the system has no solution. The strategy of the Least Squares Method to "bypass" this problem is the following one:

Strategy IV.8. (Idea of the Least Squares Method) Replace the vector of independent terms $\vec{b}$ in (2) by the vector $\vec{b}'$ of the plane span(columns of A) that is closest to $\vec{b}$. Then take a solution of the obtained linear system $A\vec{x} = \vec{b}'$.

The obtained solution is called a *least squares solution* of the system $A \cdot \vec{x} = \vec{b}$ and it can be interpreted as the closest vector to "be a solution of $A \cdot \vec{x} = \vec{b}$".

This vector $\vec{b}'$ of span(columns of A) that is closest to $\vec{b}$ is, as we have seen in the preceding subsection, the **orthogonal projection** of $\vec{b}$ onto span(columns of A), that is, the vector with red ending point of the next figure:



The whole procedure that we should follow is:

1. Compute the orthogonal projection $\vec{b}'$. As we have seen in the preceding subsection, the vector $\vec{b}'$ is $A \cdot \hat{\vec{x}}$, where $\hat{\vec{x}}$ is a solution of the system

$$A^t A \cdot \vec{x} = A^t \vec{b}.$$

2. Compute a solution of the system $A \cdot \vec{x} = \vec{b}'$.

But notice that we are doing unnecessary work: if $\hat{\vec{x}}$ is a solution of $A^tA \cdot \vec{x} = A^t\vec{b}$ then, automatically, $A \cdot \hat{\vec{x}}$ is equal to $\vec{b}'$ (the orthogonal projection), by the preceding subsection. This means that **we can skip the second step!!**. So, we only need to solve the system

$$A^tA \cdot \vec{x} = A^t\vec{b}. \tag{3}$$

This is usually called the *system of normal equations*. Let's solve it in our example. First compute the matrix of coefficients A^tA and the vector of independent terms $A^t\vec{b}$:

$$A^tA = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}, \quad \text{and} \quad A^t\vec{b} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 11 \\ 19 \\ 31 \end{bmatrix} = \begin{bmatrix} 142 \\ 61 \end{bmatrix}.$$

Now we must solve the linear system

$$\begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 142 \\ 61 \end{bmatrix}.$$

Doing it we obtain that

$$\hat{\vec{x}} = \begin{bmatrix} 10 \\ 1/3 \end{bmatrix}.$$

This is a *least squares solution* of the initial system. Hence, the Least Squares Method fit is the line

$$v = 10\ t + \frac{1}{3}.$$

Notice that this equation implies that SpongeBob accelerates towards Italian soil at 10 m/s^2 (which is an excellent approximation to reality) and that he started at a downward velocity of $\frac{1}{3}$ m/s (perhaps somebody gave him a shove...)!

As in the case of the orthogonal projections, **all the above explained works properly when $A\vec{x} = \vec{b}$ is any linear system without solutions (independently of its size)**.

Definition IV.9. Given a linear system $A\vec{x} = \vec{b}$ without solutions, a *least square solution* of it is any solution of the *system of normal equations*

$$A^tA\vec{x} = A^t\vec{b}.$$

UTILITARIAN SUMMARY IV.10. The above definition.