Practices of Discrete Mathematics: Introduction to graph theory

Session 7

1 Shortest path

2 Dijkstra's algorithm

Shortest-path problem

Definition

If $C = v_0 e_1 v_1, \ldots e_n v_n$ is a path of a weighted graph Γ, the *weight* of C, $w(C)$, is defined to be

$$w(C) = \sum_{i=1}^{n} w(e_i).$$

Shortest-path problem

To find a path joining two vertices v and v' (that must be connected) with the smallest weight.

1 Shortest path

2 Dijkstra's algorithm

Idea

- The idea is to begin at the initial vertex v and move through the graph assigning a number $L(u)$ (a "label") to each vertex u in turn which represents a length of the shortest path yet *discovered* from v to u.
- These labels $L(u)$ are initially considered temporary and may subsequently be changed if we discover a path from v to u which has length less than the currently assigned value $L(u)$.
- The algorithm constructs a subtree of the graph containing the vertices v and v'.
- A shortest path between v and v' is the **unique** path in this tree joining them.

Dijkstra's algorithm (description)

1) First assign $L(v) = 0$ to the starting vertex v. We say that v has been labelled with the value 0. Furthermore, this label is **permanent** as we will not subsequently change its value. Since we are constructing a sequence of trees, we also begin with the tree consisting of the vertex v only and no edges.
2) Let u be the vertex which has most recently been given a *permanent* label. Consider each vertex u' adjacent to u and give it a *temporary label* as follows:
    a) If u' is unlabelled, then set $L(u')$ equal to $L(U) + w(e)$, where e is the edge joining u and u'. (If there is more than one such edge e, choose the one with the smallest weight).
    b) If u' is already labelled, then again calculate $L(u) + w(e)$ as above and if this is less than the current value of $L(u')$ then change $L(u')$ to $L(u) + w(e)$; otherwise leave $L(u')$ unchanged.

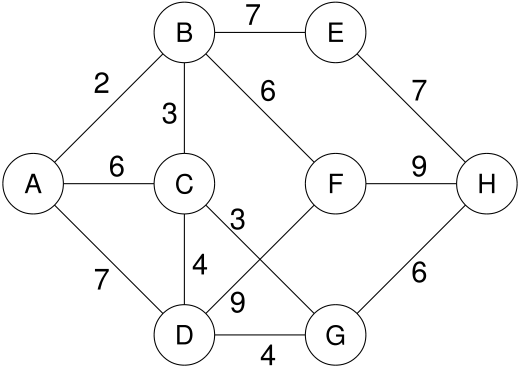
Dijkstra's algorithm (description)

3) Choose a vertex a with the *smallest* temporary label and make the label permanent. At the same time adjoint to the tree so far formed the edge which gives rise to the value $L(a)$.

4) Repeat steps 2 and 3 until the final vertex v' has been given a **permanent** label. The path of shortest length from v to v' is then the unique path in the tree thus formed joining v and v'. Its length is the permanent value of $L(v')$.

Example

We want to calculate a shortest path between the vertices A and H in the following weighted graph:



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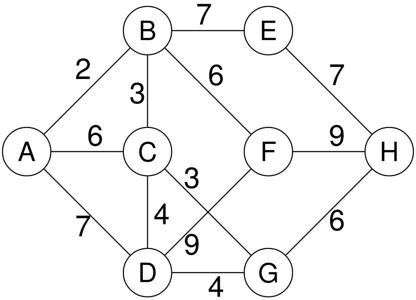
T: tree consisting of the vertex v and no edge.



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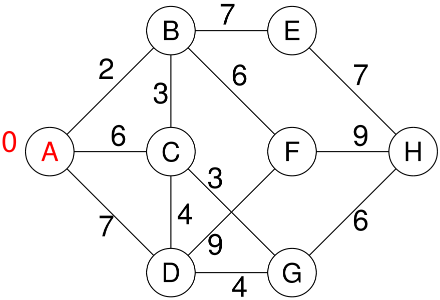

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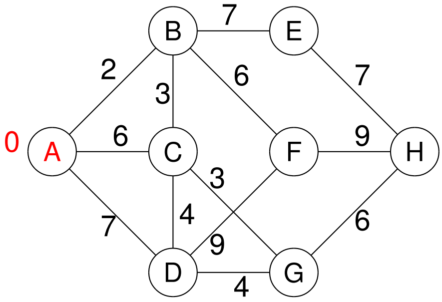

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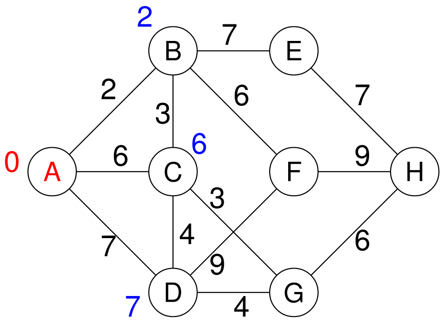

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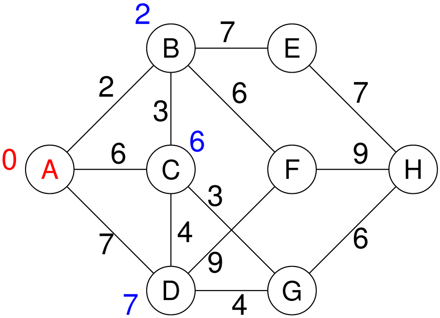

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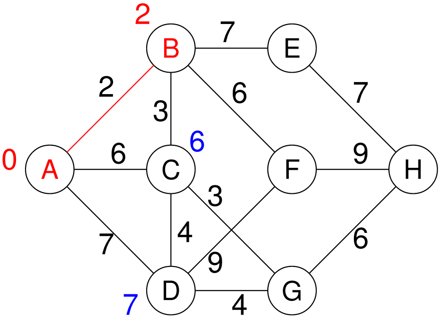

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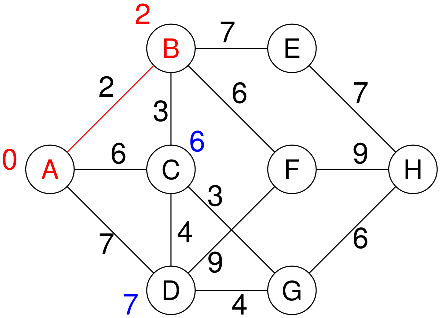
Repeat steps 2 and 3 until the final vertex v' has been given a **permanent** label.



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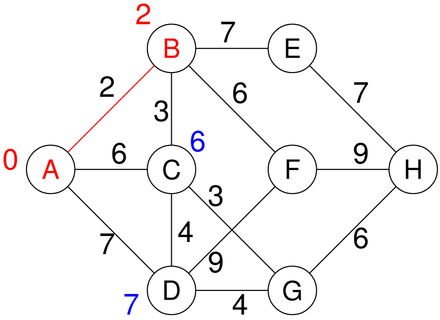

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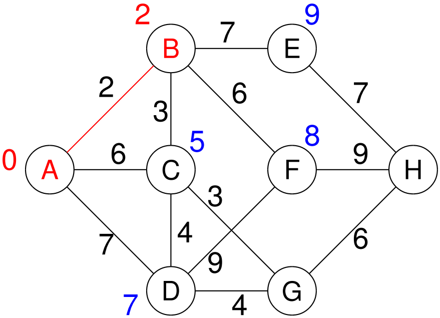

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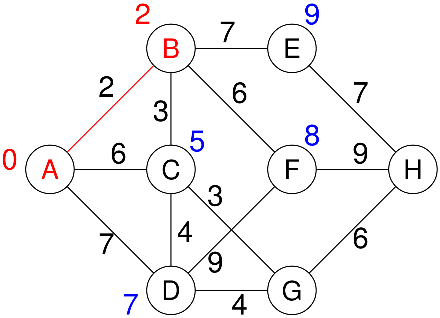

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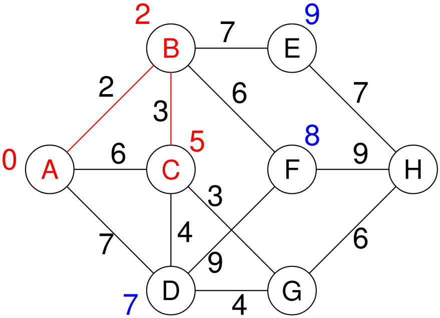

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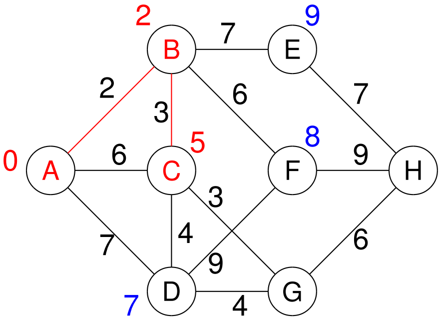

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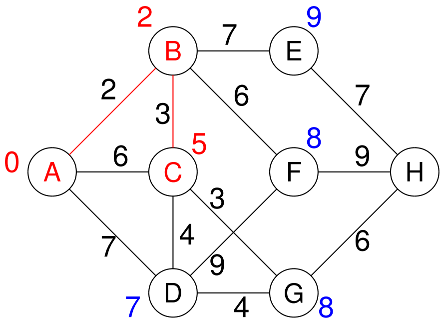

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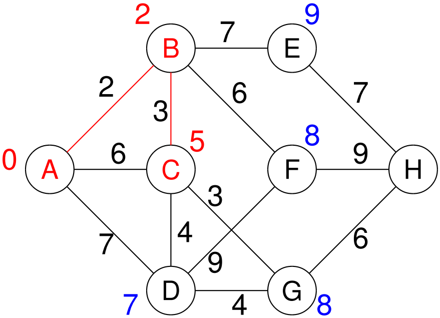

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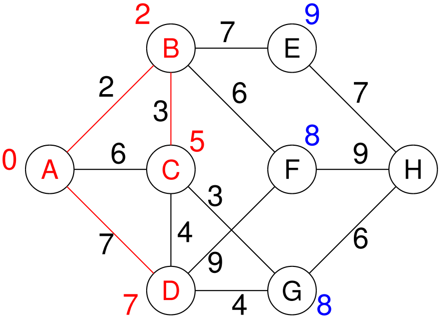

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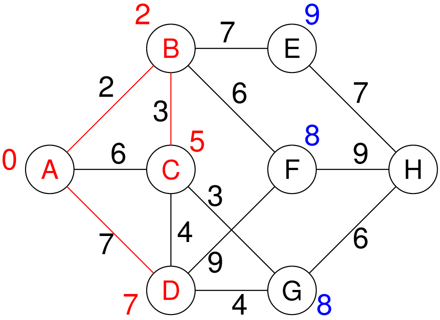

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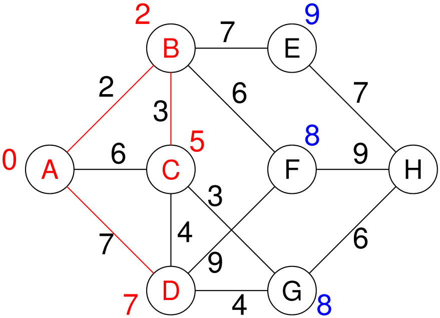

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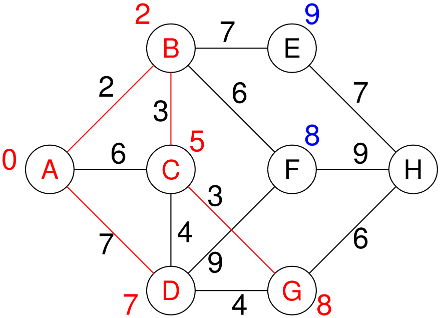

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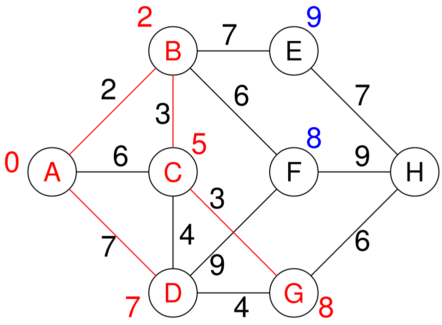

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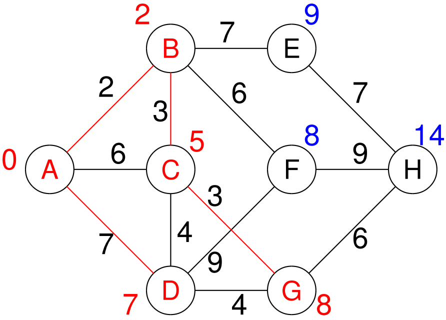

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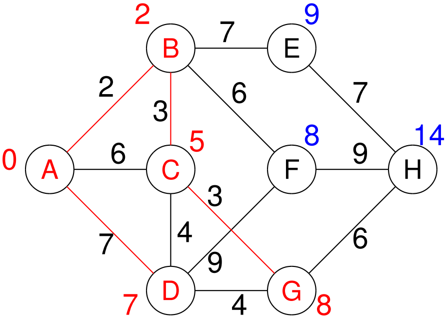

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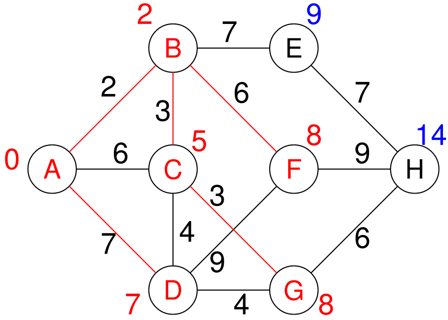

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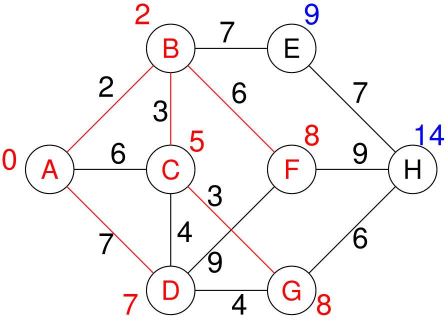

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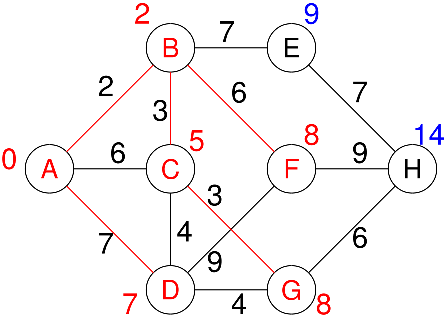

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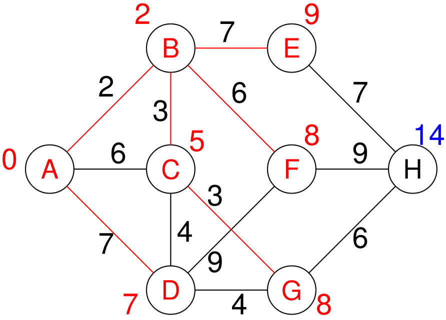

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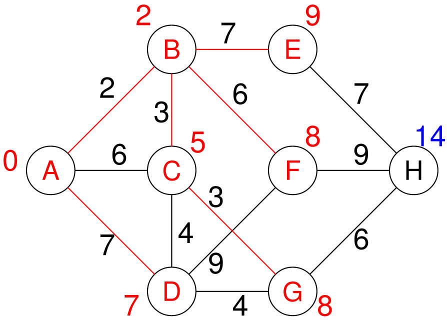

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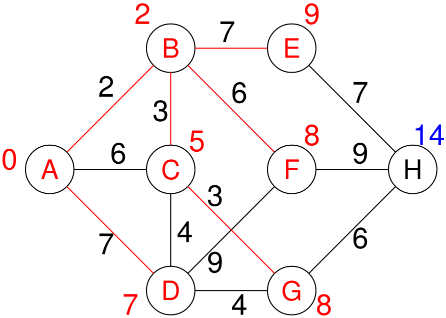

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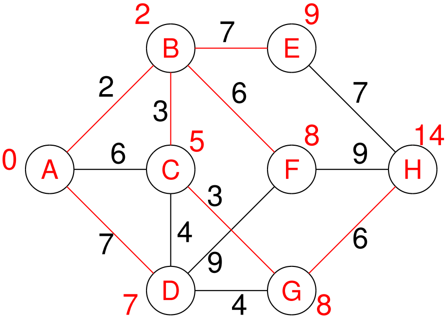

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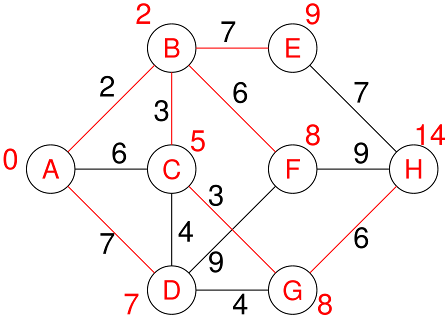

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