Session 13: Coverings and partitions. Correspondences.

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

This session has two parts. In the first one, we will define the concepts of "covering" and "partition". In the second one we will define the notion of "correspondence", "graph of a correspondence" and "composition of correspondences".

Part I

Coverings and partitions

1 Indexed families of sets

In the preceding session we defined the intersection and union of a collection of n sets. Now we turn our attention to more general "families" or "collections" of sets which will include the case where there are infinitely many sets in the family. By a family or collection of sets, we really mean a *set* of sets, although the terms "family of sets" or "collection of sets" are both in widespread use and we will use the three terms interchangeably. Before we can consider intersections and unions of arbitrary families of sets, we need first to describe carefully what we mean by such a family.

Suppose that we have the family of sets $\{A_1, A_2, \ldots A_n\}$. In the previous session we have considered the intersection $A_1 \cap A_2 \cap \cdots \cap A_n$ and the union $A_1 \cup A_2 \cup \cdots \cup A_n$. In that family, the integers $1, 2, \ldots, n$ serve as labels to distinguish the various sets in the collection. In principle, any collection of labels would be suitable; for example, if we were to choose *Alice*, *Bob*,..., *Nina* as labels, then we could write the family as

$$\{A_{Alice}, A_{Bob}, \ldots, A_{Nina}\}.$$

In practice, labels $1, 2, \ldots, n$ are usually preferable. Whatever labels we choose form an **indexing set** of **labelling set** I for the collection. For the collection $\{A_1, A_2, \ldots A_n\}$, the indexing set is $I = \{1, 2, \ldots, n\}$ and we can rite the family as

$$\{A_i \mid i \in I\} = \{A_1, A_2, \ldots, A_n\}.$$

Using this idea of indexing set, we can define more general families of seets. For example, any collection of sets that has $\mathbb{N}$ as the indexing set will contain infinitely many sets, one corresponding to each natural number:

$$\{A_r \mid r \in \mathbb{N}\} = \{A_1, A_2, A_3, \ldots\}.$$

If the set of real numbers $\mathbb{R}$ is the indexing set then the resulting family of sets $\{A_r \mid r \in \mathbb{R}\}$ also contains infinitely many sets, but this time we cannot list them even in an infinite list.

An arbitrary family of sets is of the form $\mathcal{F} = \{A_i \mid i \in I\}$, where I is **any** indexing set. In such a collection $\mathcal{F}$ there is exactly one set A_r for each element r of the indexing set I.

It is now straightforward to modify the definitions given in the preceding session and define the intersection and union of an arbitrary family of sets $\mathcal{F} = \{A_i \mid i \in I\}$ as follows:

$$\bigcap_{i\in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\},$$

$$\bigcup_{i\in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}.$$

Example 1. The definitions given above for intersection and union of arbitrary families of sets include as special cases our previous definitions for finite collections of sets. For example, let $I = \{1, 2\}$. A family of sets having I as indexing set is $\{A_1, A_2\}$. Then:

$$\bigcap_{i\in I} A_i = \{x \mid x \in A_i \text{ for } i = 1 \text{ and } i = 2\} = \{x \mid x \in A_1 \text{ and } x \in A_2\} = A_1 \cap A_2,$$

$$\bigcup_{i\in I} A_i = \{x \mid x \in A_i \text{ for } i = 1 \text{ or } i = 2\} = \{x \mid x \in A_1 \text{ or } x \in A_2\} = A_1 \cup A_2.$$

Example 2. Let $I = \mathbb{N}$ and, for each $i \in I$, let $A_i = \{i\}$. Thus $A_1 = \{1\}$, $A_2 = \{2\}$, etc. Therefore:

$$\bigcap_{i\in I} A_i = \emptyset,$$

$$\bigcup_{i\in I} A_i = \{1, 2, 3, \ldots\} = \mathbb{N}.$$

Example 3. Let $I = \mathbb{R}$ and, for each $m \in I$, let A_m be the set of points in the plane which lie on the line of slope m that passes through the origin (that is, the line with equation $y = mx$):

$$A_m = \{(x, y) \mid \ x \text{ and } y \text{ are real numbers and } y = mx\}.$$

Note that in this case we cannot list the sets in the family $\{A_m \mid m \in I\}$ even in an infinite list. This is baecause the real numbers themselves cannot be listed in an infinite list $m_1, m_2, \ldots$.

Then

$$\bigcap_{i\in I} A_i = \emptyset$$

since the origin $(0, 0)$ is the unique point common to al such lines.

The union

$$\bigcup_{i\in I} A_i$$

is the whole plane except the positive and negative parts of the y-axis. Points on the y-axis (except the origin) do not occur in the union because none of the lines A_m are vertical.

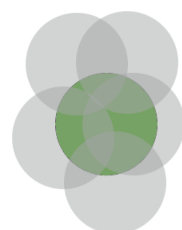
2 Coverings

Let B be a set. A **covering** of B is a family of non-empty sets $\{A_i \mid i \in I\}$ such that

$$B \subseteq \bigcup_{i\in I} A_i,$$

that is, B is contained in the union of all sets of the family.

In the following picture, the family of sets represented by "grey circles" is a **covering** of the set represented by a "green circle" (the grey circles "cover" the green one):



Example 4. Consider the set $B = \{a, b, c\}$ and the family $\{A_1, A_2, A_3\}$ where $A_1 = \{a, f\}$, $A_2 = \{b, c, d\}$ and $A_3 = \{a, e, f, h\}$. Then, the family $\{A_1, A_2, A_3\}$ is a covering of B because

$$B \subseteq A_1 \cup A_2 \cup A_3 = \{a, b, c, d, e, f, h\}.$$

Example 5. Let $B = \mathbb{R}$ and, for each $n \in \mathbb{N}$, define $A_n = [-n, n]$ (that is, the closed interval given by the set of real numbers between $-n$ and n). Then the family $\{A_n \mid n \in \mathbb{N}\}$ is a covering of B because B is included in the union of all members of the family (in fact, both sets are equal in this case).

3 Partitions

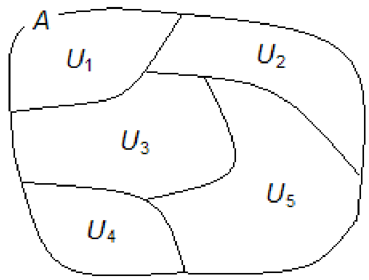
Let B be a set. A **partition** of B is a family of non-empty sets $\{A_i \mid i \in I\}$ such that

- B is is equal to the union of all sets of the family:

$$B = \bigcup_{i\in I} A_i.$$

- The sets of the family are **pairwise disjoint**, that is, $A_i \cap A_j = \emptyset$ for $i \neq j$.

In the following picture, the family $\{U_1, U_2, U_3, U_4, U_5\}$ represents a **partition** of a set A:



Notice that every partition of a set B is also a covering of B! That is, a partition is a special type of covering.

Example 6. Let $B = \{a, b, c, d, e, f, g\}$, $A_1 = \{a, b, c\}$, $A_2 = \{d, e, f\}$ and $A_3 = \{g\}$. Then the family $\{A_1, A_2, A_3\}$ is a partition of B.

Example 7. The family $\{\mathbb{Q}, \mathbb{I}\}$ is a partition of $\mathbb{R}$.

Part II

Correspondences

4 Definition of correspondence and basic notations

Given two sets A and B, we call a **correspondence** between A and B to a matching of some elements of A with some elements of B. A is the **initial set** and B is the **final set**.

Correspondences are usually denoted in the form $f : A \to B$. Here, f is the name of the correspondence (usually a letter), A is the initial set and B is the final set. The arrow indicates that the correspondence associates, to (some) elements of A, (some) elements of B.

If an element of $a \in A$ has associated another element $b \in B$, we say that b is an **image** of a, or that a is a **preimage** de b. Let us see some related terms:

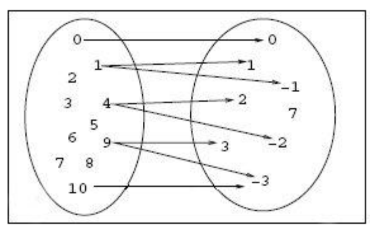
- The **image of an element** a **of** A, denoted by $f(a)$, is the set of all the images of A.
- The **preimage of** B, denoted by $f^{-1}(b)$, is the set of all preimages of b.

- The **domain of** f, denoted by $Dom(f)$, is the set of all the elements of A whose image is not empty, that is,
$$Dom(f) = \{a \in A \mid f(a) \neq \emptyset\}.$$

- The **image of** f, denoted by $Im(f)$ or $f(A)$, is the set of all the images of all the elements of A, that is,
$$Im(f) = f(A) = \bigcup_{a \in A} f(a) = \{b \in B \mid f^{-1}(b) \neq \emptyset\}.$$

- The **graph** of f, denoted by $Graph(f)$, is the subset of $A \times B$ of all ordered pairs (a, b) such that $a \in A$ and $b \in f(a)$, that is,
$$Graph(f) = \{(a, b) \in A \times B \mid b \in f(a)\}.$$

Example 8. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $B = \{0, 1, -1, 2, -2, 3, -3, 7\}$. Consider the correspondence $f : A \to B$ given by:



The sets A and B have been represented by means of Venn diagrams and the matchings using arrows. Then:

- $f(0) = \{0\}$, $f(1) = \{-1, 1\}$, $f(4) = \{2, -2\}$, $f(9) = \{3, -3\}$, $f(10) = \{-3\}$, $f(2) = f(3) = f(5) = f(6) = f(7) = f(8) = \emptyset$.
- $f^{-1}(0) = \{0\}$, $f^{-1}(1) = f^{-1}(-1) = \{1\}$, $f^{-1}(2) = f^{-1}(-2) = \{4\}$, $f^{-1}(3) = \{9\}$, $f^{-1}(-3) = \{9, 10\}$.
- $\mathrm{Dom}(f) = \{0, 1, 4, 9, 10\}$.
- $Im(f) = \{0, 1, -1, 2, -2, 3, -3\}$.
- $Graph(f) = \{(0, 0), (1, 1), (1, -1), (4, 2), (4, -2), (9, 3), (9, -3), (10, -3)\}$.

A correspondence is perfectly defined from its initial and final sets and its graph. In fact, it is usual to identify a correspondence with its graph.

5 Inverse correspondence

Given a correspondence $f : A \to B$, we call the **inverse correspondence of** f to the correspondence $f^{-1} : B \to A$ whose associated graph is

$$\mathrm{Graph}(f^{-1}) = \{(b,a) \in B \times A \mid (a,b) \in \mathrm{Graph}(f)\}.$$

In other words, f^{-1} is the correspondence whose initial set is the final set of f, whose final set is the initial set of f, and whose matchings are defined "changing the directions of the arrows".

Example 9. The inverse correspondence of the correspondence f in the previous example is the one whose graph is

$$\mathrm{Graph}(f^{-1}) = \{(0,0),(1,1),(-1,1),(2,4),(-2,4),(3,9),(-3,9),(-3,10)\}.$$

6 Composition of correspondences

Let $f : A \to B$ and $g : B \to C$ be two correspondences. It is defined the **composition of** g **and** f as the correspondence $g \circ f : A \to C$ such that $(g \circ f)(a) = g(f(a))$ for all $a \in A$. In other words, it is the correspondence whose graph is:

$$Graf(g \circ f) = \{(a,c) \in A \times C \mid \exists b \in B \text{ with } (a,b) \in Graf(f) \ \wedge \ (b,c) \in Graf(g)\}.$$

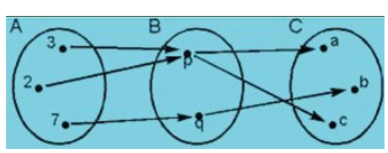
Notice that the composition $g \circ f$ exists whenever the final set of f coincides with the initial set of g:

$$A \to B \to C.$$

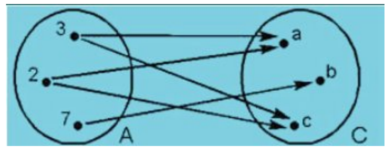
Example 10. Consider the sets $A = \{2,3,7\}$, $B = \{p,q\}$ and $C = \{a,b,c\}$ and the correspondences $f : A \to B$ and $g : B \to C$ whose graphs are

$$Graf(f) = \{(3,p),(2,p),(7,q)\} \text{ and } Graph(g) = \{(p,a),(p,c),(q,b)\}.$$

The following scheme using Venn diagrams describes the composition $g \circ f$:



The image by $g \circ f$ of an element $x \in A$ is computed by "following two consecutive arrows" (first, the arrow associated with f and, then, the arrow associated with g). This is the final description of $g \circ f$:


3
2
7
A
a
b
c
C