Practice 2: Graphic sequences and types of graphs

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Contents

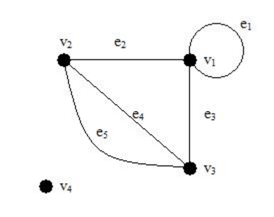
Degree of a vertex

Remember:

Definition (Degree of a vertex (non-directed graphs))

The **degree** of a vertex v, denoted as $d(v)$, is the number of edges that are incident on it (we count it two times for loops)

Example



In this graph we have:

- $d(v_1) = 4$
- $d(v_2) = 3$
- $d(v_3) = 3$
- $d(v_4) = 0.$

Degrees formula

Notation: If A is any set of, then $\mathrm{card}(A)$ denotes the number of elements in A. The cardinal of A can also be denoted as $|A|$.

Handshaking lemma

If $G=(V,E,\Psi)$ is a graph, then

$$\sum_{v\in V} d(v) = 2\cdot|E|,$$

that is, in any graph the sum of the degrees of all vertex is equal to 2 times the number of edges.

Consequences

- The sum of the degrees of the vertex of a graph is an even number.
- Every graph contains an even number of vertex of odd degree.

Graphic sequences

Definition (Graphic sequence)

A finite sequence of positive integers is said to be a **graphic sequence** , if there exists a simple (non-directed) graph without loops such that it has this finite sequence as the list of numbers of the degrees of its vertex.

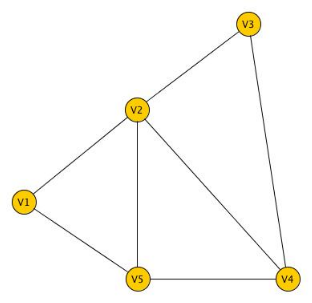
Are the following ones graphic sequences?

1. $(4, 3, 3, 2, 2)$
2. $(3, 3, 3, 2, 2)$
3. $(6, 3, 3, 2, 2)$

Graphic sequences

Are the following ones graphic sequences?

1. $(4, 3, 3, 2, 2)$ Yes. Let us give such a graph.



2. $(3, 3, 3, 2, 2)$ No. We cannot have an odd number of vertex of odd degree.
3. $(6, 3, 3, 2, 2)$ No. We cannot have a vertex of degree 6, if we only have 5 vertex, since loops are not allowed in these exercises.

Graphic sequences

Theorem (Hakimi Theorem)

A decreasing sequence of positive integers

$$(s, t_1, t_2, \dots, t_s, d_1, d_2, \dots, d_r)$$

is a graphic sequence if, and only if,

$$(t_1 - 1, t_2 - 1, \dots, t_s - 1, d_1, d_2, \dots, d_r)$$

is a graphic sequence, too.

Graphic sequences

Hakimi theorem provides an algorithm in order to determine whether a decreasing sequence of positive integers is a graphic sequence or not.

Hakimi algorithm

1. Start with a decreasing sequence of positive integers

$$(s, t_1, t_2, \ldots, t_s, d_1, d_2, \ldots, d_r)$$

   and with an empty graph with as many vertices as numbers you have in the sequence.

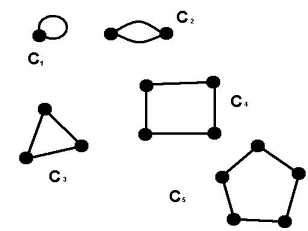
2. Remove the biggest number from the list (on the left), called it s, and substract 1 from the following s vertices. If some element of the obtained list is negative, the initial list is not a graphic sequence. Otherwise go to the next step.
3. Connect with edges the vertex associated to s with the vertices associated to $t_1, t_2, \ldots, t_s$.
4. If the obtained list only has zeroes then END (we have obtained the desired graph). Otherwise, if your list is not decreasing, reorder it (but do not mix the names of the vertices) and go to Step 2.

Types of graphs

Some definitions

- A graph without edges is said to be **void**.
- A **graph is trivial** if it consits on a unique vertex.
- Remember: a **graph** is **simple** if it has not parallel edges.
- A **graph** is **regular** if all their vertices have the same degree. If this degree is k, then we say that it is **k-regular**.

Example (All these graphs are 2-regular)

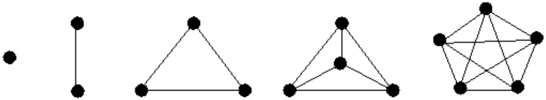


Complete graphs

Definition (Complete graph)

A **graph** is **complete** if every vertex is adjacent to the rest of vertices (that is, every pair of different vertex are joined by an edge).

We denote by $\mathbf{K_n}$ to the **simple complete graph of** n **vertices that has no loop**.

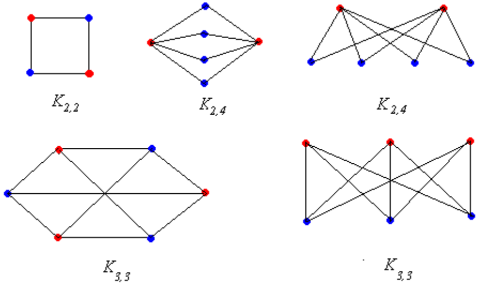


K_1 K_2 K_3 K_4 K_5

Bipartite graph

Definition (Bipartite graph)

A **graph** $G = (V, E)$ is **bipartite** if there exists a partition $\{V_1, V_2\}$ of the vertex set V such that every edge of the graph joins a vertex of V_1 with a vertex of V_2. A graph is a **complete bipartite** graph if it is simple, bipartite and verifies that there exists a partition $\{V_1, V_2\}$ of the set V such that V_1 has n vertices, V_2 has m vertices, and every vertex of V_1 is adjacent to all vertices of V_2. We denote such a graph by $\mathbf{K_{n,m}}$.



Remark: That $\{V_1, V_2\}$ was a partition of V means that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$.