Lesson 4: Vector spaces (part 2)

Algebra

Computer Science Engineering Degree

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Different types of equations

We can define a vector subspace of $\mathbb{R}^n$ by means of 3 different forms:

- IMPLICIT (OR CARTESIAN) EQUATIONS:
  $H = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$
- PARAMETRIC EQUATIONS:
  $H = \{(\alpha, \alpha - \lambda, 0) : \alpha, \lambda \in \mathbb{R}\}$
- SPAN: $H = \mathrm{span}((1, 1, 0), (0, -1, 0))$

Span $\Rightarrow$ parametric equations

To compute the implicit equations of a subspace we are going to follow the following process:

$$\text{Span} \;\Rightarrow\; \text{Parametric equations} \;\Rightarrow \text{Implicit equations}$$

Computation of parametric equations from the span

To compute the parametric equations of a subspace H defined by a set of generators:

- Compute a basis of H from the spanning set.
- Write the vectors $\vec{x} \in H$ as a linear combination of the basis. This expression is the parametric form of H.

Example Compute the paramatric equations of the vector subspace $E = \mathrm{span}((3,2,0),(1,2,0)) \subset \mathbb{R}^3$.

Every vector $(x,y,z) \in E$ is a linear combination of the vectors $(3,2,0)$, $(1,2,0)$ because it is in their span.

Therefore there exist scalars α and β such that

$$(x,y,z) = \alpha(3,2,0) + \beta(1,2,0) = (3\alpha + \beta, 2\alpha + 2\beta, 0)$$

Then $E = \{(3\alpha + \beta, 2\alpha + 2\beta, 0) : \alpha, \beta \in \mathbb{R}\}$ is the parametric form of the vector subspace.

Parametric equations $\Rightarrow$ Implicit equations

To describe a vector subspace H in terms of implicit equations, $H = \{\vec{x} \in \mathbb{R}^n : A \cdot \vec{x} = 0\}$, we must eliminate the parameters from the parametric form.
Since a vector $\vec{x}$ belongs to the subspace if and only if it satisfies the parametric equations, the system of linear equations (whose unknowns are the parameters) given by the parametric equations must be consistent.

Computation of implicit equations from parametric equations

To describe a subspace H, given in parametric form, by means of its implicit equations:

- Consider the augmented matrix of the system of parametric equations and perform elementary row operations until the coefficient matrix be row echelon.
- Study the consistency of the system. The relations among the variables making the system consistent are the implicit equations of H.

Example Compute the implicit equations of the subspace
$E = \{(3\alpha+\beta, 2\alpha+2\beta, 0) : \alpha, \lambda \in \mathbb{R}\} \subset \mathbb{R}^3$
We can consider the parametric equations defining the subspace E as the system

$$\begin{array}{rcrcl} 3\alpha & + & \beta & = & x \\ 2\alpha & + & 2\beta & = & y \\ & & 0 & = & z \end{array}$$

where α, and λ are the unknowns.
To eliminate the parameters we solve the system

$$\left[\begin{array}{cc|c} 3 & 1 & x \\ 2 & 2 & y \\ 0 & 0 & z \end{array}\right] \xrightarrow{E_1(1/3)} \left[\begin{array}{cr|r} 1 & 1/3 & x/3 \\ 2 & 2 & y \\ 0 & 0 & z \end{array}\right] \xrightarrow{E_{21}(-2)} \left[\begin{array}{cr|c} 1 & 1/3 & x/3 \\ 0 & 4/3 & y-2x/3 \\ 0 & 0 & z \end{array}\right]$$

Therefore, the system associated to the matrix

$$\left[\begin{array}{cc|c} 1 & 1/3 & x/3 \\ 0 & 4/3 & y-2x/3 \\ 0 & 0 & z \end{array}\right]$$

is consistent if and only if $z = 0$.

Then, the expression of E using its implicit equations is:
$E = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$

We have obtained 3 ways of describing the vector subspace:

$$E = \mathrm{span}((3, 2, 0), (1, 2, 0)) =$$

$$\{(3\alpha + \beta, 2\alpha + 2\beta, 0) : \alpha, \beta \in \mathbb{R}\} = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$$

Exercise 1: Compute the implicit equations of the subspace
$H = \mathrm{span}((3, -2, 1))$.

Implicit equations $\Rightarrow$ Parametric equations $\Rightarrow$ Span

Span and parametric equations from implicit equations

The implicit equations of a vector subspace H are a homogeneous system of linear equations such that their solutions define H.

- Assume that $H = \{\vec{x} \in \mathbb{R}^n : \mathrm{A} \cdot \vec{x} = 0\}$ (implicit equations). To compute the parametric equations we must solve the system $\mathrm{A} \cdot \vec{x} = 0$.
- To compute a spanning set we must "separate" the parameters in the parametric equations.

Exercise 2: Compute the parametric form of H and define it as a span:
$H = \{(x, y, z, t) \in \mathbb{R}^4 : x - z = 0,\ x + y - t = 0\} \subseteq \mathbb{R}^4$.

Observations

- The obtained system of generators will be **a basis** of the vector subspace.
- The set of solutions of a homogeneous system of m linear equations with n unknowns is a vector subspace of $\mathbb{R}^n$ and its dimension is the number of free variables.

Kernel of a matrix

If A is an $m \times n$ matrix, the **kernel** of A will be the set of solutions of the homogeneous linear system $A\vec{x} = \vec{0}$, that is:

$$Ker(A) := \{\vec{x} : A\vec{x} = \vec{0}\}$$

It is a vector subspace of $\mathbb{R}^n$ and, moreover, every vector subspace of $\mathbb{R}^n$ is the kernel of a matrix (via the implicit equations).

The dimension of the kernel is given by:

$$\boxed{\text{dim(Ker(A))} + \mathrm{rank}(\text{A}) = \text{n} = \text{number of columns of } A = \text{number of unknowns}}$$

Therefore, the system of generators obtained solving the system $A\vec{x} = \vec{0}$ is a basis of $Ker(A)$.

Row and column subspaces of a matrix

If A is a matrix of dimensions $m \times n$,

- the **row subspace** of A, denoted by $Row(A)$, is the vector subspace of $\mathbb{R}^n$ spanned by the rows of A (more exactly, by the transposed row vectors of A).
- the **column subspace** of A, denoted by $Col(A)$, is the vector subspace of $\mathbb{R}^m$ spanned by the columns of A.

Example

Example Given the matrix of dimensions 3×4:

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 2 & 2 \end{bmatrix}$$

$$\begin{aligned} Row(A) &= \mathrm{span}((1,1,2,2),(0,1,1,1),(2,0,2,2)) \\ Col(A) &= \mathrm{span}((1,0,2),(1,1,0),(2,1,2),(2,1,2)) \\ &= \mathrm{span}((1,0,2),(1,1,0),(2,1,2)) \end{aligned}$$

(As an exercise: compute bases and dimensions of $Row(A)$ and $Col(A)$)

Property

Given a matrix A:

$$\dim Row(A) = \dim Col(A).$$

Reason: $\dim Row(A) = rank(A) = rank(A^t) = \dim Col(A)$

Intersection subspace

Let H and T be two vector subspaces of a vector space V.
The **intersection subspace** of H and T, denoted by $H \cap T$, is defined by

$$H \cap T = \{\vec{v} \in V : \vec{v} \in H \text{ y } \vec{v} \in T\}$$

$H \cap T$ is a vector subspace.
To compute the intersection subspace we shall use the implicit equations.

Example Given two subspaces $H = \mathrm{span}((1,1,1),(1,0,-1))$ and $T = \mathrm{span}((3,2,1),(0,1,1))$ de $\mathbb{R}^3$, compute a basis of $H \cap T$

To compute the intersection subspace we shall find, first, the implicit equations of H and T:

Subspace H

$$(x,y,z) = \alpha_1(1,1,1) + \alpha_2(1,0,-1)$$

$$\left[\begin{array}{rr|c} 1 & 1 & x \\ 1 & 0 & y \\ 1 & -1 & z \end{array}\right] \xrightarrow{E_{3,1}(-1)E_{2,1}(-1)} \left[\begin{array}{rr|c} 1 & 1 & x \\ 0 & -1 & y-x \\ 0 & -2 & z-x \end{array}\right] \xrightarrow{E_{32}(-2)} \left[\begin{array}{rr|c} 1 & 1 & x \\ 0 & -1 & y-x \\ 0 & 0 & z+x-2y \end{array}\right]$$

The system is consistent if and only if $z + x - 2y = 0$. Then the definition of the subspace by its implicit equation is:

$$H = \{(x,y,z) \in \mathbb{R}^3 : z + x - 2y = 0\}$$

Subspace T

$$(x, y, z) = \alpha_1(3, 2, 1) + \alpha_2(0, 1, 1)$$

$$\left[\begin{array}{cc|c} 3 & 0 & x \\ 2 & 1 & y \\ 1 & 1 & z \end{array}\right] \xrightarrow{E_{13}} \left[\begin{array}{cc|c} 1 & 1 & z \\ 2 & 1 & y \\ 3 & 0 & x \end{array}\right] \xrightarrow{E_{31}(-3)E_{21}(-2)}$$

$$\left[\begin{array}{cc|c} 1 & 1 & z \\ 0 & -1 & y-2z \\ 0 & -3 & x-3z \end{array}\right] \xrightarrow{E_{32}(-3)} \left[\begin{array}{cc|c} 1 & 1 & z \\ 0 & -1 & y-2z \\ 0 & 0 & x+3z-3y \end{array}\right]$$

The system is consistent if $x + 3z - 3y = 0$. Therefore:

$$T = \{(x, y, z) \in \mathbb{R}^3 : x + 3z - 3y = 0\}$$

The vectors in $H \cap T$ are those belonging to H and T at the same time; therefore they are the vectors satisfying both implicit equations:

$$H \cap T = \{(x, y, z) \in \mathbb{R}^3 : z + x - 2y = 0,\ x + 3z - 3y = 0\}.$$

To obtain a basis of $H \cap T$ is enough to solve the system of equations formed by the implicit equations of both subspaces:

$$\begin{array}{ccccccc} x & - & 2y & + & z & = & 0 \\ x & - & 3y & + & 3z & = & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & -3 & 3 & 0 \end{array}\right] \xrightarrow{E_{21}(-1)} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

Using backward substitution:

$$\begin{array}{rcrcrcl} x & - & 2y & + & z & = & 0 \\ & - & y & + & 2z & = & 0 \end{array}$$

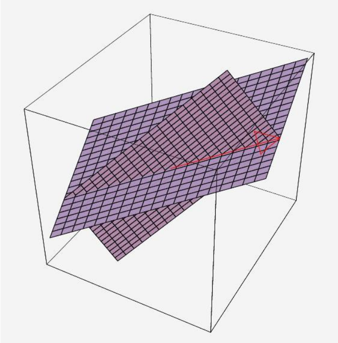
$$y = 2z = 2\alpha \quad \Rightarrow \quad x = 2y - z = 2 \cdot (2\alpha) - \alpha = 4\alpha - \alpha = 3\alpha$$

Then the solutions have the form:

$$(x, y, z) = (3\alpha, 2\alpha, \alpha) = \alpha(3, 2, 1).$$

Therefore $H \cap T = \mathrm{span}((3, 2, 1))$ and $\{(3, 2, 1)\}$ is a basis of $H \cap T$.

Graphically:



$H \cap T$ is the line of multiples of $(3, 2, 1)$.

The union of vector subspaces, why not?

Exercise 3: Given the vector subspaces of $\mathbb{R}^3$:

$$W_1 = \mathrm{span}((1,0,0)) \quad \text{and} \quad W_2 = \mathrm{span}((0,1,0)) :$$

(a) Is $W_1 \cup W_2$ a vector subspace?

(b) Which is the smallest vector subspace of $\mathbb{R}^3$ containing $W_1 \cup W_2$?

The union of subspaces IS NOT, in general, a vector subspace.

Sum of vector subspaces

Sum of vector subspaces

If F and G are two vector subspaces of a vector space V, we shall call *sum* of F and G, denoted by $F+G$, to the smallest vector subspace of V containing $F \cup G$.

Property

If F and G are two vector subspaces of a vector space V then

$$F+G=\{\vec{x}+\vec{y} \mid \vec{x} \in F \quad \text{and} \quad \vec{y} \in G\}.$$

The following property allows us to compute a system of generators of the sum of subspaces from systems of generators of the summands:

Property

If F and G are two vector subspaces of a vector space V such that $F=\operatorname{span}(S_1)$ and $G=\operatorname{span}(S_2)$ then $F+G=\operatorname{span}(S_1 \cup S_2)$.

Exercise 4: Compute bases and dimensions of the following subspaces of $\mathbb{R}^4$:

$$F = \mathrm{span}((1,0,1,0),(-2,3,1,0))$$

$$G = \mathrm{span}((0,-3,1,0),(1,1,1,1)).$$

Theorem: Grassman Formula

If F and G are two vector subspaces of a vector space V then:

$$\dim(F+G) = \dim(F) + \dim(G) - \dim(F \cap G).$$

Exercise 5: Given the following vector subspaces of $\mathbb{R}^3$:

$$U = \{(x_1,x_2,x_3) \mid x_1+x_2+x_3 = 0\},$$

$$W = \mathrm{span}((1,1,1),(1,1,0),(-1,-1,1)),$$

compute $U \cap W$ and apply the Grassman formula to compute $U + W$.

Direct sum

Under certain circumstances the sum of vector subspaces is called *direct sum* and the notation $\oplus$ is used instead of $+$.

Direct sum

If F and G are two vector subspaces of a vector space V, we shall say that the sum $F+G$ is **direct** (denoted by $F \oplus G$) if $F \cap G = \{\vec{0}\}$.

Property

If a sum of vector subspaces, $F+G$, is direct then every vector of $F+G$ is written **in a unique way** as a sum $\vec{x}+\vec{y}$, where $\vec{x} \in F$ and $\vec{y} \in G$.

Property

Let F and G be two vector subspaces of a vector space V such that the sum $F + G$ is direct. Then

$$\dim(F \oplus G) = \dim(F) + \dim(G).$$

Remark

A basis of a direct sum can be obtained as the union of bases of the summands.

Exercise 6: Prove that the following sum of vector subspaces of $\mathbb{R}^3$ is direct:

$$\mathrm{span}((-1,1,0),(-1,0,1)) + \mathrm{span}((1,1,1)).$$

Reason in 1 line that this subspace is $\mathbb{R}^3$.

Orthogonal subspaces

We shall say that a vector $\vec{u} \in \mathbb{R}^n$ is **orthogonal** to a vector subspace W of $\mathbb{R}^n$ (and we shall denote it by $\vec{u} \perp W$) if it is orthogonal to all the vectors of W, that is:

$$\vec{u} \perp W \quad \Leftrightarrow \quad \vec{u} \perp v \ \forall \, v \in W.$$

Two subspaces W_1 and W_2 of $\mathbb{R}^n$ are **orthogonal** if each vector $\vec{u}$ of W_1 is orthogonal to W_2.

Property

If W is a vector subspace of $\mathbb{R}^n$ generated by $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r$ then a vector $\vec{x} \in \mathbb{R}^n$ is orthogonal to W if and only if it is orthogonal to $\vec{u}_i$ for all $i = 1, 2, \ldots, r$.

Orthogonal complement

Given a vector subspace W of $\mathbb{R}^n$, the **orthogonal complement** of W, denoted by $W^\perp$, is the set of vectors of $\mathbb{R}^n$ that are orthogonal to W. That is:

$$W^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \perp W\}.$$

Property

Let W be a vector subspace of $\mathbb{R}^n$ and let $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r\}$ be a system of generators of W. Then:

$$W^\perp = Ker(R),$$

where R is the matrix whose **rows** are the transposed vectors of S. In particular, $W^\perp$ is a vector subspace.

Why the name "orthogonal complement"?

If W is a vector subspace of $\mathbb{R}^n$ then $\mathbb{R}^n = W \oplus W^\perp$.

Example Compute the orthogonal complement of $W = \mathrm{span}((2,5))\mathbb{R}^2$.

$$W^\perp = Ker \begin{bmatrix} 2 & 5 \end{bmatrix},$$

that is, the set of solutions of the equation

$$2x + 5y = 0.$$

Therefore $W^\perp = \mathrm{span}((5,-2))$.

Example In $\mathbb{R}^3$, compute the orthogonal complement of the XY-plane.

The XY-plane is the subspace generated by $S = \{(1,0,0),(0,1,0)\}$.
Therefore, its orthogonal complement is

$$Ker \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \mathrm{span}((0,0,1)).$$

Orthogonal projection

Orthogonal projection (revisited)

If W is a vector subspace of $\mathbb{R}^n$, since $\mathbb{R}^n = W \oplus W^{\perp}$, every vector $\vec{v} \in \mathbb{R}^n$ can be written **in a unique form** as $\vec{v} = \vec{w} + \vec{w}'$, where $\vec{w} \in W$ and $\vec{w}' \in W^{\perp}$.
We shall say, in this case, that $\vec{w}$ is the **orthogonal projection** of $\vec{v}$ over W, and we shall denote it by $Proj_W(\vec{v})$.

Orthogonal projection over a line (see Lesson 1)

PROBLEM: Compute the orthogonal projection of a vector $\vec{b} \in \mathbb{R}^n$ over the line W spanned by a non-zero vector $\vec{a}$.

$$Proj_W(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\vec{a}$$

General case (see Lesson 1)

PROBLEM: Let $\vec{b}$ be a vector of $\mathbb{R}^n$ and let W be a vector subspace of $\mathbb{R}^n$ spanned by the vectors $\vec{s}_1, \ldots, \vec{s}_k$. Compute the orthogonal projection of $\vec{b}$ over W.

1. Compute the matrix A whose **columns** are the vectors $\vec{s}_1, \ldots, \vec{s}_k$.
2. Compute a solution $\hat{\vec{x}}$ of the linear system $A^t A \cdot \vec{x} = A^t \vec{b}$.
3. The orthogonal projection $Proj_W(\vec{b})$ is $A \cdot \hat{\vec{x}}$.