Session 11: Notion of set, membership and inclusion

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

The concept of **set** is basic to understand data structures used in Programming (arrays, lists, etc.). So, although you have seen this notion in Secondary School and we have used it in the preceding lesson, it is necessary to give a more formal exposition of the basic definitions and operations involving sets.

1 Sets

The notion of **set** is one of the basic concepts of mathematics (some would say the *basic* concept). We will make no attempt to give a precise definition of a a set [1]. However, we can describe what we mean by the term:

A **set** is to be thought of as any collection of well-defined objects.

The elements contained in a given set need not have anything in common (other than the obvious common attribute that they all belong to the given set). Equally, there is no restriction on the number of elements allowed in a set; there may be an infinite number, a finite number or even no elements at all. There is, however, one restriction we insist upon: given a set and an object, we should be able to decide (in principle at least, it may be difficult in practice) whether or not the object belongs to the set.

Example 1. The following ones are examples of sets:

- A set could be defined to contain Picasso, the Eiffel Tower and the number π. This is a (rather strange) finite set.
- The set containing all the positive even integers is an infinite set.
- The "set" containing the 10 best songs of all time is not allowed unless we give a precise definition of "best". Your best? Mine? Without being more precise this fails the condition that we should be able to decide whether an element belongs to the set.

Notation: We will consider the following notations:

- The sets are denoted with upper-case letters: $A, B, C, \ldots$
- The elements are denoted with lower-case letters. (This convention will sometimes be violated, for example when elements of a particular set are themselves sets.)
- If C is a set and a denotes some object, $a \in C$ means *"a belongs to C"* (*"a is an element of C"*) and $a \notin C$ means *"a does not belong to C"* (or *"a is not an element of C"*)

[1] Set Theory has an axiomatic treatment.

2 Defining sets

2.1 Defining sets by extension

Sets can be defined in various ways. The simplest is by **listing the elements** enclosed between curly brackets or "braces" $\{\}$. In this case we say that the set is defined by **extension**. For instance, the two sets given in the previous example could be written:

$$A = \{\text{Picasso, Eiffel Tower, } \pi\},$$

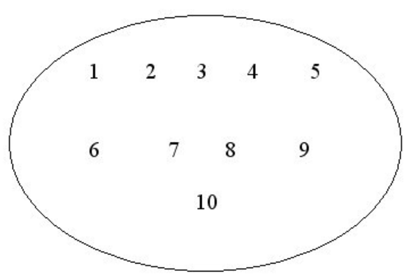
$$B = \{2, 4, 6, 8, \ldots\}.$$

In the second case we clearly cannot list *all* the elements. Instead we list enough elements to establish a pattern and use "…" to indicate that the list continues indefinitely. Other examples are the following.

For a fixed natural number n we can define $C_n = \{1, 2, \ldots, n\}$, the st of the first n natural numbers. Again we use "…" to indicate that there are elements in the list which we have omitted to write, although in this case only finitely many are missing.

$D = \{\}$ is the set which contains no element, called *empty set*. This set is usually denoted $\emptyset$.

A useful way to visualize some properties of sets is using **Venn diagrams**:



2.2 Defining sets by intension

Listing the elements of a set is impractical except for small sets or sets where there is a pattern to the elements such as B and C_n above. An alternative is to define the elements of a set by a property or predicate. More precisely, if $P(x)$ is a single-variable predicate we can form the set whose elements are all those objects a (and only those) for which $P(a)$ is a true proposition. A set defined in this way is denoted

$$A = \{x \mid P(x)\}.$$

Sometimes, to avoid ambiguities, we write also the *universe* affecting the variable x:

$$A = \{x \in U \mid P(x)\}.$$

It is also usual to write the symbol : instead of |:

$$A = \{x : P(x)\} \quad \text{and} \quad A = \{x \in U : P(x)\}.$$

This is the definition of a set by **intension**.

Example 2.

- The set B above could be defined as

  $$B = \{n \in \mathbb{N} \mid n \text{ is even}\}$$

  or

  $$B = \{n \mid n = 2m, \text{where } m \text{ is a natural number}\},$$

  or, with a slight change of notation,

  $$B = \{2m \mid m \in \mathbb{N}\}.$$

- The set C_n above could be defined as

  $$C_n = \{p : p \in \mathbb{N} \ \wedge \ 1 \leq p \leq n\}.$$

- The set $\{1, 2\}$ could alternatively be defined as $\{x \in \mathbb{R} \mid x^2 - 3x + 2 = 0\}$.
- The empty set $\emptyset$ can be defined in this way using any predicate $P(x)$ such that $P(a)$ is a contradiction for any element a in the universe. For example:

  $$\emptyset = \{x \in \mathbb{Z} \mid x > 0 \wedge x < 0\}.$$

  ($\mathbb{Z}$ denotes the set of integer numbers).

- $H = \{x \mid x \text{ is an honest politician}\}$ is not a set unless we define the term "honest" more precisely.

3 Equality of sets

We will say that two sets A and B are **equal** if they have exactly the same elements. We will denote it by $A = B$, as usual.

In other words, two sets A and B are equal if and only if the proposition

$$\forall x \ (x \in A \leftrightarrow x \in B)$$

is true.

The order in which elements are listed is immaterial. Also, it is the standard convention to disregard repeats of elements in a listing. Thus the following all define the same set:

$$\{1, -3, \pi\}$$

$$\{-3, \pi, 1\}$$

$$\{1, 1, -3, -3, -3, \pi\}.$$

4 Subsets. Inclusion between sets

A set A is a **subset** of a set B (or A **is contained in** B), denoted $A \subseteq B$, if every element of A is also an element of B. In other words, $A \subseteq B$ if and only if the proposition

$$\forall x\ (x \in A \to x \in B)$$

is true.

Clearly every set B is a subset of itself, $B \subseteq B$. Also $\emptyset \subseteq B$ for every set B. The reason is that the proposition $\forall x\ (x \in \emptyset \to x \in B)$ is true (notice that the antecedent of the conditional is always false and, therefore, the conditional is always true).

Example 3.

- $\{1, 2\} \subseteq \{3, 4, 5, 1, 6, 2\}$.
- $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$.
- Perhaps you will have some difficulty with this, but your doubts should be solved in the next class session:

  Let $X = \{1, \{2, 3\}\}$. Then $\{1\} \subseteq X$ but $\{2, 3\}$ is not a subset of X, which we can denote by $\{2, 3\} \nsubseteq X$. However $\{2, 3\}$ is an *element* of X, so $\{\{2, 3\}\} \subseteq X$. **Be careful to distinguish between set membership and inclusion, particularly when a set has elements which are themselves sets.**

To prove that two sets A and B are equal, $A = B$, it is sufficient (and frequently very convenient) to show that each is a subset of the other, that is, to prove that $A \subseteq B$ and $B \subseteq A$. In summary:

Given two sets A and B, we have

$$A = B \iff A \subseteq B \text{ and } B \subseteq A.$$

5 Power set of a given set

If C is any set, the **power set of** C, denoted by $\mathcal{P}(C)$, is the set whose elements are all the subsets of C. That is:

$$\mathcal{P}(C) = \{A : A \subseteq C\}.$$

Notice that, for any set A, the sets $\emptyset$ and A are both subsets of A and, therefore, **they are elements of** $\mathcal{P}(A)$.

Example 4.

- If $A = \{1, 2\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

- If $A = \{1, 2, 3\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

- $\mathcal{P}(\emptyset) = \{\emptyset\}$.