Session 14: Functions.

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Definition of function

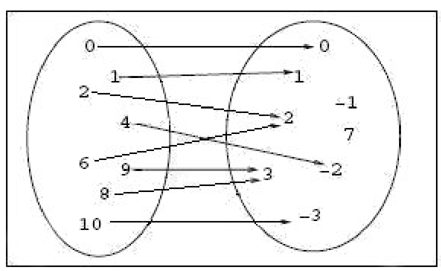
It is said that a correspondence $f : A \to B$ is a **function** (or a **map**, or a **mapping**) if all the elements of A have exactly one image.

In other words, a function is a correspondence whose domain is its initial set A and such that the set $f(a)$ has only one element for all $a \in A$.

Caution! Sometimes a function is defined as a correspondence such that every element of the initial set has, at most, one image (that is, the domain is not necessarily the whose initial set). However we are going to consider a more restrictive definition here.

Notation: If $f : A \to B$ is a function and $a \in A$, one has that $f(a) = \{b\}$ for a certain element $b \in B$. To avoid unnecessary notation, we will write $f(a) = b$ instead of $f(a) = \{b\}$.

Example 1. The following correspondence f is a function



Moreover $f(0) = 0$, $f(1) = 1$, $f(2) = f(6) = 2$, etc.

Example 2. The correspondence $g : \mathbb{R} \to \mathbb{R}$ whose graph is

$$\{(x, y) \mid x = y^2\}$$

is not a function because, for example, $g(2) = \{\sqrt{2}, -\sqrt{2}\}$ (the element 2 has more than one image).

2 Composition of functions

The following property is clear:

If $f : A \to B$ and $g : B \to C$ are two functions then the composition $g \circ f : A \to C$ is also a function.

Example 3. Consider the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{x-1} \text{ for all } x \in \mathbb{R} \setminus \{1\},$$

and the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 3x + 1$ for all $x \in \mathbb{R}$. Let us compute the expression of the function $g \circ f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$. For all $x \in \mathbb{R}$:

$$(g \circ f)(x) = g(f(x)) = 3\frac{x}{x-1} + 1 = \frac{4x-1}{x-1}.$$

3 Injective functions

A function $f : A \to B$ is said to be **injective** when all elements in A have different images, that is, if the following condition is satisfied

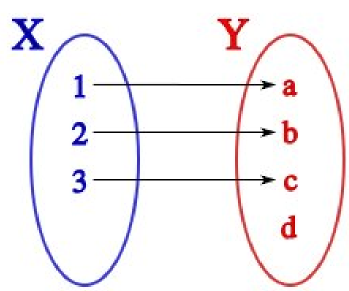
$$\forall a_1, a_2 \in A \ \ (a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)),$$

or the equivalent condition (more useful for exercises)

$$\forall a_1, a_2 \in A \ \ (f(a_1) = f(a_2) \Rightarrow a_1 = a_2).$$

In other words, a function is injective if all the elements of the domain have different images.

Example 4. The following function from X to Y is injective:



Example 5. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ is injective. Let us prove it by means of the condition:

$$\forall a_1, a_2 \in A \ \ (f(a_1) = f(a_2) \Rightarrow a_1 = a_2).$$

Consider two **arbitrary** elements of $\mathbb{R}$, a_1 and a_2, and assume that $f(a_1) = f(a_2)$. This means that $2a_1 + 1 = 2a_2 + 1$. Subtracting 1 to both sides of the equality we have $2a_1 = 2a_2$. Finally, dividing by two both sides: $a_1 = a_2$. We have proved, then, the implication $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ for every pair of arbitrary elements of the domain of f. Therefore, f is injective.

Example 6. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is not injective because, for example, $g(2) = g(-2)$ (there are two elements in the domain whose images are equal).

Proposition 1. If $f : A \to B$ and $g : B \to C$ are injective functions then $g \circ f$ is injective.

Proof. Let $a_1, a_2 \in A$ be such that $(g \circ f)(a_1) = (g \circ f)(a_2)$, that is, $g(f(a_1)) = g(f(a_1))$. Then $f(a_1) = f(a_2)$ because g is injective. And this implies that $a_1 = a_2$ because f is injective. This shows that $g \circ f$ is also injective. □

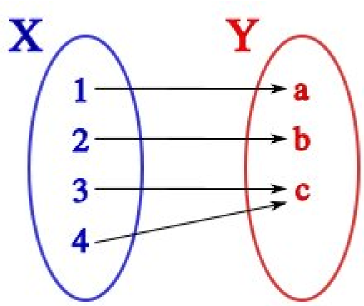
4 Surjective functions

A function $f : A \to B$ is said to be **surjective** when all elements in B have some preimage, that is, when the following condition is satisfied

$$\forall b \in B \ \ \exists a \in A \text{ such that } f(a) = b$$

Equivalently, f is surjective if $Im(f) = B$.

Example 7. The following function from X to Y is surjective:



Example 8. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is not surjective because, for example, there is no real number x such that $f(x) = -1$.

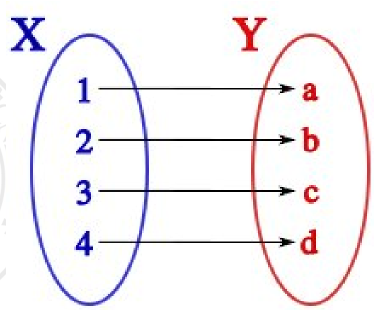
Proposition 2. If $f : A \to B$ and $g : B \to C$ are surjective functions then $g \circ f$ is surjective.

Proof. Let c be an arbitrary element of C. Since g is surjective, then there exists an element $b \in B$ such that $g(b) = c$. Moreover, since f is surjective and $b \in B$, there exists $a \in A$ such that $f(a) = b$. Then $g(f(a)) = c$, that is, $(g \circ f)(a) = c$. This shows that $g \circ f$ is surjective. $\square$

5 Bijective functions

A function $f : A \to B$ is said to be a **bijective** when it is injective and surjective.

Example 9. The following function from X to Y is bijective:



The following proposition is obvious consequence of the above ones:

Proposition 3. If $f : A \to B$ and $g : B \to C$ are two bijective functions then $g \circ f$ is bijective.

Given a set X, we define the **identity function** on X as the function $id_X : X \to X$ such that $id_X(a) = a$ for all $a \in X$.

The next result gives two interesting characterizations of the bijective functions:

Proposition 4. Let $f : A \to B$ a function.

(a) f is bijective if and only if the inverse correspondence f^{-1} is a function.

(b) f is bijective if and only if there exists a function $g : B \to A$ such that $f \circ g = id_B$ and $g \circ f = id_A$. Moreover, in this case, this function g is unique and it coincides with f^{-1}.

Proof. (a) f^{-1} is a function $\Leftrightarrow$ every element $b \in B$ has a unique image by f^{-1} $\Leftrightarrow \forall\, b \in B$ there exists a unique $a \in A$ such that $f^{-1}(b) = \{a\}$ $\Leftrightarrow \forall\, b \in B$ there exists a unique $a \in A$ such that $f(a) = \{b\}$ $\Leftrightarrow$ Every element $b \in B$ has a pre-image by f (that is, f is surjective) and, moreover, this pre-image is unique (that is, f is injective) $\Leftrightarrow$ f is bijective.

(b) $\Rightarrow$

Assume that f is bijective and take $g = f^{-1} : B \to A$ (which is a function by (a)). Then, for all $a \in A$,

$$(g \circ f)(a) = g(f(a)) = f^{-1}(f(a)) = a,$$

and for all $b \in B$,

$$(f \circ g)(b) = f(g(b)) = f(f^{-1}(b)) = b.$$

Hence $g \circ f = id_A$ and $f \circ g = id_B$.

$\Leftarrow$

Assume that there exists a function $g : B \to A$ such that $f \circ g = id_B$ and $g \circ f = id_A$.

First, let us prove that f is injective:

If $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ then $g(f(a_1)) = g(f(a_2))$, that is, $id_A(a_1) = id_A(a_2)$. Therefore $a_1 = a_2$.

Now, let us see that f is surjective:

Let b be an arbitrary element of B. Then $b = id_B(b) = (f \circ g)(b) = f(g(b))$. So, the image of $g(b)$ by f is b, and this shows that f is surjective.

Finally, let us prove the final part of the statement:

Suppose that f is bijective. And consider a function $g : B \to A$ such that $f \circ g = id_B$ and $g \circ f = id_A$. We will prove that $g = f^{-1}$.

Let b be an arbitrary element of B. Then, $(b, a) \in Graph(g) \Leftrightarrow g(b) = a \Leftrightarrow f(g(b)) = f(a)$ $\Leftrightarrow (f \circ g)(b) = f(a) \Leftrightarrow b = f(a) \Leftrightarrow (a, b) \in Graph(f)$. This means that $g = f^{-1}$.

□