(3) CHARACTERISTIC POLYNOMIAL:

$$P_A(\lambda) = |A - \lambda I| = \begin{vmatrix} 1-\lambda & b & 0 \\ 0 & b-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = (b-\lambda)(1-\lambda)(2-\lambda)$$

Set of eigenvalues: $\{1, 2, b\}$

WE DISTINGUISH 3 CASES:

$\boxed{\text{CASE 1}}$ $b \notin \{1,2\}$. IN THIS CASE, A HAS 3 DISTINCT EIGENVALUES $\Longrightarrow$ A IS DIAGONALIZABLE

$\boxed{\text{CASE 2}}$ $b = 1$. IN THIS CASE, A HAS 2 DISTINCT EIGENVALUES

$\lambda_1 = 1$	$\alpha_1 = 2$	$d_1 = ?$
$\lambda_2 = 2$	$\alpha_2 = 1$	$d_2 = 1$

$\longrightarrow$ BECAUSE $1 \le d_2 \le \alpha_2 = 1$

NOTICE THAT $\alpha_1 + \alpha_2 = 2 + 1 = 3$.

$$d_1 = \dim V_{\lambda_1} = \dim \operatorname{Ker}(A - \lambda_1 I) = 3 - \operatorname{rank}(A - \lambda_1 I) =$$

$$= 3 - \operatorname{rank} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} = 3 - 2 = \boxed{1} \neq \alpha_1 \Longrightarrow \text{A IS NOT DIAGONALIZABLE}$$

$\boxed{\text{CASE 3}}$ $b = 2$.

$\lambda_1 = 1$	$\alpha_1 = 1$	$d_1 = 1$
$\lambda_2 = 2$	$\alpha_2 = 2$	$d_2 = ?$

$\longrightarrow$ BECAUSE $1 \le d_1 \le \alpha_1 = 1$

$$d_2 = \dim V_{\lambda_2} = \dim \operatorname{Ker}(A - \lambda_2 I) = 3 - \operatorname{rank}(A - \lambda_2 I) =$$

$$= 3 - \operatorname{rank} \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} = 3 - 2 = \boxed{1} \neq \alpha_2 \longrightarrow \text{A IS NOT DIAGONALIZABLE}$$

FOR THE CASE $b=3$: EIGENVALUES $= \{ \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \}$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$V_{\lambda_1} = Ker(A - \lambda_1 I) = Ker \begin{bmatrix} 0 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} = span\left(\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right)$$

$$V_{\lambda_2} = Ker(A - \lambda_2 I) = Ker \begin{bmatrix} -1 & 3 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix} = span\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$V_{\lambda_3} = Ker(A - \lambda_3 I) = Ker \begin{bmatrix} -2 & 3 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} = span\left(\begin{bmatrix} 1 \\ 2/3 \\ 2 \end{bmatrix} \right)$$

THEN:

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2/3 \\ -2 & 1 & 2 \end{bmatrix}$$

$$A^n = A \cdot A \overset{n)}{\cdots} A = PDP^{-1}PDP^{-1} \cdots PDP^{-1} = PD^nP^{-1} =$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2/3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2/3 \\ -2 & 1 & 2 \end{bmatrix}^{-1}$$

④ a) $\mathrm{Col}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix} \right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix} \right)$

$= \mathrm{span}\left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right)$ because $\begin{bmatrix} 2 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

THEN $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right\}$ IS A BASIS OF $\mathrm{Col}(A)$ BECAUSE

IT IS A SPANNING SET AND LINEARLY INDEPENDENT.

So, $\dim \mathrm{Col}(A) = 2$.

$\mathrm{Row}(A^t) = \mathrm{Col}(A)$ AND, THEREFORE, THE SAME BASIS IS VALID.

b) SINCE $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A) = \mathrm{rank}(A)$, WE HAVE THAT:

$\dim \mathrm{Row}(A) = 2$

$\mathrm{rank}(A) = 2$

c) $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 4 & 2 \end{bmatrix} \xrightarrow{\rho_2 - 3\rho_1} \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{bmatrix} \xrightarrow{\rho_3 - 2\rho_2} \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

THEREFORE: $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ IS A BASIS OF $\mathrm{Row}(A)$.

d) $\underbrace{\text{NUMBER OF COLUMNS}}_{\underset{4}{\shortparallel}} = \underbrace{\text{rank}(A)}_{\underset{2}{\shortparallel}} + \dim \text{Ker}(A)$

THEN, $\dim \text{Ker}(A) = \boxed{2}$.

e) $$\left[\begin{array}{cccc|c} 0 & 1 & 2 & 2 & 0 \\ 0 & 3 & 8 & 7 & 0 \\ 0 & 0 & 4 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{cases} y + 2z + 2t = 0 & (\bullet) \\ 2z + t = 0 & (\bullet\bullet) \end{cases}$$

FROM $(\bullet\bullet)$:

$z = -\frac{1}{2}t$

FROM $(\bullet)$: $y - t + 2t = 0 \longrightarrow y = -t$

PARAMETRIC EQUATIONS OF THE SOLUTION SET:

$$\begin{cases} x = \alpha \\ y = -\beta \\ z = -\frac{1}{2}\beta \\ t = \beta \end{cases} \quad \alpha, \beta \in \mathbb{R}. \iff \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ -1 \\ -1/2 \\ 1 \end{bmatrix} \quad \alpha, \beta \in \mathbb{R}$$

THEREFORE:

$$\text{Ker}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1/2 \\ 1 \end{bmatrix} \right)$$

AND $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1/2 \\ 1 \end{bmatrix} \right\}$ IS A BASIS OF $\text{Ker}(A)$.

f)

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \in \text{Row}(A) \iff \exists\, \alpha, \beta \in \mathbb{R} \text{ SUCH THAT } \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\iff \text{THE SYSTEM } \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \text{ HAS SOLUTION.}$$

$$\left[\begin{array}{cc|c} 0 & 0 & x \\ 1 & 0 & y \\ 2 & 2 & z \\ 2 & 1 & t \end{array}\right] \xrightarrow[\substack{\rho_2 \leftrightarrow \rho_3 \\ \rho_3 \leftrightarrow \rho_4}]{\rho_1 \leftrightarrow \rho_2} \left[\begin{array}{cc|c} 1 & 0 & y \\ 2 & 2 & z \\ 2 & 1 & t \\ 0 & 0 & x \end{array}\right] \xrightarrow[\rho_3 - 2\rho_1]{\rho_2 - 2\rho_1} \left[\begin{array}{cc|c} 1 & 0 & y \\ 0 & 2 & z-2y \\ 0 & 1 & t-2y \\ 0 & 0 & x \end{array}\right]$$

$$\xrightarrow{\rho_3 - \frac{1}{2}\rho_2} \left[\begin{array}{cc|c} 1 & 0 & y \\ 0 & 2 & z-2y \\ 0 & 0 & -y - \frac{1}{2}z + t \\ 0 & 0 & x \end{array}\right]$$

THEREFORE:

$$\text{Row}(A) = \{ (x, y, z, t) \in \mathbb{R}^4 \,/\, 2y + z - 2t = 0 \ \wedge \ x = 0 \}$$

g) $(0, 1, 1, 1) \notin \text{Row}(A)$ BECAUSE IT DOES NOT SATISFY THE IMPLICIT EQUATIONS OF $\text{Row}(A)$.

h) WE NEED IMPLICIT EQUATIONS OF $\text{Col}(A)$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \text{Col}(A) \iff \exists\, \alpha, \beta \in \mathbb{R} \text{ SUCH THAT } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\iff \text{THE SYSTEM } \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ HAS SOLUTION.}$$

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ 3 & 4 & y \\ 0 & 2 & z \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & y-3x \\ 0 & 2 & z \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & x \\ 0 & 1 & y-3x \\ 0 & 0 & 6x-2y+z \end{array}\right]$$

THEREFORE: $Col(A) = \{(x,y,z) \in \mathbb{R}^3 / 6x-2y+z=0\}$

WE NEED IMPLICIT EQUATIONS OF $Span((1,5,4),(-1,1,1))$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in Span\left(\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) \iff \exists\, \alpha, \beta \in \mathbb{R} \text{ SUCH THAT } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$\iff$ THE SYSTEM $\begin{bmatrix} 1 & -1 \\ 5 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ HAS SOLUTION.

$$\left[\begin{array}{cc|c} 1 & -1 & x \\ 5 & 1 & y \\ 4 & 1 & z \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 6 & -5x+y \\ 0 & 5 & z-4x \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1 & x \\ 0 & 6 & -5x+y \\ 0 & 0 & \frac{1}{6}x - \frac{5}{6}y + z \end{array}\right]$$

THEREFORE:

$$Span\left(\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \{(x,y,z) \in \mathbb{R}^3 / x-5y+6z=0\}$$

THEN: $Col(A) \cap Span\left(\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 / \begin{array}{c} 6x-2y+z=0 \\ \wedge \\ x-5y+6z=0 \end{array} \right\}$

SOLVING THE SYSTEM WE WILL FIND A BASIS:

$$\left[\begin{array}{ccc|c} 1 & -5 & 6 & 0 \\ 6 & -2 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -5 & 6 & 0 \\ 0 & 28 & -35 & 0 \end{array}\right] \quad \begin{cases} x-5y+6z=0 \\ 4y-5z=0 \end{cases}$$

$$y = \frac{5}{4}z, \quad x - \frac{25}{4}z + 6z = 0 \longrightarrow x = \left(-6 + \frac{25}{4}\right)z = \frac{1}{4}z$$

$$\begin{cases} x = \frac{1}{4}\alpha \\ y = \frac{5}{4}\alpha \\ z = \alpha \end{cases} \alpha \in \mathbb{R} \implies \left\{\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}\right\} \text{ IS A BASIS OF } Col(A) \cap Span([\;],[\;]).$$

i) IF $W = Col(A) \cap Span\left(\begin{bmatrix}1\\r\\4\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}\right)$, USING GRASSMAN FORMULA WE HAVE:

$$\dim\left(Col(A) + W\right) = \underbrace{\dim Col(A)}_{2 \text{ By (a)}} + \underbrace{\dim W}_{2 \text{ By (b)}} - \underbrace{\dim(Col(A) \cap W)}_{= 1 \text{ BY (h)}}$$

THEN $Col(A) + W = \mathbb{R}^3$ AND WE CAN TAKE, FOR INSTANCE, $\{(1,0,0), (0,1,0), (0,0,1)\}$ AS A BASIS OF $Col(A) + W$.