Lesson 3

Determinants

Machine computing a determinant of order 1000

March 16, 2016

Contents

Initial advertisement

- The proofs marked with an asterisk $*$ are rather technical. We recommend to read them if an only if you are a curious unbeliever. In other case skip them.
- Although it is not essential, it is interesting that you read the proofs without $*$.

I Definition of determinant and expansions by cofactors

The definition of the determinant that we will give is recursive, that is, the determinant of a matrix is defined in terms of the determinant of smaller matrices. To this end, we will make a few definitions.

Let A be an arbitrary square matrix of order n:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & & & \\ \cdots & & & \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}.$$

Definition I.1. A *submatrix* of A is any matrix that is obtained removing some rows and some columns from A.

Definition I.2. Let $a_{i,k}$ be an entry of A. The *complementary submatrix* of $a_{i,k}$, denoted by $A_{i,k}$, is the $(n-1) \times (n-1)$ submatrix obtained removing the row i and the column k from A.

Example I.3. Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & -2 & 0 \\ 3 & 5 & 2 \end{bmatrix}$$

The complementary submatrices of $a_{2,3}$ and $a_{3,1}$ are, respectively,

$$A_{2,3} = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \qquad A_{3,1} = \begin{bmatrix} -2 & 3 \\ -2 & 0 \end{bmatrix}$$

Definition I.4. Suppose A is a square matrix. Then its determinant, $\det(A) = |A|$, is a scalar defined recursively by:

1) If A is a 1×1 matrix then $\det(A) := a_{1,1}$
2) If A is an $n \times n$ matrix, with $n \geq 2$, then

$$\begin{aligned}\det(A) = {} & a_{1,1}\det(A_{1,1}) - a_{1,2}\det(A_{1,2}) + a_{1,3}\det(A_{1,3}) - a_{1,4}\det(A_{1,4}) \\ & + \cdots + (-1)^{1+k} a_{1,k}\det(A_{1,k}) + \cdots + (-1)^{n+1} a_{1,n}\det(A_{1,n})\end{aligned}$$

The *order* of the determinant is defined as n (the order of the matrix).

Each number $(-1)^{1+k}\det(A_{1,k})$ is called the *cofactor* of the element $a_{1,k}$ and the expression given in 2) is called *expansion of the determinant by the cofactors of the first row.*

So to compute the determinant of a 5×5 matrix we must build 5 submatrices, each of size 4. To compute the determinants of each the 4×4 matrices we need to create 4 submatrices each, these now of size 3 and so on.

Example I.5. Consider the matrix:

$$A=\begin{bmatrix} 3 & 2 & -1\\ 4 & 1 & 6\\ -3 & -1 & 2\end{bmatrix}.$$

Then

$$\begin{aligned}\det(A)=A&=\begin{vmatrix} 3 & 2 & -1\\ 4 & 1 & 6\\ -3 & -1 & 2\end{vmatrix}\\ &=3\begin{vmatrix} 1 & 6\\ -1 & 2\end{vmatrix}-2\begin{vmatrix} 4 & 6\\ -3 & 2\end{vmatrix}+(-1)\begin{vmatrix} 4 & 1\\ -3 & -1\end{vmatrix}\\ &=3\left(1\left|2\right|-6\left|-1\right|\right)-2\left(4\left|2\right|-6\left|-3\right|\right)-\left(4\left|-1\right|-1\left|-3\right|\right)\\ &=3\left(1(2)-6(-1)\right)-2\left(4(2)-6(-3)\right)-\left(4(-1)-1(-3)\right)\\ &=24-52+1\\ &=-27\end{aligned}$$

Caution! Here $|\ \ |$ means “determinant” and not “absolute value”.

Determinants of order 2

If $A=\begin{bmatrix} a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{bmatrix}$ then:

$$\det(A)=\begin{vmatrix} a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix}=a_{11}\det(A_{1,1})-a_{12}\det(A_{1,2})=a_{1,1}a_{2,2}-a_{1,2}a_{2,1}$$

Example I.6.

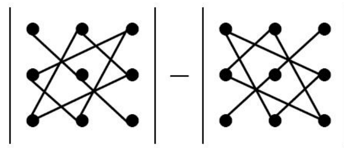
$$\begin{vmatrix} 1 & 3\\ 2 & 4\end{vmatrix}=1\cdot 4-3\cdot 2=-2$$

Determinants of order 3 (Sarrus’ rule)

When A is a matrix of order 3, its determinant is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}=$$

$$a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{21}a_{12} =$$



This mnemonic rule to compute determinants of order 3 is called *Sarrus' rule*.

Example I.7.

$$\begin{vmatrix} 4 & 2 & 3 \\ 1 & -3 & 7 \\ 2 & 0 & -1 \end{vmatrix} = 4 \cdot (-3) \cdot (-1) + 1 \cdot 0 \cdot 3 + 2 \cdot 7 \cdot 2 - 3 \cdot (-3) \cdot 2 - 7 \cdot 0 \cdot 4 - (-1) \cdot 1 \cdot 2 = 60$$

Expansion by the cofactors of any row

Theorem I.8. *Suppose that A is a square matrix of size n. Then, for $1 \leq i \leq n$:*

$$\begin{aligned} \det(A) &= (-1)^{i+1} a_{i,1} \det(A_{i,1}) + (-1)^{i+2} a_{i,2} \det(A_{i,2}) \\ &\quad + (-1)^{i+3} a_{i,3} \det(A_{i,3}) + \cdots + (-1)^{i+n} a_{i,n} \det(A_{i,n}) \end{aligned}$$

PROOF*:
We will proceed by induction on n. The case $n = 1$ corresponds to the definition of determinant. For the case $n = 2$:

$$\begin{aligned} (-1)^{2+1} a_{21} \det(A_{2,1}) &+ (-1)^{2+2} a_{22} \det(A_{2,2}) \\ &= -a_{21}a_{12} + a_{22}a_{11} \\ &= a_{11}a_{22} - a_{12}a_{21} \\ &= \det(A) \end{aligned} .$$

So, the theorem is true for matrices of order $n = 1$ or $n = 2$. Now assume the result is true for all matrices of order $n - 1$ (induction hypotheses) as we derive an expression for expansion by cofactors of row i for a matrix of order n.

We need to introduce some notation here (only for this proof): A_{i_1,i_2,k_1,k_2} will denote the matrix formed by removing rows i_1 and i_2, along with removing columns k_1 and k_2. By definition of determinant we have

$$\det(A) = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det(A_{1,k}).$$

Since, for each k, $\det(A_{1,k})$ is a determinant of order $n - 1$ we can apply induction hypotheses:

$$\det(A_{1,k}) = \sum_{k=1}^{k-1} (-1)^{i-1+k} a_{i,k} \det(A_{1,i,k,k}) + \sum_{k=k+1}^{n} (-1)^{i+k-2} a_{i,k} \det(A_{1,i,k,k})$$

Therefore

$$\det(A) = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \left(\sum_{k=1}^{k-1} (-1)^{i-1+k} a_{i,k} \det(A_{1,i,k,k}) + \sum_{k=k+1}^{n} (-1)^{i+k-2} a_{i,k} \det(A_{1,i,k,k}) \right)$$

$$= \sum_{k=1}^{n}(-1)^{1+k}a_{1k}\sum_{k=1}^{k-1}(-1)^{i-1+k}a_{i,k}\det(A_{1,i,k,k}) + \sum_{k=1}^{n}(-1)^{1+k}a_{1k}\sum_{k=k+1}^{n}(-1)^{i+k-2}a_{i,k}\det(A_{1,i,k,k})$$

$$= \sum_{k=1}^{n}\sum_{k=1}^{k-1}(-1)^{i+k+k}a_{1k}a_{i,k}\det(A_{1,i,k,k}) + \sum_{k=1}^{n}\sum_{k=k+1}^{n}(-1)^{i+k+k-1}a_{1k}a_{i,k}\det(A_{1,i,k,k})$$

In the above first summand, the pairs of indices (k, k) run over the set

$$\{(k, k) \mid k \in \{1, \ldots, n\}, k \in \{1, \ldots, n\} \quad \text{and} \quad k > k\}$$

and the pairs of indices (k, k) of the second summand run over the set

$$\{(k, k) \mid k \in \{1, \ldots, n\}, k \in \{1, \ldots, n\} \quad \text{and} \quad k < k\}.$$

Therefore

$$\det(A) = \sum_{k=1}^{n}\sum_{k=k+1}^{n}(-1)^{i+k+k}a_{1k}a_{i,k}\det(A_{1,i,k,k}) + \sum_{k=1}^{n}\sum_{k=1}^{k-1}(-1)^{i+k+k-1}a_{1k}a_{i,k}\det(A_{1,i,k,k})$$

$$= \sum_{k=1}^{n}(-1)^{i+k}a_{i,k}\left(\sum_{k=k+1}^{n}(-1)^{k}a_{1k}\det(A_{1,i,k,k}) + \sum_{k=1}^{k-1}(-1)^{k-1}a_{1k}\det(A_{1,i,k,k})\right)$$

$$= \sum_{k=1}^{n}(-1)^{i+k}a_{i,k}\left(\sum_{k=1}^{k-1}(-1)^{1+k}a_{1k}\det(A_{1,i,k,k}) + \sum_{k=k+1}^{n}(-1)^{1+k-1}a_{1k}\det(A_{1,i,k,k})\right)$$

$$= \sum_{k=1}^{n}(-1)^{i+k}a_{i,k}\det(A_{i,k})$$

QED

Definition I.9. Each number $(-1)^{i+k}\det(A_{i,n})$ is called *cofactor* of the element $a_{i,k}$ and the expression of the determinant given in Theorem I.8 is called *expansion by the cofactors of the ith row.*

Determinant of the transpose

Theorem I.10. *If A is a square matrix then* $\det(A) = \det(A^t)$.

PROOF*:

We proceed by induction over n. For $n = 1$, the transpose of a matrix is identical to the original matrix; so the determinants are equal. Now assume the result is true for matrices of order $n - 1$. Then

$$\det(A^t) = \frac{1}{n}\sum_{i=1}^{n}\det(A^t)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n}(-1)^{i+k}a_{ki}\det(A^t_{i,k})$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n}(-1)^{i+k}a_{ki}\det((A_{k,i})^t) \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n}(-1)^{i+k}a_{ki}\det(A_{k,i}) && \text{by induction hypothesis} \\
&= \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{n}(-1)^{k+i}a_{ki}\det(A_{k,i}) \\
&= \frac{1}{n}\sum_{k=1}^{n}\det(A) && \text{by Theorem I.8} \\
&= \det(A)
\end{aligned}
$$

QED

Expansion by the cofactors of any column

The next result is a consequence of Theorem I.10. In some sense, it means that, when we are computing a determinant, rows and columns play the same role.

Theorem I.11. *Suppose that A is a square matrix of size n. Then, for $1 \leq k \leq n$:*

$$
\begin{aligned}
\det(A) = {} & (-1)^{1+k}a_{1,k}\det(A_{1,k}) + (-1)^{2+k}a_{2,k}\det(A_{2,k}) \\
& + (-1)^{3+k}a_{3,k}\det(A_{3,k}) + \cdots + (-1)^{n+k}a_{n,k}\det(A_{n,k})
\end{aligned}
$$

PROOF*:

$$
\begin{aligned}
\det(A) &= \det(A^t) && \text{by Theorem I.10} \\
&= \sum_{i=1}^{n}(-1)^{k+i}a_{ik}\det(A^t_{k,i}) && \text{by Theorem I.8} \\
&= \sum_{i=1}^{n}(-1)^{i+k}a_{ik}\det(A_{i,k}) && \text{by Theorem I.10}
\end{aligned}
$$

QED

Definition I.12. The expression of the determinant given in Theorem I.11 is called *expansion by the cofactors of the kth column.*

The definition of determinant that we have given and the above results give rise to a first method to compute a determinant:

Strategy I.13. (Computing a determinant by cofactors expansion) Let A be an $n \times n$ matrix. The expansion by the cofactors of any row or column allows us to reduce the computation of $\det(A)$ to n determinants of order $n-1$.

Example I.14. We compute the following determinant using its expansion by cofactors of the third column:

$$\begin{vmatrix} 3 & 2 & 1 & -1 \\ 2 & 1 & 0 & -2 \\ 3 & -4 & 0 & 5 \\ 1 & 2 & -3 & 2 \end{vmatrix} =$$

$$1 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 2 & 1 & -2 \\ 3 & -4 & 5 \\ 1 & 2 & 2 \end{vmatrix} + 0 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 3 & 2 & -1 \\ 3 & -4 & 5 \\ 1 & 2 & 2 \end{vmatrix} +$$

$$0 \cdot (-1)^{3+3} \cdot \begin{vmatrix} 3 & 2 & -1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{vmatrix} + (-3) \cdot (-1)^{4+3} \cdot \begin{vmatrix} 3 & 2 & -1 \\ 2 & 1 & -2 \\ 3 & -4 & 5 \end{vmatrix} = -147.$$

The expansion by cofactors has been able to reduce the computation of the initial determinant (of order 4) to determinants of order 3, **which can be computed using either cofactor expansion again (reducing them to determinants of order 2) or by Sarrus' rule**.

INTERESTING OBSERVATION: Notice that it is highly interesting to use the expansion by cofactors of the row or column with the maximum number of zeros.

UTILITARIAN SUMMARY I.15.

- Any determinant can be computed using its expansion by cofactors of *any* row or column.
- There is an easy formula to compute determinants of order 2. Determinants of order 3 can be computed using a mnemonic rule: *Sarrus' rule.*
- The determinant of a square matrix and the determinant of its transpose are equal.

II Properties

Consider, as in the beginning of the lesson, an arbitrary matrix A of order n.

Proposition II.1. *Suppose that A has a row where every entry is zero, or a column where every entry is zero. Then* $\det(A) = 0$.

PROOF:
Suppose that every entry of the ith row of A is zero. If we compute the determinant by the cofactors of this row, it is clear that it is zero. The same argument is valid if A has a zero column instead of a zero row. QED

Proposition II.2. *Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then* $\det(B) = -\det(A)$.

PROOF*:

Begin with the special case where A is a square matrix of order n and we form B by swapping adkacent rows i and $i+1$ for some $1 \leq i \leq n-1$. Notice that the assumption about swapping adkacent rows means that $B_{i+1,k} = A_{i,k}$ for all $1 \leq k \leq n$, and $b_{i+1,k} = a_{i,k}$ for all $1 \leq k \leq n$. We compute $\det(B)$ using the expansion by cofactors of the row i:

$$\begin{aligned}
\det(B) &= \sum_{k=1}^{n} (-1)^{(i+1)+k} b_{i+1,k} \det(B_{i+1,k}) \\
&= \sum_{k=1}^{n} (-1)^{(i+1)+k} a_{ik} \det(A_{i,k}) \\
&= \sum_{k=1}^{n} (-1)^{1} (-1)^{i+k} a_{ik} \det(A_{i,k}) \\
&= (-1) \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det(A_{i,k}) \\
&= -\det(A)
\end{aligned}$$

Now consider a swap of two arbitrary rows, say rows s and t with $1 \leq s < t \leq n$. Begin with row s, and repeatedly swap it with each row kust below it, including row t and stopping there. This will total $t-s$ swaps. Now swap the former row t, which currently lives in row $t-1$, with each row above it, stopping when it becomes row s. This will total another $t-s-1$ swaps. In this way, we create B through a sequence of $2(t-s)-1$ swaps of adkacent rows, each of which adkusts $\det(A)$ by a multiplicative factor of -1. So

$$\det(B) = (-1)^{2(t-s)-1} \det(A) = \left((-1)^2\right)^{t-s} (-1)^{-1} \det(A) = -\det(A)$$

The proof for the case of swapping two columns is entirely similar. QED

Remark II.3. Proposition II.2 shows that, if we perform to a square matrix A an elementary operation of type 1, then its determinant only changes of sign.

Proposition II.4. *Suppose that A has two equal rows, or two equal columns. Then* $\det(A) = 0$.

PROOF:
The proof is very easy. If we interchange the two rows or columns of A that are equal, we obtain A again. Therefore, by Proposition II.2, $\det(A) = -\det(A)$, and this means that $\det(A) = 0$. QED

Proposition II.5. *Let B be the square matrix obtained from A by multiplying a single row a the scalar α, or by multiplying a single column a the scalar α. Then $\det(B) = \alpha \det(A)$.*

PROOF:

We form the square matrix B by multiplying each entry of row i of A by α. Notice that the other rows of A and B are equal, so $A_{i,k} = B_{i,k}$, for all $1 \leq k \leq n$. Now we compute $\det(B)$ using the expansion by cofactors of row i:

$$\begin{aligned}
\det(B) &= \sum_{k=1}^{n} (-1)^{i+k} b_{ik} \det(B_{i,k}) \\
&= \sum_{k=1}^{n} (-1)^{i+k} \alpha a_{ik} \det(A_{i,k}) \\
&= \alpha \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det(A_{i,k}) \\
&= \alpha \cdot \det(A)
\end{aligned}$$

The proof for the case of a multiple of a column is similar. QED

Example II.6.

$$\begin{vmatrix} 4 & 2 & 3 \\ 5 \cdot 1 & 5 \cdot (-3) & 5 \cdot 7 \\ 0 & 2 & -1 \end{vmatrix} = 5 \cdot \underbrace{\begin{vmatrix} 4 & 2 & 3 \\ 1 & -3 & 7 \\ 0 & 2 & -1 \end{vmatrix}}_{60} = 5 \cdot 60 = 300$$

Remark II.7. Propositon II.5 shows that, if we perform to a square matrix an elementary operation of type 2, its determinant is multiplied by the operation's factor.

Proposition II.8. *If A has either two proportional rows or two proportional columns then $\det(A) = 0$.*

PROOF:

Suppose that A has two proportional rows and let α be the factor of proportionality. By Proposition II.5 one has that $\det(A) = \alpha \det(B)$, where B is a matrix with two equal rows. Now, by Proposition II.4, $\det(B) = 0$.

If A has two proportional columns the proof is similar. QED

Proposition II.9. *Suppose that A, B, and C are all square matrices and that they differ by only a row, say the kth row. Let's further suppose that the kth row of C can be found by adding the corresponding entries from the kth rows of A and B. Then in this case we will have that*

$$\det(C) = \det(A) + \det(B).$$

The same result will hold if we replace the word row with column above.

Schematically (for columns):

$$\underbrace{\begin{vmatrix} a_{1,1} & \cdots & a_{1,k}+b_{1,k} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,k}+b_{2,k} & \cdots & a_{2,n} \\ \cdots & & & \cdots & \\ \cdots & & & \cdots & \\ a_{n,1} & \cdots & a_{n,k}+b_{n,k} & \cdots & a_{n,n} \end{vmatrix}}_{|C|} =$$

$$\underbrace{\begin{vmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,k} & \cdots & a_{2,n} \\ \cdots & & & \cdots & \\ \cdots & & & \cdots & \\ a_{n,1} & \cdots & a_{n,k} & \cdots & a_{n,n} \end{vmatrix}}_{|A|} + \underbrace{\begin{vmatrix} a_{11} & \cdots & b_{1,k} & \cdots & a_{1,n} \\ a_{21} & \cdots & b_{2,k} & \cdots & a_{2,n} \\ \cdots & & & \cdots & \\ \cdots & & & \cdots & \\ a_{n,1} & \cdots & b_{n,k} & \cdots & a_{n,n} \end{vmatrix}}_{|B|}$$

PROOF:
Let us consider the above scheme with columns (the proof for rows is similar). If we compute the determinant of C using its expansion by cofactors of the kth column we obtain that it is equal to

$$\det(C) = \sum_{i=1}^{k}(a_{i,k}+b_{i,k})(-1)^{i+k}\det(C_{i,k}) = \sum_{i=1}^{k} a_{i,k}(-1)^{i+k}\det(C_{i,k}) + \sum_{i=1}^{k} b_{i,k}(-1)^{i+k}\det(C_{i,k}).$$

But notice that $\det(C_{i,k}) = \det(A_{i,k}) = \det(B_{i,k})$ for all $i = 1\ldots,n$. Therefore

$$\det(C) = \sum_{i=1}^{k}(a_{i,k}(-1)^{i+k}\det(A_{i,k}) + \sum_{i=1}^{k}(b_{i,k}(-1)^{i+k}\det(B_{i,k}) = \det(A)+\det(B).$$

QED

Proposition II.10. *If we add to a row (respectively, column) of A a multiple of another row (respectively, column) then the determinant does not change.*

PROOF:
Consider the matrix

$$A = \begin{matrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,k} & \cdots & a_{2,n} \\ \cdots & & & \cdots & \\ \cdots & & & \cdots & \\ a_{n,1} & \cdots & a_{n,k} & \cdots & a_{n,n} \end{matrix}$$

and let us compute de determinant of the matrix obtained by adding the jth column to the kth column (with $j \neq k$):

$$\begin{vmatrix} a_{1,1} & \cdots & a_{1,k}+\alpha a_{1,j} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,k}+\alpha a_{2,j} & \cdots & a_{2,n} \\ \cdots & & & \cdots & \\ \cdots & & & \cdots & \\ a_{n,1} & \cdots & a_{n,k}+\alpha a_{n,j} & \cdots & a_{n,n} \end{vmatrix} =$$

$$\underbrace{\begin{vmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,k} & \cdots & a_{2,n} \\ \cdots & & & \cdots & \\ \cdots & & & \cdots & \\ a_{n,1} & \cdots & a_{n,k} & \cdots & a_{n,n} \end{vmatrix}}_{|A|} + \underbrace{\begin{vmatrix} a_{1,1} & \cdots & \alpha a_{1,j} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & \alpha a_{2,j} & \cdots & a_{2,n} \\ \cdots & & & \cdots & \\ \cdots & & & \cdots & \\ a_{n,1} & \cdots & \alpha a_{n,j} & \cdots & a_{n,n} \end{vmatrix}}_{0}.$$

The above equality follows from Proposition II.9 and the last determinant is zero because its kth column is proportional to its jth column (Proposition II.8).

The argument for rows is similar. QED

Remark II.11. Notice that the above Proposition implies that, when we apply an elementary row operation of type 3 to a determinant, it does not change.

Proposition II.12. *If a row of A is a linear combination of the other rows of A then* $\det(A) = 0$*. The same happens replacing "row" by "column".*

PROOF:

Assume that A is an $n \times n$ matrix. By Proposition II.9, if a row of A is a linear combination of other rows of A then it splits into a sum of determinants having proportional rows. By Proposition II.8, all these summands are zero. Then $\det(A) = 0$.

The same reasoning is valid for columns. QED

Proposition II.13. *The determinant of a triangular square matrix is the product of the elements in the main diagonal.*

PROOF:

Consider an upper triangular matrix:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n} \\ & & \ddots & \\ 0 & 0 & \dots & a_{n,n} \end{bmatrix}.$$

Using the expansion by cofactors of the first column several times we have that

$$\det(A) = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ 0 & a_{3,3} & \dots & a_{3,n} \\ & & \ddots & \\ 0 & 0 & \dots & a_{n,n} \end{vmatrix} = a_{1,1}a_{2,2} \begin{vmatrix} a_{3,3} & a_{3,4} & \dots & a_{3,n} \\ 0 & a_{4,4} & \dots & a_{4,n} \\ & & \ddots & \\ 0 & 0 & \dots & a_{n,n} \end{vmatrix} = \cdots = a_{1,1}a_{2,2}\cdots a_{n,n}.$$

If the matrix is lower triangular, the proof is similar. QED

As a trivial consequence of the above proposition we have

Corollary II.14. *The determinant of any identity matrix is* 1*.*

Remark II.15. (A kind of "Gauss' Method" to compute a determinant)

Using the properties of the determinants that we have seen, a determinant can be transformed, by elementary operations (**by rows or by columns**), into the determinant of a triangular matrix multiplied by a scalar. But, by Proposition II.13, this last determinant is the product of the diagonal elements.

UTILITARIAN SUMMARY II.16. Let A be a square matrix.

- If either
  - a row of A has only zero entries or
  - a row of A is a linear combination of other rows of A

  then $\det(A) = 0$. The same occurs replacing "row" by "column".
- The determinant is *linear* with respect to a row or a colum:
  - If we multiply a row (or column) of A by a factor, the determinant is multiplied also by that factor.
  - If a row (or column) of A splits into the sum of two row (or column) vectors then the determinant splits also into a sum of two determinants.
- Behavior of the determinant with respect to elementary row operations:
  - If we perform an elementary row operation of type 1 to A, then its determinant only changes of sign.
  - If we perform an elementary row operation of type 2 (with factor α) to A, then its determinant is multiplied by α.
  - If we perform an elementary row operation or type 3 to A, then its determinant does not change.
- The determinant of a triangular matrix is the product of the elements in the main diagonal.
- The determinant of an identity matrix is 1.
- A determinant can be transformed, by elementary operations (by rows or columns) into a scalar multiplied by a triangular matrix. We can take advantage of this to compute the determinant.

III Determinants of the elementary matrices

The following proposition shows the value of the determinants of the elementary matrices:

Proposition III.1. *Let A a $n \times n$ matrix, $i, j \in \{1, \dots, n\}$ with $i \neq j$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then:*

- $|E_{i,j}| = -1$
- $|E_i(\alpha)| = \alpha$.
- $|E_{i,j}(\alpha)| = 1$.

PROOF:

The matrix $E_{i,j}$ is obtained swapping the rows i and j in the identity matrix $I_{n\times n}$. Then, by Proposition II.2, $\det(E_{i,j}) = -\det(I_{n\times n}) = -1$ (notice that the determinant of an identity matrix is 1 by Corollary II.14).

The matrix $E_i(\alpha)$ is obtained from the identity matrix multiplying the ith row by α. Then, by Proposition II.5, $\det(E_i(\alpha)) = \alpha \det(I_{n\times n}) = \alpha$.

Finally, the matrix $E_{i,j}(\alpha)$ is obtained applying an elementary row operation of type 3 to the identity matrix. Then, by Remark II.11, $\det(E_{i,j}(\alpha)) = \det(I_{n\times n}) = 1$. QED

IV Determinant of a product of matrices

Proposition IV.1. *If E is an $n \times n$ elementary matrix and A is an $n \times n$ matrix then*

$$\det(EA) = \det(E)\det(A).$$

PROOF:

- If $E = E_{i,j}$ is an elementary matrix of type 1 then, on the one hand, EA is the matrix obtained by swapping rows i and j of A. So, by Proposition II.2 we have that $\det(EA) = -\det(A)$. On the other hand, by the above proposition, $\det(E) = -1$ and, therefore, $\det(E)\det(A) = -\det(A)$. Hence the equality

  $$\det(EA) = \det(E)\det(A)$$

  holds.

- If $E = E_i(\alpha)$ is an elementary matrix of type 2 then, on the one hand, EA is the matrix obtained by multiplying the ith row of A by α. So, by Proposition II.5 we have that $\det(EA) = \alpha\det(A)$. On the other hand, by the above proposition, $\det(E) = \alpha$ and, therefore, $\det(E)\det(A) = \alpha\det(A)$. Hence the equality

  $$\det(EA) = \det(E)\det(A)$$

  holds also in this case.

- Finally, if $E = E_{i,j}(\alpha)$ is an elementary matrix of type 3 then, on the one hand, EA is the matrix obtained by adding, to the ith row of A, the jth row multplied by α. So, by Proposition II.11 we have that $\det(EA) = \det(A)$. On the other hand, by the above proposition, $\det(E) = 1$ and, therefore, $\det(E)\det(A) = \det(A)$. Hence the equality

$$\det(EA) = \det(E)\det(A)$$

is true also in this case.

QED

Remark IV.2. Notice that a successive application of the above proposition shows that, if $E_1, E_2, \ldots, E_r$ are elementary matrices, then

$$\det(E_1 \cdot E_2 \cdots E_r \cdot A) = \det(E_1) \cdot \det(E_2) \cdots \det(E_r) \cdot \det(A).$$

Corollary IV.3. *Let A be an nn matrix. A is invertible if and only if* $\det(A) \neq 0$.

PROOF:

$\Rightarrow$

We saw in Lesson 2 that the matrix A is invertible if and only if

$$A = E_1 \cdot E_2 \cdots E_r,$$

where each E_i is an elementary matrix.

Therefore, if we assume that A is invertible, applying the above proposition (and Remark) we have that

$$\det(A) = \det(E_1) \cdot \det(E_2) \cdots \det(E_r) \neq 0$$

because, by Proposition III.1, the determinants of the elementary matrices are not zero.

$\Leftarrow$

Assume now that $\det(A) \neq 0$. We saw in Lesson 2 that there exist elementary matrices $E_1, E_2, \ldots, E_r$ such that

$$E_1 \cdot E_2 \cdots E_r \cdot A = R,$$

where R is the RREF of A. Since $\det(A) \neq 0$ and the determinants of the elementary matrices are not zero, one has that

$$\det(R) = \det(E_1) \cdot \det(E_2) \cdots \det(E_r) \cdot \det(A) \neq 0.$$

But R is an upper triangular matrix (because A is square) and, therefore, its determinant is the product of the diagonal elements. Since $\det(R) \neq 0$ we conclude that the diagonal elements of R are, all of them, equal to 1. So, all of them are pivots. This shows that $\operatorname{rank}(A) = n$ and, therefore, A is invertible. QED

Corollary IV.4. *Let A and B be any pair of $n \times n$ matrices. Then* $\det(AB) = \det(A)\det(B)$.

PROOF:

We distinguish two cases:

- If A is not invertible then $\det(A) = 0$ by Corollary IV.3. Moreover AB is not invertible (we saw this in Lesson 2) and, then, $\det(AB) = 0$ also. Hence the equality $\det(AB) = \det(A)\det(B)$ holds in this case.

- If A is invertible then, as we saw in Lesson 2, it is a product of elementary matrices: $A = E_1E_2\cdots E_r$. Then, applying Proposition IV.1 (and the remark):

$$\det(AB) = \det(E_1E_2\cdots E_rB) = \underbrace{\det(E_1)\det(E_2)\cdots\det(E_r)}_{\det(A)}\det(B) = \det(A)\det(B).$$

QED

Remark IV.5. Applying successively the above corollary we have also that, if $A_1, \ldots, A_r$ are any number of matrices of the same order,

$$\det(A_1A_2\cdot A_r) = \det(A_1)\det(A_2)\cdots\det(A_r).$$

UTILITARIAN SUMMARY IV.6.

- A square matrix A is invertible if and only if $\det(A) \neq 0$.
- The determinant of a product of square matrices is the product of their determinants.

V Applications

V.1 Computation of inverses

The following theorem gives a formula to compute the inverse of an invertible matrix using determinants.

Theorem V.2. *If A is an invertible matrix then*

$$A^{-1} = \frac{1}{|A|}\mathrm{Cof}(A)^t.$$

Example V.3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}.$$

$|A| = -1 \neq 0$. Therefore, A is invertible. Let us compute its cofactors:

$$\begin{array}{lll} \alpha_{11} = \begin{vmatrix} 3 & 4 \\ 4 & 6 \end{vmatrix} = 2 & \alpha_{12} = -\begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix} = 0 & \alpha_{13} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1 \\ \alpha_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 0 & \alpha_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} = -3 & \alpha_{23} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2 \\ \alpha_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1 & \alpha_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 2 & \alpha_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1 \end{array}$$

Therefore

$$\mathrm{Cof}(A) = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \tfrac{1}{|A|}\mathrm{Cof}(A)^t = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

V.2 Resolution of linear systems: Cramer's rule

Theorem V.4. (Cramer's rule) *Let A be a square matrix of order n. The system of linear equations $A\vec{x} = \vec{b}$ has a unique solution if and only if $|A| \neq 0$. Moreover, in this case, the solution is given by*

$$x_i = \frac{|A(i)|}{|A|} \quad \forall i = 1, 2, \ldots, n,$$

where $A(i)$ denotes the matrix obtained after replacing the column i of A by the vector of independent terms $\vec{b}$.

Example V.5. Consider the following system:

$$\begin{cases} 3x + 3y + z = 0 \\ x + 3z = 3 \\ 3x + 3y + 4z = 3 \end{cases}$$

We compute first the determinant of the coefficient matrix:

$$\begin{vmatrix} 3 & 3 & 1 \\ 1 & 0 & 3 \\ 3 & 3 & 4 \end{vmatrix} = -9$$

By Cramer's rule, the system has a unique solution and it is

$$x = \frac{\begin{vmatrix} 0 & 3 & 1 \\ 3 & 0 & 3 \\ 3 & 3 & 4 \end{vmatrix}}{-9} = 0; \; y = \frac{\begin{vmatrix} 3 & 0 & 1 \\ 1 & 3 & 3 \\ 3 & 3 & 4 \end{vmatrix}}{-9} = -1/3; \; z = \frac{\begin{vmatrix} 3 & 3 & 0 \\ 1 & 0 & 3 \\ 3 & 3 & 3 \end{vmatrix}}{-9} = 1$$

V.3 Computation of the rank of a matrix

Definition V.6. Let A be any matrix (non-necessarily square). The *minors* of A are the determinants of the square submatrices of A.

The following proposition shows that the rank of a matrix can be computed using determinants.

Proposition V.7. *The rank of a matrix A (non-necessarily square) coincides with the highest order of the non-zero minors of A.*

Example V.8. Let

$$A = \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 2 \end{bmatrix}.$$

Since $\det(A) = 0$ and the order 3 minor obtained deleting the last row and the first column is non-zero, A has rank 3.

Here we have a very interesting (and surprising!) application of this result:

Corollary V.9. *For any matrix A it holds that* $\operatorname{rank}(A) = \operatorname{rank}(A^t)$.

PROOF:

We know that the determinant of a matrix and the determinant of its transpose coincide. Therefore the matrices A and A^t have exactly the same minors. Hence, by the above proposition, $\operatorname{rank}(A) = \operatorname{rank}(A^t)$. QED

UTILITARIAN SUMMARY V.10.

- The inverse of a square invertible matrix can be computed by means of the coffactor matrix.
- The Cramer's rule allows us to solve a linear system with the same number of equations than unknowns and with only one solution.
- The rank of a matrix is the highest order or its non-zero minors.
- **The rank of a matrix A coincides with the rank of its transpose A^t**