Mathematical Analysis

- Inequalities

Exercise: Find $x \in \mathbb{R}$ such as $\left||x|-2\right| \le 1$

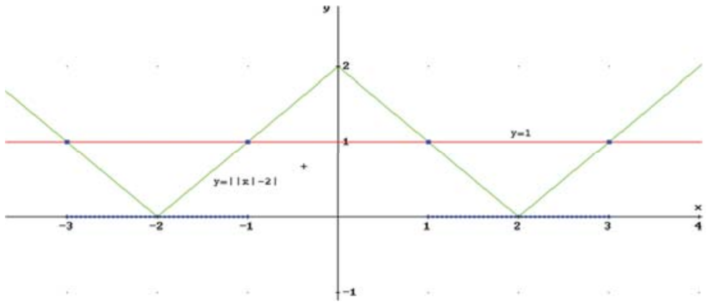
by second property of absolute value

$$\left||x|-2\right| \le 1 \iff 1 \le |x| \le 3 \iff |x| \le 3 \wedge |x| \ge 1$$

$$|x| \le 3 \iff x \in [-3,3]$$

third property of absolute value $|x| \ge 1 \iff x \in]-\infty,-1] \cup [1,+\infty[$

$$[-3,3] \cap (]-\infty,-1] \cup [1,+\infty[) = [-3,-1] \cup [1,3]$$



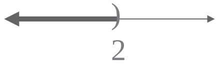
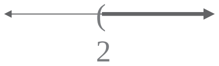
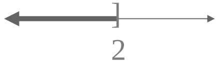
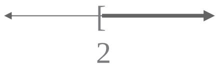
Inequality symbols

- **<** Less Than
- **>** Greater Than
- ≤ Less Than or Equal to
- ≥ Greater Than or Equal to
- ≠ Not Equal

Inequalities

- a > b if *a* is to the right of *b*
- a < b if *a* is to the left of *b*
- The inequality sign always points to the smaller value.

Examples:

Inequality	Graph	Interval
x < 2		(-∞, 2)
x > 2		(2, ∞)
x ≤ 2		(-∞, -2]
x ≥ 2		[2, ∞)

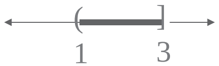
Examples:

Inequality	Graph	Interval
3 < x < 7		(3,7)
3 ≤ x < 7		[3,7)
3 < x ≤ 7		(3,7]
3 ≤ x ≤ 7		[3,7]

Open Interval

1 < x < 3	

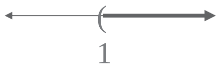
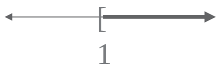
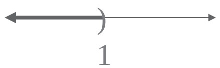
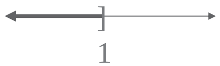
Half Open Intervals

1 ≤ x < 3	
1< x ≤ 3	

Closed Intervals

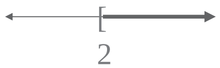
1 ≤ x ≤ 3	

Unbounded Intervals

x > 1		(1,∞)
x ≥ 1		[1,∞)
x < 1		(-∞, 1)
x ≤ 1		(-∞, 1]
-∞ < x < ∞		(-∞,∞)

Compound Inequality “AND”

$x \geq 2$ and $x < 7$



equivalent to

$2 \leq x < 7$

Intersection of the two graphs

[2,7)

$\boxed{\text{and} \equiv \wedge}$

Compound Inequality“OR”

$x < -2$ or $x \geq 3$

Graph:

- Union of two graphs

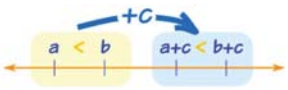
Interval Notation:

$(-\infty, -2) \cup [3, \infty)$

$\boxed{\text{or} \equiv \vee}$

Addition and Subtraction

Adding **c** to both sides of an inequality just **shifts everything along**, and the inequality stays the same.



1.- If a < b, then a **+ c** < b **+ c**

Addition and Subtraction

Likewise:

2.- If a < b, then a − c < b − c

3.- If a > b, then a + c > b + c, and

4.- If a > b, then a − c > b − c

So adding (or subtracting) the same value to both a and b **will not change** the inequality

Additive Inverse

As we just saw, putting “minuses” in front of a and b **changes the direction** of the inequality.
This is called the "Additive Inverse":

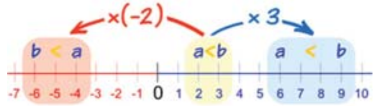
5.- If $a < b$ then $-a > -b$

6.- If $a > b$ then $-a < -b$

This is really the same as multiplying by (-1), and that is why it changes direction.

Multiplication and Division

When you multiply both a and b by a **positive number**, the inequality **stays the same**.
But when you multiply both a and b by a **negative number**, the inequality **swaps over**!



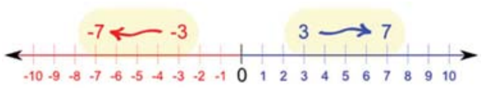
Notice that $\mathbf{a<b}$ becomes $\mathbf{b<a}$ after multiplying by (-2)
But the inequality stays the same when multiplying by +3
Here are the rules:

7.- If $a < b$, and **c is positive**, then $\mathbf{ac < bc}$

8.- If $a < b$, and **c is negative**, then $\mathbf{ac > bc}$ (inequality swaps over!)

Why does multiplying by a negative reverse the sign?

Well, just look at the number line!
For example, from 3 to 7 is **an increase**, but from -3 to -7 is **a decrease.**

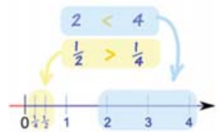


Multiplicative Inverse

Taking the reciprocal (1/value) of both a and b **can change the direction** of the inequality.
When a and b are **both positive** or **both negative**:

9.- If $a < b$ then $1/a > 1/b$

10.- If $a > b$ then $1/a < 1/b$



Summary

- A statement that two quantities are not equal is an ***inequality***; $3 \neq 7$ (read "3 is not equal to 7").
- When two numbers are not equal, one must be less than the other.
  - the symbol $<$ means "is less than"; $8 < 9$, $-2 < 5$, and $-9 < -4$
  - the symbol $>$ means "is greater than"; $11 > 7$, $2 > -3$, and $3/4 > 0$

Notice that in each case, the symbol "points toward the smaller number.

- The smaller of two numbers is always to the left of the other on a number line.

Inequalities on a Number Line

On a number line,
$a < b$ if a is to the left of b; $a > b$ if a is to the right of b;

Inequality Symbols

Symbol	*Meaning*	*Example*
$\neq$	is not equal to	$3 \neq 7$
$<$	is less than	$-5 < -4$
$>$	is greater than	$2 > -3$
$\leq$	is less than or equal to	$4 \leq 4$
$\geq$	is greater than or equal to	$-3 \geq -5$

Summary

Graph sets of real numbers.

- A ***compound inequality*** is two or more inequalities connected by the word *and* or *or*.

Interval Notation for Unbounded Intervals

Set-builder notation	Interval notation	Type	Graph
$\{x \mid x > a\}$	(a, ∞)	Open	
$\{x \mid x < a\}$	$(-\infty, a)$	Open	
$\{x \mid x \geq a\}$	$[a, \infty)$	Half Open	
$\{x \mid x \leq a\}$	$(-\infty, a]$	Half Open	
Real Numbers	$(-\infty, \infty)$	Open	
$\{x \mid x \leq a$ or $x > b\}$	$(-\infty, a] \cup (b, \infty)$	Compound Inequality	

Interval Notation for Bounded Intervals (Compound Inequalities)

Set-builder notation	Interval notation	Type	Graph
$\{x \mid a < x < b\}$	(a, b)	Open	
$\{x \mid a \leq x \leq b\}$	$[a, b]$	Closed	
$\{x \mid a \leq x < b\}$	$[a, b)$	Half Open	
$\{x \mid a < x \leq b\}$	$(a, b]$	Half Open	

Summary

1.- If a < b, then a **+ c** < b **+ c**
2.- If a < b, then a − c < b − c
3.- If a > b, then a + c > b + c
4.- If a > b, then a − c > b − c
5.- If a < b then -a > -b
6.- If a > b then -a < -b
7.- If a < b, and **c is positive**, then **ac < bc**
8.- If a < b, and **c is negative**, then **ac > bc** (inequality swaps over!)
9.- If a < b then 1/a > 1/b
10.- If a > b then 1/a < 1/b