Practice 3

Activities sheet

Activity 1. Determine which of the following matrices are stochastic. For the stochastic matrices, compute the set of stationary vectors.

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \qquad \begin{bmatrix} 2/5 & -2/5 \\ 3/5 & 7/5 \end{bmatrix} \qquad \begin{bmatrix} 1/3 & 1/6 & 1/4 \\ 1/3 & 2/3 & 1/4 \\ 1/3 & 1/6 & 1/2 \end{bmatrix}$$

Solution:

It is obvious that only the third matrix is stochastic. Let A be this matrix. Recall that a stationary vector is any non-zero vector $\vec{x}$ such that $A\vec{x} = \vec{x}$ or, equivalently, $(A - I)\vec{x} = \vec{0}$. Therefore, the set of stationary vectors are those non-zero vectors in the kernel of $A - I$:

```
-->A=[1/3 1/6 1/4; 1/3 2/3 1/4; 1/3 1/6 1/2];

-->kernel(A-eye(3,3))
 ans  =

    0.3841106
    0.7682213
    0.5121475
```

This means that the set of stationary vectors for A is the set of all non-zero multiples of the vector $(0.3841106,\ \ 0.7682213,\ \ 0.5121475)$.

Activity 2. Consider the stochastic matrix $A = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0.25 & 0 & 0 & 0 \\ 0.5 & 0.25 & 1 & 0 \\ 0.25 & 0.25 & 0 & 1 \end{bmatrix}$.

(a) Compute the set of stationary vectors.

(b) Compute a probability stationary vector. Is it unique?

(c) Is A a regular stochastic matrix?

Solution:

(a) As in the previous activity:

```
-->A=[0 0.5 0 0; 0.25 0 0 0; 0.5 0.25 1 0; 0.25 0.25 0 1];

-->kernel(A-eye(4,4))
```

```
ans  =

   0.    0.
   0.    0.
   0.    1.
   1.    0.
```

Then, the set of stationary vectors is

$$\operatorname{span}((0,0,0,1),(0,0,1,0)) \setminus \{\vec{0}\}.$$

(b) Recall that a probability vector is a vector whose coordinates sum 1. In our case, it is clear that $(0,0,0,1)$ and $(0,0,1,0)$ are two different probability stationary vectors.

(c) By Theorem 1 (see the practice's bulletin) any regular stochastic matrix has a unique probability stationary vector. Since our matrix has, at least, two of them, **it cannot be regular**.

Activity 3. Consider the matrix

$$B = \begin{bmatrix} 0.05 & 0.85 & 0.5 \\ 0.1 & 0.05 & 0.1 \\ 0.85 & 0.1 & 0.4 \end{bmatrix}$$

(a) Check that B is a regular stochastic matrix.

(b) Compute the set of stationary vectors for B.

(c) Compute a stationary probability vector.

(d) Write the 3 first terms of the Markov chain with transition matrix B and initial vector of states $x_0 = (0.3, 0.5, 0.2)$.

(e) Is the chain convergent?

Solution:

(a) B is stochastic because its columns are probability vectors. It is regular because all the entries of $B^1 = B$ are strictly positive.

(b) With Scilab:

```
-->B=[0.05 0.85 0.5; 0.1 0.05 0.1; 0.85 0.1 0.4];

-->kernel(B-eye(3,3))
```

```
 ans  =

    0.5591810
    0.1447880
    0.8163045
```

Therefore, the set of stationary vectors is:

$$\text{span}(0.5591810, 0.1447880, 0.8163045) \setminus \{\vec{0}\}.$$

(c) The unique (becuase B is stochastic and regular) probability stationary vector can be obtained dividing any stationary vector by the sum of its components:

```
-->ans/sum(ans)
 ans  =

    0.3678161
    0.0952381
    0.5369458
```

Then $(0.3678161, 0.0952381, 0.5369458)$ is the probability stationary vector.

(d)

```
-->x0=[0.3; 0.5; 0.2];

-->x1=B*x0
 x1  =

    0.54
    0.075
    0.385

-->x2=B*x1
 x2  =

    0.28325
    0.09625
    0.6205

-->x3=B*x2
 x3  =

    0.406225
    0.0951875
    0.4985875
```

(e) Of course, the Markov chain is convergent because the transition matrix B is stochastic and regular (see Theorem 1). Moreover, also by Theorem 1, its limit is the (unique) probatility stationary vector: $(0.3678161, 0.0952381, 0.5369458)$.

Activity 4. In a country elections are held every four years and the results of each choice depend only on the results of the previous election. The presented parties are: the Democratic (D), the Liberal (L) and Conservative (C). 70 % of voters for D will vote again for D, 10 % of voters for D will vote for L, and 20 % of them will vote for C; 80 % of the voters for L will continue voting for L, 5 % will be voting for D and 15 % will vote for C; and finally, 70 % of voters for C will vote again for C and 30 % will vote for L.

(a) Write the matrix P that corresponds to this process and check that it is stochastic.

(b) If the percentages of votes in an election are 55% for D, 40% for L and 5% for C, determine the result of the next election.

(c) What percentage of votes have to get each of the parties in an election if we want, in the next election, the result to be the same?

Solution:

(a) The matrix corresponding to this process is the following one:

```
-->P=[0.7 0.05 0; 0.1 0.8 0.3; 0.2 0.15 0.7]
 P  =

    0.7     0.05     0.
    0.1     0.8      0.3
    0.2     0.15     0.7
```

Clearly this matrix is stochastic. Moreover it is regular because, although the element $(1,3)$ of P is zero, all the entries of P^2 are strictly positive (you may check it).

(b) To obtain the percentage of votes in the next elections we can multiply the matrix P by the vector $\vec{x}_0$ that represents the percentage obtained in the present elections:

```
-->x0=[0.55; 0.40; 0.05]
 x0  =

    0.55
    0.4
    0.05

-->P*x0
```

```
ans  =

    0.405
    0.39
    0.205
```

Therefore D, L and C will obtain, respectively, the following percentages of votes: 40.5, 39 and 20.5.

(c) We need to find a non-zero vector $\vec{v}$ such that $P\vec{v} = \vec{v}$ and such the sum of its components is 1. That is, we need to find a stationary probability vector of the matrix P. We know that it exists and it is unique because P is stochastic and regular.

```
-->x=kernel(P-eye(3,3))
 x  =

    0.1407970
    0.8447819
    0.5162556
```

This vector $\vec{x}$ and all its non-zero multiples are the stationary vectors of P. To obtain the probability stationary vector we need to divide $\vec{x}$ by the sum if its components:

```
-->v=x/sum(x)
 v  =

    0.09375
    0.5625
    0.34375
```

Therefore, the answer to the question is: 9.375% (D), 56.24% (L) and 34.375% (C).