Practice 4

Activities sheet

Activity 1. *Given the matrix*

$$A = \begin{bmatrix} -2 & 4 & 2 & 1 \\ 4 & 2 & 1 & -2 \\ 2 & 1 & -2 & 4 \\ 1 & -2 & 4 & 2 \end{bmatrix}$$

a) Compute A^2.
b) Without doing any computation, determine the inverse of A.

SOLUTION:

```
-->A=[-2 4 2 1; 4 2 1 -2; 2 1 -2 4; 1 -2 4 2];

-->A^2
 ans  =

    25.    0.     0.     0.
    0.     25.    0.     0.
    0.     0.     25.    0.
    0.     0.     0.     25
```

Therefore we have that $A \cdot A = 25I$, where I is the identity matrix. Then, dividing both sides by 25, we have $\frac{1}{25}A \cdot A = I$ and we can conclude that $\frac{1}{25}A$ is the inverse of A.

Activity 2. *Compute with Scilab, using two different methods, the inverses of the following matrices. If some of the results is not correct, explain why.*

$$A = \begin{bmatrix} 1 & - & 2 \\ -1 & 2 & -1 \\ 2 & -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 4 & 3 & 5 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$

SOLUTION:

One method consists of applying the command *inv*. The other consists of computing the RREF of the matrix $[A \quad I]$, where A is the matrix we are considering and I is the identity matrix of the same order than A:

```
-->A=[1 -1 2; -1 2 -1; 2 -3 1]; B=[2 1 0 1; 1 1 1 2; 2 1 3 1; 4 3 5 5];
 C=[2 4 6; 8 10 12; 14 16 18];
```

```
-->inv(A)
 ans  =

    0.5    2.5    1.5
    0.5    1.5    0.5
    0.5  - 0.5  - 0.5

-->rref([A eye(3,3)])
 ans  =

    1.    0.    0.    0.5    2.5    1.5
    0.    1.    0.    0.5    1.5    0.5
    0.    0.    1.    0.5  - 0.5  - 0.5
```

For A it is clear that it is an invertible matrix and it is clear which is the inverse (both methods).

```
-->inv(B)
       !--error 19
El problema es singular.


-->rref([B eye(4,4)])
 ans  =

    1.    0.    0.  - 1.    0.6666667    0.    0.8333333  - 0.5
    0.    1.    0.    3.  - 0.3333333    0.  - 1.6666667    1.
    0.    0.    1.    0.  - 0.3333333    0.    0.3333333    0.
    0.    0.    0.    0.    0.           1.    0.5        - 0.5
```

For B, the first method shows that B is a singular (that is, non-invertible) matrix and, in the second method, we see that $[B \quad I]$ is row equivalent to a matrix $[D \quad E]$ where D is not the identity matrix; this implies that B is not invertible.

```
-->inv(C)
Advertencia :
la matriz esta cerca de la singularidad o mal escalada. rcond =    0.0000D+00

 ans  =

   1.0D+15 *
```

```
  - 2.2517998     4.5035996   - 2.2517998
    4.5035996   - 9.0071993     4.5035996
  - 2.2517998     4.5035996   - 2.2517998

-->rref([C eye(3,3)])
 ans  =

    1.    0.  - 1.    0.  - 1.3333333    0.8333333
    0.    1.    2.    0.    1.1666667  - 0.6666667
    0.    0.    0.    1.  - 2.           1.
```

For C, the first method returns a result but, previously, Scilab warns us that C is either "near to be singular" or "it is bad scaled". This means that we must be careful with the returned matrix because it is possible that it is not the inverse of C. Scilab detects that all computations can be done but, due to rounding errors, may be the result is not correct.

Using the second method we see clearly that the matrix is, actually, non-invertible.

Activity 3. *Consider the matrix*

$$D = \begin{bmatrix} 7 & 1 & 2 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 4 & 2 \end{bmatrix}$$

a) Compute the matrix T such that $TD = R$, where R is the RREF of D.
b) Solve the matrix equation $TX + X = DD^t$ (where X is an unknown matrix).

SOLUTION:

```
-->D=[7 1 2; 4 2 1; 0 1 -2; 0 4 2];

-->rref([D eye(4,4)])
 ans  =

    1.    0.    0.    0.    0.25  - 1.388D-17  - 0.125
    0.    1.    0.    0.    0.      0.2          0.2
    0.    0.    1.    0.    0.    - 0.4          0.1
    0.    0.    0.    1.  - 1.75    0.6          0.475

-->clean(ans)
 ans  =

    1.    0.    0.    0.    0.25    0.   - 0.125
```

```
    0.    1.    0.    0.    0.         0.2    0.2
    0.    0.    1.    0.    0.       - 0.4    0.1
    0.    0.    0.    1.  - 1.75       0.6    0.475
```

Therefore

$$T = \begin{bmatrix} 0 & 0.25 & 0. & -0.125 \\ 0 & 0. & 0.2 & 0.2 \\ 0 & 0. & -0.4 & 0.1 \\ 1 & -1.75 & 0.6 & 0.475 \end{bmatrix}$$

We can "capture" easily T with Scilab:

```
-->T=ans(:,[4 5 6 7])
 T  =

    0.    0.25    0.    - 0.125
    0.    0.      0.2     0.2
    0.    0.    - 0.4     0.1
    1.  - 1.75    0.6     0.475
```

The equation $TX + X = DD^t$ is equivalent to the equation

$$(T + I)X = DD^t,$$

where I is the identity matrix of order 4. We check with Scilab that $T + I$ is invertible:

```
-->H=T+eye(4,4)
 H  =

    1.    0.25    0.    - 0.125
    0.    1.      0.2     0.2
    0.    0.      0.6     0.1
    1.  - 1.75    0.6     1.475

-->inv(H)
 ans  =

    0.9090909  - 0.0681818  - 0.0681818    0.0909091
    0.0909091    0.8181818  - 0.1818182  - 0.0909091
    0.0909091  - 0.1818182    1.8181818  - 0.0909091
  - 0.5454545    1.0909091  - 0.9090909    0.5454545
```

Then we can obtain easily X: $X = (T + I)^{-1}DD^t$. With Scilab:

```
-->inv(H)*D*D'
 ans  =
```

```
   47.840909    28.568182  - 3.0681818    8.4090909
   30.909091    19.181818  - 1.1818182    7.0909091
 - 7.0909091  - 1.8181818    8.8181818  - 2.9090909
   12.545455    10.909091  - 2.9090909    17.454545
```

Activity 4. *Given the matrix*

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -1 & -1 \\ -3 & 13 & 4 \end{bmatrix}$$

a) Compute (by hand) an LU decomposition of A.
b) Solve (by hand) the following system using the LU decomposition that you have obtained:

$$\begin{aligned} -x + 3y + 2z &= 2 \\ x - y + -z &= -2 \\ -3x + 13y + 4z &= -2 \end{aligned}$$

c) Compute with Scilab the LU decomposition of A. If it is not the same matrix than the one that you have obtained in a), explain why.
d) Compute the inverse of A and its determinant using the LU factorization that you have obtained in c). Check that the same results are obtained using the commands inv(A) and det(A), respectively.

SOLUTION:

(a) First we must transform A into an upper triangular matrix U using elementary row operations. It can be done performing the following operations: (1) $\rho_2 + \rho_1$, (2) $\rho_3 - 3\rho_1$, (3) $\rho_3 - 2\rho_2$. The obtained equivalent matrix is

$$U = \begin{bmatrix} -1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

This means that

$$E_{3,2}(-2)E_{3,1}(-3)E_{2,1}(1)A = U$$

and, therefore:

$$A = (E_{3,2}(-2)E_{3,1}(-3)E_{2,1}(1))^{-1}U$$

. Then

$$L = (E_{3,2}(-2)E_{3,1}(-3)E_{2,1}(1))^{-1} = E_{2,1}(1)^{-1}E_{3,1}(-3)^{-1}E_{3,2}(-2)^{-1}$$

$$= E_{2,1}(-1)E_{3,1}(3)E_{3,2}(2) = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

You can check that

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -4 \end{bmatrix}}_{U}.$$

(b) The system that we want to solve is $A\vec{x} = \vec{b}$, where $\vec{x} = x, y, z)$. Replacing A by its LU decomposition we have the equivalent system

$$LU\vec{x} = \vec{b}. \tag{1}$$

Then we name $\vec{y} = U\vec{x}$. Observe that $\vec{y}$ must be a vector with 3 components. So, we put

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Replacing, in (**??**), $U\vec{x}$ by $\vec{y}$ we get a system whose vector of unknowns is $\vec{y}$:

$$L\vec{y} = \vec{b} \Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$$

Translating into equations:

$$y_1 = 2$$
$$-y_1 + y_2 = -2$$
$$3y_1 + 2y_2 + y_3 = -2$$

and solving the system by forward substitution we obtain: $y_1 = 2$, $y_2 = 0$ and $y_3 = -8$, that is:

$$\vec{y} = \begin{bmatrix} 2 \\ 0 \\ -8 \end{bmatrix}.$$

Now, since $U\vec{x} = \vec{y}$, replacing $\vec{y}$ by the obtained solution we have the following linear system:

$$\underbrace{\begin{bmatrix} -1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & -4 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \\ x \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ -8 \end{bmatrix}}_{\vec{y}}.$$

Translating into equations:

$$-x + 3y + 2z = 2$$
$$2y + z = 0$$
$$-4z = -8$$

Solving it by backward substitution we obtain the following solution: $x = -1$, $y = -1$, $z = 2$.

(c) With Scilab:

```
-->A=[-1 3 2; 1 -1 -1; -3 13 4]
 A  =

  - 1.    3.     2.
    1.  - 1.   - 1.
  - 3.    13.    4.

-->[L,U]=lu(A)
 U  =

  - 3.    13.          4.
    0.    3.3333333    0.3333333
    0.    0.           0.8
 L  =

    0.3333333  - 0.4    1.
  - 0.3333333    1.     0.
    1.           0.     0.
```

Notice that the result given by Scilab does not coincide with our result. The reason is that Scilab always uses the **partial pivoting method** when it applies elementary row operations (to minimize the rounding errors). This fact makes that some permutations of rows occur in the process and, as a consequence, the result is not the same than the one obtained by hand. (Notice that the matrix L that is obtained by Scilab is not lower triangular, but it would be lower triangular after performing some permutations of rows).

(d) Since $A = LU$ then $A^{-1} = U^{-1}L^{-1}$. Then one can compute U^{-1} and L^{-1} (that is easy because the matrix are triangular) and, then, one can obtain A^{-1} doing the product. This is the way that, actually, Scilab uses when the **inv** command is applied:

```
-->inv(U)*inv(L)
 ans  =

    1.125    1.75  - 0.125
  - 0.125    0.25    0.125
    1.25     0.5   - 0.25

-->inv(A)
 ans  =

    1.125    1.75  - 0.125
  - 0.125    0.25    0.125
    1.25     0.5   - 0.25
```

To compute the determinant of A, we have that $\det(A) = \det(LU) = \det(L)\det(U)$. When Scilab applies the command $\mathbf{det}$, it computes first an LU decomposition of A and, then, it computes its determinant using the above equality. Notice that the determinants of L and U are very easy to compute.

```
-->det(L)*det(U)
 ans  =

    8.

-->det(A)
 ans  =

    8.
```