Practice 1
Activities sheet

Activity 1. *Study the number of solutions of these systems of linear equations and solve them using* ***rref****.*

$$a) \left.\begin{array}{rcl} 3x+4y-7t+5u+3v &=& -9 \\ x-3y-7z+8t+9u-12v &=& 1 \\ 2x-4y+7z-11t-v &=& 2 \\ x-y+2z-3t+5u-3v &=& -2 \\ 2x+2y-4z+2t+u+v &=& -5 \\ x+2y+z-4t-7u+3v &=& 2 \end{array}\right\} \quad b) \left.\begin{array}{rcl} 3x+4y-7t+5u+3v &=& -9 \\ x-3y-7z+8t+9u-12v &=& 1 \\ 2x-4y+7z-11t-v &=& 2 \\ x-y+2z-3t+5u-3v &=& -2 \\ 2x+2y-4z+2t+u+v &=& -5 \\ x+2y+z-4t-7u+3v &=& 2 \\ 2x-6y-7z-5t-15u-11v &=& 8 \end{array}\right\}.$$

SOLUTION:

First, we introduce in Scilab the augmented matrices of the first system:

```
-->Ab=[3 4 0 -7 5 3 -9; 1 -3 -7 8 9 -12 1; 2 -4 7 -11 0 -1 2;
 1 -1 2 -3 5 -3 -2; 2 2 -4 2 1 1 -5; 1 2 1 -4 -7 3 2];
```

Applying **rref** we obtain:

```
-->rref(Ab)
 ans  =

    1.    0.    0.    0.    0.    0.  - 1.7325784
    0.    1.    0.    0.    0.    0.  - 0.5007919
    0.    0.    1.    0.    0.    0.  - 0.5246278
    0.    0.    0.    1.    0.    0.  - 0.5818023
    0.    0.    0.    0.    1.    0.  - 0.7336079
    0.    0.    0.    0.    0.    1.  - 0.7345581
```

We see that the rank of the coefficient matrix and the rank of the augmented matrix are both equal to 6, which coincides with the number of unknowns. Applying Rouché-Fröbenius Theorem we deduce that the system has only one solution. "Passing to equations" the RREF of the augmented matrix we obtain that the solution is:

$$\begin{bmatrix} x \\ y \\ z \\ t \\ u \\ v \end{bmatrix} = \begin{bmatrix} -1.7325784 \\ -0.5007919 \\ -0.5246278 \\ -0.5818023 \\ -0.7336079 \\ -0.7345581 \end{bmatrix}.$$

With respect to the second system, observe that it contains the equations of the first one and one more equation. So, we can compute easily the augmented matrix:

```
-->Bb=[Ab; [2 -6 -7 -5 -15 -11 8] ]
 Bb  =

    3.    4.    0.  - 7.     5.     3.   - 9.
    1.  - 3.  - 7.    8.     9.   - 12.    1.
    2.  - 4.    7.  - 11.    0.   - 1.     2.
    1.  - 1.    2.  - 3.     5.   - 3.   - 2.
    2.    2.  - 4.    2.     1.     1.   - 5.
    1.    2.    1.  - 4.   - 7.     3.     2.
    2.  - 6.  - 7.  - 5.   - 15.  - 11.    8.
```

Now, we apply **rref**:

```
-->rref(Bb)
 ans  =

    1.    0.    0.    0.    0.    0.    0.
    0.    1.    0.    0.    0.    0.    0.
    0.    0.    1.    0.    0.    0.    0.
    0.    0.    0.    1.    0.    0.    0.
    0.    0.    0.    0.    1.    0.    0.
    0.    0.    0.    0.    0.    1.    0.
    0.    0.    0.    0.    0.    0.    1.
```

The rank of the coefficient matrix is 6 and the rank of the augmented matrix is 7. Therefore, by Rouché-Fröbenius Theorem, the system has no solution. Also, this obvious from the fact that the equation associated to the last row of the obtained RREF is $0 = 1$.

Activity 2. *Study the number of solutions of these systems of linear equations and solve them using **rref** (write the vectorial expressions of the solutions, whenever they exist).*

$$a)\quad \left.\begin{aligned} x - y &= 2 \\ x + 2y &= 8 \\ x - y + z &= 3 \end{aligned}\right\} \quad b)\quad \left.\begin{aligned} x + y - z + 2t &= 1 \\ 2x + 3y + 4t &= 2 \\ y + z + 3t &= -4 \\ -x - 2y - z - 2t &= -1 \end{aligned}\right\} \quad c)\quad \left.\begin{aligned} x - 2z &= 2 \\ -x - 2y + 2z &= -2 \\ 2x + 2y - 4z &= 3 \end{aligned}\right\}.$$

$d)$ *Are the vectors* $\vec{s_1} = (33/2, -11, 5/2, 1/2)$ *and* $\vec{s_2} = (33/2, -11, 11/2, 1/2)$ *solutions of the linear system given in* $b)$?

SOLUTION:

(a) Instead of introducing directly the augmented matrix into Scilab, we will define separately the coefficient matrix and the vector of independent terms and, from these, we will construct the augmented matrix. We will proceed in this way because, in Activity 3, we will need to use the coefficient matrix and the vector of independent terms.

```
-->A1=[1 -1 0; 1 2 0; 1 -1 1]
 A1  =

    1.  - 1.    0.
    1.    2.    0.
    1.  - 1.    1.

-->b1=[2; 8; 3]
 b1  =

    2.
    8.
    3.

-->rref([A1 b1])
 ans  =

    1.    0.    0.    4.
    0.    1.    0.    2.
    0.    0.    1.    1.
```

The rank of the coefficient matrix and the rank of the augmented matrix coincide with the number of unknowns. Then, applying Rouché-Fröbenius Theorem, we concluIde that the system has only one solution, which is $(x, y, z) = (4, 2, 1)$.

(b) We proceed analogously:

```
-->A2=[1 1 -1 2; 2 3 0 4; 0 1 1 3; -1 -2 -1 -2];

-->b2=[1; 2; -4; -1];

-->rref([A2 b2])
 ans  =
```

```
   1.    0.    0.  - 7.    13.
   0.    1.    0.    6.  - 8.
   0.    0.    1.  - 3.    4.
   0.    0.    0.    0.    0.
```

The rank of the coefficient matrix and the rank of the augmented matrix are both equal to 3, that is less than the number of unknowns (4). Therefore the system has infinitely many solutions. To compute them, pass the RREF into equations:

$$x - 7t = 13$$

$$y + 6t = -8$$

$$z - 3t = 4$$

Replacing the free variable (t) by a parameter we obtain the parametric expression of the solutions:

$$x = 13 + 7\lambda,$$

$$y = -8 - 6\lambda,$$

$$z = 4 + 3\lambda,$$

$$t = \lambda,$$

where λ varies in $\mathbb{R}$. Then, the vectorial expression of the solution set is:

$$(x, y, z, t) = (13, -8, 4, 0) + \lambda(7, -6, 3, 1), \quad \lambda \in \mathbb{R}.$$

(c)

```
-->A3=[1 0 -2; -1 -2 2; 2 2 -4];

-->b3=[2; -2; 3];

-->rref([A3 b3])
 ans  =

   1.    0.  - 2.    0.
   0.    1.    0.    0.
   0.    0.    0.    1.
```

The rank of the coefficient matrix is 2 and the rank of the augmented matrix is 3. Therefore, by Rouché-Fröbenius Theorem, the system has no solution.

(d)

```
-->s1=[33/2; -11; 5/2; 1/2]; s2=[33/2; -11; 7/2; 1/2];

-->clean(A2*s1-b2)
 ans  =
```

```
    3.
    0.
  - 3.
    3.

-->clean(A2*s2-b2)
 ans  =

    0.
    0.
    0.
    0.
```

Then, the first vector is not a solution but the second one is.

Activity 3. *Study the number of solutions of the systems of the preceding activity and compute the solutions (if they exist) using \ and **kernel**. Compare your results with those obtained in Activity 2.*

SOLUTION:

(a) First we apply \ to obtain a "tentative" solution:

```
-->x=A1\b1
 x  =

    4.
    2.
    1.
```

Now we check if this is actually a solution or not:

```
-->clean(A1*x-b1)
 ans  =

    0.
    0.
    0.
```

We see that it is. Therefore the system has, at least, one solution. To determine the whose solution set, we use the command **kernel** to compute the kernel of the coefficient matrix:

```
-->kernel(A1)
 ans  =

     []
```

The kernel is trivial (that is, $\{\vec{0}\}$). Therefore the obtained vector $(4, 2, 1)$ is the unique solution of the system.

(b) We proceed analogously:

```
-->x=A2\b2
Advertencia :
la matriz esta cerca de la singularidad o mal escalada. rcond =    0.0000D+0
calculando la solución de mínimos cuadrados. (vea Isq).

 x  =

    0.
    3.1428571
  - 1.5714286
  - 1.8571429

-->clean(A2*x-b2)
 ans  =

    0.
    0.
    0.
    0.
```

Then the system has, at least, one solution.

```
-->kernel(A2)
 ans  =

  - 0.7181848
    0.6155870
  - 0.3077935
  - 0.1025978
```

Therefore the solution set is:

$$\{\begin{bmatrix} 0 \\ 3.1428571 \\ -1.5714286 \\ -1.8571429 \end{bmatrix} + \lambda \begin{bmatrix} -0.7181848 \\ 0.6155870 \\ -0.3077935 \\ -0.1025978 \end{bmatrix} \text{ such that } \lambda \in \mathbb{R}\} =$$

$$\begin{bmatrix} 0 \\ 3.1428571 \\ -1.5714286 \\ -1.8571429 \end{bmatrix} + \text{span}(\begin{bmatrix} -0.7181848 \\ 0.6155870 \\ -0.3077935 \\ -0.1025978 \end{bmatrix})\}$$

(c) Similarly:

```
 ->x=A3\b3
Advertencia :
la matriz esta cerca de la singularidad o mal escalada. rcond =    0.0000D+0
calculando la solución de mínimos cuadrados. (vea Isq).

 x  =

    0.
  - 5.692D-16
  - 0.8333333

-->clean(A3*x-b3)
 ans  =

  - 0.3333333
    0.3333333
    0.3333333
```

Therefore the obtained vector x is not a solution. We can conclude, then, that the system has no solution.

Activity 4. *A carrier has three trucks (C1, C2 and C3) that have to tow containers of three types (A, B and C). The truck C1 has capacity for 5 containers of type A, 2 containers of type B and 4 of type C. The truck C2 has capacity for 3 containers of type A, 5 of type B and 3 of type C. The truck C3 has capacity for 4 containers of type A, 5 of type B and 6 of type C. It is needed to transport 45 containers of type A, 46 of type B and 54 of type C. How many (fully booked) trips of each truck are needed?*

SOLUTION:

The unknowns are:

$x =$ Number of trips of the truck C1.
$y =$ Number of trips of the truck C2.
$z =$ Number of trips of the truck C3.

On the one hand, since the number of containers of type A that can be carried by C1, C2 and C3 in one travel are, respectively, 5, 3 and 4, the total number of transported containers of type A is $5x + 3y + 4z$.

On the other hand, the statement says that the total number of transported containers of type A is 45. Therefore the following equation must be satisfied:

$$5x + 3y + 4z = 45.$$

Reasoning similarly with containers of types B and C we obtain the equations:

$$2x + 5y + 5z = 46$$

and

$$4x + 3y + 6z = 54.$$

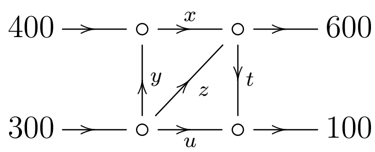
The solution to our problem will be given by a solution of the linear system formed by the three equations. Solving it with Scilab:

```
-->A=[5 3 4; 2 5 5; 4 3 6]
 A  =

    5.    3.    4.
    2.    5.    5.
    4.    3.    6.

-->b=[45; 46; 54]
 b  =

    45.
    46.
    54.

-->x=A\b
 x  =

    3.
    2.
```

```
    6.

-->clean(A*x-b)
 ans  =

    0.
    0.
    0.


-->kernel(A)
 ans  =

     []
```

Then, this system has only one solution, that is: $x = 3$, $y = 2$, $z = 6$. Therefore the number of trips of C1, C2 and C3 are, respectively, 3, 2 and 6.

Activity 5. *The following figure shows the traffic flow (in vehicles per hour) in a street network (nodes of the graph represent the intersections). Find the dependencies among the flows. What is the traffic flow when $u = 50$ and $z = 150$?*



SOLUTION:

The equations that describe the traffic flow are:

$$y + 400 = x$$

$$x + z = 600 + t$$

$$300 = y + z + u$$

$$u + t = 100$$

that is,

$$x - y = 400$$

$$x + z - t = 600$$

$$y + z + u = 300$$

$$t + u = 100$$

Solving this system with Scilab we have

```
-->Aug=[1 -1 0 0 0 400; 1 0 1 -1 0 600; 0 1 1 0 1 300; 0 0 0 1 1 100]
 Aug  =

    1.  - 1.    0.    0.    0.    400.
    1.    0.    1.  - 1.    0.    600.
    0.    1.    1.    0.    1.    300.
    0.    0.    0.    1.    1.    100.

-->rref(Aug)
 ans  =

    1.    0.    1.    0.    1.    700.
    0.    1.    1.    0.    1.    300.
    0.    0.    0.    1.    1.    100.
    0.    0.    0.    0.    0.    0.
```

Using Rouché-Fröbenius Theorem we obtain that the system has infinitely many solutions. The leading variables are x, y and t. The free variables are z and u. The parametric expression of the solution is

$$\begin{aligned} x &= 700 - \alpha - \beta \\ y &= 300 - \alpha - \beta \\ z &= \alpha \\ t &= 100 - \beta \\ u &= \beta \end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$. When $u = 50$ and $z = 150$ we have that:

$$\begin{aligned} x &= 700 - 150 - 50 = 500 \\ y &= 300 - 150 - 50 = 100 \\ z &= 150 \\ t &= 100 - 50 = 50 \\ u &= 50 \end{aligned}$$