Practice 2
Activities sheet

Activity 1. *Determine if the coefficient matrices of the following systems of linear equations are (or not) strictly diagonally dominant.*

$$a)\quad \left.\begin{array}{rcl} 10x+y+2z &=& 3 \\ 4x-6y-z &=& 9 \\ -2x+3y+8z &=& 51 \end{array}\right\} \quad b)\quad \left.\begin{array}{rcl} 2x+y+t &=& 1 \\ x+y+z+2t &=& 1 \\ 2x+y+3z+t &=& 1 \\ x+2y+z-3t &=& 2 \end{array}\right\}$$

SOLUTION:

(a) The coefficient matrix is strictly diagonally dominant because:

$$|10| > |1| + |2|$$

$$|-6| > |4| + |-1|$$

$$|8| > |-2| + |3|$$

(b) The coefficient matrix is not strictly diagonally dominant in this case because, for instance, the absolute value of the diagonal element of its first row is not <u>strictly</u> greater than the sum of the absolute values of the elements in that row:

$$|2| \not> |1| + |1|.$$

Activity 2. *a) Compute directly the solutions of the linear systems given in Activity 1.*

b) Apply Jacobi and Gauss-Seidel's methods to the same linear systems peforming only 6 *iterations and taking the zero vector as initial vector. Are these methods convergent?*

SOLUTION:

(a) We can use, for example, a combination of \ and **kernel**. We begin with the system of a):

```
-->A=[10 1 2; 4 -6 -1; -2 3 8];

-->b1=[3; 9; 51]
 b1  =


    3.
    9.
```

```
    51.

-->A=[10 1 2; 4 -6 -1; -2 3 8];

-->b1=[3; 9; 51];

-->x=A\b1
 x  =

  - 0.85
  - 3.3
    7.4

-->clean(A*x-b1)
 ans  =

    0.
    0.
    0.
```

This means that the system has solution and that the vector $(-0.85,\ -3.3,\ 7.4)$ is a particular solution. Now we compute the kernel of A to obtain the whole set of solutions:

```
-->kernel(A)
 ans  =

     []
```

This means that $(-0.85,\ -3.3,\ 7.4)$ is the unique solution.

Using the same procedure to solve b):

```
-->B=[1 1 0 1; 1 1 1 2; 2 1 3 1; 1 2 1 -3]; b2=[1; 1; 1; 2];

-->x=B\b2
 x  =

    0.
    1.
    0.
    0.

-->clean(B*x-b2)
 ans  =
```

```
    0.
    0.
    0.
    0.

-->kernel(B)
 ans  =

     []
```

Then we conclude that the vector $(0, 1, 0, 0)$ is the unique solution of the system.

(b) First, we apply Jacobi's method to a):

- Compute the decomposition $\mathsf{A} = \mathsf{L} + \mathsf{D} + \mathsf{U}$:

  ```
  -->D=diag([diag(A)])
   D  =

      10.    0.     0.
      0.   - 6.     0.
      0.     0.     8.

  -->L=tril(A)-D
   L  =

      0.     0.     0.
      4.     0.     0.
    - 2.     3.     0.

  -->U=triu(A)-D
   U  =

      0.     1.     2.
      0.     0.   - 1.
      0.     0.     0.
  ```

- Compute the inverse of D:

  ```
  -->F=inv(D);
  ```

- Compute 6 iterations using the formula $\vec{x}_{k+1} = \mathsf{D}^{-1}[\vec{b} - (\mathsf{L} + \mathsf{U})]\vec{x}_k$:

  ```
  -->x=[0; 0; 0];

  -->for i=1:6 x=F*(b1-(L+U)*x); end;
  ```

```
-->clean(x)
 ans  =

  - 0.8541810
  - 3.2981296
    7.383166
```

You see that the obtained vector is an approximation to the actual solution.

Now we are going to apply Gauss-Seidel method to a) using the formula $(\mathsf{L}+\mathsf{D})\vec{x}_{k+1} = \vec{b} - \mathsf{U}\vec{x}_k$ and solving this system by forward substitution:

```
-->x=[0; 0; 0];

-->for i=1:6 x=SustitucionProgresiva(L+D,b1-U*x); end;

-->clean(x)
 ans  =

  - 0.8500021
  - 3.3000002
    7.3999996
```

You can see that the obtained vector is also an approximation to the actual solution.

Now we are going to do the same with the system in b):

```
-->D=diag([diag(B)]); L=tril(B)-D; U=triu(B)-D;

-->F=inv(D);

-->x=[0; 0; 0; 0];

-->for i=1:6 x=F*(b2-(L+U)*x); end;

-->clean(x)
 ans  =

  - 1.6460905
  - 0.8765432
```

```
  - 0.6803841
    0.1865569
```

You can see that the obtained vector is far to be the solution. Doing more iterations you can convince youself that, in this case, Jacobi's method is not convergent:

```
-->x=[0; 0; 0; 0];

-->for i=1:100 x=F*(b2-(L+U)*x); end;

-->clean(x)
 ans  =

   1.0D+08 *

    9.1458357
    17.535724
  - 13.013743
    32.548355
```

Let's apply now the Gauss-Seidel Method:

```
-->x=[0; 0; 0; 0];

-->for i=1:6 x=SustitucionProgresiva(L+D,b2-U*x); end;

-->clean(x)
 ans  =

  - 0.0457586
    0.7979136
    0.1101515
  - 0.1132600
```

You can see that the obtained vector is not a good approximation of the actual solution. We need more iterations to "see" if, in this case, the Gauss-Seidel Method is convergent or not:

```
-->x=[0; 0; 0; 0];

-->for i=1:100 x=SustitucionProgresiva(L+D,b2-U*x); end;

-->clean(x)
 ans  =
```

```
0.
1.
0.
0.
```

And... it is convergent!

Activity 3. *Consider the linear system*

$$\begin{aligned} 0.2x + 2.2y + 4.5z &= 0.7 \\ 1.3x + 3.7y + 2.1z &= 1.2 \\ 4.2x + 3.1y + 0.4z &= 5.2 \end{aligned}$$

(a) Taking the zero vector as initial approximation, get 20 approximations applying Jacobi's method. Is this method convergent?

(b) Arrange the equations of this system in such a way that the associated matrix be strictly diagonally dominant. Check that, in this case, Jacobi's method is convergent and compute an approximation using 20 iterations.

SOLUTION:

(a)

```
-->D=diag([diag(A)]); L=tril(A)-D; U=triu(A)-D;

-->F=inv(D);

-->x=[0; 0; 0];

-->for i=1:20 x=F*(b-(L+U)*x); end;

-->clean(x)
 ans  =

    1.0D+22 *

  - 98.673183
  - 3.3001795
  - 27.427204
```

You can see that the method is not convergent (try with more iterations to convice yourself, if you need it).

(b) Observe that the coefficient matrix of this system is not strictly diagonally dominant. However, we can get an equivalent system with strictly diagonally dominant coefficient matrix swapping the equations 1 and 3:

$$\begin{aligned} 4.2x + 3.1y + 0.4z &= 5.2 \\ 1.3x + 3.7y + 2.1z &= 1.2 \\ 0.2x + 2.2y + 4.5z &= 0.7 \end{aligned}$$

Applying Jacobi's Method with the new matrix we obtain the following:

```
-->B=[A(3,:); A(2,:); A(1,:)]
 B  =

    4.2    3.1    0.4
    1.3    3.7    2.1
    0.2    2.2    4.5

-->b1=[b(3); b(2); b(1)]
 b1  =

    5.2
    1.2
    0.7

-->D=diag([diag(B)]); L=tril(B)-D; U=triu(B)-D;

-->F=inv(D);

-->for i=1:20 x=F*(b1-(L+U)*x); end;

-->clean(x)
 ans  =

    1.4596445
  - 0.3318754
    0.2534168

-->clean(B*x-b1)
 ans  =

    0.0030599
    0.0017741
    0.0021785
```

You can see that that the obtained vector is not so far to be a solution. We can try with more iterations to convince ourselves:

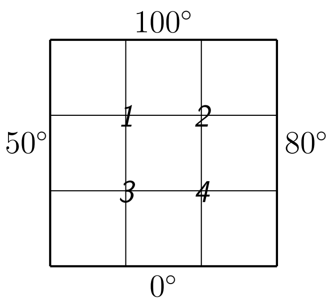
```
-->x=[0; 0; 0];

-->for i=1:100 x=F*(b1-(L+U)*x); end;

-->clean(x)
 ans  =

    1.4589329
  - 0.3318375
    0.2529458

-->clean(B*x-b1)
 ans  =

    0.
    0.
    0.
```

Then we see that, this time, the method is convergent (as it should be, taking into account Theorem 1).

Activity 4. *A square metal plate has a constant temperature in each of its four edges. To approximate the temperature of the points inside, a 3×3 grid with 4 inner points is superimposed, as shown in figure below. It is assumed that the temperature at each interior point of the grid is the average of the temperatures of the 4 nearest neighbor points of the grid. For example, if T_i denotes the temperature at the point i, one has that:*

$$T_1 = \frac{1}{4}(50 + 100 + T_2 + T_3).$$

Give an approximation of the temperature distribution at the four inner grid points after 11 iterations using the Jacobi's method and the Gauss-Seidel's method (taking the zero vector as initial approximation).



SOLUTION:

In addition to the equation given in the statement, we can deduce the following equations:

$$T_2 = \frac{1}{4}(T_1 + 100 + 80 + T_4)$$

$$T_3 = \frac{1}{4}(T_1 + 50 + 0 + T_4)$$

$$T_4 = \frac{1}{4}(T_3 + T_2 + 80 + 0)$$

Multiplying by 4 and reordering we get the following linear system:

$$\begin{aligned}
4T_1 - T_2 - T_3 &= 150 \\
-T_1 + 4T_2 - T_4 &= 180 \\
-T_1 + 4T_3 - T_4 &= 50 \\
-T_2 - T_3 + 4T_4 &= 80
\end{aligned}$$

Now we apply Jacobi and Gauss-Seidel's Methods with the help of Scilab. Notice that this methods are convergent because the coefficient matrix is strictly diagonally dominant.

```
-->D=diag([diag(A)]); L=tril(A)-D; U=triu(A)-D;

-->x=[0; 0; 0; 0];

-->F=inv(D);

-->x=[0; 0; 0; 0];

-->for i=1:11 x=F*(b-(L+U)*x); end;

-->clean(x)
 ans  =

    66.221924
    73.721924
    41.221924
    48.721924

-->x=[0; 0; 0; 0];

-->for i=1:11 x=SustitucionProgresiva(L+D,b-U*x); end;
```

```
-->x
 x  =

    66.249963
    73.749982
    41.249982
    48.749991
```

Then

$$T_1 \approx 66.249963$$
$$T_2 \approx 73.749982$$
$$T_3 \approx 41.249982$$
$$T_4 \approx 48.749991$$