Lesson 2

Elementary matrices and invertible matrices

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I Elementary matrices

Definition I.1. We call $n \times n$ *elementary matrices* to those that result from applying a row elementary operation to the $n \times n$ identity matrix. There are three types or elementary matrices, according with the type of applied elementary row operation:

(1) Type 1: $I \overset{\rho_i \leftrightarrow \rho_j}{\longrightarrow} E_{i,j}$ for $i \neq j$

(2) Type 2: $I \overset{k\rho_i}{\longrightarrow} E_i(k)$ for $k \neq 0$

(3) Type 3: $I \overset{\rho_i + k\rho_j}{\longrightarrow} E_{i,j}(k)$ for $i \neq j$

Example I.2.

$$E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ E_2(1/5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E_{32}(-2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

The following lemma, whose proof is very easy, will be **extremely important** for us.

Lemma I.3. The **left** multiplication of a matrix A by an elementary matrix has **the same effect** that the corresponding elementary operation applied to A. That is:

(1) If $H \overset{\rho_i \leftrightarrow \rho_j}{\longrightarrow} G$ then $E_{i,j}H = G$.

(2) If $H \overset{k\rho_i}{\longrightarrow} G$ then $E_i(k)H = G$.

(3) If $H \overset{\rho_i + k\rho_j}{\longrightarrow} G$ then $E_{i,j}(k)H = G$.

As a consequence of this lemma, we can interpret the Gauss' and Gauss-Jordan Methods by means of **left** multiplication by elementary matrices.

Example I.4. Let us show the interpretation of the process to compute an equivalent echelon form of the following matrix

$$\begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & 0 & 3 \end{bmatrix}$$

in terms of matrix multiplication.

Swap the first and third rows,

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{E_{1,3}} \begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

triple the first row,

$$\underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1(3)} \begin{bmatrix} 1/3 & 2 & 0 & 3 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

and then add -1 times the first row to the second,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{2,1}(-1)} \begin{bmatrix} 1 & 6 & 0 & 9 \\ 1 & 5 & -2 & 2 \\ 0 & 0 & 3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{bmatrix}.$$

The last obtained matrix is a row echelon form of the initial matrix. Summarizing, we have seen that:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{2,1}(-1)} \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_1(3)} \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{E_{1,3}} \begin{bmatrix} 0 & 0 & 3 & 9 \\ 1 & 5 & -2 & 2 \\ 1/3 & 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 0 & 9 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & 3 & 9 \end{bmatrix}.$$

As a consequence of Lemma I.3 we have

Theorem I.5. For any matrix H and any matrix G that is row equivalent to H, there are elementary matrices $E_1, \ldots, E_r$ such that $E_r \cdot E_{r-1} \cdots E_1 \cdot H = G$

UTILITARIAN SUMMARY I.6.

- The *elementary matrices* are those that result from applying a row elementary operation to an identity matrix. There are three types or elementary matrices, according with the type of applied elementary row operation.
- **Applying a elementary row operation to a matrix is equivalent to multiplying the matrix (on the left) by the corresponding elementary matrix**.

II Inverse of a square matrix

II.1 Definition and unicity of the inverse matrix

Definition II.1. An $n \times n$ matrix A is *invertible* or *non-singular* if there exists an $n \times n$ matrix B such that $AB = BA = I_{n\times n}$. Otherwise we will say that A is *non-invertible* or *singular*.

Example II.2. The matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is invertible because

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$$

Let us see now an example of a non-invertible matrix:

Example II.3. If the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ were invertible then

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for some matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then:

$$\begin{bmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

that is

$$\left.\begin{array}{l} a + 2c = 1 \\ 2a + 4c = 0 \\ b + 2d = 0 \\ 2b + 4d = 1 \end{array}\right\}.$$

This is a system without solution. Therefore, A is not invertible.

Proposition II.4. *If A is an $n \times n$ invertible matrix then there exists a* **unique** *$n \times n$ matrix B such that $AB = BA = I_{n\times n}$.*

PROOF:

Suppose that B and C are two matrices such that: $AB = BA = I_{n\times n}$ and $AC = CA = I_{n\times n}$. Then

$$B = BI_{n\times n} = B(AC) = (BA)C = I_{n\times n}C = C.$$

QED

This proposition allows us to give the following definition:

Definition II.5. If A is an invertible matrix, the unique matrix B such that $AB = BA = I$ is called *inverse* of A and it is denoted by A^{-1}.

II.2 Further properties of the inverse

Theorem II.6. *Let A, B be two $n \times n$ matrices.*

(a) If A is invertible then A^{-1} is also invertible and $(A^{-1})^{-1} = A$.

(b) If A and B are invertible then AB are also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(c) The above property can be extended to the product of k invertible matrices of the same order: $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}$.

(d) If A is invertible and k is a positive integer then A^k is also invertible and $(A^k)^{-1} = (A^{-1})^k$.

(e) If A is invertible and α is a non-zero scalar then αA is also invertible and $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$.

(f) If A is invertible then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

PROOF:

(a) Evident

(b) Applying the associative property of matrix multiplication and the fact that the identity matrix is the unit of the product of square matrices, we have that

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_{n\times n}A^{-1} = AA^{-1} = I_{n\times n},$$
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_{n\times n}B = B^{-1}B = I_{n\times n}.$$

(c) We will prove the result by induction on the number of factors k. The case $k = 2$ is Part (b). Assume, then, that the result is true for a certain k, that is, $(A_1A_2\cdots A_k)^{-1} = A_k^{-1}\cdots A_2^{-1}A_1^{-1}$. Let us show that it is also true for $k+1$:

$$(\underbrace{A_1A_2\cdots A_k}_{B}A_{k+1})^{-1} = (BA_{k+1})^{-1}$$

Since the result is true for a product of two matrices, this is equal to

$$= A_{k+1}^{-1}B^{-1},$$

using the induction hypotheses this is equal to

$$= A_{k+1}^{-1}(A_k^{-1}\cdots A_2^{-1}A_1^{-1}) = A_{k+1}^{-1}A_k^{-1}\cdots A_2^{-1}A_1^{-1}.$$

(d) It is a direct consequence of (c).

(e)

$$(\alpha A)(\frac{1}{\alpha}A^{-1}) = \alpha\frac{1}{\alpha}AA^{-1} = 1\cdot I_{n\times n} = I_{n\times n}$$

$$(\frac{1}{\alpha}A^{-1})(\alpha A) = \frac{1}{\alpha}\alpha A^{-1}A = 1\cdot I_{n\times n} = I_{n\times n}$$

(f) Using properties of the transpose:

$$A^t(A^{-1})^t = (A^{-1}A)^t = I_{n\times n}^t = I_{n\times n},$$
$$(A^{-1})^tA^t = (AA^{-1})^t = I_{n\times n}^t = I_{n\times n}.$$

QED

II.3 Inverses of the elementary matrices

Proposition II.7. *The elementary matrices are invertible and their inverses are:*

$$E_{i,j}^{-1} = E_{i,j}$$
$$E_i(k)^{-1} = E_i(1/k)$$
$$E_{i,j}(k)^{-1} = E_{i,j}(-k)$$

PROOF:

The matrix $E_{i,j}$ is the identity matrix with the rows i an j interchanged. By Lemma I.3, if we multiply (on the left) $E_{i,j}$ by $E_{j,i}$ we interchange again the rows i and j. Therefore $E_{j,i}E_{i,j} = I$. The same reasoning shows that $E_{i,j}E_{j,i} = I$. Therefore $E_{j,i}$ is the inverse of $E_{i,j}$.

The proof of the remaining equalities is similar. QED

II.4 How to compute inverses?

Let us explain how to compute inverse matrices with an example. Consider the square matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. We want to determine if A has (or not) an inverse. Denote by

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix}$$

to the "candidate" to be an inverse of A. If X were, actually, the inverse of A, X should satisfy the following relation:

$$AX = I_{3\times 3}, \tag{1}$$

that is,

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

Multiplying on the left we have

$$\begin{bmatrix} \underbrace{\begin{matrix} 1\cdot x_{1,1} + 2\cdot x_{2,1} - 1\cdot x_{3,1} \\ 3\cdot x_{1,1} + 4\cdot x_{2,1} + 0\cdot x_{3,1} \\ 0\cdot x_{1,1} - 2\cdot x_{2,1} + 1\cdot x_{3,1} \end{matrix}}_{\overbrace{A\cdot \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix}}^{\Downarrow}} & \underbrace{\begin{matrix} 1\cdot x_{1,2} + 2\cdot x_{2,2} - 1\cdot x_{3,2} \\ 3\cdot x_{1,2} + 4\cdot x_{2,2} + 0\cdot x_{3,2} \\ 0\cdot x_{1,2} - 2\cdot x_{2,2} + 1\cdot x_{3,2} \end{matrix}}_{\overbrace{A\cdot \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ x_{3,2} \end{bmatrix}}^{\Downarrow}} & \underbrace{\begin{matrix} 1\cdot x_{1,3} + 2\cdot x_{2,3} - 1\cdot x_{3,3} \\ 3\cdot x_{1,3} + 4\cdot x_{2,3} + 0\cdot x_{3,3} \\ 0\cdot x_{1,3} - 2\cdot x_{2,3} + 1\cdot x_{3,3} \end{matrix}}_{\overbrace{A\cdot \begin{bmatrix} x_{1,3} \\ x_{2,3} \\ x_{3,3} \end{bmatrix}}^{\Downarrow}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This means that Condition (2) is satisfied if and only if these three conditions are satisfied (simultaneously!):

$$A \cdot \begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A \cdot \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ x_{3,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A \cdot \begin{bmatrix} x_{1,3} \\ x_{2,3} \\ x_{3,3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3}$$

These are three linear systems. Solving them we will compute the three columns of X. But notice that **the three linear systems have the same coefficient matrix**. This means that **we can solve them simultaneously**, according with the method described in Lesson 1:

1. Write the matrix $[A|I_{3\times3}]$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \tag{4}$$

2. Apply Gauss-Jordan Method to the left hand side matrix, but performing the same elementary operations at the right hand side matrix:

$$\cdots \to \underbrace{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -3/4 & 1/4 & -3/4 \\ 0 & 0 & 1 & -3/2 & 1/2 & 1/2 \end{array}\right]}_{(*)}.$$

This gives, automatically, the solutions of the above three linear systems:

$$\begin{bmatrix} x_{1,1} \\ x_{2,1} \\ x_{3,1} \end{bmatrix} = \begin{bmatrix} 1 \\ -3/4 \\ -3/2 \end{bmatrix}, \quad \begin{bmatrix} x_{1,2} \\ x_{2,2} \\ x_{3,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/4 \\ 1/2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} x_{1,3} \\ x_{2,3} \\ x_{3,3} \end{bmatrix} = \begin{bmatrix} 1 \\ -3/4 \\ 1/2 \end{bmatrix}.$$

Then, the matrix X that satisfies Equation (1) is

$$X = \begin{bmatrix} 1 & 0 & 1 \\ -3/4 & 1/4 & -3/4 \\ -3/2 & 1/2 & 1/2 \end{bmatrix}.$$

Observe that this is the right hand side matrix of (*). Therefore, once we have applied Gauss-Jordan Method to (4) we have, on the left, the RREF of A (that coincides with the identity matrix $I_{3\times3}$) and, on the right, the matrix X.

Notice that, **if A were invertible, its inverse should be, necessarily, the matrix X**. But, to be sure that X is the inverse of A, we must check also the equality $XA = I_{3\times3}$. Certainly, it is satisfied (you can check it). But... why? What is the reason? Let's see the actual reason of this:

In the above process we have computed the RREF of the matrix A, but performing, at the same time, the same elementary operations to the matrix $I_{3\times3}$. Let's think about this process (applied to A) but interpreting it in terms of products (on the left) by elementary matrices (as explained in Section I). What we have is

$$\underbrace{E_{13}(-2)E_{23}(3/2)E_{12}(-2)E_3(-1/2)E_2(-1/2)E_{32}(-1)E_{21}(-3)}_{Y} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{RREF of } A}$$

Notice that Y satisfies the condition $YA = I_{3\times 3}$.

The key point is the following: the matrix Y (that is, the product of all these elementary matrices) is formed applying, successively, the same elementary row operations that we have applied to A. Then, this matrix Y is formed exactly in the same way that the right hand side matrix of (*). That is, Y **coincides with** X**!** And therefore X satisfies also the condition $XA = I_{3\times 3}$. We can conclude, then, that

$$X = A^{-1}.$$

Observe that **we can apply the above procedure with any square matrix** A. Moreover we can deduce the following important conclusions:

1) In the example, the RREF of A is the identity matrix $I_{3\times 3}$. This is equivalent to say there are 3 pivots or, equivalently, that the rank of A is 3 (the maximum possible). Applying Rouché-Fröbenius Theorem, this implies that the three linear systems given in (3) have (any of them) a unique solution and, then, all works properly.

However, imagine for a moment that the rank of A is not 3. This implies that the systems (3) have not a unique solution (in fact, it is not difficult to see that at least one of these systems must have no solution). And this means that A has no inverse.

We can generalize this reasoning to any square matrix A obtaining the following important result:

Theorem II.8. *An $n \times n$ matrix A is invertible if and only if* $\operatorname{rank}(A) = n$.

2) We can deduce, then, the following strategy to compute inverses:

Strategy II.9. Let A be an $n \times n$ matrix.

(1) Write the matrix $[A \mid I_{n\times n}]$.

(2) Perform to A elementary row operations to compute **a row echelon form** of A, applying the elementary row operations to the whole matrix $[A \mid I_{n\times n}]$. At this stage, we can compute the rank of A. If $\operatorname{rank}(A) < n$ then conclude that A is not invertible and stop. Otherwise conclude that A is invertible and continue to step (3).

(3) Continue the process until applying the complete Gauss-Jordan Method to A. At the end of the process you will have $[I_{n\times n} \mid A^{-1}]$.

3) Notice that, in our example, if we focus our attention on the interpretation in terms of elementary matrices, we have that

$$A^{-1} = Y = E_{13}(-2)E_{23}(3/2)E_{12}(-2)E_3(-1/2)E_2(-1/2)E_{32}(-1)E_{21}(-3),$$

that is, A^{-1} **can be written as a product of elementary matrices**. But, using this and taking into account some properties of the inverse and the inverses of the elementary matrices, one has that A **can also be written as a product of elementary matrices**:

$$\begin{aligned} A = (A^{-1})^{-1} &= (E_{13}(-2)E_{23}(3/2)E_{12}(-2)E_3(-1/2)E_2(-1/2)E_{32}(-1)E_{21}(-3))^{-1} \\ &= E_{21}(-3)^{-1}E_{32}(-1)^{-1}E_2(-1/2)^{-1}E_3(-1/2)^{-1}E_{12}(-2)^{-1}E_{23}(3/2)^{-1}E_{13}(-2)^{-1} \\ &= E_{21}(3)E_{32}(1)E_2(-2)E_3(-2)E_{12}(2)E_{23}(-3/2)E_{13}(2) \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that this holds, not only in this particular example, but in general. The unique requirement is that the matrix A be invertible. Then we can conclude that **if a matrix A is invertible then A (and A^{-1}) can be written as a product of elementary matrices**.

But the converse fact is also true: **if a matrix A is a product of elementary matrices then it is invertible**. It is obvious because the product of invertible matrices is invertible, and we have seen that the elementary matrices are invertible. Therefore we have the following theorem:

Theorem II.10. *A square matrix A is invertible if and only if it is a product of elementary matrices.*

There is one more characterization of invertible matrices that we would like to show. It is related with linear systems:

Theorem II.11. *An $n \times n$ matrix is invertible if and only if* any *linear system $A\vec{x} = \vec{b}$ has a unique solution.*

PROOF:

By the Rouché-Fröbenius Theorem, *any* linear system $A\vec{x} = \vec{b}$ has a unique solution $\Leftrightarrow$ $\text{rank}(A) = n \Leftrightarrow A$ is invertible. The last equivalence follows by Theorem II.8. QED

Remark II.12. If A is an invertible matrix, a possibility to compute the unique solution of a linear system $A\vec{x} = \vec{b}$ is the following one: Since A is invertible, it has an inverse A^{-1}. Multiplying (on the left) by A^{-1} both sides of the linear system we have that:

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}.$$

Since the matrix multiplication is associative, we have that

$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}.$$

But $A^{-1}A = I_{n\times n}$. Then

$$I_{n\times n}\vec{x} = A^{-1}\vec{b},$$

that is,

$$\vec{x} = A^{-1}\vec{b}.$$

And we are done: the **unique** solution is the vector $A^{-1}\vec{b}$.

A specially interesting property is the following:

Proposition II.13. *Let A and B be two $n \times n$ matrices. If AB is an invertible matrix then A and B are invertible.*

PROOF:

Assume that AB is invertible. Let us first show that A is invertible. Reasoning by contradiction, suppose that A is not invertible. Then, by Theorem II.8, $\text{rank}(A) < n$. This means that the RREF of A, which we call R, has, at least, one zero row. By Corollary I.5, there are elementary matrices $E_1, \ldots, E_r$ such that

$$E_r \cdot E_{r-1} \cdots E_1 \cdot A = R.$$

Then, on the one hand, the matrix $E_r \cdot E_{r-1} \cdots E_1(AB)$ is equal to RB and, since R has a zero row, RB has also a zero row. Therefore $\operatorname{rank}(E_r \cdot E_{r-1} \cdots E_1(AB)) < n$.

But on the other hand, the matrix $E_r \cdot E_{r-1} \cdots E_1(AB)$ is row equivalent to AB and, therefore, it has the same rank than AB. This is a contradiction with Theorem II.8 because AB is invertible.

So, we conclude that A is invertible. Let us see now that B is also invertible. But, taking into account the associativity of matrix multiplication:

$$B = IB = (A^{-1}A)B = A^{-1}(AB).$$

Then B is the product of A^{-1} and AB, which are two invertible matrices. Then, by Clause (b) of Theorem II.6, we conclude that B is invertible. QED

The following corollary may be very useful to solve certain exercises:

Corollary II.14. *Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix B such that either $BA = I$ or $AB = I$ then A is invertible and $B = A^{-1}$.*

PROOF:

Suppose, first, that $BA = I$. Since the identity matrix I is obviously invertible, by Proposition II.13 we have that A is invertible; so it has an inverse A^{-1}. Then, multiplying at both sides (on the right) by A^{-1}:

$$(BA)A^{-1} = IA^{-1} \quad \Rightarrow \quad B(AA^{-1}) = A^{-1} \quad \Rightarrow BI = A^{-1} \quad \Rightarrow \quad B = A^{-1}.$$

If we suppose that $AB = I$ the proof is analogous (but multiplying by A^{-1} on the left instead of on the right). QED

UTILITARIAN SUMMARY II.15.

- An invertible matrix has a unique inverse A^{-1}.
- Be careful: The inverse of the product of several matrices is the product of inverses **but with the order of the factors reversed** (in Lesson 1 we saw a similar property for the transpose).
- The elementary matrices are invertible and their inverses are also elementary matrices of the same type.
- To compute the inverse of an invertible matrix A perform Gauss-Jordan Method to the matrix $[A \mid I]$ until obtaining $[I \mid A^{-1}]$.
- Let A be an $n \times n$ matrix. The following assertions are equivalent:
  (1) A is invertible.
  (2) $\operatorname{rank}(A) = n$.

(3) A can be written as a product of elementary matrices.

(4) Any linear system $A \cdot \vec{x} = \vec{b}$ has a unique solution.

- Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix B such that either $BA = I$ or $AB = I$ then A is invertible and $B = A^{-1}$. This means that, to check that B is the inverse of A, one does not need to check both equalities: $AB = I$ and $BA = I$; it is enough to check only one of them.