AMA - Formulas

ABSOLUTE VALUE $\mathbb{R}$ (L1)

$$|x| \leq a \Leftrightarrow -a \leq x \leq a \ \ ; \ \ |x| \geq b \ \Leftrightarrow (x \geq b \text{ o } x \leq -b) \ \ ; \ \ |x+y| \leq |x| + |y|$$

REAL VALUED FUNCTIONS (L2)

Even function: $f(-x) = f(x)$, Odd function: $f(-x) = -f(x)$
Exponential function: $a^x > 0$, $a^0 = 1$, $a^x \cdot a^y = a^{x+y}$, $a^x / a^y = a^{x-y}$, $(a^x)^y = a^{xy}$
Logarithmic function:

$$\begin{aligned} \log_a(1) &= 0, \quad \log(e) = 1, \quad \log_a(x \cdot y) = \log_a(x) + \log_a(y), \log_a(x/y) = \log_a(x) - \log_a(y), \\ \log_a(x^y) &= y \log_a(x), \quad \log_a(x) = \frac{\log_b(x)}{\log_b(a)} \end{aligned}$$

Trigonometric function:

$$\cos^2(x) + \sin^2(x) = 1; \quad |\cos(x)| \leq 1; \quad |\sin(x)| \leq 1$$

Properties of derivatives:

$$\begin{aligned} (f \pm g)'(x) &= f'(x) \pm g'(x), \quad (\alpha f)'(x) = \alpha f'(x), \\ (f \cdot g)'(x) &= f'(x)g(x) + f(x)g'(x), \quad \left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \end{aligned}$$

Chain rule: $(f \circ g)'(x) = f'(g(x))g'(x)$
Derivatives of elemental functions:

$$\begin{array}{ll} (x^n)' = nx^{n-1} & (u(x)^n)' = nu(x)^{n-1}(u(x))' \\ (\sqrt{x})' = \frac{1}{2\sqrt{x}} & \left(\sqrt{u(x)}\right)' = \frac{1}{2\sqrt{u(x)}}(u(x))' \\ \log_a'(x) = \frac{1}{x\log(a)} & \log_a'(u(x)) = \frac{1}{u(x)'\log(a)}(u(x))' \\ (a^x)' = a^x \log(a) & \left(a^{u(x)}\right)' = a^{u(x)}\log(a)\,(u(x))' \\ \sin'(x) = \cos(x) & \sin'(u(x)) = \cos(u(x))\,(u(x))' \\ \cos'(x) = -\sin(x) & \cos'(u(x)) = -\sin(u(x))\,(u(x))' \\ \arctan'(x) = \frac{1}{1+x^2} & \arctan'(u(x)) = \frac{1}{1+u(x)^2}(u(x))' \\ \arcsin'(x) = \frac{1}{\sqrt{1-x^2}} & \arcsin'(u(x)) = \frac{1}{\sqrt{1-u(x)^2}}(u(x))' \end{array}$$

Please note: If $a = e$ then $\log(a) = \log(e) = 1$

Increasing/Decreasing: $f'(x) > / < 0 \Rightarrow f$ strictly increasing/decreasing

Local maximum and minimum: $f'(x) = 0$ and $f''(x) > / < 0 \Rightarrow f$ has a local minimum or maximum at x

Concavity up/down: $f''(x) > / < 0 \Rightarrow f$ concave up/down

Points of inflection: f has a point of inflection in $x \Rightarrow f''(x) = 0$

RIEMANN's INTEGRATION (L3)

f monotonic in $[a,b] \Rightarrow f$ integrable in $[a,b]$, f continuous in $[a,b] \Rightarrow f$ integrable in $[a,b]$

f, g integrable in $[a,b] \Rightarrow \alpha f + \beta g$ and $f \cdot g$ integrable in $[a,b]$ but $\int_a^b f \cdot g \neq \left(\int_a^b f\right)\left(\int_a^b g\right)$

f, g integrable in $[a,b]$ and $f \leq g \Rightarrow \int_a^b f \leq \int_a^b g$; $\left|\int_a^b f\right| \leq \int_a^b |f|$

Area (plane) of the figure limited by $y = f(x)$ and OX, between $x = a$ and $x = b$: $A = \int_a^b |f|$

Barrow's : f integrable in $[a,b]$ and $h' = f$ in $[a,b]$ $\Rightarrow \int_a^b f = h(b) - h(a)$

Integration by parts : $\int_a^b f \cdot g' = [f \cdot g]_a^b - \int_a^b f' \cdot g = (f(b)g(b) - f(a)g(a)) - \int_a^b f' \cdot g$

Integration by changing variable: $\int_{a=g(c)}^{b=g(d)} f = \int_c^d (f \circ g) g'$; $x = g(t)$ change

Immediate Integrals:

$\int k dx = kx + c$

$\int x^p dx = \frac{x^{p+1}}{p+1} + c,\ (p \neq -1)$ $\quad\quad$ $\int u^p(x) u'(x) dx = \frac{u^{p+1}(x)}{p+1} + c,\ (p \neq -1)$

$\int \frac{dx}{x} = \ln|x| + c$ $\quad\quad$ $\int \frac{u'(x)}{u(x)} dx = \ln|u(x)| + c$

$\int e^x dx = e^x + c$ $\quad\quad$ $\int e^{u(x)} u'(x) dx = e^{u(x)} + c$

$\int a^x dx = \frac{a^x}{\ln(a)} + c,\ (a > 0,\ a \neq 1)$ $\quad\quad$ $\int a^{u(x)} u'(x) dx = \frac{a^{u(x)}}{\ln(a)} + c,\ (a > 0,\ a \neq 1)$

$\int \cos(x) dx = \sin(x) + c$ $\quad\quad$ $\int \cos(u(x)) u'(x) dx = \sin(u(x)) + c$

$\int \sin(x) dx = -\cos(x) + c$ $\quad\quad$ $\int \sin(u(x)) u'(x) dx = -\cos(u(x)) + c$

$\int \frac{dx}{\cos^2 x} = \int \left(1 + \tan^2 x\right) dx = \tan(x) + c$ $\quad\quad$ $\int \frac{u'(x)}{\cos^2 u(x)} dx = \tan(u(x)) + c$

$\int \frac{dx}{\sin^2 x} = \int \left(1 + \cot^2 x\right) dx = -\cot(x) + c$ $\quad\quad$ $\int \frac{u'(x)}{\sin^2 u(x)} dx = -\cot((u(x)) + c$

$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + c$ $\quad\quad$ $\int \frac{u'(x)}{\sqrt{1-u^2(x)}} dx = \arcsin(u(x)) + c$

$\int \frac{dx}{1+x^2} = \arctan(x) + c$ $\quad\quad$ $\int \frac{u'(x)}{1+u^2(x)} dx = \arctan(u(x)) + c$

APPROXIMATED INTEGRATION (L3)

Trapezoidal rule: $T_n f = \frac{h}{2}\left(f(a) + 2\sum_{k=1}^{n-1} f(a+kh) + f(b)\right)$; $h = \frac{b-a}{n}$

$$\text{Error: } E_n = \left|\int_a^b f - T_n f\right| \leq \frac{nh^3}{12} M_2 = \frac{(b-a)^3}{12n^2} M_2 \ ;\ M_2 \geq \max_{[a,b]} |f''|$$

Simpson's formula : $S_n f = \frac{h}{3}\left(f(a) + 4\sum_{k=0}^{n/2-1} f(a+(2k+1)h) + 2\sum_{k=1}^{n/2-1} f(a+2kh) + f(b)\right)$
(n even)

$$\text{Error: } E_n = \left|\int_a^b f - S_n f\right| \leq \frac{nh^5}{180} M_4 = \frac{(b-a)^5}{180n^4} M_4 \ ;\ M_4 \geq \max_{[a,b]} \left|f^{(iv)}\right|$$

SEQUENCES (L4)

$a>1 \Rightarrow \lim(a^n)=+\infty \quad ; \quad |a|<1 \Rightarrow \lim(a^n)=0$

Euler's formula: $(a_n)\to 1,\ (b_n)\to \pm\infty \ \Rightarrow \lim a_n^{b_n} = e^{\lim\{b_n(a_n-1)\}}$

Stolz (quotient): (b_n) increasing, $(b_n)\to+\infty \Rightarrow \lim \frac{a_n}{b_n} = \lim \frac{(a_{n+1}-a_n)}{(b_{n+1}-b_n)}$

Magnitude order (a_n and b_n positive and divergent to $+\infty$):

$$\lim \frac{a_n}{b_n} = \begin{cases} l\in\mathbb{R}^+ \Longrightarrow a_n \in \Theta(b_n) a_n \approx b_n \text{ (same order)} \\ 0 \Longrightarrow a_n \in O(b_n) a_n \ll b_n \\ +\infty \Longrightarrow a_n \in \Omega(b_n) a_n \gg b_n \end{cases}$$

Lineal recurrences (2^o order and constant coefficients; this method can be applied to first order):

Homogenous case: $a_{n+2}+p\cdot a_{n+1}+q\cdot a_n=0$; characteristic eq.: $r^2+pr+q \overset{r_1,\,r_2}{=} 0$

$r_1\neq r_2\in\mathbb{R} \quad\Rightarrow\quad a_n^h = c_1 r_1^n + c_2 r_2^n \ ;\ c_1,c_2\in\mathbb{R}$

$r_1=r_2=r\in\mathbb{R} \quad\Rightarrow\quad a_n^h = c_1 r^n + c_2 n\cdot r^n \ ;\ c_1,c_2\in\mathbb{R}$

$r_1=\rho_\alpha\ ,\ r_2=\rho_{-\alpha}\in\mathbb{C} \quad\Rightarrow\quad a_n^h=\rho^n(c_1\cos(n\alpha)+c_2\sin(n\alpha)) \ ;\ c_1,c_2\in\mathbb{R}$

No homogenous case: $a_{n+2}+p\cdot a_{n+1}+q\cdot a_n=t_n$; $(t_n=P(n),\ k^n,\ P(n)k^n)$

$a_n^c = a_n^h + u_n$; with u_n particular solution *similar* to t_n (indeterminate coefficients)

CONVERGENCE OF NUMERICAL SERIES (L5)

Remainder criterium: $\sum a_n$ convergent $\Rightarrow (a_n)\to 0$

$\sum \frac{1}{n^\alpha}$ (generalized harmonic) convergent $\Leftrightarrow \alpha>1$

Leibniz's criterium: $\sum(-1)^{n+1}a_n\ ,\ a_n>0$; (a_n) decreasing and $(a_n)\to 0 \Rightarrow \sum(-1)^{n+1}a_n$ convergent

SUM OF NUMERICAL SERIES (L5)

$$\text{Exact sums:}\begin{cases} \text{Geometric: } \sum_{n=p}^\infty r^n = \frac{r^p}{1-r}, \\ \text{Arithmetic-geometric : } \sum_{n=p}^\infty n\cdot r^n = \frac{p\cdot r^p}{1-r}+\frac{r^{p+1}}{(1-r)^2},\ \text{if } |r|<1 \end{cases}$$

Number e: $\sum\limits_{n=0}^{\infty} \frac{1}{n!} = e$

App. sums:
$\sum_{n=1}^\infty (-1)^{n+1}a_n = s$ (is convergent with Leibniz's criterium) $\Rightarrow E_n=|s-s_n|\le a_{n+1}$
$\sum_{n=1}^\infty A_n = s$, con $|A_n|\le cK^n \ \Rightarrow\ E_n=|s-s_n|\le \frac{cK^{n+1}}{1-K}$

POWER SERIES (L6)

$$\text{McLaurin's formula:}\quad f(x) = \overbrace{f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n}^{P_n f(x)} + \overbrace{\frac{f^{(n+1)}(\alpha x)}{(n+1)!}x^{n+1}}^{R_n f(x)}$$

Power series $\sum\limits_{n\ge 0} a_n x^n$ are convergent at the interval $I=\,]-\rho,\rho[\ ,\ \rho\in[0,+\infty]$

$$f(x)=\sum_{n\ge 0} a_n x^n\ ,\ |x|<\rho \Rightarrow \begin{cases} f'(x)=\sum_{n\ge 1} n a_n x^{n-1}\ ,\ |x|<\rho\ .\ \text{Using derivatives: } a_n=\frac{f^{(n)}(0)}{n!} \\ \int f = \sum_{n\ge 0}\frac{a_n}{n+1}x^{n+1}+C\ ,\ |x|<\rho \end{cases}$$

$$\frac{1}{1-x}=\sum_{n=0}^\infty x^n\ ,\ |x|<1 \quad ; \quad e^x=\sum_{n=0}^\infty \frac{x^n}{n!}\ ,\ x\in\mathbb{R} \quad ; \quad \sin(x)=\sum_{n=0}^\infty (-1)^n\frac{x^{2n+1}}{(2n+1)!}\ ,\ x\in\mathbb{R}$$