Mathematical Analysis

Numerical sequences

Sequences of real numbers

Sequences of real numbers

Everyone knows how to add two numbers together…but

How do you add infinitely many numbers?

Introduction

Definition: A **sequence** is a function whose domain D is a subset of N ={0, 1, 2, …} and range is a subset of real number. Usually:

$$f:\mathbb{N}\to\mathbb{R}\ ;\ f(n)=a_n$$

$$\{a_n\}_{n=1}^{+\infty}\ ,\ \{a_n\}_{n\geq 1}\ ,\ \{a_n\}$$

- $\boldsymbol{a_n}$, is called a ***term*** of the sequence and denotes the image of the integer *n*.
- $\boldsymbol{\{a_n\}}$ denotes the sequence.
  - Do not confuse the above notation with set notation. This is a “misuse” of the set notation.

Definitions

Explicit form: $a_n = \varphi(n)$

$$\left\{\frac{1}{n+1}\right\}_{n\geq 1} , \ \{n!\}_{n\geq 0} , \ \left\{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right\}_{n\geq 1}$$

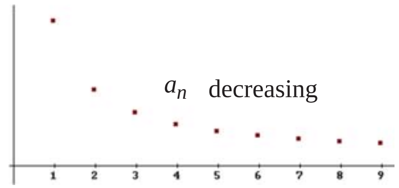
Recurrent form: $a_n = \varphi(a_1, a_2, ..., a_{n-1}, n)$

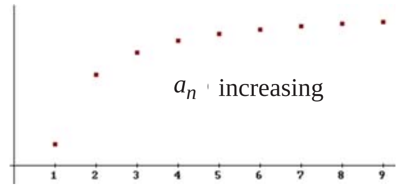
$$\begin{cases} a_n = a_{n-1} + \dfrac{1}{n} \\ a_1 = 1 \end{cases} , \quad \begin{cases} a_n = n \cdot a_{n-1} \\ a_0 = 1 \end{cases} , \quad \begin{cases} a_{n+1} = a_n + a_{n-1} \\ a_1 = a_2 = 1 \end{cases} \text{(Fibonacci)}$$

Monotonic sequences

$\{a_n\}_{n\geq 1}$ decreasing $\Leftrightarrow a_n \geq a_{n+1} , \forall n \in \mathbb{N}$

$\{a_n\}_{n\geq 1}$ increasing $\Leftrightarrow a_n \leq a_{n+1} , \forall n \in \mathbb{N}$





Example $\{n^2+1\}_{n\geq 1}$ increasing , $\begin{cases} a_{n+2} = a_{n+1} + a_n \\ a_1 = a_2 = 1 \end{cases}$ increasing

$\left\{\dfrac{1}{n^2}\right\}_{n\geq 1}$ decreasing , $\begin{cases} a_{n+2} = a_{n+1} - a_n \\ a_1 = a_2 = 1 \end{cases}$ no increasing and no decreasing

Why is necessary to calculate the difference between to consecutive terms in order to know if the sequence is increasing or decreasing

The character of a sequence is defined by the behaviour when n tends to infinity. In some sequences the behaviour is not the same in a finite number of terms at the beginning of the sequence. It is a common behaviour of polynomial sequences. Usually polynomial sequences have as many roots as the degree of the polynomial and the behaviour oscillates between the roots

$$a(n) := n^2 - 14n + 40$$

Why is necessary to calculate the difference between to consecutive terms in order to know if the sequence is increasing or decreasing

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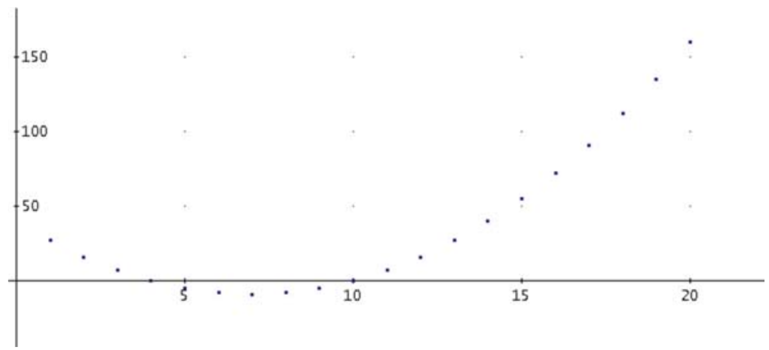
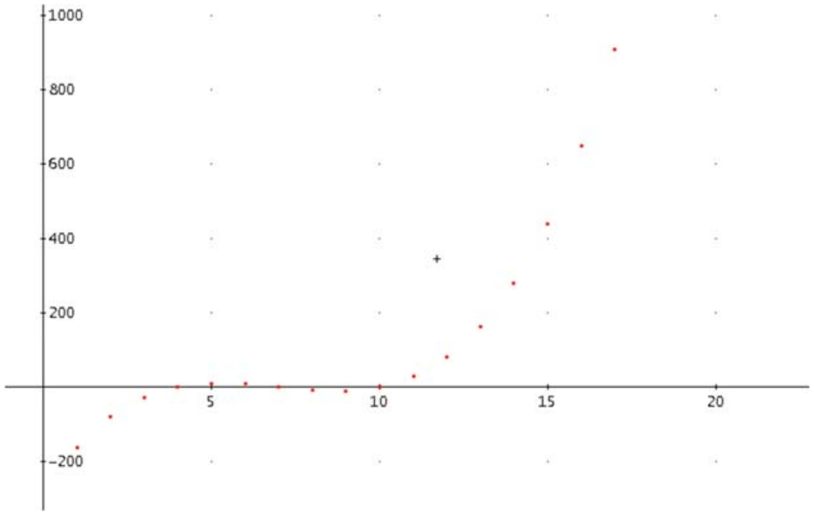


$a(n) := n^3 - 21n^2 + 138n - 280$



Exercise

- Verify that $\{a_n\}_{n\geq 1}$ such as $a_n = \dfrac{2n+4}{1-3n}$ is increasing

$$\left.\begin{array}{l} a_n = \dfrac{2n+4}{1-3n} \\ a_{n+1} = \dfrac{2(n+1)+4}{1-3(n+1)} \end{array}\right\} \Rightarrow a_{n+1} - a_n = \frac{2(n+1)+4}{1-3(n+1)} - \frac{2n+4}{1-3n} = \cdots = \frac{14}{9n^2+3n-2} > 0$$

- Verify that the sequence: $\begin{cases} a_{n+1} = \sqrt{2+a_n} \\ a_1 = 7 \end{cases}$ is decreasing

$a_1 \geq a_2$ because of $a_1 = 7$, $a_2 = \sqrt{2+7} = 3$

suppose that $a_n \geq a_{n+1}$

and finally

$$a_{n+1} = \sqrt{2+a_n} \geq \sqrt{2+a_{n+1}} = a_{n+2}$$

Bounded sequences

$\{a_n\}_{n\geq 1}$ upper bounded $\Leftrightarrow a_n \leq K \ , \forall n \in \mathbb{N}$

$\{a_n\}_{n\geq 1}$ lower bounded $\Leftrightarrow a_n \geq K \ , \forall n \in \mathbb{N}$

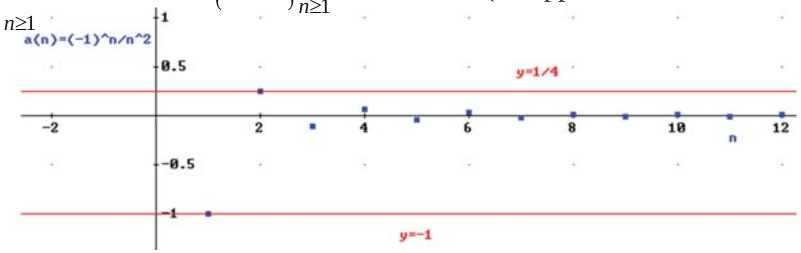
$\{a_n\}_{n\geq 1}$ bounded $\Leftrightarrow \{|a_n|\}_{n\geq 1}$ upper bounded

Examples:

$\{n^2+1\}_{n\geq 1}$ lower bounded but no upper bounded (no bounded)

$\{\mathrm{sen}(n) - n\}_{n\geq 1}$ upper bounded but no lower bounded (no bounded)

$\left\{\dfrac{(-1)^n}{n^2}\right\}_{n\geq 1}$ bounded ; $\{(-n)^n\}_{n\geq 1}$ no bounded (no upper and lower bounded)



Exercises:

Exercise Show that $\left\{\dfrac{2n+4}{1-3n}\right\}_{n\geq 1}$ is upper bounded by $-\dfrac{2}{3}$

$$\frac{2n+4}{1-3n} \leq \frac{-2}{3} \Leftrightarrow 3(2n+4) \geq -2(1-3n) \Leftrightarrow 6n+12 \geq -2+6n \Leftrightarrow 14 \geq 0$$

Exercise: Show that $\begin{cases} a_{n+1} = \sqrt{2+a_n} \\ a_1 = 7 \end{cases}$ is lower bounded by 2

$a_1 \geq 2$ because $a_1 = 7 \geq 2$

Suppose that $a_n \geq 2$

and finally:

$$a_{n+1} = \sqrt{2+a_n} \geq \sqrt{2+2} = 2$$

Convergent sequences

$\{a_n\}_{n\geq 1}$ is convergent at $\alpha \in \mathbb{R}$ if, for all $\varepsilon > 0, \exists n_0 \in \mathbb{N}$ such as

$$n \geq n_0 \quad \Rightarrow \quad |a_n - \alpha| < \varepsilon$$

α is the limit (the only one) of the sequence

Notation: $\lim\limits_{n\to+\infty} a_n = \alpha$, $\lim a_n = \alpha$, $\{a_n\} \to \alpha$

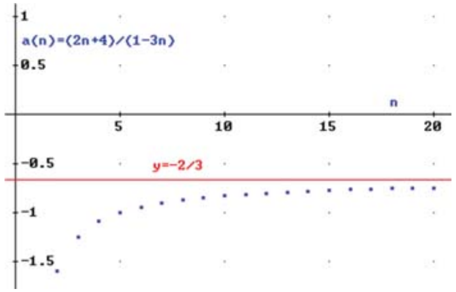
Examples:

$a_n = c \quad \Rightarrow \quad \{a_n\} \to c$

$\left\{\dfrac{1}{n}\right\} \to 0$

$\left\{\dfrac{2n+4}{1-3n}\right\} \to -\dfrac{2}{3}$

$0 < a < 1 \Rightarrow \{a^n\} \to 0$



Divergent sequence

$\{a_n\}_{n\geq 1}$ if it isn't convergent , it's obviously divergent

- $\{a_n\} \to +\infty$ if, for $K > 0$, exits n_0 such as $a_n > K$, $n \geq n_0$
- $\{a_n\} \to -\infty$ if $\{-a_n\} \to +\infty$
- $\{a_n\} \to \infty$ if $\{|a_n|\} \to +\infty$

divergent sequence $\begin{cases} \text{divergent} & \infty\ (+\infty, -\infty) \\ \text{oscillating} & \begin{cases} \text{bounded} \\ \text{no bounded} \end{cases} \end{cases}$

Example:

$\{n!\} \to +\infty$; $\{-n\} \to -\infty$

$\{(-n)^n\} \to \infty$ (oscillating, no bounded)

$a > 1 \Rightarrow \{a^n\} \to +\infty$

$\{(-1)^n\}$ divergent (oscillating, no bounded)

Notation : $\lim a_n \in \overline{\mathbb{R}} \Leftrightarrow \{a_n\}$ convergent or divergent $\pm\infty$

Arithmetic of limits

If $\lim a_n = \mathrm{a} \in \overline{\mathbb{R}}$ and $\lim b_n = \mathrm{b} \in \overline{\mathbb{R}}$, (if we haven't any indeterminate case)

- $\lim(a_n + b_n) = \lim a_n + \lim b_n = \mathrm{a} + \mathrm{b}$
- $\lim(\lambda \cdot a_n) = \lambda \cdot \lim a_n = \lambda \cdot \mathrm{a}$
- $\lim(a_n \cdot b_n) = \lim a_n \cdot \lim b_n = \mathrm{a} \cdot \mathrm{b}$
- $\lim(a_n / b_n) = \lim a_n / \lim b_n = \mathrm{a/b}$
- $\lim\left((a_n)^{b_n}\right) = (\lim a_n)^{\lim b_n} = \mathrm{a}^{\mathrm{b}}$
- If f is continuous(in an interval at a), $\lim f(a_n) = f(\lim a_n) = f(\mathrm{a})$
- $\{a_n\}$ bounded $\lim b_n = 0 \Rightarrow \lim(a_n \cdot b_n) = 0$
- $\lim |a_n| = 0 \Leftrightarrow \lim a_n = 0$,

Indetermination: $\infty - \infty$, $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $0 \cdot \infty$, ∞^0 , 0^0 , 1^∞ , $\cancel{0^\infty}$, $\cancel{\infty^\infty}$

Examples

$$\lim_{n\to\infty} \frac{6n^2+3n-5}{3n^2-1} = \lim_{n\to\infty} \frac{\frac{6n^2+3n-5}{n^2}}{\frac{3n^2-1}{n^2}} = \lim_{n\to\infty} \frac{6+\frac{3}{n}-\frac{5}{n^2}}{3-\frac{1}{n^2}} = \frac{6+0-0}{3-0} = 2$$

$$\lim_{n\to\infty} \frac{n^3+3n^2-5}{3n^2+2} = \lim_{n\to\infty} \frac{\frac{n^3+3n^2-5}{n^3}}{\frac{3n^2+2}{n^3}} = \lim_{n\to\infty} \frac{1+\frac{3}{n}-\frac{5}{n^3}}{\frac{3}{n}+\frac{2}{n^2}} = \frac{1+0-0}{0+0} = \infty$$

$$\lim_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right) = \lim_{n\to\infty} \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\left(\sqrt{n+1}+\sqrt{n}\right)} = \lim_{n\to\infty} \frac{1}{\left(\sqrt{n+1}+\sqrt{n}\right)} = 0$$

$$\lim_{n\to\infty}(\log(2n+1)-\log(n)) = \lim_{n\to\infty}\left(\log\left(\frac{2n+1}{n}\right)\right) = \log(2)$$

Sequence of real numbers

1. General concepts
   - Explicit and recurrent form
   - Monotone and bounded sequences
   - Convergent and divergent sequences
   - Arithmetic of limits. Indetermination
2. Calculus of limits
   - Limit of quotient
   - Euler formula
   - Stolz criterion.
3. Magnitude order
4. Lineal recurrence
   - Direct resolution in primer order
   - Second order. Characteristic equation method
   - No homogeneous case

Calculating limits of sequences

Limits of quotients:

$$\lim \frac{a_p n^p + a_{p-1} n^{p-1} + \cdots + a_2 n^2 + a_1 n + a_0}{b_q n^q + b_{q-1} n^{q-1} + \cdots + b_2 n^2 + b_1 n + b_0} = \lim \frac{a_p n^p}{b_q n^q} \ ; a_p, b_q \neq 0$$

Examples:

$$\lim \frac{6n^2+3n-5}{3n^2-1} = \lim \frac{6n^2}{3n^2} = 2$$

$$\lim \frac{n^3+3n^2-5}{3n^2+2} = \lim \frac{n^3}{3n^2} = \lim \frac{n}{3} = +\infty$$

$$\lim \frac{n^2\sqrt{n}+3n^2-5\sqrt{n^2+1}}{3\sqrt{n^5}+n-2\sqrt{n+1}} = \lim \frac{n^2\sqrt{n}}{3\sqrt{n^5}} = \frac{1}{3} \qquad \lim \frac{3^{n+1}+2^n}{5^{n+2}-8\cdot 3^n} = \lim \frac{3^{n+1}}{5^{n+2}} = 0$$

Euler's formula

$$\lim \left(1+\frac{1}{n}\right)^n = e$$

$$\lim a_n = 1 \ \text{ y } \ \lim b_n = \pm\infty \ \Rightarrow \lim (a_n)^{b_n} = e^{\lim \ b_n(a_n-1)}$$

Examples:

$$\lim \left(\frac{3n-5}{3n+2}\right)^{n-1} = \cancel{\left(1^\infty\right)} = e^{\lim(n-1)\left(\frac{3n-5}{3n+2}-1\right)} = e^{\lim(n-1)\left(\frac{-7}{3n+2}\right)} = e^{-\frac{7}{3}} = \frac{1}{\sqrt[3]{e^7}}$$

$$\lim \left(\frac{3n^2+3n-5}{3n^2+2}\right)^{\sqrt{n^2-3n}} = \cancel{\left(1^\infty\right)} = e^{\lim \sqrt{n^2-3n}\cdot\left(\frac{3n^2+3n-5}{3n^2+2}-1\right)} = e^{\lim \sqrt{n^2-3n}\cdot\left(\frac{3n-7}{3n^2+2}\right)} = e$$

Note: $\lim \left(\frac{6n^2+3n-5}{3n^2-1}\right)^{n+3} = \left(\frac{6}{3}\right)^{+\infty} = +\infty$

$$\lim \left(\frac{n^2+3n-5}{3n^2+2}\right)^{n^2-3} = \left(\frac{1}{3}\right)^{+\infty} = 0$$

Stolz criterion (quotient):

$$b_n \ \text{ increasing }, \ b_n \to +\infty \ \text{ y } \ \lim \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lambda \in \overline{\mathbb{R}} \ \Rightarrow \ \lim \frac{a_n}{b_n} = \lambda$$

Remark: $\nexists \lim \frac{a_{n+1}-a_n}{b_{n+1}-b_n} \not\Leftarrow \nexists \lim \frac{a_n}{b_n}$ $\left(\textbf{Example: } \frac{(-1)^n}{n}\right)$

Examples:

$$\lim \frac{\log(n)}{n} = \lim \frac{\log(n+1)-\log(n)}{(n+1)-n} = \lim\left(\log(n+1)-\log(n)\right) = \lim \log\left(\frac{n+1}{n}\right) = \log(1) = 0$$

$$\lim \frac{1+2+3+\cdots+n}{n^2} = \lim \frac{(1+2+3+\cdots+(n+1))-(1+2+3+\cdots+n)}{(n+1)^2-n^2} = \lim \frac{n+1}{2n+1} = \frac{1}{2}$$

$$\lim \frac{1+2+3+\cdots+2n}{n^2} = \lim \frac{(1+2+\cdots+2(n+1))-(1+2+\cdots+2n)}{(n+1)^2-n^2} = \lim \frac{4n+3}{2n+1} = 2$$

Limits with logarithmic (roots)

$$\left.\begin{array}{r} a_n > 0, \\ b_n \text{ increasing } b_n \to +\infty \end{array}\right\} \Rightarrow \lim (a_n)^{\frac{1}{b_n}} = e^{\log\left(\lim (a_n)^{\frac{1}{b_n}}\right)} = e^{\lim\left(\frac{\log(a_n)}{b_n}\right)}$$

Remark: $\lim \sqrt[b_n]{a_n} = \lim (a_n)^{\frac{1}{b_n}}$

If P and Q are polynomials

$\lim \sqrt[Q(n)]{P(n)} = 1$

Examples:

$$\lim \sqrt[n]{n} = e^{\lim\left(\frac{\log(n)}{n}\right)} \underset{SC}{=} e^{\lim\left(\log\left(\frac{n+1}{n}\right)\right)} = e^{\log(1)} = 1$$

$$\lim \sqrt[2n+3]{3^n+1} = e^{\lim\left(\frac{\log(3^n+1)}{2n+3}\right)} \underset{SC}{=} e^{\lim\left(\frac{\log\left(\frac{3^{n+1}+1}{3^n+1}\right)}{2n+5-(2n+3)}\right)} = e^{\frac{1}{2}\log\left(\lim\left(\frac{3^{n+1}+1}{3^n+1}\right)\right)} = \sqrt{3}$$

$$\lim \sqrt[n^2]{n!} = e^{\lim\left(\frac{\log(n!)}{n^2}\right)} \underset{SC}{=} e^{\lim\left(\frac{\log\left(\frac{(n+1)!}{n!}\right)}{(n+1)^2-n^2}\right)} = e^{\lim\left(\frac{\log(n+1)}{2n+1}\right)} \underset{SC}{=} e^{\frac{1}{2}\lim\left(\log\left(\frac{n+2}{n+1}\right)\right)} = 1$$

Sequence of real numbers

Order of magnitude

$\{a_n\}$, $\{b_n\}$ sequences of positive terms which tend to $+\infty$

Notation O:

$a_n \in O(b_n)$ if $\left[\lim\left(\frac{a_n}{b_n}\right) = 0 \Rightarrow a_n \in O(b_n)\right]$ and we write $a_n \ll b_n$

Notation Ω:

$a_n \in \Omega(b_n)$ if $\left[\lim\left(\frac{a_n}{b_n}\right) = +\infty \Rightarrow a_n \in \Omega(b_n)\right]$ and we write $a_n \gg b_n$

Notation Θ:

$a_n \in \Theta(b_n)$ if $\left[\lim\left(\frac{a_n}{b_n}\right) = \alpha > 0 \Rightarrow a_n \in \Theta(b_n)\right]$ and we write $a_n \approx b_n$

Properties

$$a_n \in O(b_n) \Leftrightarrow b_n \in \Omega(a_n)$$

$$a_n \in \Theta(b_n) \Leftrightarrow a_n \in O(b_n) \wedge a_n \in \Omega(b_n)$$

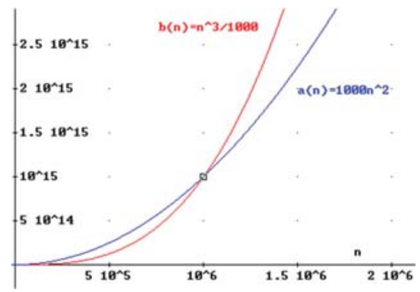
$$\Theta(a_n + b_n) = \Theta(\max(a_n, b_n))$$

Examples:

$$27n^2 + \frac{355}{113}n + 12 \in \Theta(n^2)$$

$$\sqrt{n} \approx \frac{1}{\sqrt{n+1} - \sqrt{n}}$$

$$1000n^2 \in O\left(\frac{n^3}{1000}\right)$$



$$\boxed{c} \ll \log(n) \ll \sqrt{n} \ll n \ll n\log(n) \ll n^2 \ll e^n \ll n! \ll n^n$$

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How to solve lineal recurrences

First order: $\left(a_{n+1} = s_n \cdot a_n + t_n \ , n \in \mathbb{N}\right)$:

Important cases: $s_n = c_1 \ ; \ t_n = c_2 \cdot n \ , \ c_3 \cdot k^n \big)$

Examples

$$\begin{cases} a_{n+1} = a_n + k \\ a_1 = k \end{cases} \Rightarrow \ a_n = n \cdot k$$

$$\begin{cases} a_{n+1} = k \cdot a_n \\ a_1 = k \end{cases} \Rightarrow \ a_n = k^n$$

$$\begin{cases} a_n = n \cdot a_{n-1} \\ a_1 = 1 \end{cases} \Rightarrow \ a_n = n!$$

```
RSolve[eqn, a[n], n]
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```
RSolve[{a[n+1] == a[n] + k, a[1] == k}, a[n], n]
{{a[n] → k n}}

RSolve[{a[n+1] == k a[n], a[1] == k}, a[n], n]
{{a[n] → k^n}}

RSolve[{a[n] == n a[n-1], a[1] == 1}, a[n], n]
{{a[n] → n!}
```

Linear recurrence

How to solve lineal recurrences

Examples of direct resolution *(recursive algorithm)*:

a) Hanoi's Towers: $\begin{cases} a_{n+1} = 2a_n + 1 \\ a_1 = 1 \end{cases}$

Geometric series

$$\left(1 + r + r^2 + \cdots + r^m = \frac{r^{m+1} - 1}{r - 1} \right)$$

$$\downarrow$$

$$a_2 = 2a_1 + 1 = 2 + 1$$

$$a_3 = 2a_2 + 1 = 2(2+1) + 1 = 2^2 + 2 + 1$$

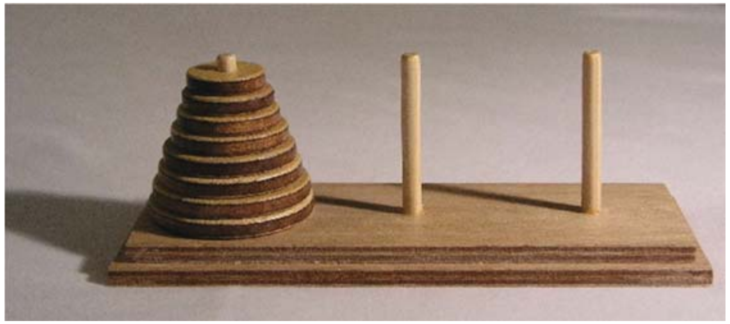
$$\vdots$$

$$a_n = 2a_{n-1} + 1 = 2(2^{n-2} + 2^{n-3} + \cdots + 2 + 1) + 1 = 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 = \frac{1 - 2^{n-1}2}{1 - 2} = 2^n - 1$$

induction

$$a_n \in \Theta(2^n)$$

http://www.dynamicdrive.com/dynamicindex12/towerhanoi.htm



$a(8) = 2^8 - 1 = 255$

$a(4) = 2^4 - 1 = 15$

$a(64) = 2^{64} - 1 = 18446744073709551615$

500.000 millions years long if each movement lasts one second

THE LEGEND OF HANOI TOWERS

According to legend, God placed on Earth three diamond rods and sixty-four gold records in creating the world. The discs are all of different size and were initially placed in descending order on the first diameter of the rods. God also created a monastery whose monks are tasked to move all the discs from the first bar to the third. The only operation permitted to move a disc of a rod any other, but with the condition that there can be placed one above another larger diameter disk. Legend also says that when the monks finish their task, the world will end.

How to solve linear recurrences

b) $\begin{cases} a_{n+1} = a_n + 3n \\ a_1 = 1 \end{cases}$

$$a_2 = 1 + 3\cdot 1$$
$$a_3 = 1 + 3\cdot 1 + 3\cdot 2$$
$$a_4 = 1 + 3\cdot 1 + 3\cdot 2 + 3\cdot 3$$
$$\vdots$$

Arithmetic series

$$1 + 2 + 3 + \cdots + m = \frac{m\cdot(m+1)}{2}$$

$$a_n = 1 + 3\cdot 1 + 3\cdot 2 + 3\cdot 3 + \cdots + 3\cdot(n-1) =$$
$$= 1 + 3(1 + 2 + 3 + \cdots + (n-1)) = 1 + 3\left(\frac{(n-1)n}{2}\right) = \frac{3n^2 - 3n + 2}{2}$$

$$a_n \in \Theta(n^2)$$

Second order and constant coefficients

$$a_{n+2} + p\cdot a_{n+1} + q\cdot a_n = t_n \ ;\ p, q(\neq 0) \in \mathbb{R},\quad n \in \mathbb{N}$$

(More interesting cases: $t_n = 0$ homogeneous, $t_n = P(n)$; $t_n = k^n$; $t_n = P(n)\cdot k^n$)

Homogeneous case : $\left(a_{n+2} + p\cdot a_{n+1} + q\cdot a_n = 0 \ ;\ n \in \mathbb{N}\right)$

Properties of the solutions:

Any solution may be written as a linear combination of:

$$a_n = c_1\cdot a_n^1 + c_2\cdot a_n^2 \ ;\ c_1, c_2 \in \mathbb{R},$$

a_n^1 , a_n^2 are linearly independent solutions (particular one)

Equation characteristic's method

Suppose that $a_n = r^n$ $(r \neq 0)$ satisfies $a_{n+2} + p\cdot a_{n+1} + q\cdot a_n = 0$

$$\Downarrow$$

$$P(r) \equiv r^2 + pr + q = 0$$

P is called characteristic polynomial, and the equation, characteristic equation

Case 1 $(r_1 \neq r_2$, real roots of $P(r))$:

Particular solutions linearly independent: r_1^n , r_2^n

General solutions: $a_n = c_1 r_1^n + c_2 r_2^n$

Case 2 $(r_1 = r_2 = r$, double real roots):

Particular solutions linearly independent r^n , $n\cdot r^n$

General solutions: $a_n = c_1 r^n + c_2 n r^n$ ***verify it***

Case 3 $(r_1 = \rho_\alpha, r_2 = \rho_{-\alpha}$ conjugate complex roots):

Particular (real) solutions linearly independent: $\rho^n\cos(n\alpha)$, $\rho^n\,\mathrm{sen}(n\alpha)$

General solutions: $a_n = \rho^n\left(c_1\cos(n\alpha) + c_2\,\mathrm{sen}(n\alpha)\right)$

Example

Example: Solve the recurrence $\begin{cases} a_{n+2} = a_{n+1} + a_n \\ a_1 = 1,\ a_2 = 1 \end{cases}$ (a_n Fibonacci sequence)

Now $p = q = -1$ and the characteristic equation is $r^2 - r - 1 = 0$

with solutions $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$ (case 1)

General solution: $c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$

Constant are calculated using the initial values

$$\left.\begin{aligned} a_1 = 1 &\Rightarrow c_1\left(1+\sqrt{5}\right) + c_2\left(1-\sqrt{5}\right) = 2 \\ a_2 = 1 &\Rightarrow c_1\left(3+\sqrt{5}\right) + c_2\left(3-\sqrt{5}\right) = 2 \end{aligned}\right\} \Rightarrow c_1 = \frac{1}{\sqrt{5}}\ ,\ c_2 = -\frac{1}{\sqrt{5}}$$

$$a_n\ (\text{Fibonacci}) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$\Theta(a_n) = \left(\frac{1+\sqrt{5}}{2}\right)^n$$

Example

Example: Solve the recurrence $a_{n+2}=2\,a_{n+1}-a_n$

Now characteristic equation is $r^2-2r+1=(r-1)^2=0$ with the double solution $r=1$

General solution: $a_n=c_1+c_2n$

If $a_1=2$ and $a_2=3$ $\quad a_1=c_1+c_2 1=2$

$a_2=c_1+c_2 2=3$

Solving the linear system it can be obtained:

$c_1=1$

$c_2=1$ And the general solution: $a_n=1+n$

Example

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General solution: $a_n=c_1+c_2n$

Example: Solve the recurrence $a_{n+2}=a_{n+1}-a_n$

Now characteristic equation is $r^2-r+1=0$ with solutions:

$$r_1=\frac{1}{2}+\frac{\sqrt{3}}{2}i=\rho_\alpha\ ,\ r_2=\frac{1}{2}-\frac{\sqrt{3}}{2}i=\rho_{-\alpha} \qquad \text{(case 3)}$$

then $\rho=1,\ \cos(\alpha)=\frac{1}{2},\ \operatorname{sen}(\alpha)=\frac{\sqrt{3}}{2}\ \left(\alpha=\frac{\pi}{3}\right)$

General solution: $a_n=c_1\cos\left(\frac{n\pi}{3}\right)+c_2\operatorname{sen}\left(\frac{n\pi}{3}\right)$

Example

Exercise (Generalization) Solve the recurrence $\begin{cases} a_{n+3}=5a_{n+2}-8a_{n+1}+4a_n \\ a_1=0\ ,\ a_2=1\ ,\ a_3=2 \end{cases}$

This case is homogenous third order problem:

$a_{n+3}+m\cdot a_{n+2}+p\cdot a_{n+1}+q\cdot a_n=0 \qquad m=-5,\ p=8,\ q=-4$

The characteristic equation is: $r^3-5r^2+8r-4=(r-1)(r-2)^2$ with solutions $r_1=1$ and $r_2=2$ (double)

General solution: $c_1+c_2 2^n+c_3 n\cdot 2^n$

Using the initial values

$$\left.\begin{aligned} a_1=0 &\Rightarrow c_1+2c_2+2c_3=0 \\ a_2=1 &\Rightarrow c_1+4c_2+8c_3=1 \\ a_3=2 &\Rightarrow c_1+8c_2+24c_3=2 \end{aligned}\right\}\Rightarrow c_1=-2\ ,\ c_2=\frac{5}{4}\ ,\ c_3=-\frac{1}{4}$$

Solution: $a_n=-2+(5-n)2^{n-2}$

Non homogeneous case

Whole equation (non homogeneous) $\left(a_{n+2}+p\cdot a_{n+1}+q\cdot a_n=t_n\ ;\ n\in\mathbb{N}\right)$

If u_n is a solution of the non homogeneous equation (whole equation) and a_n any solution

$$\left.\begin{aligned} u_{n+2}+p\cdot u_{n+1}+q\cdot u_n=t_n \\ a_{n+2}+p\cdot a_{n+1}+q\cdot a_n=t_n \end{aligned}\right\}\Rightarrow (a_{n+2}-u_{n+2})+p(a_{n+1}-u_{n+1})+q(a_n-u_n)=0$$

Then a_n-u_n is any solution of homogeneous equation

General solution (non homogeneous case): $a_n=\underbrace{c_1\cdot a_n^1+c_2\cdot a_n^2}_{a_n-u_n}+u_n\ ;\ c_1,c_2\in\mathbb{R}$

- a_n^1 and a_n^2 are obtained from the characteristic equation (homogeneous case)
- u_n (particular) is searched with indeterminate coefficients' method
- First order recurrences can be handled in the same way

$a_n=c_1\cdot a_n^1+c_2\cdot a_n^2+u_n; c_1,c_2\in\mathbb{R}$

(characteristic equation: $r-p=0$ $\longrightarrow$ general solution $a_n=c\cdot p^n+u_n$)

Non homogeneous case

•u_n (particular) is searched with indeterminate coefficients' method

$F(n)$	$a_n^{(p)}$
C,constant	C_0,constant
n	C_0+C_1n
n^2	$C_0+C_1n+C_2n^2$
$n^t, t\in Z^+$	$C_0+C_1n+\cdots+C_tn^t$
$r^n, r\in R$	C_0r^n
n^tr^n	$r^n(C_0+C_1n+C_2n^2)$
$sen(An), A\in R$	$C_0sen(An)+C_1cos(An)$
$cos(An), A\in R$	$C_0sen(An)+C_1cos(An)$
$r^n sen(An), A\in R$	$C_0r^nsen(An)+C_1r^ncos(An)$
$r^n cos(An), A\in R$	$C_0r^nsen(An)+C_1r^ncos(An)$

Non homogeneous case

•u_n (particular) is searched with indeterminate coefficients' method

$F(n)$	$a_n^{(p)}$
C,constant	C_0,constant
n	C_0+C_1n
n^2	$C_0+C_1n+C_2n^2$
$n^t, t\in Z^+$	$C_0+C_1n+\cdots+C_tn^t$
$r^n, r\in R$	C_0r^n

Example

Example: Solve the recurrence $\begin{cases} a_{n+2}-5a_{n+1}+6a_n=2 \\ a_1=1\ ,\ a_2=-1 \end{cases}$

Now characteristic equation is $r^2 -5r +6 = 0 \;\; r_1=2, r_2=3$, *case 1*

General solution of homogeneous: $c_1 2^n + c_2 3^n$

Particular solution (no homogeneous case): $u_n = k$ (now t_n is constant)

(Plugging into the equation, $k - 5k + 6k=2 \longrightarrow k = 1$)

General solution (no homogeneous, whole equation): $c_1 2^n + c_2 3^n + 1$

using initial conditions

$$\left.\begin{array}{l} a_1=1 \;\Rightarrow\; 2c_1+3c_2=0 \\ a_2=-1 \;\Rightarrow\; 4c_1+9c_2=-2 \end{array}\right\} \Rightarrow\; c_1=1\,,\, c_2=-\frac{2}{3}$$

solution: $a_n = 2^n - 2\cdot 3^{n-1}+1$

Example

Example: Solve the recurrence $a_{n+2}-a_{n+1}-a_n=-n^2$

Now characteristic equation is $r^2 -r -1 = 0$ is the Fibonacci's case

General solution of homogeneous: $c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$

Particular solution (no homogeneous case): $u_n=a\cdot n^2 + b\cdot n + c$
(now t_n is a second grade polynomial)

General solution (no homogeneous, whole equation):

$$a_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n + n^2+2n+5$$