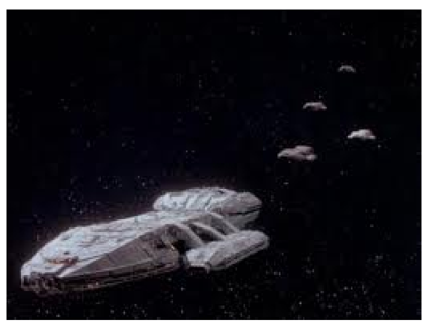
Lesson 4

Vector spaces
(Part 1)



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Contents

I Definition of vector space and basic properties

We begin the section with several examples.

Example I.1. In $\mathbb{R}^n$ we have defined two operations. On the one hand, the sum of vectors:

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$

and, on the other hand, the multiplication of a vector by an scalar:

$$\lambda \cdot (x_1, x_2, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)$$

These operations satisfy the following 8 properties:

- S1: $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ for every couple of vectors $\vec{x}$ and $\vec{y}$.
- S2: $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$ for all $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$.
- S3: The operation $+$ has an identity element: the zero vector, $\vec{0}$. That is, $\vec{x} + \vec{0} = \vec{x}$ for any vector $\vec{x}$.
- S4: Any vector $\vec{x}$ has an inverse with respect to the operation $+$: $-\vec{x}$. That is, $\vec{x} + (-\vec{x}) = \vec{0}$.
- M1: $\lambda \cdot (\vec{x} + \vec{y}) = \lambda \cdot \vec{x} + \lambda \cdot \vec{y}$ for any scalar λ and for any pair of vectors $\vec{x}$ and $\vec{y}$.
- M2: $(\lambda + \mu) \cdot \vec{x} = \lambda \cdot \vec{x} + \mu \cdot \vec{x}$ for any pair of scalars λ and μ, and for any vector $\vec{x}$.
- M3: $(\lambda\mu) \cdot \vec{x} = \lambda \cdot (\mu \cdot \vec{x})$ for any pair of scalars λ and μ and for any vector $\vec{x}$.
- M4: $1 \cdot \vec{x} = \vec{x}$ for any vector $\vec{x}$.

Example I.2. In the set of $m \times n$ matrices $\mathcal{M}_{m\times n}$ we have defined the following two operations: sum of matrices and multiplication of a matrix by an scalar. These operations satisfy the above 8 properties.

Example I.3. In the set of polynomials (with one variable x) with real coefficients (that we will denote by $\mathbb{R}[x]$) we consider the usual operations: "sum of polynomials" and "product of a polynomial by a scalar". For example:

$$(3 + 2x + 5x^3) + (-2 - 4x^2 + x^3 - x^4) = 1 + 2x - 4x^2 + 6x^3 - x^4,$$

$$3(3 + 2x + 5x^3) = 9 + 6x + 15x^3.$$

These operations satisfy also the above 8 properties.

Example I.4. Let n be a non-negative integer and let $\mathbb{R}_n[x]$ be the set of polynomials (with one variable x and with real coefficients) of degree less than or equal to n, that is, the set of expressions of the type

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \quad a_0, a_1, \dots, a_n \in \mathbb{R}.$$

If we sum two polynomials of degree less than or equal to n, or if we multiply such a polynomial by a scalar, the result is a polynomial of degree less than or equal to n. Therefore these operations are well-defined in $\mathbb{R}_n[x]$. Moreover they satisfy the properties of Example I.1.

Example I.5. Let I be a real interval. In the set of all real functions defined in I we consider the operations given by the "sum of functions" and the "product of a function by a scalar". That is, if f and g are two functions and α is an scalar, we define $f+g$ and αf as follows:

$$(f+g)(x) := f(x)+g(x) \quad \text{i} \quad (\alpha f)(x) := \alpha f(x) \quad \forall x \in I.$$

These operations satisfy also the 8 properties of Example I.1.

Definition I.6. A *vector space* (over $\mathbb{R}$) consists of a non-empty set V with two operations '+' and '·' such that:

- '+' is an *internal* operation on V, that is, for any two elements $\vec{u}, \vec{v} \in V$, its *sum* $\vec{u}+\vec{v}$ is an element of V,
- '·' is an *external* operation, that is, for any scalar $\alpha \in \mathbb{R}$ and for any element $\vec{v} \in V$, the *product* $\alpha \cdot \vec{v}$ is an element of V, and
- the operations '+' and '·' satisfy the 8 properties S1, S2, S3, S4, M1, M2, M3 and M4 given in Example I.1.

We will call *vectors* to the elements of a given vector space V. Usually, they will be denoted by letters with arrows. Given a vector $\vec{v} \in V$ and a scalar α, many times we will write $\alpha\vec{v}$ instead of $\alpha \cdot \vec{v}$ (omitting the dot).

We can say now that the sets of the Examples I.1, I.2, I.3, I.4 and I.5 (with the defined operations) are vector spaces.

Example I.7. The singleton set

$$\{\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\} \subseteq \mathbb{R}^n$$

is a vector space under the operations

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \alpha \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

that it inherits from $\mathbb{R}^4$.

A vector space must have at least one element, its zero vector. Thus a one-element vector space is the smallest possible.

Definition I.8. A one-element vector space is a *trivial* space.

Proposition I.9. *Let V be a vector space. For every pair of scalars α and β and for every pair of vectors $\vec{a}, \vec{b} \in V$:*

(1) $0\vec{a} = \vec{0}$.

(2) $\alpha\vec{0} = \vec{0}$.

(3) *If* $\alpha\vec{a} = \vec{0}$ *then either* $\alpha = 0$ *or* $\vec{a} = \vec{0}$*.*

(4) $-(\alpha\vec{a}) = (-\alpha)\vec{a} = \alpha(-\vec{a})$*.*

(5) *If* $\alpha\vec{a} = \beta\vec{a}$ *and* $\vec{a} \neq \vec{0}$ *then* $\alpha = \beta$*.*

(6) $\alpha\vec{a} = \alpha\vec{b}$ *and* $\alpha \neq 0$ *then* $\vec{a} = \vec{b}$*.*

PROOF:

(1) Using Property M2 we have: $0\vec{a} = (0+0)\vec{a} = 0\vec{a} + 0\vec{a}$. Adding $-0\vec{a}$ to both sides of the equality $0\vec{a} = 0\vec{a} + 0\vec{a}$ we obtain $0\vec{a} = \vec{0}$.

(2) Since $\vec{0}$ is the identity element of the operation '+' and applying Property M1 we have that $\alpha\vec{0} = \alpha(\vec{0}+\vec{0}) = \alpha\vec{0} + \alpha\vec{0}$. Adding $-\alpha\vec{0}$ to both sides of the equality $\alpha\vec{0} = \alpha\vec{0} + \alpha\vec{0}$ we have $\alpha\vec{0} = \vec{0}$.

(3) If $\alpha\vec{a} = \vec{0}$ i $\alpha \neq 0$, multiplying by α^{-1} and applying (2) we have $\alpha^{-1}(\alpha\vec{a}) = \alpha^{-1}\vec{0} = \vec{0}$. Now, applying M3: $(\alpha^{-1}\alpha)\vec{a} = \vec{0}$; therefore $\vec{a} = \vec{0}$ as a consequence of M4.

(4) $\alpha\vec{a} + (-\alpha)\vec{a} = (\alpha + (-\alpha))\vec{a} = 0\vec{a} = \vec{0}$ by Property M2 and (1). Similarly $\alpha\vec{a} + \alpha(-\vec{a}) = \alpha(\vec{a} + (-\vec{a})) = \alpha\vec{0} = \vec{0}$ by M1 and (2). Therefore $(-\alpha)\vec{a}$ and $\alpha(-\vec{a})$ coincide with $-\alpha\vec{a}$.

(5) Equality $\alpha\vec{a} = \beta\vec{a}$ is equivalent to $\alpha\vec{a} - \beta\vec{a} = \vec{0}$ by adding $-\beta\vec{a}$ to both sides. By M2: $(\alpha - \beta)\vec{a} = \vec{0}$ and, by (3), one has that $\alpha - \beta = 0$.

(6) Equality $\alpha\vec{a} = \alpha\vec{b}$ is equivalent to $\alpha\vec{a} - \alpha\vec{b} = \vec{0}$ by adding $-\alpha\vec{b}$ to both sides. By M1 we have that $\alpha(\vec{a} - \vec{b}) = \vec{0}$. Therefore $\vec{a} - \vec{b} = \vec{0}$ as a consequence of(3). QED

UTILITARIAN SUMMARY I.10.

- A *vector space* (over $\mathbb{R}$) is any non-empty set V with two operations (sum and multiplication by a scalar) satisfying the properties S1, S2, S3, S4, M1, M2, M3 and M4 given in Example I.1.
- Basic properties that one may expect (as, for instance, $0\vec{a} = \vec{0}$ for any vector $\vec{a}$) are, actually, consequences of properties S1, S2, S3, S4, M1, M2, M3 and M4.

II Subspaces and spanning sets

Subspaces

Definition II.1. For any vector space, a *subspace* (or *vector subspace*) is a subset that is itself a vector space, under the inherited operations.

Example II.2. The plane

$$P = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y + z = 0 \}$$

is a subspace of $\mathbb{R}^3$. As specified in the definition, the operations are the ones that are inherited from the larger space, that is, vectors add in P as they add in $\mathbb{R}^3$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

and scalar multiplication is also the same as it is in $\mathbb{R}^3$. To show that P is a subspace, we need only note that it is a subset and then verify that it is a space. Checking that P satisfies the conditions in the definition of a vector space is routine. For instance, for closure under addition, note that if the summands satisfy that $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$ then the sum satisfies that $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) = 0$.

Example II.3. The x-axis in $\mathbb{R}^2$ is a subspace where the addition and scalar multiplication operations are the inherited ones.

$$\begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 0 \end{bmatrix} \qquad r \cdot \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} rx \\ 0 \end{bmatrix}$$

As above, to verify that this is a subspace we simply note that it is a subset and then check that it satisfies the conditions in definition of a vector space. For instance, the two closure conditions are satisfied: (1) adding two vectors with a second component of zero results in a vector with a second component of zero, and (2) multiplying a scalar times a vector with a second component of zero results in a vector with a second component of zero.

Example II.4. Another subspace of $\mathbb{R}^2$ is its trivial subspace.

$$\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$$

Any vector space has a trivial subspace $\{ \vec{0} \}$. At the opposite extreme, any vector space has itself for a subspace. These two are the *improper* subspaces. Other subspaces are *proper*.

Example II.5. As we have said in Exmple I.4, the set of polynomials of degree $\leq n$, $\mathbb{R}_n[x]$, is a vector space with the usual sum of polynomials and product of a scalar by a polynomial. Therefore $\mathbb{R}_n[x]$ is a vector subspace of the larger vector space $\mathbb{R}[x]$.

Example II.6. Being vector spaces themselves, subspaces must satisfy the closure conditions. The set $\mathbb{R}^+$ is not a subspace of the vector space $\mathbb{R}^1$ because with the inherited operations it is not closed under scalar multiplication: if $\vec{v} = 1$ then $-1 \cdot \vec{v} \notin \mathbb{R}^+$.

Definition II.7. Let S be a subset of a vector space V. A *linear combination* of S is any vector $\vec{v}$ such that

$$\vec{v} = \lambda_1 \vec{s}_1 + \cdots \lambda_r \vec{s}_r,$$

where $\lambda_1, \ldots, \lambda_r$ are scalars and $\vec{s}_1, \ldots, \vec{s}_r \in S$.

The next result says that Example II.6 is prototypical. The only way that a subset can fail to be a subspace, if it is nonempty and under the inherited operations, is if it isn't closed. This is shown in the following result.

Theorem II.8. *For a nonempty subset S of a vector space, under the inherited operations, the following are equivalent statements:*

(1) S is a subspace of that vector space

(2) S is closed under linear combinations of pairs of vectors: for any vectors $\vec{s}_1, \vec{s}_2 \in S$ and scalars r_1, r_2 the vector $r_1\vec{s}_1 + r_2\vec{s}_2$ is in S

(3) S is closed under linear combinations of any number of vectors: for any vectors $\vec{s}_1, \ldots, \vec{s}_n \in S$ and scalars $r_1, \ldots, r_n$ the vector $r_1\vec{s}_1 + \cdots + r_n\vec{s}_n$ is in S.

Briefly, a subset is a subspace if it is closed under linear combinations.

PROOF:

'The following are equivalent' means that each pair of statements are equivalent.

$$(1) \iff (2) \qquad (2) \iff (3) \qquad (3) \iff (1)$$

We will prove the equivalence by establishing that $(1) \implies (3) \implies (2) \implies (1)$. This strategy is suggested by the observation that $(1) \implies (3)$ and $(3) \implies (2)$ are easy and so we need only argue the single implication $(2) \implies (1)$.

Assume that S is a nonempty subset of a vector space V that is S closed under combinations of pairs of vectors. We will show that S is a vector space by checking the conditions.

- The first item in the vector space definition (DefinitionI.6) is the closure under addition. But, if $\vec{s}_1, \vec{s}_2 \in S$ then $\vec{s}_1 + \vec{s}_2 \in S$, as $\vec{s}_1 + \vec{s}_2 = 1 \cdot \vec{s}_1 + 1 \cdot \vec{s}_2$.

- The second item is the closure under multiplication by a scalar. But, if $\alpha \in \mathbb{R}$ and $\vec{s} \in S$ then $\alpha\vec{s} \in S$ because $\alpha\vec{s} = 0 \cdot \vec{s} + \alpha \cdot \vec{s}$.

- The third item says that the operations in S must satisfy properties S1, S2, S3, S4, M1, M2, M3 and M4. But it is clear that these properties are inherited from V because the operations in S are the same as the operations in V. QED

Proposition II.9. *If S is a subspace of a vector space V then $\vec{0} \in S$.*

PROOF:

Since S is nonempty (because it is a subspace), there exists an element $\vec{s} \in S$. By Theorem II.8 we have that $\vec{0} = 0 \cdot \vec{s} + 0 \cdot \vec{s} \in S$. QED

UTILITARIAN SUMMARY II.10.

- A *subspace* (or *vector subspace*) of a vector space V is a subset of V that is itself a vector space, under the inherited operations.

- **Characterization of subspaces**: A nonempty subset S of a vector space V is a subspace of V if and only if any linear combination of vectors in S belongs to S.

- Every subspace contains the zero vector $\vec{0}$.

Spanning sets

Definition II.11. The *span* (or *linear closure*) of a nonempty subset S of a vector space, denoted by $\operatorname{span}(S)$, is the set of all linear combinations of vectors from S:

$$\operatorname{span}(S) := \{c_1\vec{s}_1 + \cdots + c_n\vec{s}_n \bigm| c_1, \ldots, c_n \in \mathbb{R} \text{ and } \vec{s}_1, \ldots, \vec{s}_n \in S\}$$

The span of the empty subset of a vector space is defined as the trivial subspace.

Lemma II.12. *In a vector space, the span of any subset is a subspace.*

PROOF:

If the subset S is empty then by definition its span is the trivial subspace. If S is not empty then, by Theorem II.8, we need only check that the span $\operatorname{span}(S)$ is closed under linear combinations. For a pair of vectors from that span, $\vec{v} = c_1\vec{s}_1 + \cdots + c_n\vec{s}_n$ and $\vec{w} = c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m$, a linear combination

$$\begin{aligned} &p \cdot (c_1\vec{s}_1 + \cdots + c_n\vec{s}_n) + r \cdot (c_{n+1}\vec{s}_{n+1} + \cdots + c_m\vec{s}_m) \\ &\qquad\qquad = pc_1\vec{s}_1 + \cdots + pc_n\vec{s}_n + rc_{n+1}\vec{s}_{n+1} + \cdots + rc_m\vec{s}_m \end{aligned}$$

(p, r scalars) is a linear combination of elements of S and so is in $\operatorname{span}(S)$ (possibly some of the $\vec{s}_i$'s from $\vec{v}$ equal some of the $\vec{s}_j$'s from $\vec{w}$, but it does not matter). QED

The converse of the lemma holds: any subspace is the span of some set, because a subspace is obviously the span of the set of its members. Thus **a subset of a vector space is a subspace if and only if it is a span**.

Taken together, Theorem II.8 and Lemma II.12 show the following:

Proposition II.13. *The span of a subset S of a vector space is the smallest subspace containing all the members of S.*

Definition II.14. A *spanning set* of a vector space V is any set of vectors S such that $V = \operatorname{span}(S)$.

Example II.15. In any vector space V, for any vector $\vec{v}$ the set $\{r \cdot \vec{v} \bigm| r \in \mathbb{R}\}$ is a subspace of V because it is just the span of $\vec{v}$. For instance, for any vector $\vec{v} \in \mathbb{R}^3$ the line through the origin containing that vector $\{k\vec{v} \bigm| k \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$. This is true even when $\vec{v}$ is the zero vector, in which case the subspace is the degenerate line, the trivial subspace.

Example II.16. The span of this set is all of $\mathbb{R}^2$.

$$\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\}$$

To check this we must show that any member of $\mathbb{R}^2$ is a linear combination of these two vectors. So we ask: for which vectors (with real components x and y) are there scalars c_1 and c_2 such that this holds?

$$c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

Gauss' Method

$$\begin{array}{rcl} c_1 + c_2 = x \\ c_1 - c_2 = y \end{array} \xrightarrow{\rho_2 - \rho_1} \begin{array}{rcl} c_1 + \quad c_2 &=& x \\ -2c_2 &=& -x+y \end{array}$$

with back substitution gives $c_2 = (x-y)/2$ and $c_1 = (x+y)/2$. These two equations show that for any x and y there are appropriate coefficients c_1 and c_2 making the above vector equation true. For instance, for $x = 1$ and $y = 2$ the coefficients $c_2 = -1/2$ and $c_1 = 3/2$ will do. That is, we can write any vector in $\mathbb{R}^2$ as a linear combination of the two given vectors. Therefore we can say that the set

$$\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}$$

is a **spanning set** of $\mathbb{R}^2$.

Example II.17. Every vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ of $\mathbb{R}^n$ can be written in the following form:

$$\vec{x} = x_1(1,0,0,\ldots,0) + x_2(0,1,0,\ldots,0) + \cdots x_n(0,0,\ldots,0,1).$$

Therefore, any vector of this vector space is a linear combination of the vectors in the set

$$S_1 = \{(1,0,0,\ldots,0), (0,1,0,\ldots,0), \ldots, (0,0,\ldots,0,1)\}.$$

Therefore $\mathbb{R}^n = \mathrm{span}(S_1)$ and, then, S_1 is a spanning set of $\mathbb{R}^2$.

Example II.18. Let us consider the vector space $\mathcal{M}_{2\times 3}$ of 2×3 matrices. Any of these matrices A can be written:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+a_{13} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + a_{23} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

that is, it is a linear combination of the set of vectors

$$S_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Therefore $\mathcal{M}_{2\times 3} = \mathrm{span}(S_2)$, that is, S_2 is a spanning set of $\mathcal{M}_{2\times 3}$.

Example II.19. We consider now the vector space $\mathbb{R}_n[x]$ of real polynomials of degree $\leq n$. An arbitrary vector $\vec{p}$ of this vector space has the following form:

$$\vec{p} = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n,$$

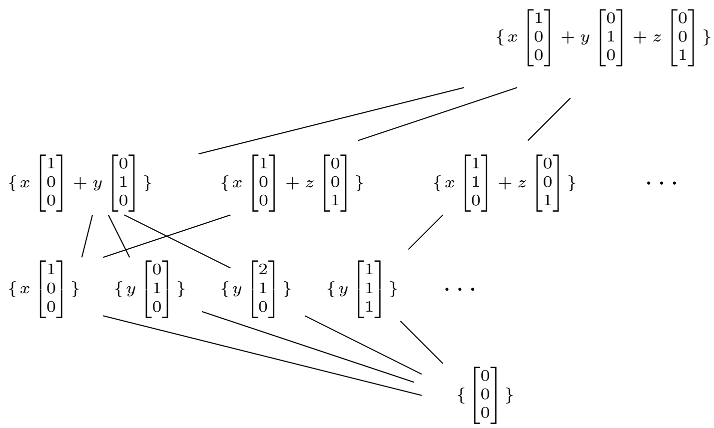
where $a_1, a_2, \ldots, a_n$ are scalars. It is clear that $\vec{p}$ is a linear combination of the vectors in the set $S_3 = \{1, x, x^2, \ldots, x^n\}$ and, therefore, $\mathbb{R}_n[x] = \mathrm{span}(S_3)$. Then S_3 is a spanning set of $\mathbb{R}_n[x]$.

Example II.20. Consider the vector space $\mathbb{R}[x]$ of real polynomials. An arbitrary vector $\vec{p}$ of this vector space has the following form:

$$\vec{p} = a_0 + a_1 x + a_2 x^2 + \cdots a_d x^d,$$

where $a_1, a_2, \ldots, a_n$ are scalars **and d is a non-negative integer**. It is clear that $\vec{p}$ is a linear combination of the vectors in the set $S_4 = \{1, x, x^2, \ldots, \}$ and, therefore, $\mathbb{R}[x] = \mathrm{span}(S_4)$. Then S_4 is a spanning set of $\mathbb{R}[x]$. **Notice that S_4 is an infinite spanning set**. Moreover it is clear that it is not possible to obtain a finite spanning set of $\mathbb{R}[x]$.

Example II.21. These are the subspaces of $\mathbb{R}^3$ that we now know of, the trivial subspace, the lines through the origin, the planes through the origin, and the whole space (of course, the picture shows only a few of the infinitely many subspaces). Afterwards we will prove that $\mathbb{R}^3$ has no other type of subspaces, so in fact this picture shows them all.



Notice that all the vector spaces we have considered until now, except $\mathbb{R}[x]$ (see Example II.20), admit a finite spanning set. That is, all of them are *finitely generated* vector spaces. But...

From now on, all the vector spaces that we will consider will be finitely generated and all the spanning sets that we will use will be finite.

UTILITARIAN SUMMARY II.22. Let V be a vector space.

- The subspaces of V are the spans.
- A spanning set of a subspace W of V is a set of vectors S such that $W = \text{span}(S)$.

III Linear independence

Definition III.1. A subset S of a vector space is *linearly independent* if $S \neq \{\vec{0}\}$ and none of its elements is a linear combination of the others. Otherwise it is *linearly dependent*.

Observe that, although this way of writing one vector as a combination of the others

$$\vec{s}_0 = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n$$

visually sets $\vec{s}_0$ off from the other vectors, algebraically there is nothing special about it in that equation. For any $\vec{s}_i$ with a coefficient c_i that is non-0 we can rewrite the relationship to set off $\vec{s}_i$.

$$\vec{s}_i = (1/c_i)\vec{s}_0 + \cdots + (-c_{i-1}/c_i)\vec{s}_{i-1} + (-c_{i+1}/c_i)\vec{s}_{i+1} + \cdots + (-c_n/c_i)\vec{s}_n$$

When we don't want to single out any vector by writing it alone on one side of the equation we will instead say that $\vec{s}_0, \vec{s}_1, \ldots, \vec{s}_n$ are in a *linear relationship* and write the relationship with all of the vectors on the same side. The next result rephrases the linear independence definition in this style. It is how we usually compute whether a finite set is dependent or independent.

Lemma III.2. *A subset $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ of a vector space is linearly independent if and only if the only linear relationship*

$$c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0} \qquad c_1, \ldots, c_n \in \mathbb{R}$$

is the trivial one $c_1 = 0, \ldots, c_n = 0$.

PROOF:

If S is linearly independent then no vector $\vec{s}_i$ is a linear combination of other vectors from S so there is no linear relationship where some of the $\vec{s}$'s have nonzero coefficients.

If S is not linearly independent then some $\vec{s}_i$ is a linear combination $\vec{s}_i = c_1\vec{s}_1 + \cdots + c_{i-1}\vec{s}_{i-1} + c_{i+1}\vec{s}_{i+1} + \cdots + c_n\vec{s}_n$ of other vectors from S. Subtracting $\vec{s}_i$ from both sides gives a relationship involving a nonzero coefficient, the -1 in front of $\vec{s}_i$. QED

Strategy III.3. Let $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ be a subset of a vector space V. To determine if S is linearly independent, take a linear relationship

$$c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$$

with **arbitrary** coefficients $c_1, \ldots, c_n \in \mathbb{R}$ and compute the possible values of $c_1, \ldots, c_n$. If the unique possibility is $c_1 = 0, \ldots, c_n = 0$ then S is linearly independent. Otherwise S is linearly dependent.

Example III.4. In $\mathbb{R}^2$, the two-element set $\{\begin{bmatrix}40\\15\end{bmatrix}, \begin{bmatrix}-50\\25\end{bmatrix}\}$ is linearly independent. To check this, take

$$c_1 \cdot \begin{bmatrix}40\\15\end{bmatrix} + c_2 \cdot \begin{bmatrix}-50\\25\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

and solving the resulting system

$$\begin{array}{l} 40c_1 - 50c_2 = 0 \\ 15c_1 + 25c_2 = 0 \end{array} \xrightarrow{\rho_2-(15/40)\rho_1} \begin{array}{rl} 40c_1 - & 50c_2 = 0 \\ & (175/4)c_2 = 0 \end{array}$$

shows that both c_1 and c_2 are zero. So the only linear relationship between the two given vectors is the trivial relationship.

In the same vector space, $\{\begin{bmatrix}40\\15\end{bmatrix}, \begin{bmatrix}20\\7.5\end{bmatrix}\}$ is linearly dependent since we can satisfy

$$c_1 \begin{bmatrix}40\\15\end{bmatrix} + c_2 \cdot \begin{bmatrix}20\\7.5\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

with $c_1 = 1$ and $c_2 = -2$.

Example III.5. The set $\{1+x, 1-x\}$ is linearly independent in $\mathbb{R}_2[x]$, the space of quadratic polynomials with real coefficients, because

$$0 + 0x + 0x^2 = c_1(1+x) + c_2(1-x) = (c_1+c_2) + (c_1-c_2)x + 0x^2$$

gives

$$\begin{array}{rl} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{array} \xrightarrow{-\rho_1+\rho_2} \begin{array}{rl} c_1 + c_2 = 0 \\ 2c_2 = 0 \end{array}$$

since polynomials are equal only if their coefficients are equal. Thus, the only linear relationship between these two members of $\mathbb{R}_2[x]$ is the trivial one.

Example III.6. In $\mathbb{R}^3$, where

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 9 \\ 2 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 4 \\ 18 \\ 4 \end{bmatrix}$$

the set $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent because this is a relationship

$$0 \cdot \vec{v}_1 + 2 \cdot \vec{v}_2 - 1 \cdot \vec{v}_3 = \vec{0}$$

where not all of the scalars are zero (the fact that some of the scalars are zero doesn't matter). Notice that this is not the unique linear relationship among the vectors: for example, if we multiply the above equation by any non-zero scalar, we get another relation: $0 \cdot \vec{v}_1 + 4 \cdot \vec{v}_2 - 2 \cdot \vec{v}_3 = \vec{0}$, $0 \cdot \vec{v}_1 + 6 \cdot \vec{v}_2 - 3 \cdot \vec{v}_3 = \vec{0}$, etc.

Example III.7. The empty subset of a vector space is linearly independent. There is no nontrivial linear relationship among its members as it has no members.

Example III.8. In any vector space, any subset containing the zero vector is linearly dependent because the zero vector is a linear combination of any set of vectors.

Example III.9. Let us consider the subset of $\mathbb{R}^2$ given in Example II.16:

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

This system is linearly independent. To check it, we must prove that the linear relationship

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

can only happen when $c_1 = c_2 = 0$. The above relationship gives rise to the system

$$\begin{array}{rl} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{array} \xrightarrow{\rho_2-\rho_1} \begin{array}{rl} c_1 + c_2 = 0 \\ -2c_2 = 0 \end{array}$$

whose unique solution is, clearly, $c_1 = c_2 = 0$.

Example III.10. The subset of $\mathbb{R}^n$ given in Example II.17,

$$S_1 = \{(1,0,0,\dots,0), (0,1,0,\dots,0), \dots, (0,0,\dots,0,1)\},$$

is linearly independent because the linear relationship

$$c_1(1,0,0,\dots,0) + \cdots + c_n(0,0,\dots,0,1) = (0,0,\dots,0,0)$$

is equivalent to a system whose unique solution is $c_1 = \cdots = c_n = 0$.

Example III.11. Let us consider the vector space $\mathcal{M}_{2\times 3}$ of 2×3 matrices and the set of vectors

$$S_2 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

given in Example II.18. This system is linearly independent because the linear relationship

$$c_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+c_3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c_5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + c_6 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

forces to the equalities $c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0$.

Example III.12. We consider now the vector space $\mathbb{R}_n[x]$ of real polynomials of degree $\leq n$ and the set $S_3 = \{1, x, x^2, \ldots, x^n\}$ given in Example II.19. A linear relationship

$$c_0 + c_1 x + c_2 x^2 + \cdots c_n x^n = 0$$

foces to $c_0 = c_1 = c_2 = \cdots = c_n = 0$ (because the zero polynomial is the one whose coefficients are zero). Therefore S_3 is linearly independent.

UTILITARIAN SUMMARY III.13.

- A subset of a vector space is *linearly independent* if none of its elements is a linear combination of the others. Otherwise it is *linearly dependent*.
- **Useful characterization of linearly independent sets**: A subset $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ of a vector space is linearly independent if and only if the only linear relationship

  $$c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0} \qquad c_1, \ldots, c_n \in \mathbb{R}$$

  is the trivial one $c_1 = 0, \ldots, c_n = 0$.

IV Basis and dimension

Basis

Definition IV.1. A *basis* of a vector space W is a set of vectors of W that is:

(1) a spanning set of W and

(2) linearly independent.

Example IV.2. We have seen that the sets of Examples II.16, II.17, II.18 and II.19 are spanning sets of their respective vector spaces and, moreover, they are linearly independent sets. Therefore they are bases.

Example IV.3. This is a basis of $\mathbb{R}^2$:

$$\{\begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\}.$$

Indeed, it is linearly independent:

$$c_1\begin{bmatrix}2\\4\end{bmatrix} + c_2\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad\Longrightarrow\quad \begin{aligned}2c_1 + 1c_2 = 0\\ 4c_1 + 1c_2 = 0\end{aligned} \quad\Longrightarrow\quad c_1 = c_2 = 0,$$

and it spans $\mathbb{R}^2$:

$$\begin{aligned}2c_1 + 1c_2 = x\\ 4c_1 + 1c_2 = y\end{aligned} \quad\Longrightarrow\quad c_2 = 2x - y \text{ and } c_1 = (y - x)/2$$

Notice that we have seen several bases of the space $\mathbb{R}^2$: this one and those of Examples II.16 and II.17.

Definition IV.4. For any $\mathbb{R}^n$

$$\mathcal{C}_n := \{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \dots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}\}$$

is the *canonical* (or *standard*) (or *natural*) basis. We denote these vectors $\vec{e}_1, \dots, \vec{e}_n$.

Example IV.5. The set $W = \{a \cdot \cos(x) + b \cdot \sin(x) \mid a, b \in \mathbb{R}\}$ is a subspace of the vector space of functions of the real variable x (because it is the span of the functions $\cos(x)$ and $\sin(x)$). A basis of W is

$$\{\cos(x), \sin(x)\}$$

Another basis is $\{\cos(x) - \sin(x), 2\cos(x) + 3\sin(x)\}$. The verification that these two sets are bases is left as an exercise.

Example IV.6. The trivial space $\{\vec{0}\}$ has only one basis, the empty one $\varnothing$.

Example IV.7. We have seen bases before. In Lesson 1 we described the solution set of homogeneous systems such as this one

$$\begin{aligned}x + y \quad - w &= 0\\ z + w &= 0\end{aligned}$$

by parametrizing:

$$\{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}\alpha + \begin{bmatrix}1\\0\\-1\\1\end{bmatrix}\beta \mid \alpha, \beta \in \mathbb{R}\}$$

Thus the vector space of solutions is the span of a two-element set. This two-vector set is also linearly independent; that is easy to check. Therefore the solution set is a subspace of $\mathbb{R}^4$ with a basis comprised of the above two elements.

Example IV.8. Parametrization helps find bases for other vector spaces, not just for solution sets of homogeneous systems. To find a basis for this subspace of $\mathcal{M}_{2\times 2}$

$$\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \mid a+b-2c=0 \}$$

we rewrite the condition as $a=-b+2c$.

$$\{ \begin{bmatrix} -b+2c & b \\ c & 0 \end{bmatrix} \mid b,c\in\mathbb{R} \} = \{ b \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \mid b,c\in\mathbb{R} \}$$

Thus, this is a natural candidate for a basis.

$$\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \}$$

The above work shows that it spans the space. Linear independence is also easy.

UTILITARIAN SUMMARY IV.9.

- Basis = linearly independent set + spanning set.
- The vector space $\mathbb{R}^n$ has a basis with n elements: the canonical basis.

Coordinates with respect to a basis

A basis of a vector space is, in particular, a spanning set. Therefore any vector can be written as a linear combination of the vectors of the basis. The next result shows that this fact is, actually, stronger: any vector can be written, **in a unique way**, as a linear combination of the vectors of the basis.

Theorem IV.10. *In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in a unique way.*

PROOF:

A subset is a basis if and only if it is a spanning set and linearly independent. And, a subset is a spanning set if and only if each vector in the space is a linear combination of elements of that subset in at least one way. Thus we need only show that a spanning subset is linearly independent if and only if every vector in the space is a linear combination of elements from the subset in at most one way.

Consider two expressions of a vector as a linear combination of the members of the subset. We can rearrange the two sums and if necessary add some $0\cdot\vec{\beta}_i$ terms so that the two sums combine the same $\vec{\beta}$'s in the same order: $\vec{v}=c_1\vec{\beta}_1+c_2\vec{\beta}_2+\cdots+c_n\vec{\beta}_n$ and $\vec{v}=d_1\vec{\beta}_1+d_2\vec{\beta}_2+\cdots+d_n\vec{\beta}_n$. Now

$$c_1\vec{\beta}_1+c_2\vec{\beta}_2+\cdots+c_n\vec{\beta}_n=d_1\vec{\beta}_1+d_2\vec{\beta}_2+\cdots+d_n\vec{\beta}_n$$

holds if and only if

$$(c_1-d_1)\vec{\beta}_1+\cdots+(c_n-d_n)\vec{\beta}_n=\vec{0}$$

holds. So, asserting that each coefficient in the lower equation is zero is the same thing as asserting that $c_i = d_i$ for each i, that is, that every vector is expressible as a linear combination of the $\vec{\beta}$'s in a unique way. QED

Definition IV.11. In a vector space with basis B the *representation of $\vec{v}$ with respect to B* (or *vector of coordinates of $\vec{v}$ with respect to B*) is the column vector of the coefficients used to express $\vec{v}$ as a linear combination of the basis vectors:

$$\mathrm{Rep}_B(\vec{v}) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where $B = \{\vec{\beta}_1, \dots, \vec{\beta}_n\}$ and $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$. The c's are the *coordinates of $\vec{v}$ with respect to B*.

Remark IV.12. We are "abusing" slightly of notation because the above definition requires that a basis be a sequence (ordered set), and not only a set. Notice that the order of the basis elements matters, in order to make this definition possible. We must consider, implicitly, the bases as "ordered" sets.

Example IV.13. In $\mathbb{R}_3[x]$, with respect to the basis $B = \{1, 2x, 2x^2, 2x^3\}$, the representation of $x + x^2$ is

$$\mathrm{Rep}_B(x + x^2) = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

(note that the coordinates are scalars, not vectors). With respect to a different basis $D = \langle 1 + x, 1 - x, x + x^2, x + x^3 \rangle$, the representation

$$\mathrm{Rep}_D(x + x^2) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

is different.

Remark IV.14. In $\mathbb{R}^n$ and with respect to the canonical basis $\mathcal{C}_n$, the vector starting at the origin and ending at $(v_1, \dots, v_n)$ has this representation:

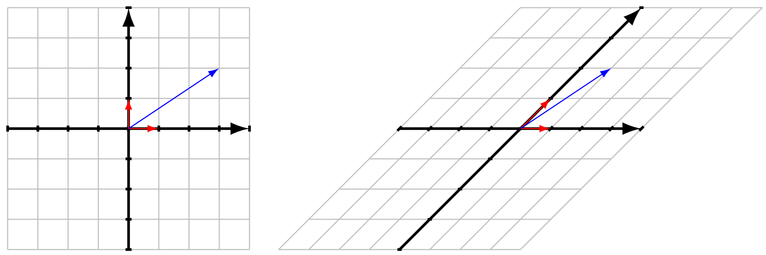
$$\mathrm{Rep}_{\mathcal{C}_n}\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

That is, the components of the vector can be "interpreted" as the coordinates of that vector with respect to the canonical basis of $\mathbb{R}^n$.

The following example shows the graphical meaning of representing a vector in different bases.

Example IV.15. In the figure on the left we have represented the canonical basis $\mathcal{C}_2 = \{(1,0),(0,1)\}$ of $\mathbb{R}^2$ and the vector $\vec{v} = (3,2)$; its coordinates with respect to $\mathcal{C}_2$ are 3 and 2. In the figure on the right we have represented the basis $B = \{(1,0),(1,1)\}$ and the same vector $\vec{v}$; the coordinates of $\vec{v}$ with respect to this basis are 1 and 2 because $\vec{v} = \mathbf{1}(1,0) + \mathbf{2}(1,1)$.

$$\mathrm{Rep}_{\mathcal{C}_2}(\vec{v}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathrm{Rep}_B(\vec{v}) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$



The following example shows a strategy to compute the representation of a vector of $\mathbb{R}^2$ with respect to a certain basis. Notice that the procedure is not only for $\mathbb{R}^2$, but it is general.

Example IV.16. In $\mathbb{R}^2$, consider the vector $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. To find the coordinates of that vector with respect to the basis

$$B = \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}\}$$

we solve

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

to get that $c_1 = 3$ and $c_2 = 1/2$. Then we have this:

$$\mathrm{Rep}_B(\vec{v}) = \begin{bmatrix} 3 \\ -1/2 \end{bmatrix}.$$

Strategy IV.17. To compute the coordinates of a vector $\vec{v}$ with respect to a basis $B = \{\vec{\beta}_1, \ldots, \vec{\beta}_n\}$ consider an expression of $\vec{v}$ as a linear combination of the vectors of B (but with indeterminate coefficients),

$$\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n,$$

and compute the coefficients $c_1, \ldots, c_n$ satisfying this relation.

UTILITARIAN SUMMARY IV.18. Any vector $\vec{v}$ of a vector space can be written, **in a unique way**, as a linear combination of the vector of a basis B. The coefficients of this linear combination are called *coordinates* of $\vec{v}$ with respect to B. The vector whose components are these coefficients is called *representation* of $\vec{v}$ with respect to B (or *vector of coordinates* of $\vec{v}$ with respect to B) and it is denoted $\mathrm{Rep}_B(\vec{v})$.

Any vector space has a basis

Lemma IV.19. (Elimination Lemma) *Let S be a finite set of vectors of V and let $\vec{v}$ be a vector of S. Then* $\operatorname{span}(S) = \operatorname{span}(S \setminus \{\vec{v}\})$ *if and only if $\vec{v}$ is a linear combination of some other vectors in S.*

(That is, we can "eliminate" a vector $\vec{v}$ from S without changing the span if and only if $\vec{v}$ is a linear combination of other vectors in S).

PROOF:

$\Rightarrow$

Suppose that $\operatorname{span}(S) = \operatorname{span}(S \setminus \{\vec{v}\})$. Therefore

$$\vec{v} \in S \subseteq \operatorname{span}(S) = \operatorname{span}(S \setminus \{v\}).$$

This means that there exist vectors $\vec{s}_1, \ldots, \vec{s}_r \in S$, different from $\vec{v}$, and such that $\vec{v}$ is a linear combination of them.

$\Leftarrow$

Supppose now that there exist vectors $\vec{s}_1, \ldots, \vec{s}_r \in S \setminus \{v\}$ such that $\vec{v}$ is a linear combination of them, that is,

$$\vec{v} = \sum_{i=1}^{r} \alpha_i \vec{s}_i$$

for some scalars $\alpha_1, \ldots, \alpha_r$.

We must prove that $\operatorname{span}(S) \subseteq \operatorname{span}(S \setminus \{v\})$ (because the other inclusion is obvious). Take, then, any vector $\vec{x} \in \operatorname{span}(S)$ and let us show that $\vec{x}$ also belongs to $\operatorname{span}(S \setminus \{\vec{v}\})$. We have that

$$\vec{x} = \lambda \vec{v} + \sum_{i=1}^{m} \beta_i \vec{t}_i$$

for some vectors $\vec{t}_1, \ldots, \vec{t}_m \in S \setminus \{\vec{v}\}$ and some scalars $\lambda, \beta_1, \ldots, \beta_m$. But then

$$\vec{x} = \lambda \sum_{i=1}^{r} \alpha_i \vec{s}_i + \sum_{i=1}^{m} \beta_i \vec{t}_i = \sum_{i=1}^{r} \lambda \alpha_i \vec{s}_i + \sum_{i=1}^{m} \beta_i \vec{t}_i \in \operatorname{span}(S \setminus \{v\}).$$

QED

Theorem IV.20. (Existence of bases) *Any vector space W admits a basis.*

PROOF:

Let S be a finite spanning set of W (recall that we are assuming that all considered vector spaces have a finite spanning set). Consider the following algorithm:

(1) If no vector in S is not linear combination of the others, then S is linearly independent and, therefore, S is a basis of W (and we stop and return S).

(2) If there exists a vector $\vec{v} \in S$ such that $\vec{v}$ is a linear combination of the other vectors in S then, by the Elimination Lemma, $S \setminus \{\vec{v}\}$ is also a spanning set of W.

(3) Replace S by $S \setminus \{v\}$ and go to Step (1).

Since the initial set S is finite, it is clear that this algorithm stops and that the returned set of vectors is a basis of W. QED

Any vector space can be identified with some space $\mathbb{R}^n$

Let us consider an arbitrary vector space W and let $B = \{\vec{u}_1, \dots, \vec{u}_n\}$ be a basis of W. Using this basis we can define a map $\mathrm{Rep}_B : W \to \mathbb{R}^n$ by

$$\vec{v} \mapsto \mathrm{Rep}_B(\vec{v}).$$

That is, it assigns, to each vector of $\vec{v}$ of W, its representation with respect to the basis B (notice that it is a vector of $\mathbb{R}^n$ whose components are the coordinates of $\vec{v}$ with respect to the basis B). Since the coordinates of a vector of W are unique, this map is well-defined. Let us prove that Rep_B is bijective:

- Rep_B is injective:

  Suppose that $\vec{v}$ and $\vec{w}$ are two vectors of W such that $\mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_B(\vec{w})$. Let $(c_1, \dots, c_n)$ be the vector of $\mathbb{R}^n$ such that

  $$\mathrm{Rep}_B(\vec{v}) = \mathrm{Rep}_B(\vec{w}) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

  Then, since $c_1, \dots, c_n$ are the coordinates with respect to B of both vectors $\vec{v}$ and $\vec{w}$ we have that

  $$\vec{v} = \sum_{i=1}^{n} c_i \vec{u}_i = \vec{w}.$$

  Therefore Rep_B is injective.

- Rep_B is surjective:

  For any vector $(c_1, \dots, c_n) \in \mathbb{R}^n$ it is clear that

  $$\mathrm{Rep}_B(\sum_{i=1}^{n} c_i \vec{u}_i) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

  Therefore Rep_B is surjective.

In fact, the map Rep_B is more than a bijection:

Proposition IV.21. *The map* Rep_B *is "compatible with linear combinations", that is, for any linear combination* $\alpha_1\vec{w}_1 + \alpha_2\vec{w}_2 + \cdots + \alpha_r\vec{w}_r$ *it holds that*

$$\mathrm{Rep}_B(\alpha_1\vec{w}_1 + \alpha_2\vec{w}_2 + \cdots + \alpha_r\vec{w}_r) = \alpha_1\mathrm{Rep}_B(\vec{w}_1) + \alpha_2\mathrm{Rep}_B(\vec{w}_2) + \cdots + \alpha_r\mathrm{Rep}_B(\vec{w}_r).$$

PROOF* :

Suppose that, for each $j \in \{1, \ldots, r\}$:

$$\mathrm{Rep}_B(\vec{w}_j) = \begin{bmatrix} c_{1,j} \\ c_{2,j} \\ \vdots \\ c_{n,j} \end{bmatrix}.$$

This means that

$$\vec{w}_j = c_{1,j}\vec{u}_1 + \cdots + c_{n,j}\vec{u}_n.$$

Therefore:

$$\alpha_1\vec{w}_1 + \alpha_2\vec{w}_2 + \cdots + \alpha_r\vec{w}_r = \alpha_1(c_{1,1}\vec{u}_1 + \cdots + c_{n,1}\vec{u}_n) + \alpha_2(c_{1,2}\vec{u}_1 + \cdots + c_{n,2}\vec{u}_n) + \cdots$$

$$+\alpha_r(c_{1,r}\vec{u}_1 + \cdots + c_{n,r}\vec{u}_n) = (\sum_{j=1}^{r}\alpha_j c_{1,j})\vec{u}_1 + \cdots + (\sum_{j=1}^{r}\alpha_j c_{n,j})\vec{u}_n.$$

Then

$$\mathrm{Rep}_B(\alpha_1\vec{w}_1 + \alpha_2\vec{w}_2 + \cdots + \alpha_r\vec{w}_r) = \begin{bmatrix} \sum_{j=1}^{r}\alpha_j c_{1,j} \\ \vdots \\ \sum_{j=1}^{r}\alpha_j c_{n,j} \end{bmatrix} = \sum_{j=1}^{r}\alpha_j \begin{bmatrix} c_{1,j} \\ c_{2,j} \\ \vdots \\ c_{n,j} \end{bmatrix} = \sum_{j=1}^{r}\alpha_j \mathrm{Rep}_B(\vec{w}_j).$$

QED

Proposition IV.22. *Let* $S = \{\vec{w}_1, \ldots, \vec{w}_r\}$ *be a subset of* W*.*

(a) S *is a spanning set of* W *if and only if* $\mathrm{Rep}_B(S)$ *is a spanning set of* $\mathbb{R}^n$*.*

(b) S *is linearly independent if and only if* $\mathrm{Rep}_B(S)$ *is linearly independent.*

(c) S *is a basis of* W *if and only if* $\mathrm{Rep}_B(S)$ *is a basis of* $\mathbb{R}^n$*.*

PROOF* :

(a) Since the map Rep_B is bijective, it is enough to prove the equality

$$\mathrm{Rep}_B(\mathrm{span}(S)) = \mathrm{span}(\mathrm{Rep}_B(S)). \tag{1}$$

But
$\vec{c} = (c_1, \ldots, c_r) \in \mathrm{Rep}_B(\mathrm{span}(S)) \Leftrightarrow$
there exist scalars $\alpha_1, \ldots, \alpha_r$ such that $\vec{c} = \mathrm{Rep}_B(\sum_{j=1}^{r}\alpha_j\vec{w}_j) \Leftrightarrow$
(by Proposition IV.21) there exist scalars $\alpha_1, \ldots, \alpha_r$ such that $\vec{c} = \sum_{j=1}^{r}\alpha_j\mathrm{Rep}_B(\vec{w}_j) \Leftrightarrow$
$\vec{c} \in \mathrm{span}(\mathrm{Rep}_B(S))$
Then, equality (1) holds.

(b) Let us prove the following equivalent statement: S is linearly dependent if and only if $\mathrm{Rep}_B(S)$ is linearly dependent.

S is linearly dependent $\Leftrightarrow$

There exist $\alpha_1, \ldots, \alpha_r$ (with some $\alpha_j \neq 0$) such that $\sum_{j=1}^{r} \alpha_j \vec{w}_j = \vec{0}$.

But, since Rep_B is bijective, this is equivalent to the assertion:

There exist $\alpha_1, \ldots, \alpha_r$ (with some $\alpha_j \neq 0$) such that $\mathrm{Rep}_B(\sum_{j=1}^{r} \alpha_j \vec{w}_j) = \mathrm{Rep}_B(\vec{0})$.

But, taking into account that $\mathrm{Rep}_B(\vec{0}) = \vec{0}$ and Lemma IV.21, this is equivalent to say that:

There exist $\alpha_1, \ldots, \alpha_r$ (with some $\alpha_j \neq 0$) such that $\sum_{j=1}^{r} \alpha_j \mathrm{Rep}_B(\vec{w}_j) = \vec{0} \Leftrightarrow$

$\mathrm{Rep}_B(S)$ is linearly dependent.

(c) It follows from (a) and (b). QED

Remark IV.23. A map between two vector spaces satisfying the compatibility condition given in Proposition IV.21 is called a *linear map* (we will study linear maps in Lesson 5). A linear map that is bijective is called an **isomorphism**. Therefore, what we have deduced above is that the map Rep_B is an isomorphism. The existence of this isomorphism means that W and $\mathbb{R}^n$ "have the same structure" as vector spaces. In other words, we can "identify" the vectors of W with the vectors of $\mathbb{R}^n$ given by their images by Rep_B.

All the bases of a vector space have the same number of elements

Proposition IV.24.

(1) Every spanning set of $\mathbb{R}^n$ has, at least, n vectors.

(2) Every linearly independent set of $\mathbb{R}^n$ has, at most, n vectors.

PROOF:

(1) (Informal proof with an example)

We consider a set of vectors of $\mathbb{R}^4$ with cardinality < 4, for example the set $S = \{\vec{u}_1 = (1, 2, 0, -2), \vec{u}_2 = (1, 0, 3, 2), \vec{u}_3 = (0, 2, 1, 0)\}$, and we are going to see the reasons for which S cannot be a spanning set of $\mathbb{R}^4$.

Given an arbitrary vector $\vec{b} = (\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4) \in \mathbb{R}^4$ we wonder if there exist scalars x_1, x_2, x_3 such that

$$x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 = \vec{b}.$$

This is equivalent to wonder if the solution set of the system of linear equations $A \cdot \vec{x} = \vec{b}$ is not empty **independently of the vector** $\vec{b}$, where $A = [\vec{u}_1 \;\; \vec{u}_2 \;\; \vec{u}_3]$ (the matrix whose columns are the vectors $\vec{u}_i$). Computing an echelon form of the augmented matrix:

$$\left[\begin{array}{ccc|l} 1 & 1 & 0 & b_1 \\ 0 & -2 & 2 & -2b_1+b_2 \\ 0 & 3 & 1 & b_3 \\ 0 & 4 & 0 & 2b_1+b_4 \end{array}\right] \sim \left[\begin{array}{ccc|l} 1 & 1 & 0 & b_1 \\ 0 & -2 & 2 & -2b_1+b_2 \\ 0 & 0 & 4 & -3b_1+\frac{3}{2}b_2+b_3 \\ 0 & 0 & 4 & -2b_1+2b_2+b_4 \end{array}\right] \sim \left[\begin{array}{ccc|l} 1 & 1 & 0 & b_1 \\ 0 & -2 & 2 & -2b_1+b_2 \\ 0 & 0 & 4 & -3b_1+\frac{3}{2}b_2+b_3 \\ 0 & 0 & 0 & b_1+\frac{1}{2}b_2-b_3+b_4 \end{array}\right]$$

For those vectors $\vec{b}$ such that $b_1 + \frac{1}{2}b_2 - b_3 + b_4 \neq 0$ the system has no solution and, therefore, S **cannot be a spanning set** of $\mathbb{R}^4$. The problem is that the obtained echelon form of A has a zero row, and it is clear that this will happen whenever the number of vectors in S is < 4. Hence, any spanning set of $\mathbb{R}^4$ must have, at least, 4 vectors.

Since this reasoning is valid in general we conclude that any spanning set of $\mathbb{R}^n$ must have, at least, n vectors.

(2) Reasoning by contradiction, suppose that $S = \{\vec{u}_1, \dots, \vec{u}_m\}$ is a linearly independent set of vectors of $\mathbb{R}^n$ such that $m > n$. We take an arbitrary linear relationship

$$c_1\vec{u}_1 + \cdots + c_m\vec{u}_m = \vec{0},$$

that is

$$c_1 \begin{bmatrix} | \\ \vec{u}_1 \\ | \end{bmatrix} + c_2 \begin{bmatrix} | \\ \vec{u}_2 \\ | \end{bmatrix} + \cdots + c_m \begin{bmatrix} | \\ \vec{u}_m \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since the set S is linearly independent, the unique vector $(c_1, \dots, c_m)$ satisfying this linear relation is the zero vector. But this means that the zero vector is the unique solution of the homogeneous linear system $A \cdot \vec{x} = \vec{0}$, where A is the $n \times m$ matrix whose columns are the vectors in S. But this is a contradiction because the number of unknowns m is greater than the number of equations n. QED

Corollary IV.25. *Let W be a vector space and let $B = \{\vec{u}_1, \dots, \vec{u}_n\}$ be a basis of W. Then*

(1) Every spanning set of W has, at least, n vectors.

(2) Every linearly independent set of W has, at most, n vectors.

PROOF:

(1) If S is a spanning set of W then $\mathrm{Rep}_B(S)$ is a spanning set of $\mathbb{R}^n$ (by Proposition IV.22) and, therefore, S must have, at least, n vectors (by Proposition IV.24).

(2) If S is a linearly independent set of W then $\mathrm{Rep}_B(S)$ is a linearly independent set of $\mathbb{R}^n$ (by Lemma IV.22) and, therefore, S must have, at most, n vectors (by Proposition IV.24). QED

Corollary IV.26. *All the bases of a vector space W have the same number of vectors.*

PROOF:

Let $B_1 = \{\vec{u}_1, \dots, \vec{u}_n\}$ and $B_2 = \{\vec{v}_1, \dots, \vec{v}_m\}$ be two bases of W.

On the one hand, since B_2 is basis of W and B_1 is a spanning set of W, by Corollary IV.25 (applied to $B = B_2$ and $S = B_1$) we have that $n \geq m$.

On the other hand, since B_1 is basis of W and B_2 is a spanning set of W, by Corollary IV.25 (applied to $B = B_1$ and $S = B_2$) we have that $m \geq n$.

Therefore $n = m$. QED

This result allows us to state the following definition:

Definition IV.27. The *dimension* of a vector space W, denoted by $\dim(W)$, is the number of vectors of any basis of W.

Example IV.28. The dimension of $\mathbb{R}^n$ is n.

The next proposition, that is direct consequence of Corollary IV.25 and Corollary IV.26, provides an interesting characterization of the dimension.

Proposition IV.29. *The dimension of a vector space W is*

- *the minimum number of elements of a spanning set of W and*
- *the maximum number of elements of a linearly independent set of W.*

Working with vector spaces different from $\mathbb{R}^n$

We have seen in the preceding subsection that any (finitely generated) vector space W can be identified with some space $\mathbb{R}^n$. We can take advantage of this fact to solve, in a comfortable way, problems involving linear operations in W:

Strategy IV.30. If S is a set of vectors of W with which we want to do something:

- Step 1: Fix a basis B of W (say, with n vectors).
- Step 2: Compute the representations of the vectors in S with respect to the fixed basis B. These are vectors in $\mathbb{R}^n$.
- Step 3: Do what you want to do with the obtained representations.
- Step 4: Perhaps you get, as a result, some vectors in $\mathbb{R}^n$ that must be interpreted as representations (with respect to B) of certain vectors of W. Obtain these vectors of W from which they come from.

Example IV.31. Consider the following subset of vectors of $\mathbb{R}_3[x]$:

$$S = \{5, x - x^3, x^2\}$$

We want to give an answer to the following question: does the vector $1 + x + x^2 + x^3$ belong to $\text{span}(S)$?

Following the above instructions, fix first a basis of $\mathbb{R}_3[x]$. May be the "easiest" one is

$$B = \{1, x, x^2, x^3\}.$$

Now consider the set of representations of the vectors in S with respect to this basis,

$$S' = \{ \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \},$$

and the representation of the vector $1 + x + x^2 + x^3$:

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The "translated" problem is: is $\vec{b}$ a linear combination of the vectors in S'? We need to check if there exist scalars c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

But this is equivalent to solve the linear system $A \cdot \vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

This system has no solution and, therefore, $\vec{b}$ is not a linear combination of the vectors in S. So, we conclude that the vector $1 + x + x^2 + x^3$ is not a linear combination of the vectors in S and, hence, it does not belong to $\mathrm{span}(S)$.

UTILITARIAN SUMMARY IV.32.

- Any vector space admits a basis.
- All the bases of a vector space V have the same number of elements. This number is called *dimension* of V, $\dim(V)$.
- Moreover, the vectors of any vector space "can be identified" with their representations with respect to a certain basis. This means that all the computations with vectors of any vector space can be translated in terms of computations with vectors of $\mathbb{R}^n$ where n is the dimension.

Computing a basis using elementary row operations

Definition IV.33. Let $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ be an (ordered) set of vectors of a vector space V. We will say that we are performing an *elementary operation* to S if one of the following operations is done:

- (Elementary operation of type 1): We swap two vectors of S.
- (Elementary operation of type 2): We multiply one of the vectors in S by a non-zero scalar.
- (Elementary operation of type 3): We add, to a vector of S, a multiple of a different vector of S.

The next lemma shows that, applying elementary operations to a set of vectors, the span remains invariant.

Lemma IV.34. *If S' is the (ordered) set obtained after performing an elementary operation to the set $S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ then* $\mathrm{span}(S) = \mathrm{span}(S')$.

PROOF* :

If the elementary operation is of type 1 then it is evident that $\mathrm{span}(S) = \mathrm{span}(S')$ (S and S' correspond to the same unordered set!).

Assume now that the operation is of type 2, that is, $S' = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{j-1}, \lambda\vec{u}_j, \vec{u}_{j+1} \ldots, \vec{u}_n\}$ for some j and for some scalar $\lambda \neq 0$.

- On the one hand, $\vec{u}_i \in \mathrm{span}(S')$ if $i \neq j$ and $\vec{u}_j = \frac{1}{\lambda}\lambda\vec{u}_j$ belongs also to $\mathrm{span}(S')$ (because it is a multple of a vector of S'). Therefore $S \subseteq \mathrm{span}(S')$; then $\mathrm{span}(S')$ also contain all linear combinations of the vectors in S, that is, $\mathrm{span}(S) \subseteq \mathrm{span}(S')$.
- On the other hand, $\vec{u}_i \in \mathrm{span}(S)$ if $i \neq j$ and $\lambda\vec{u}_j \in \mathrm{span}(S)$. Reasoning as before we deduce the other inclussion $\mathrm{span}(S') \subseteq \mathrm{span}(S)$.

Then we have the equality $\mathrm{span}(S') = \mathrm{span}(S)$.

Finally, assume that the elementary operation is of type 3, that is, that S' has been obtained by adding, to a certain vector $\vec{u}_j$, a non-zero multiple of a different vector $\vec{u}_k$. Without loss of generality we can sssume that $j = 1$ and $k = 2$ (we can change the order, if necessary). Therefore:

$$S' = \{\vec{u}_1 + \lambda\vec{u}_2, \vec{u}_2, \ldots, \vec{u}_n\}.$$

It is clear that $S' \subseteq \mathrm{span}(S)$ and, therefore, $\mathrm{span}(S') \subseteq \mathrm{span}(S)$ (reasoning as before).

Let us show that $S \subseteq \mathrm{span}(S')$. Since $\vec{u}_i \in \mathrm{span}(S')$ if $i \neq 1$, it is enough to prove that $\vec{u}_1 \in \mathrm{span}(S')$. But

$$\vec{u}_1 = (\vec{u}_1 + \lambda\vec{u}_2) - \lambda\vec{u}_2 \in \mathrm{span}(S').$$

Hence we have the inclusion $S \subseteq \mathrm{span}(S')$, and this implies that $\mathrm{span}(S) \subseteq \mathrm{span}(S')$. QED

Example IV.35. (Revealing example) Consider the following subset of 4 vectors of $\mathbb{R}^4$:

$$S = \{\vec{u}_1 = (1,3,2,4), \vec{u}_2 = (2,9,3,0), \vec{u}_3 = (3,2,0,-2), \vec{u}_4 = (4,15,7,8)\}$$

and the subspace $W = \text{span}(S)$. Is it possible to get a spanning set of W with less vectors? To give an answer to this question, we apply the above strategy. The matrix R whose rows are the transposed vectors of S is

$$R = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 9 & 3 & 0 \\ 3 & 2 & 0 & -2 \\ 4 & 15 & 7 & 8 \end{bmatrix}$$

Applying elementary row operations to this matrix we obtain

$$R \to \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 3 & -1 & -8 \\ 0 & -7 & -6 & -14 \\ 0 & 3 & -1 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 3 & -1 & -8 \\ 0 & 0 & -25/3 & -98/3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to R' = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 3 & 1 & 8 \\ 0 & 0 & 25 & 98 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, by Lemma IV.34, the set

$$S' = \{(1,3,2,4), (0,3,1,8), (0,0,25,98)\}$$

is a spanning set of W with only 3 vectors. We can remove the last zero row because the zero vector is a linear combination of S'.

Now we are going to see that, alternatively, we can choose also the first three vectors of S as a spanning set. Taking into account the elementary row operations that we have done to the matrix R we can obtain the row vectors of R' as linear combinations of the row vectors of R:

In the first step we have subtracted 2 times the first row to the second one, we have subtracted 3 times the first row to the row 3 and, finally, we have subtracted 4 times the first row to the last one. This means that the row vectors in the second matrix are the transposes of $\vec{u}_1' = \vec{u}_1$, $\vec{u}_2' = \vec{u}_2 - 2\vec{u}_1$, $\vec{u}_3' = \vec{u}_3 - 3\vec{u}_1$ y $\vec{u}_4' = \vec{u}_4 - 4\vec{u}_1$.

Similarly, the rows of the third matrix correspond to the transposes of the following vectors:

$$\vec{u}_1'' = \vec{u}_1' = \vec{u}_1$$

$$\vec{u}_2'' = \vec{u}_2' = \vec{u}_2 - 2\vec{u}_1$$

$$\vec{u}_3'' = \vec{u}_3' + \frac{7}{3}\vec{u}_2' = \frac{5}{3}\vec{u}_1 - \frac{7}{3}\vec{u}_2 + \vec{u}_3,$$

$$\vec{u}_4'' = \vec{u}_4' - \vec{u}_2' = -2\vec{u}_1 - \vec{u}_2 + \vec{u}_4$$

And, for the rows of the last matrix:

$$\vec{u}_1''' = \vec{u}_1'' = \vec{u}_1' = \vec{u}_1$$

$$\vec{u}_2''' = \vec{u}_2'' = \vec{u}_2' = \vec{u}_2 - 2\vec{u}_1$$

$$\vec{u}_3''' = 3\vec{u}_3'' = 5\vec{u}_1 - 7\vec{u}_2 + 3\vec{u}_3,$$

$$\vec{u}_4''' = \vec{u}_4'' = \vec{u}_4' - \vec{u}_2' = -2\vec{u}_1 - \vec{u}_2 + \vec{u}_4$$

Since $\vec{u}_4''' = \vec{0}$ (because the last row of R' is zero) we deduce the following non-trivial linear relationship among the vectors of S:

$$-2\vec{u}_1 - \vec{u}_2 + \vec{u}_4 = \vec{0}.$$

This implies that $\vec{u}_4$ is a linear combination of $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$. Then, by the Elimination Lemma, $S'' = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ **is also a spanning set of W with only 3 vectors**.

We have obtained two spanning sets of W smaller than S: S' and S''. But we can deduce a stronger fact: S' **and** S'' **are, in fact, two bases of W**. Why? Let's see the reason:

- On the one hand S' is an "echelon system of non-zero vectors" (in the sense that their transposes form an echelon matrix) and **any "echelon system of non-zero vectors" is linearly independent** (no vector can be written as a linear combination of the others; think about it for a while!). So, S' is a basis of S because it is linearly independent and spanning set.
- On the other hand a similar reasoning to the one given in the proof of Theorem IV.20 shows that S'' contains a basis of W. But, since all the bases of W have the same number of vectors and S' is a basis with 3 vectors, the whole set S'' must be a basis.

The reasoning used in this example can be done in general and it suggests an easy strategy to compute a basis of a vector subspace of $\mathbb{R}^n$ (in fact, not only one, but two bases!):

Strategy IV.36. (Computing a basis) Let $S = \{\vec{u}_1, \ldots, \vec{u}_r\}$ be a spanning set of a subspace W of $\mathbb{R}^n$. To get a basis of W:

(1) Write the matrix R whose **rows** are the transposed vectors of S.

(2) Compute, performing elementary row operations, a row echelon form R' of R.

(3) Let S' be the set of vectors of $\mathbb{R}^n$ given by the transposed **non-zero rows** of R'. This is a basis of W.

- A different basis of W is given by the vectors of the initial set S associated with the non-zero rows of R' (**be careful if you have interchanged rows!**).

Remark IV.37. To compute a basis of a vector subspace of a space different from some $\mathbb{R}^n$ follow the procedure explained in the preceding subsection.

Example IV.38. Consider the vector space $\mathcal{M}_{2\times 3}$ of 2×3 matrices and let W be the vector subspace of $\mathcal{M}_{2\times 3}$ given by $W = \operatorname{span}(S)$ where

$$S = \{\vec{u}_1 = \begin{bmatrix} 0 & 3 & 8 \\ 1 & 3 & 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 3 & 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} 4 & 5 & 6 \\ 2 & 2 & 1 \end{bmatrix}\}.$$

Problem: find a basis of W.

First, we fix a basis of $M_{2\times 3}$. For example:

$$B := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Taking coordinates with respect to B we can identify W with the subspace $\operatorname{Rep}_B(W)$ of $\mathbb{R}^6$:

$$\operatorname{Rep}_B(W) = \operatorname{span}(\begin{bmatrix} 0 \\ 3 \\ 8 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \\ 2 \\ 2 \\ 1 \end{bmatrix}).$$

Now we compute a basis of $\operatorname{Rep}_B(W)$ following the steps given in the above strategy: we write the transposed vectors of the spanning set of $\operatorname{Rep}_B(W)$ in a matrix, and then we compute an echelon form of this matrix:

$$R = \begin{bmatrix} 0 & 3 & 8 & 1 & 3 & 1 \\ 0 & 2 & 3 & 1 & 3 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \\ 4 & 5 & 6 & 2 & 2 & 1 \end{bmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_4} \begin{bmatrix} 4 & 5 & 6 & 2 & 2 & 1 \\ 0 & 2 & 3 & 1 & 3 & 0 \\ 0 & 1 & 5 & 0 & 0 & 1 \\ 0 & 3 & 8 & 1 & 3 & 1 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 4 & 5 & 6 & 2 & 2 & 1 \\ 0 & 2 & 3 & 1 & 3 & 1 \\ 0 & 0 & 7 & -1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R'$$

Notice that the unique elementary operation of type 1 is done at the beginning (swapping rows 1 and 4). Therefore, the first row of the final matrix R' is "associated" with $\vec{u}_4$, the last one with $\vec{u}_1$ and the second and third rows with $\vec{u}_2$ and $\vec{u}_3$. Taking into account the above strategy, we can consider the following two bases of $\mathrm{Rep}_B(W)$:

$$\{\mathrm{Rep}_B(\vec{u}_4) = \begin{bmatrix} 4 \\ 5 \\ 6 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \mathrm{Rep}_B(\vec{u}_2) = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \mathrm{Rep}_B(\vec{u}_3) = \begin{bmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}\} \quad \text{and}$$

$$\{\begin{bmatrix} 4 \\ 5 \\ 6 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 7 \\ -1 \\ -3 \\ 2 \end{bmatrix}\}.$$

The corresponding bases of W are:

$$S' = \{\vec{u}_4, \vec{u}_2, \vec{u}_3\} \quad \text{and}$$

$$S'' = \{\begin{bmatrix} 4 & 5 & 6 \\ 2 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 3 \\ 1 & 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 7 \\ -1 & -3 & 2 \end{bmatrix}\}.$$

Notice that $\dim(W) = 3$.

A useful result

The following result is very useful for exercises. It shows that, when we know the dimension of a vector space V, to prove that a certain set of vectors is a basis one only needs to check one of the conditions (1) and (2) of the definition of basis.

Theorem IV.39. *Let V be a vector space of dimension $n \geq 1$ and let B be a subset of B with n vectors. Then the following properties are equivalent:*

(a) B is a basis of V.

(b) B is linearly independent.

(c) B is a spanning set of V.

This means that, when we have a set of n vectors (where n is the dimension), to determine whether it is a basis one only needs to check either that it is linearly independent of that it is a spanning set. (Only one condition instead of both).

PROOF:

It is clear that (a) implies (b) and (c). We only need to prove the implications $(b) \Rightarrow (a)$ and $(c) \Rightarrow (a)$.

$(b) \Rightarrow (a)$

We will reason by contradiction, assuming that B is linearly independent but it is not a spanning set of V. Then, there exists a vector $\vec{v} \in V$ such that $\vec{v} \notin \mathrm{span}(B)$. Then $B \cup \{\vec{v}\}$ is linearly independent. But this is a contradiction with Proposition IV.29 (n is the maximum number of linearly independent vectors of V).

$(c) \Rightarrow (a)$

As before, we will reason by contradiction assuming that B is a spanning set but it is not linearly independent. Then there exists some vector $\vec{v} \in B$ such that $\vec{v}$ is a linear combination of other elements of $\vec{v}$. Then, by the Elimination Lemma, $\mathrm{span}(B \setminus \{\vec{v}\}) = \mathrm{span}(B) = V$. But this is a contradiction because, by Proposition IV.29, n is the minimum number of vectors that span V. QED

Example IV.40. Consider the set of vectors of $\mathbb{R}^3$ given by $B = \{(1,0,0),(0,3,5),(0,0,9)\}$. Since the matrix whose rows are the transposed vectors of B is row echelon, B is linearly independent. Since B has 3 vectors and the dimension of $\mathbb{R}^3$ is 3, applying the above theorem we deduce that B is a basis of $\mathbb{R}^3$.

UTILITARIAN SUMMARY IV.41.

- Given a finite set of vectors S of $\mathbb{R}^n$, consider the matrix R whose **rows** are the transposed vectors of S. The transposed rows of any matrix that is (row) equivalent to R span the same subspace than S. The transposed non-zero rows of any row echelon form of R form a basis of $\mathrm{span}(S)$. As a consequence, the dimension of $\mathrm{span}(S)$ is equal to the rank of R.
- When we have a set of n vectors (where n is the dimension), to determine whether it is a basis one only needs to check either that it is linearly independent of that it is a spanning set. (Only one condition instead of both).

V Change of basis

Let $B = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ and $B' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ two bases of a non-trivial vector space V. Let $\vec{x}$ be a vector of V. Assume that representants of $\vec{x}$ with respect to these bases are:

$$\mathrm{Rep}_B(\vec{x}) = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad \mathrm{Rep}_{B'}(\vec{x}) = (x'_1, x'_2, \ldots, x'_n).$$

The question that we are going to solve in this section is the following: which is the relationship between the representants $\mathrm{Rep}_B(\vec{x})$ and $\mathrm{Rep}_{B'}(\vec{x})$?

Assume that we know the coordinates of the vectors in $\mathcal{B}'$ with respect to the basis $\mathcal{B}$:

$$\vec{v}_j = p_{1j}\vec{u}_1 + p_{2j}\vec{u}_2 + \cdots + p_{nj}\vec{u}_n, \quad j = 1, 2, \ldots, n.$$

Then

$$(x_1, x_2, \ldots, x_n) = \mathrm{Rep}_B(\vec{x}) = \mathrm{Rep}_B(x'_1\vec{v}_1 + x'_2\vec{v}_2 + \cdots + x'_n\vec{v}_n) = x'_1\mathrm{Rep}_B(\vec{v}_1) + x'_2\mathrm{Rep}_B(\vec{v}_2) + \cdots + x'_n\mathrm{Rep}_B(\vec{v}_n)$$

$$= x'_1\begin{bmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{n1} \end{bmatrix} + x'_2\begin{bmatrix} p_{12} \\ p_{22} \\ \vdots \\ p_{n2} \end{bmatrix} + \cdots + x'_n\begin{bmatrix} p_{1n} \\ p_{2n} \\ \vdots \\ p_{nn} \end{bmatrix} = \underbrace{\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}}_{M_{B'\mathcal{B}}} \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix}.$$

$M_{B'B}$ is the **change of basis matrix from B' to $\mathcal{B}$. Its columns are the representants of the vectors of B' with respect to B.**

We have deduced the following:

Proposition V.1. *Given two bases B and B' of a vector space V, every vector $\vec{x} \in V$ satisfies the following relation:*

$$\mathrm{Rep}_B(\vec{x}) = M_{B'B}\mathrm{Rep}_{B'}(\vec{x}).$$

Similarly we can construct the change of basis matrix from B to B', $M_{\mathcal{B}B'}$. Every vector $\vec{x}$ satisfies the following:

$$\mathrm{Rep}_B(\vec{x}) = M_{B'B}\mathrm{Rep}_{B'}(\vec{x}) = M_{B'B}(M_{BB'}\mathrm{Rep}_B(\vec{x})) = (M_{B'B}M_{BB'})\mathrm{Rep}_B(\vec{x}).$$

Taking $\vec{x} = \vec{u}_i$ one has that $\mathrm{Rep}_B(\vec{x}) = \mathrm{Rep}_B(\vec{u}_i) = \vec{e}_i$, where $\vec{e}_i = (0, 0, \ldots, 1, \ldots, 0)$, with a 1 in the ith component; then, applying the above equality to $\vec{u}_i$ we obtain that

$$\vec{e}_i = (M_{B'B}M_{BB'})\vec{e}_i = (\text{column } i \text{ of } M_{B'B}M_{BB'}).$$

Therefore $M_{B'B}M_{\mathcal{B}B'}$ is the identity matrix $I_{n\times n}$. As a consequence we have the following proposition:

Proposition V.2. *The change of basis matrices $M_{B'B}$ and $M_{BB'}$ and invertible and*

$$M_{B'B} = M_{BB'}^{-1}.$$

Example V.3. Let us consider the following bases of $\mathbb{R}^2$: $B_1 = \{(1,1), (5,-1)\}$, $B_2 = \{(0,3), (2,5)\}$. We are going to compute the change of basis matrices $M_{B_1B_2}$ and $M_{B_2B_1}$.

To obtain $M_{B_1B_2}$ we have to compute t he representants of the vectors of B_1 with respect to the basis $\mathcal{B}_2$ (we must "write the vectors of $\mathcal{B}_1$ in $\mathcal{B}_2$") and then we must arrange them by columns. This means that we must solve two systems of linear equations with the same coefficient matrix:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = x_1\begin{bmatrix} 0 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \qquad \begin{bmatrix} 5 \\ -1 \end{bmatrix} = x_1\begin{bmatrix} 0 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We can solve both systems simultaneously (see Lesson 1):

$$\left[\begin{array}{cc|cc} 0 & 2 & 1 & 5 \\ 3 & 5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 3 & 5 & 1 & -1 \\ 0 & 2 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 5/3 & 1/3 & -1/3 \\ 0 & 1 & 1/2 & 5/2 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -1/2 & -9/2 \\ 0 & 1 & 1/2 & 5/2 \end{array}\right].$$

Then, the representats that we are looking for are

$$\mathrm{Rep}_{B_2}(1,1) = (-1/2, 1/2) \qquad \text{and} \qquad \mathrm{Rep}_{B_2}(5,-1) = (-9/2, 5/2)$$

and the change of basis matrix from B_1 to B_2 is

$$M_{B_1B_2} = \begin{bmatrix} -1/2 & -9/2 \\ 1/2 & 5/2 \end{bmatrix}.$$

The change of basis matrix from B_2 to B_1 can be obtained by a similar procedure (computing the representants of the vectors of $\mathcal{B}_2$ with respect to B_1 and arranging them by columns) or also, using the above proposition, computing the inverse of the matrix $M_{B_1B_2}$:

$$M_{B_2B_2} = M_{B_1B_2}^{-1} = \begin{bmatrix} -1/2 & -9/2 \\ 1/2 & 5/2 \end{bmatrix}^{-1} = \begin{bmatrix} 5/2 & 9/2 \\ -1/2 & -1/2 \end{bmatrix}.$$