Session 12: Operations with sets

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Intersection and union

Given two sets A and B we can define two new sets as follows:

The **intersection** of A and B is the set of all elements which belong both to A and B. It is denoted $A \cap B$.

The **union** of A and B is the set of all elements which belong to A or B (or to both). It is denoted $A \cup B$.

Symbolically:

$$A \cap B = \{x \mid x \in A \ \wedge \ x \in B\},$$
$$A \cup B = \{x \mid x \in A \ \vee \ x \in B\}.$$

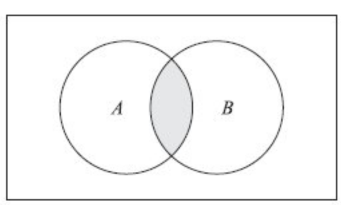
There are obvious connections between intersection of sets and conjunction of propositions and between union of sets and disjunction of propositions.Indeed, if A and B are defined by intension by means of the predicates $P(x)$ and $Q(x)$ with a common universe U, that is, $A = \{x \in U \mid P(x)\}$ and $B = \{x \in U \mid Q(x)\}$, then

$$A \cap B = \{x \in U \mid P(x) \wedge Q(x)\},$$

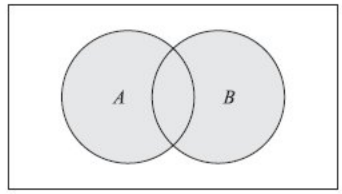
$$A \cup B = \{x \in U \mid P(x) \vee Q(x)\}.$$

These sets can best be visualized by the following Venn diagrams:

- The shaded region represents the intersection of A and B:



- The shaded region represents the union of A and B:



Clearly we can extend the definition of intersection and union to more than two sets. Let $A_1, A_2, \ldots, A_n$ be sets.

The intersection is:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cdots \cap A_n = \{x \mid x \in A_1 \ \wedge \ x \in A_2 \ \wedge \cdots \wedge \ x \in A_n\}.$$

The union is:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cdots \cup A_n = \{x \mid x \in A_1 \ \vee \ x \in A_2 \ \vee \cdots \vee \ x \in A_n\}.$$

Sets A and B are said **disjoint** if they have no elements in common; that is, if $A \cap B = \emptyset$.

Example 1. If $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 3, 7, 8, 9\}$ we have that $A \cap B = \{1, 2, 3\}$ and $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Example 2. If $\mathbb{Q}$ denotes the set of rational numbers and $\mathbb{I}$ the set of irrational numbers, then $\mathbb{Q} \cup \mathbb{I} = \mathbb{R}$ and $\mathbb{Q} \cap \mathbb{I} = \emptyset$ (that is, these sets are disjoint).

2 Complement

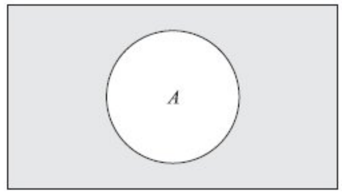
Assume that we have defined a **universal set**, that is, a set containing all sets that we are going to consider. We can define the **complement** of a set with respect to this universal set:

Given a set A inside a universal set U, the **complement of A with respect to U** consists of all the elements of U which do not belong to A. Usually we will assume that the universal set is known and well defined and we will simply say **complement of A**. The complement of A is denoted A^c.

In other words,

$$A^c = \{x \in U \mid x \notin A\}.$$

Using Venn diagrams, the complement of A is represented by the shaded region of the next figure. The rectangle represents the universal set U.



Example 3. If the universal set is $U = \{a, b, c, d, e, f, g, h\}$ and $A = \{a, b, c\}$ then

$$A^c = \{d, e, f, g, h\}.$$

Example 4. If the universal set is $U = \mathbb{R}$ and $A = \mathbb{Q}$ then

$$A^c = \mathbb{I}.$$

3 Properties of the union, intersection and complement

The following properties are evident but we are going to prove them formally:

Property. If A and B are sets then:

(a) $A \subseteq A \cup B$

(b) $A \cap B \subseteq A$

(c) $A \subseteq B \Leftrightarrow A \cap B = A \Leftrightarrow A \cup B = B$

Proof.

(a) Notice that $A \cup B = \{x \mid x \in A \vee x \in B\}$. To check that $A \subseteq A \cup B$ we must choose an *arbitrary* element of A and we must check that it belongs to $A \cup B$. Let us proceed:

Take an arbitrary element $x \in A$. By *addition* (the inference law), it holds that the proposition $x \in A \ \vee \ x \in B$ is true. Therefore $x \in A \cup B$.

(b) Notice that $A \cap B = \{x \mid x \in A \wedge x \in B\}$. To check that $A \cap B \subseteq A$ we must choose an *arbitrary* element of $A \cap B$ and we must check that it belongs to A. Let us proceed:

Take an arbitrary element $x \in A \cap B$. Then $x \in A \ \wedge \ x \in B$. By *simplification* (the inference law), it holds that $x \in A$ is true.

(c) Let us prove first the *equivalence* $A \subseteq B \Leftrightarrow A \cap B = A$. We can use *biconditional inference*, that is, we can prove first the direct implication $(A \subseteq B \Rightarrow A \cap B = A)$ and, then, the converse implication $(A \cap B = A \Rightarrow A \subseteq B)$.

Proof of the direct implication $(A \subseteq B \Rightarrow A \cap B = A)$:

Assume the hypothesis $A \subseteq B$. Taking into account Section 4 of Session 11, to prove the equality of sets $A \cap B = A$ we can prove both inclusions ($A \cap B \subseteq A$ and $A \subseteq A \cap B$). Notice that the first one is already proved in (b); therefore we only need to check that $A \subseteq A \cap B$.

To do that, take an *arbitrary* element x of A. Since, by hypothesis, $A \subseteq B$, it holds that x also belongs to B. Therefore $x \in A \ \wedge \ x \in B$, and this means that $x \in A \cap B$.

Proof of the converse implication ($A \cap B = A \Rightarrow A \subseteq B$):

Assume, as hypothesis, that $A \cap B = A$. To check that $A \subseteq B$ we only need to verify that an arbitrary element of A is also in B.

Take an arbitrary element $x \in A$. Since, by hypothesis, $A = A \cap B$, it holds that $x \in A \cap B$ and, therefore, $x \in B$ (by definition of the intersection).

The proof of the *equivalence* $A \subseteq B \Leftrightarrow A \cup B = B$ is left as an exercise.

3.1 Boolean properties

Similarly, the following properties concerning the union, intersection and complement can be proved:

Let A, B, C be subsets of a universal set U.

1. Associative properties

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

2. Commutative properties

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

3. Distributive properties

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. Identity element

$$A \cup \emptyset = A$$
$$A \cap U = A$$

5. Inverse element

$$A \cup A^c = U$$
$$A \cap A^c = \emptyset$$

These properties are called *Boolean* properties. Notice that they are totally analogous to the Boolean properties concerning propositional forms (where the union corresponds to the disjunction, the intersection corresponds to the conjunction, "taking complement" corresponds to the negation, the universal set U to a tautology and the empty set $\emptyset$ to a contradiction):

Sets vs. propositional forms

1. Associative properties

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

2. Commutative properties

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

3. Distributive properties

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. Identity elements

$$A \cup \emptyset = A$$
$$A \cap U = A$$

5. Inverse elements

$$A \cup A^c = U$$
$$A \cap A^c = \emptyset$$

1. Associative properties

$$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$$
$$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$$

2. Commutative properties

$$P \vee Q \equiv Q \vee P$$
$$P \wedge Q \equiv Q \wedge P$$

3. Distributive properties

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$
$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$$

4. Identity elements

$$P \vee \phi \equiv P$$
$$P \wedge \tau \equiv P$$

5. Inverse elements

$$P \vee \neg P \equiv \tau$$
$$P \wedge \neg P \equiv \phi$$

3.2 Other properties

There are other interesting properties involving union, intersection and complement. They can be proved directly (as before), and also can be deduced from the Boolean properties:

Let A, B, C be subsets of a universal set U.

- Absorption properties

$$U \cup A = U, \quad \emptyset \cap A = \emptyset$$

- Simplification properties

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

- Idempotent properties

$$A \cup A = A$$
$$A \cap A = A$$

- De Morgan's laws

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

- Double complement property

$$(A^c)^c = A$$

As in the case of Boolean properties, these are analogous to properties of propositional forms:

Sets vs. propositional forms

- Absorption property

$$U \cup A = U, \quad \emptyset \cap A = \emptyset$$

- Simplification property

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

- Idempotent properties

$$A \cup A = A$$
$$A \cap A = A$$

- De Morgan's laws

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

- Double complement property

$$(A^c)^c = A$$

- Absorption property

$$\tau \vee P \equiv \tau, \quad \phi \wedge P \equiv \phi$$

- Simplification property

$$P \vee (P \wedge Q) \equiv P, \quad P \wedge (P \vee Q) \equiv P$$

- Idempotent properties

$$P \vee P \equiv P$$
$$P \wedge P \equiv P$$

- De Morgan's laws

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$
$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

- Double negation property

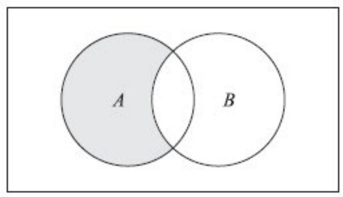
$$\neg(\neg P) \equiv P$$

4 Difference

The **difference** of two sets A and B, denoted by $A \setminus B$ (or also by $A - B$) is the set of all elements of A which do not belong to B, that is,

$$A \setminus B = \{x \mid x \in A \ \wedge \ x \notin B\}.$$

Using Venn diagrams, the difference $A \setminus B$ is represented by the shaded region:



Example 5. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 6, 7, 8\}$ then $A \setminus B = \{3, 4, 5\}$.

A useful property is the following one (**prove it!**):

Useful property:

Let A and B be two sets inside a universal set U. Then

$$A \setminus B = A \cap B^c.$$

5 Cartesian product

The order in which the elements of a (finite) set are listed is immaterial; in particular, $\{x, y\} = \{y, x\}$. In some circumstances, however, the order is significant. For instance, in coordinate geometry the points with coordinates $(1, 2)$ and $(2, 1)$, respectively, are distinct. We therefore wish to define, in the context of sets, something similar to the coordinates of points used in analytical geometry.

In order to deal with situations where order is important, we define the **ordered pair** (x, y) of objects x and y to be such that

$$(x, y) = (x', y') \text{ if and only if } x = x' \text{ and } y = y'.$$

With this definition it is clear that (x, y) and (y, x) are different (unless $x = y$), so the order is significant.

We are now in position to define the Cartesian product of two sets, a concept which is fundamental:

The cartesian product $X \times Y$ of two sets X and Y, is the set of all ordered pairs (x, y) where x belongs to X and y belongs to Y:

$$X \times Y = \{(x, y) \mid x \in X \ \wedge \ y \in Y\}.$$

Example 6. If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$ then

$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}.$$

Example 7. If $X = Y = \mathbb{R}$, the set of real numbers, then $X \times Y = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, which is the coordinate geometry representation of the (two-dimensional) plane.

Example 8. Let X be the set of main courses offered by a certain restaurant, and Y the set of desserts offered by the same restaurant. Then $X \times Y$ can be identified with the set of all (two-course) meals which can be ordered at the restaurant.

When $X = Y$, it is usual to denote $X \times X$ by X^2. This is read as "X two" and not "X squared".

Note that, if either X or Y (or both) is the empty set then $X \times Y$ is also the empty set. For example, if $X = \emptyset$ then there are no elements x to place in the first position of the ordered pair (x, y), so there are no ordered pairs in $X \times Y$.

If X and Y are both non-empty then $X \times Y = Y \times X$ if and only if $X = Y$. (**Why?**)

The ordered pair (x, y) may be generalized to an **ordered n-tuple** $(x_1, x_2, \ldots, x_n)$ with the property that

$$(x_1, x_2, \ldots, x_n) = (x_1', x_2', \ldots, x_n') \text{ if and only if } x_1 = x_1', x_2 = x_2', \ldots, x_n = x_n'.$$

The cartesian product of n sets if now a natural generalization of the case of two sets:

The cartesian product of n sets $X_1, X_2, \ldots, X_n$ is

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) \mid x_1 \in X_1 \ \wedge \ x_2 \in X_2 \ \wedge \cdots \wedge \ x_n \in X_n\}.$$

It is evident that the cardinal of the cartesian product of n (finite) sets is the product of their respective cardinals, that is,

If $X_1, X_2 \ldots, X_n$ are finite sets then

$$|X_1 \times X_2 \times \cdots \times X_n| = |X_1| \cdot |X_2| \cdot \cdots \cdot |X_n|.$$