Tema 4:
Linear functions (from $\mathbb{R}^n$ to $\mathbb{R}^m$)

Algebra

Computer Science Engineering Degree

May 21, 2015

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Linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$

Linear function

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if there exists a real matrix A, of size $m \times n$, such that

$$f(\vec{x}) = \mathrm{A}\vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n.$$

The matrix A will be called **canonical matrix** of f.

Example

Example: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as

$$f(x_1, x_2) = (2x_1 + x_2, x_1 + x_2, 7x_1 + 5x_2) \quad \forall\, (x_1, x_2) \in \mathbb{R}^2.$$

Observe that f admits the following matrix expression:

$$f(x_1, x_2) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and, therefore, it is a linear function

The **image** by f of the vector $(3, -2)$ is

$$f(3, -2) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 11 \end{bmatrix}$$

Equivalently, we can also say that $(3, -2)$ is a **pre-image** of $(4, 1, 11)$.

Example

Let us see the images of the canonical basis of $\mathbb{R}^2$:

$$f(1,0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix},$$

$$f(0,1) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}.$$

They **are the columns of** A. This is true in general:

Property

Given a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\vec{x}) = \mathrm{A}\vec{x}$, the column vectors of A are the images of the canonical basis of $\mathbb{R}^n$.

Therefore, to determine a linear function it is enougg to know the images of the canonical basis.

More Examples

Null function

If A is the $m \times n$ null matrix therefore $f(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$. It is the **null function** from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Identity function

If A is the identity matrix of order n then $f(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. It is **the identity map** from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Linearity conditions

Linearity conditions

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function defined by a matrix A of dimensions $m \times n$ then the following conditions are satisfied:

(1) $f(\vec{x}+\vec{y}) = \mathrm{A}(\vec{x}+\vec{y}) = \mathrm{A}\vec{x}+\mathrm{A}\vec{y} = f(\vec{x})+f(\vec{y})$ for all $\vec{x},\vec{y} \in \mathbb{R}^n$.

(2) $f(\lambda\vec{x}) = \mathrm{A}(\lambda\vec{x}) = \lambda(\mathrm{A}\vec{x}) = \lambda f(\vec{x})$ for every scalar λ and for all vector $\vec{x} \in \mathbb{R}^n$.

Characterization

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if the linearity conditions are satisfied, that is:

(1) $f(\vec{x}+\vec{y}) = f(\vec{x})+f(\vec{y})$ for all $\vec{x},\vec{y} \in \mathbb{R}^n$.

(2) $f(\lambda\vec{x}) = \lambda f(\vec{x})$ for every scalar λ and for each vector $\vec{x} \in \mathbb{R}^n$.

Endomorphisms

If a linear map is defined between two spaces of the same dimension, it is an **endomorphism**.

Endomorphism

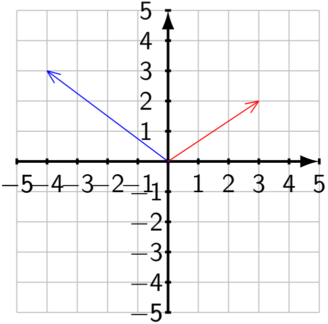
A linear map $\mathbb{R}^n \to \mathbb{R}^n$, that is, defined by a square matrix, is an **endomorphism**.

Projection over the x-axis

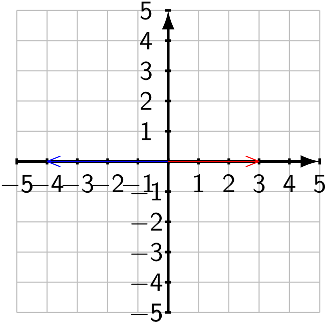
Example: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function with canonical map

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

For whichever vector $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$, $f(x_1, x_2) = (x_1, 0)$. The function f transforms each vector into its orthogonal projection over the x-axis.



$\xrightarrow{f}$

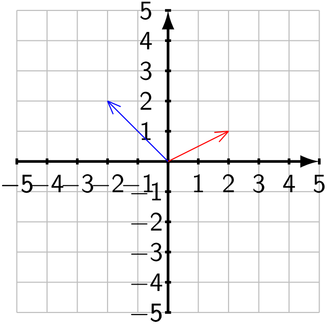


Homotecies

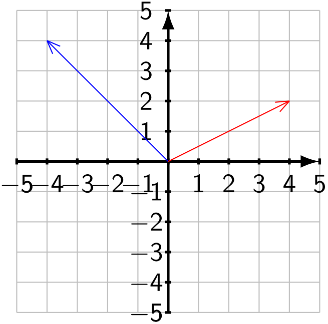
Example: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ the linear function with canonical matrix

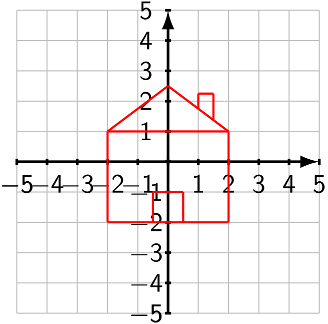
$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

For every vector $\vec{x} = (x_1, x_2) \in \mathbb{R}^2, f(x_1, x_2) = (2x_1, 2x_2)$. The function f transforms each vector into its double.

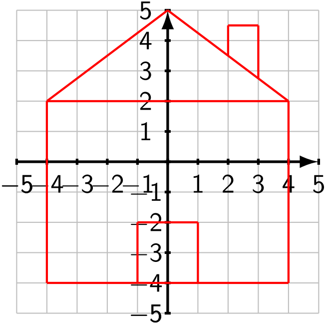


$\xrightarrow{f}$





$\xrightarrow{f}$



Rotation of angle α

Rotation of angle α

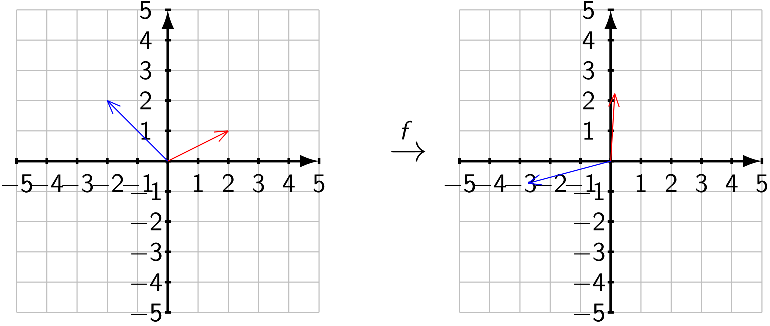
The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ of canonical matrix

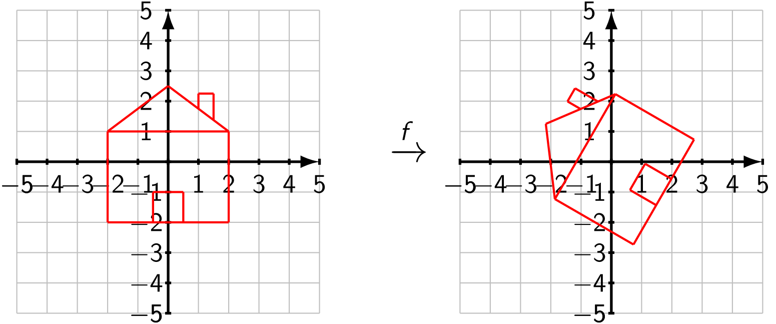
$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

applies, to each vector, a rotation of angle α anticlockwise, centered at the origin.

Rotation of angle α

Example: The following figure shows the effect of applying a rotation of angle $\alpha = \frac{\pi}{3}$, that is, the function with canonical matrix: $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$





5
4
3
2
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−5 −4 −3 −2 −1 1 2 3 4 5
−1
−2
−3
−4
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f
5
4
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2
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−5 −4 −3 −2 −1 1 2 3 4 5
−1
−2
−3
−4
−5

Reflection with respect to a line

Reflection with respect to a line

The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the reflection with respect to a line has a canonical matrix of this form

$$\begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{bmatrix},$$

where

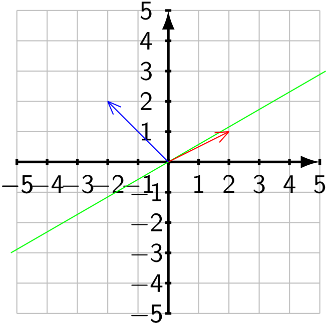
$$\sin\left(\frac{\alpha}{2}\right) x_1 - \cos\left(\frac{\alpha}{2}\right) x_2 = 0$$

is the equation of the line of symmetry.

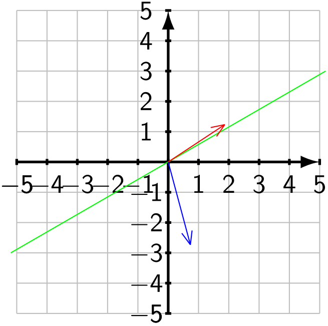
Reflection with respect to a line

Example: Taking $\alpha = \frac{\pi}{3}$ we obtain the following matrix: $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$,

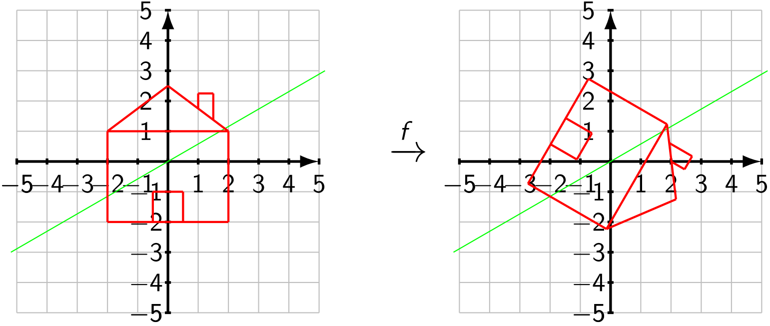
where the line $x_1 - \sqrt{3}x_2 = 0$ is the symmetry line.



$\xrightarrow{f}$



Reflection with respect to a line



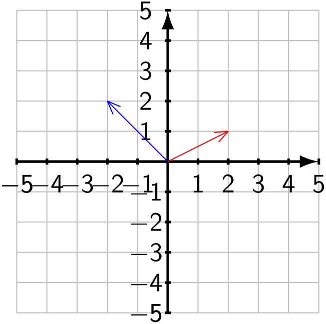
General

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear function with canonical matrix

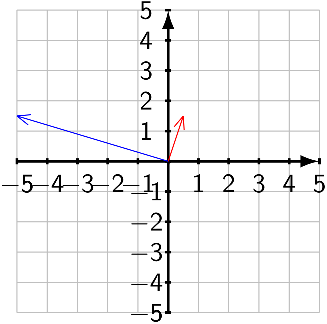
$$A = \begin{bmatrix} 1 & 1/4 \\ -3/2 & 1 \end{bmatrix}.$$

For each vector $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$,

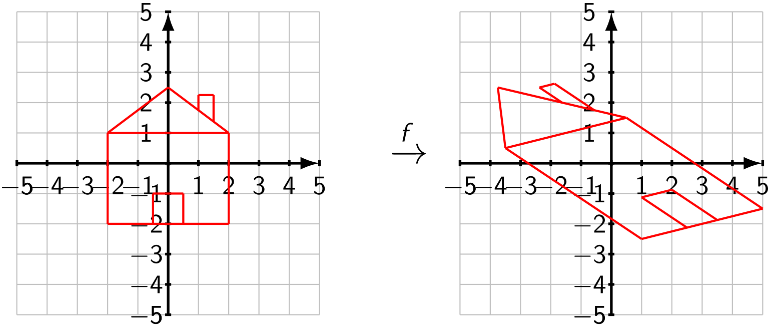
$$f(x_1, x_2) = A(x_1, x_2) = (x_1 + (1/4)x_2, -(3/2)x_1 + x_2).$$



$\xrightarrow{f}$



Deformation



Kernel

Definition

Given a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$, lthe **kernel** of f is the set:

$$\mathrm{Ker}(f) := \{\vec{x} \in \mathbb{R}^n \mid f(\vec{x}) = \vec{0}\},$$

that is, the set of pre-images of $\vec{0}$.

If A is the canonical matrix of f, $f(\vec{x}) = \vec{0}$ is equivalent to $A\vec{x} = 0$ and, therefore, the kernel of f coincides with the kernel of the matrix A:

$$\mathrm{Ker}(f) = \mathrm{Ker}(A)$$

consequence: $\mathrm{Ker}(f)$ is a vector subspace.

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If A is the canonical matrix of f, $f(\vec{x}) = \vec{0}$ is equivalent to $\mathsf{A}\vec{x} = 0$ and, therefore, the kernel of f coincides with the kernel of the matrix A:

$$\mathrm{Ker}(f) = \mathrm{Ker}(\mathsf{A})$$

consequence: $\mathrm{Ker}(f)$ is a vector subspace.

Image

Definition

Given a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$, the **image** of f is the set:

$$\mathrm{Im}(f) := \{f(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\},$$

that is, the set formed by the images by f of all vectors of $\mathbb{R}^n$.

If A denotes the canonical matriz of f therefore, for each vector $\vec{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$:

$$f(\vec{x}) = \mathrm{A}\vec{x} = x_1 Col(1) + x_2 Col(2) + \cdots + x_n Col(n).$$

Then:

$$\mathrm{Im}(f) = \{\text{linear combinations of the column vectors of A}\} = Col(\mathrm{A}).$$

consequence: $\mathrm{Im}(f)$ is a vector subspace of $\mathbb{R}^m$. Moreover:

$$\dim \mathrm{Im}(f) = \mathrm{rg}(\mathrm{A}).$$

Interesting formula

The known formula $n = \dim \mathrm{Nuc}(A) + \mathrm{rg}(A)$ can be written in terms of linear maps:

$$n = \dim \mathrm{Ker}(f) + \dim \mathrm{Im}(f)$$

Exercise 1: Consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^4$ such that its canonical basis is

$$A = \begin{bmatrix} 1 & 2 & -3 \\ -1 & 2 & 0 \\ 0 & 4 & -3 \\ 1 & 6 & -6 \end{bmatrix}.$$

Computes its kernel and its image. Verify the above formula.

Types of functions

Concepts involving whichever function (no necessarily linear) $f : A \to B$:

- f is **injective** if different elements of A have different images, that is:

$$f \text{ is injective} \;\Leftrightarrow\; \text{if } x \neq y \text{ then } f(x) \neq f(y)$$

- f is **surjective** if every element of B is the image of some element of A, that is:

$$f \text{ is surjective} \;\Leftrightarrow \forall\, b \in B \;\text{ there exists } a \in A \text{ such that } f(a) = b.$$

- f is **bijective** if it is injective and surjective.

Property

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function.

(1) f is injective if and only if $\mathrm{Ker}(f) = \{\vec{0}\}$.

(2) f is surjective if and only if $\mathrm{Im}(f) = \mathbb{R}^m$.

Exercise 2: Determine which is the type of the fuction of Exercise 1.

Exercise 3: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function.

(a) If f is injective, it may be true the inequality $n > m$?

(b) If f is surjective, it can be true the inequality $n < m$?

Isomorphism

Definition

A **isomorphism** is a bijective linear map.

Exercise 4: Show that the identity function $\mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism.

Property

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is an isomorphism then $n = m$.

Exercise 5: Determine if the linear function $f : \mathbb{R}^3 \to \mathbb{R}^2$ with canonical matrix

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

is either injective, surjective or isomorphism.

Exercise 6: Idem with the linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with canonical matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Exercise 7: Classify the linear function $f : \mathbb{R}^5 \to \mathbb{R}^4$ such that $f(1,0,0) = (1,0,0,1), f(0,1,0) = (-2,1,0,1)$ and $f(0,0,1) = (5,-2,5,1)$

Exercise 8: Classify the linear function $f : \mathbb{R}^5 \to \mathbb{R}^4$ whose canonical matrix is

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 5 & 1 & 0 \\ 1 & 1 & -2 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Property

A endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with canonical matrix A is an isomorphism if and only if A is invertible.

Exercise 8: Classify the linear function $f : \mathbb{R}^5 \to \mathbb{R}^4$ whose canonical matrix is

$$A = \begin{bmatrix} 1 & 0 & 5 & 1 & 0 \\ 1 & 1 & -2 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Property

A endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with canonical matrix A is an isomorphism if and only if A is invertible.

Inverse function

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, since f is bijective, there exists an inverse function $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that the functions $f \circ f^{-1}$ and $f^{-1} \circ f$ are the identity function. Moreover, the inverse function is also linear:

Property

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism with canonical matrix A then $f^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is also linear, A^{-1} being its canonical basis.

Exercise 9: Show that the endomorphism $f : \mathbb{R}^3 \to \mathbb{R}^3$ with associated canonical matrix

$$\mathrm{A} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

is an isomorphism. Compute the canonical matrix of f^{-1}.

Exercise 10: Classify the linear functions of the examples of the second section.

Definition

An **isometry** of $\mathbb{R}^n$ is an endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ preserving the norm of the vectors, that is, $\|\vec{x}\| = \|f(\vec{x})\|$ for every vector $\vec{x} \in \mathbb{R}^n$.

Property

An isometry is an isomorphism.

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Property

An isometry is an isomorphism.

Exercise 11: Which linear functions corresponding to the examples of the second section are isometries?

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Property

An isometry is an isomorphism.

Exercise 11: Which linear functions corresponding to the examples of the second section are isometries?

Property

An endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ preserves the norms of the vectors if and only if it **preserves the scalar product**, that is: $\vec{x} \cdot \vec{y} = f(\vec{x}) \cdot f(\vec{y})$ for all pair of vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Since the angle between two vectors is defined from the scalar product and the norms, one has the following consequence:

Corollary

The isometries preserve the angles.

Definition

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Property

An isometry is an isomorphism.

Exercise 11: Which linear functions corresponding to the examples of the second section are isometries?

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Corollary

The isometries preserve the angles.

Isometries and orthogonal matrices

Property

An endomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with canonical matrix A is an isometry if and only if A is orthogonal.

Exercise 12: Show that the linear function $f : \mathbb{R}^3 \to \mathbb{R}^3$ of canonical basis

$$\mathrm{A} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

is an isometry.

Given two linear functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$, with canonical matrices A and B, we can consider:

- The sum function: $f + g : \mathbb{R}^n \to \mathbb{R}^m; \quad (f+g)(\vec{x}) = f(\vec{x}) + g(\vec{x})$.
- The "product by a scalar function" α:
  $\alpha f : \mathbb{R}^n \to \mathbb{R}^m; \quad (\alpha f)(\vec{x}) = \alpha f(\vec{x})$.

It is easy to prove that both functions are linear functions and that their canonical matrices are, respectively, $\mathrm{A} + \mathrm{B}$ and $\alpha \mathrm{A}$.
Además:

Property

Given two linear functions $f : \mathbb{R}^m \to \mathbb{R}^n$ y $g : \mathbb{R}^n \to \mathbb{R}^k$ with associated canonical matrices A and B respectively, the composition $g \circ f$ is also a linear function and its canonical matrix is BA.