Sessions 7 and 8: Inference laws with predicates

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Equivalences (with predicates)

In addition to the equivalences between propositional forms that we have seen, we are going to introduce **two specific** equivalences involving quantifiers:

Negation of quantifiers:

1. $\neg \forall x\ P(x) \equiv \exists x\ \neg P(x)$
2. $\neg \exists x\ P(x) \equiv \forall x\ \neg P(x)$

Let us see the reason of the logical coherence of these equivalences:

For any predicate $P(x)$, the proposition $\forall x\ P(x)$ estates that, for all x in the universe, x has the property defined by the predicate P. The negation of this proposition, $\neg \forall x\ P(x)$, states that "it is not the case that all x have the property defined by P", that is, there is at least one x that does not have the property P. This is symbolized by $\exists x\ \neg P(x)$. So, the propositions $\neg \forall x\ P(x)$ and $\exists x\ \neg P(x)$ have the same meaning and, therefore, they can be declared to be equivalent.

A similar reasoning proves the second equivalence.

2 Inference laws (with predicates)

The following inference laws are specific for propositional forms involving predicates:

Universal specification:

From $\forall x\ P(x)$ we can deduce $P(a)$ for **any** element a of the universe. This element is said to be **arbitrary**.

Existential specification:

From $\exists x\ P(x)$ we can deduce $P(a)$ for **some specific** element a of the universe.

Universal generalization:

If $P(y)$ is true **for any** element y of the universe (that is for an **arbitary** element), then we can deduce $\forall x\ P(x)$.

Existential generalization:

If $P(a)$ is true for **certain** (or **specific**) element a of the universe, then we can deduce $\exists x\ P(x)$.

We can perform an inference process with predicates using all the inference laws we knew before this session, and these new specific rules and equivalences. This example illustrates it.

Example:

Let us see that the following argument is right:

Some soccer supporters are also basketball supporters.
Soccer supporters don't go to the cinema on Sunday afternoon.
Therefore, there are some basketball supporters who don't go to the cinema on Sunday afternoon.

Let us define the following predicates:

$S(x) :=$ "x is a soccer supporter"
$B(x) :=$ "x is a basketball supporter"
$C(x) :=$ "x goes to the cinema on Sunday afternoon"

A suitable universe for the variable x is the set of human beings. These are the hypotheses:

- $\exists x\ S(x) \wedge B(x)$
- $\forall x\ S(x) \to \neg C(x)$

and the conclusion will be $\exists x\ B(x) \wedge \neg C(x)$.

Let us see the inference process:

H1:	$\exists x\ S(x) \wedge B(x)$	
H2:	$\forall x\ S(x) \rightarrow \neg C(x)$	
3:	$S(a) \wedge B(a)$	Existential specification of (1) (for certain $x = a$ specific)
4:	$S(a) \rightarrow \neg C(a)$	Universal specification of (2) (for the $x = a$ specified in (1))
5:	$S(a)$	Simplification (3)
6:	$\neg C(a)$	Modus ponens (4,5)
7:	$B(a)$	Simplification (3)
8:	$B(a) \wedge \neg C(a)$	Conjunction (7,6)
C:	$\exists x\ B(x) \wedge \neg C(x)$	Existential generalization (8)