Session 5: Inference by contradiction

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

In this session we are going to talk about an extremely useful method of inference (or proof): *inference by contradiction* (or *proof by contradiction* or *reductio ad absurdum.* It allows us to give proofs that, using direct, conditional or biconditional inference, would be impossible. A beautiful example is the proof of the following statement: *There are infinitely many prime numbers.*

1 Foundation of the method

Informally speaking:

This method consists of **assuming as a new hypothesis the negation of the conclusion**. Then after applying inference and equivalence rules **we arrive to a contradiction** (that, usually, is of the type $R \wedge \neg R$). Therefore, our additional hypothesis cannot be fulfilled and, then, its contrary must be true.

From a more rigorous point of view:

The method is based on the following equivalence (**prove it!**):

$$P \to Q \equiv (P \wedge \neg Q) \to \phi.$$

If we want to deduce, from some hypotheses $H_1, H_2, \ldots H_n$, a conclusion Q, what we want is to prove the following implication:

$$H_1 \wedge H_2 \wedge \cdots \wedge H_n \Rightarrow Q. \tag{1}$$

Now, replacing P by $H_1 \wedge H_2 \wedge \cdots \wedge H_n$ in the above equivalence, we can transform (1) into

$$H_1 \wedge H_2 \wedge \cdots \wedge H_n \wedge \neg Q \Rightarrow \phi. \tag{2}$$

Inference by contradiction consists of proving the implication (2) instead of (1).

Summarizing:

How to apply inference by contradiction? Basic steps:

1. We add to the hypotheses the negation of the conclusion and we replace the conclusion by a contradiction ϕ.
2. We apply inference and equivalence rules until we get a contradiction (usually $R \wedge \neg R$ for some propositional form R).
3. We say that we arrive to a contradiction, and we affirm that our initial conclusion is true (assuming the original hypotheses).

To convince you that the method agrees with your usual *way of thinking* consider the following informal example:

Let us assume the following hypotheses:

H_1 : "I do not spend the afternoon watching TV and play football at the same time"

H_2 :"If it rains then I spend the afternoon watching TV"

H_3 :"This afternoon I play football"

My question for you is: *Is it raining this afternoon?*

I'm sure that you have deduced that *it is not raining* and that your way of thinking has been the following one: It is not raining because if *it rains* then *I spend the afternoon watching TV* (by H_2), but it cannot be the case by H_3 and H_1. Then, you conclude that *It is not raining this afternoon.*

Observe that, in addition to the hypotheses H_1, H_2 and H_3, you have assumed the contrary of the conclusion (*it rains this afternoon*) and you have got a *contradiction*: on the one hand, *I spend the afternoon watching TV* (by H_2) and, on the other hand, *I do not spend the afternoon watching TV* (by H_3 and H_1). So, you conclude that you assumption (*it rains this afternoon*) cannot be true; then it should be true that *It is not raining this afternoon.*

Hence *you have used inference by contradiction in your reasoning!*

Let us formalize this inference. The "atomic" propositions that we use are the following:

$R =$"It rains this afternoon"

$W =$"I spend the afternoon watching TV"

$F =$"I play football this afternoon"

The hypothesis and conclusion are given by the following scheme:

H1:	$\neg(W \wedge F)$
H2:	$R \rightarrow W$
H3:	F
C:	$\neg R$

The induction process using *inference by contradiction* can be described with the following scheme:

H1:	$\neg(W \wedge F)$		
H2:	$R \rightarrow W$		
H3:	F		
4:		R	Additional hypothesis (inference by contradiction)
5:		W	MP(4,2)
6:		$\neg W \vee \neg F$	DM(1)
7:		$\neg F$	DS(5,6)
8:		$F \wedge \neg F$	Conj(3,7)
9:		ϕ	Inv(8)
10:	$\neg R$		Conclusion (inference by contradiction)

In H_4 we have assumed the contrary of what we want to prove and we have got a contradiction ($H9$). So, we establish that R must not be true and, therefore, $\neg R$ must be true ($H10$).

Observe that all the steps in the above inference process have been done *internally* in your reasoning.