Practice 6

Activities sheet

Activity 1. *Given the vectors* $\vec{u} = (3, 5, -1, 0)$ *and* $\vec{v} = (1/2, 1/4, 1/3, -3)$, *compute*

(a) $\vec{u} \cdot \vec{v}$, $\|\vec{u}\|$ *and* $\|\vec{v}\|$

(b) the distance between $\vec{u}$ *and* $\vec{v}$

(c) a unitary vector with the direction of $\vec{u}$.

SOLUTION:

(a) With Scilab:

```
-->u=[3; 5; -1; 0]; v=[1/2; 1/4; 1/3; -3];

-->u'*v
 ans  =

    2.4166667

-->norm(u)
 ans  =

    5.9160798

-->norm(v)
 ans  =

    3.0697901
```

(b) The distance between two vectors is the norm of the difference. With Scilab:

```
-->norm(u-v)
 ans  =

    6.2920806
```

(c) A unitary vector with the direction of $\vec{u}$ can be obtained dividing $\vec{u}$ by its norm:

```
-->u/norm(u)
 ans  =

    0.5070926
    0.8451543
  - 0.1690309
    0.
```

Activity 2. *Given the vectors $\vec{b} = (1, 2, 3)$ and $\vec{c} = (1, 0, 2)$.*

(a) Determine the value of m such that the vector $\vec{y} = (m, -1, 2)$ is orthogonal to $\vec{b}$ and $\vec{c}$.

(b) Compute $H^{\perp}$, where $H = \mathrm{span}(\vec{b}, \vec{c})$.

(c) Check that the vector $\vec{y}$ that you have obtained in (a) belongs to $H^{\perp}$.

SOLUTION:

(a) $\vec{y}$ is orthogonal to $\vec{b}$ if and only if the dot product $b \cdot \vec{y}$ is equal to zero. This is equivalent to:

$$m - 2 + 6 = 0.$$

Similarly, $\vec{y}$ is orthogonal to $\vec{c}$ if and only if

$$m + 4 = 0.$$

The unique value of m satisfying both conditions is $m = -4$.

(b) A vector $\vec{x} = (x, y, z)$ belongs to $H^{\perp}$ if and only if it is orthogonal to $\vec{b}$ and $\vec{c}$. Imposing both conditions we get the system

$$x + 2y + 3z = 0$$

$$x + 2z = 0$$

And solving it we obtain that

$$H^{\perp} = \mathrm{span}(-2, -1/2, 1).$$

(c) The vector $\vec{y}$ that we have obtained in (a) (with $m = -4$) is $\vec{y} = (-4, -1, 2)$. Notice that $\vec{y} = 2(-2, -1/2, 1)$ and, therefore, $\vec{y} \in \mathrm{span}(-2, -1/2, 1) = H^{\perp}$.

Activity 3. *Let $\vec{r} = (1, -2, 4, -1)$ and let $W = \mathrm{span}(\vec{r})$*

(a) Compute the orthogonal projection of the vector $\vec{x} = (3, 0, -3, 5)$ over W.

(b) Compute a basis of $W^{\perp}$.

(c) Check that the vector that you have obtained in (a) is orthogonal to the vectors of the basis of $W^{\perp}$.

SOLUTION:

(a) Since W is a line, we need to obtain, first, a unitary vector $\vec{q}$ in the direction of the line (that is, we need to make $\vec{r}$ unitary). With Scilab:

```
-->r=[1;-2;4;-1];

-->q=r/norm(r)
 q  =

    0.2132007
  - 0.4264014
    0.8528029
  - 0.2132007
```

Now we need to apply the formula $Proj_W(\vec{x}) = (\vec{q}^t\vec{x})\vec{q}$. With Scilab:

```
-->x=[3; 0; -3; 5];

-->p=(q'*x)*q
 p  =

  - 0.6363636
    1.2727273
  - 2.5454545
    0.6363636
```

(b) A vector (x, y, z, t) belongs to $W^\perp$ if and only if it is orthogonal to $\vec{r}$, that is, if and only if

$$x - 2y + 4z - t = 0.$$

Solving this equation we have that:

$$W^\perp = \text{span}(\begin{bmatrix}2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-4\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\1\end{bmatrix}).$$

The obtained spanning set is a basis of $W^\perp$.

(c) We only need to check that the dot products of the vector that you have obtained in (a) and the vectors in the obtained spanning set of $W^\perp$ is zero. With Scilab:

```
-->v1=[2; 1; 0; 0]; v2=[-4; 0; 1; 0]; v3=[1; 0; 0; 1];

-->v1'*p
 ans  =

    0.
```

```
-->v2'*p
 ans  =

    0.

-->v3'*p
 ans  =

    0.
```

Activity 4. *Consider* $W = \text{span}(\vec{u}_1, \vec{u}_2)$ *where* $\vec{u}_1 = (-1, 2, 4)$ *and* $\vec{u}_2 = (4, -5, 1)$

(a) Write the orthogonal projection of the vector $\vec{x} = (2, 2, 3)$ *over* W*,* $Proj_W(\vec{x})$*, as a linear combination of the vectors* $\vec{u}_1$ *and* $\vec{u}_2$*.*

(b) Compute $Proj_W(\vec{x})$ *by means of the projection matrix* P_W*. Check that it is obtained the same result given in (a).*

(c) Compute $Proj_W(\vec{z})$ *and* $Proj_W(\vec{t})$*, where* $\vec{z} = (-6, 9, 7)$ *and* $\vec{t} = (-22/3, -17/3, 1)$*. Can you deduce a conclusion from the obtained results?*

SOLUTION:

(a) First we define in Scilab the needed vectors and the matrix $M(S)$, where $S = \{\vec{u}_1, \vec{u_2}\}$:

```
-->x=[2; 2; 3]; u1=[-1; 2; 4]; u2=[4; -5; 1]; MS=[u1 u2];
```

Now we need to compute a solution $\vec{y}$ of the system

$$M(S)^t M(S)\vec{y} = M(S)^t \vec{x}.$$

Using the \ command:

```
-->y=(MS'*MS)\(MS'*x)
 y  =

    0.7647059
    0.2058824
```

And this means that

$$Proj_W(\vec{x}) = 0.7647059\, \vec{u}_1 + 0.2058824\, \vec{u}_2.$$

This vector can be computed also as the product $M(S)\vec{y}$. With Scilab:

```
-->MS*y
 ans  =

    0.0588235
    0.5
    3.2647059
```

(b) Notice that $S = \{\vec{u}_1, \vec{u_2}\}$ is linearly independent and, therefore, we can also use the projection matrix to compute $Proj_W(\vec{x})$. With Scilab:

```
-->P=MS*inv(MS'*MS)*MS'
 P  =

    0.3810742  - 0.4782609    0.0843990
  - 0.4782609    0.6304348    0.0652174
    0.0843990    0.0652174    0.9884910

-->P*x
 ans  =

    0.0588235
    0.5
    3.2647059
```

Activity 5. *Let W be a vector subspace of $\mathbb{R}^n$. Check that any projection matrix P_W is symmetric and idempotent ($P_W^2 = P_W$).*

SOLUTION:

To simplify notation, let's name $M = M(S)$, where S is a fixed basis of W. Then

$$P_W = M(M^tM)^{-1}M^t.$$

Let's check that P_W is symmetric:

$$P_W^t = [M(M^tM)^{-1}M^t]^t = (M^t)^t((M^tM)^{-1})^tM^t = M((M^tM)^t)^{-1}M^t$$

$$= M(M^t(M^t)^t)^{-1}M^t = M(M^tM)^{-1}M^t = P_W.$$

Let's check that P_W is idempotent:

$$P_W^2 = [M(M^tM)^{-1}M^t][M(M^tM)^{-1}M^t] = M\underbrace{(M^tM)^{-1}M^tM}_{I}(M^tM)^{-1}M^t$$

$$= MI(M^tM)^{-1}M^t = M(M^tM)^{-1}M^t = P_W.$$