Practice 1: Introduction

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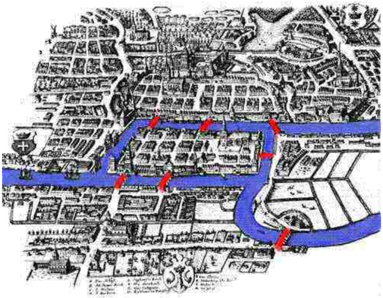
Leonhard Euler (1707-1783)

Graph theory is one of the most recent branches of Mathematics. The first monograph on this topic appeared in 1936 and it is due to König. However, it started a long time ago. Leonhard Euler was the "father" of the theory. He solved the celebrated problem of the **Königsberg bridges**.

Problem of the Königsberg bridges

The Pregel River crosses the ancient city of Königsberg (nowadays called Kaliningrad, in Russia). In the 18th Century there were 7 bridges that comunicate the island in the middle of the river with the two banks:



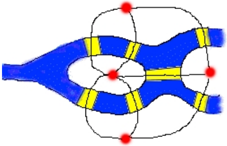
Problem

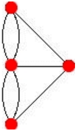
Is it possible to cross the 7 bridges without passing two times by the same bridge and coming back to the departure point?

It is said that this question used to amuse the inhabitants of Königsberg :).

Interpretation using graph theory

Euler represented every bank and the island with points (vertex), and every bridge with a line (edge) that joins two points:





Then the problem of the Königsberg bridges is reduced to the following question:

Problem

Is it possible to draw this picture without repeating any line and coming back to the departure point without without lifting the pen from the paper?

In 1736 Euler published a paper where he solved this problem in a more general case. This is considered as the birth of the Graph Theory.

Definition of a graph

Let us recall again the definition of a graph:

Definition (Graph (non-directed))

A **graph** G consists on a 3-tuple (V, E, Ψ) given by V, E, Ψ where:

1. V is a non-empty set of elements, that we will call **vertex, nodes** or **points**.
2. E is a finite set whose elements are called **edges** (or **arcs**).
3. Ψ is an application, that we will call of **incidence**, that to every element $e \in E$ it assigns a non-ordered pair of elements of V (that can coincide or not).

When at the incidence application we consider ordered pairs of vertices, we will speak of **directed graphs**.
In general, we will omit the explicit reference to the incidency application Ψ, having in mind that every edge has assigned a concrete pair of vertex.

Example

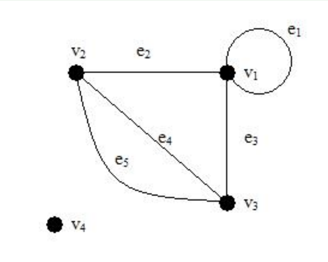
Example (Consider the following graph given by $G = (V, E, \psi)$, where)

Let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ and let ψ be the incidency mapping ψ defined as follows:

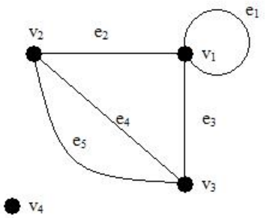
$$\psi(e_1) = \{v_1, v_1\},\ \ \psi(e_2) = \{v_1, v_2\},\ \ \psi(e_3) = \{v_1, v_3\},$$

$$\psi(e_4) = \{v_2, v_3\},\ \ \psi(e_5) = \{v_2, v_3\}$$

This graph can be represented with the following diagram:



Example and more definitions



More definitions

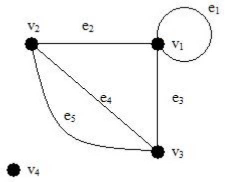
- The edge e_1 is a **loop**, that is, *an edge that joins a vertex with itself.*
- e_4 and e_5 are **parallel edges**, that is, *2 edges that join the same vertex*. A graph *without parallel edges* will be called **simple**.
- v_4 is an **isolated vertex**, that is, *it is a vertex that it is not incident with any edge*.

Adjacency matrix of a graph

Definition (Adjacency matrix of a graph)

Let $G = (V, E, \Psi)$ be a graph, with $V = \{v_1, v_2, \ldots, v_n\}$. The **adjacency matrix** of a graph G is the square matrix $M_A = (m_{ij})$ of size $n \times n$ whose element located at the position (i, j) is the number of edges joining v_i and v_j.

Example



$$M_A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In this case, on the one hand, the 1 at the position $(1,1)$ is given by a loop at vertex v_1. On the other hand, the 2's at the positions $(2,3)$ and $(3,2)$ appear because the vertex v_2 and v_3 are connected by two edges.

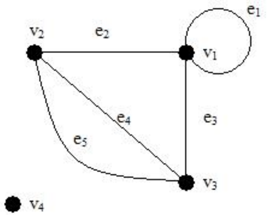
Incidency matrix

Definition

Let G be a graph with set of vertices $V = \{v_1, v_2, \ldots, v_m\}$ and set of edges $E = \{e_1, e_2, \ldots, e_n\}$. The **incidency matrix** of G is defined as the matrix $M_I = (m_{ij})$, of size $m \times n$, given by:

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \text{ and } e_j \text{ is not a loop,} \\ 0 & \text{if } v_i \text{ is not incident with } e_j, \\ 2 & \text{if } e_j \text{ is a loop that joins } v_i \text{ with itself.} \end{cases}$$

Example:



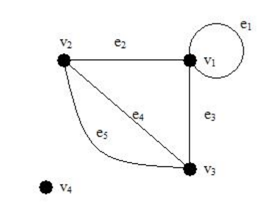
$$M_I = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Degree of a vertex

Definition (Degree of a vertex (non-directed graphs))

The **degree** of a vertex v, denoted as $d(v)$, is the number of edges that are incident on it (we count it two times for loops)

Example



In this graph we have:

- $d(v_1) = 4$
- $d(v_2) = 3$
- $d(v_3) = 3$
- $d(v_4) = 0.$

Degrees formula

Notation: If A is any set of, then card(A) denotes the number of elements in A. The cardinal of A can also be denoted as $|A|$.

Handshaking lemma

If $G = (V, E)$ is a graph, then

$$\sum_{v \in V} d(v) = 2 \cdot |E|,$$

that is, in any graph the sum of the degrees of all vertex is equal to 2 times the number of edges.

Consequences

- The sum of the degrees of the vertex of a graph is an even number.
- Every graph contains an even number of vertex of odd degree.

Weighted graphs

Definition

A **weighted graph** is a graph such that each edge has an associated number, called *weight*.

Polimedia

- Introduction:

  https://polimedia.upv.es/visor/?id=2f31ebc4-81e3-bc41-b93a-8aa9e489d6b2
- Firsts concepts:

  https://polimedia.upv.es/visor/?id=8a2f7ba0-6bfa-1f42-9843-1f1c5d5f7310
- Adjacency matrix:

  https://polimedia.upv.es/visor/?id=6316a576-46a0-4b43-8f80-392cc3bec9bf
- Adjacency matrices in 4 situations:

  https://polimedia.upv.es/visor/?id=a4cc8fd6-ad4e-5a4a-a871-cc3cef685d9e