

Exercise

Compute the probability that the Markov Model in page 17 (Chapter 5 slides) generates the string “bcba”.

b	c	b	a
1	1	1	2
2	2	2	3

b	c	b	a
1	1	1	
	2	2	3

, and state 2 cannot repeat itself.

Valid sequence of states: **1 1 1 3, 1 1 2 3, 1 2 1 3**

$$P(\text{bcba} | M) = P(\text{bcba}, 1113F) + P(\text{bcba}, 1123F) + P(\text{bcba}, 1213F) =$$

$$P(\text{bcba} | 1113F) * P(1113F) + P(\text{bcba} | 1123F) * P(1123F) + P(\text{bcba} | 1213F) * P(1213F) = 0.0073386$$

In this resolution, we have first to calculate all the possible sequences of states that generate the string. We might leave out some sequence of states ...

Forward algorithm

The forward algorithm is used to compute $P(y|M)$ without need to enumerate explicitly the sequence of states that derive the string 'y'.

Instead of calculating a priori the sequence of states, the Forward algorithm performs a series of recursive calculations to compute $P(y|M)$ and also obtains the sequence of states that derive the string 'y'.

Basic concept

$\alpha(q,t)$: where 'q' is a state and 't' is a sequence of 't' symbols; e.g. $\alpha(1,b)$, $\alpha(1,bc)$, $\alpha(2,bca)$,....

For example, for string $y \equiv bcaa$

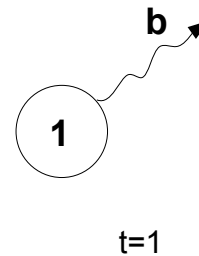
states \ sequence of symbols	t=1 (b) b	t=2 (c) bc	t=3 (a) bca	t=4 (a) bcaa
1	$\alpha(1,b)$	$\alpha(1,bc)$	$\alpha(1,bca)$	$\alpha(1,bcaa)$
2	$\alpha(2,b)$	$\alpha(2,bc)$	$\alpha(2,bca)$	$\alpha(2,bcaa)$
3	$\alpha(3,b)$	$\alpha(3,bc)$	$\alpha(3,bca)$	$\alpha(3,bcaa)$

$\alpha(q, t)$ is the probability that a Markov model (M) reaches state 'q' at instant 't' and emits the string y_1, y_2, \dots, y_t

More specifically, let's assume we have three states (1,2,3):

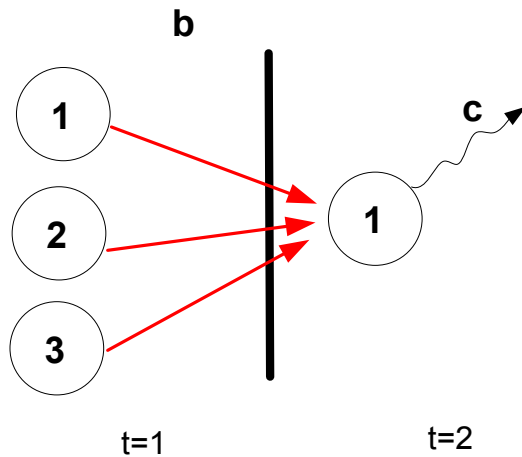
t=1

$\alpha(1, b)$: probability that M reaches state 1 at t=1 and emits symbol 'b': $\alpha(1, b) = P(1)P(b | 1)$



t>1

$\alpha(1, bc)$: probability that M reaches state 1 at t=2 and emits symbol 'c' having emitted before symbol 'b' with any other state. The emission before is recursively computed.



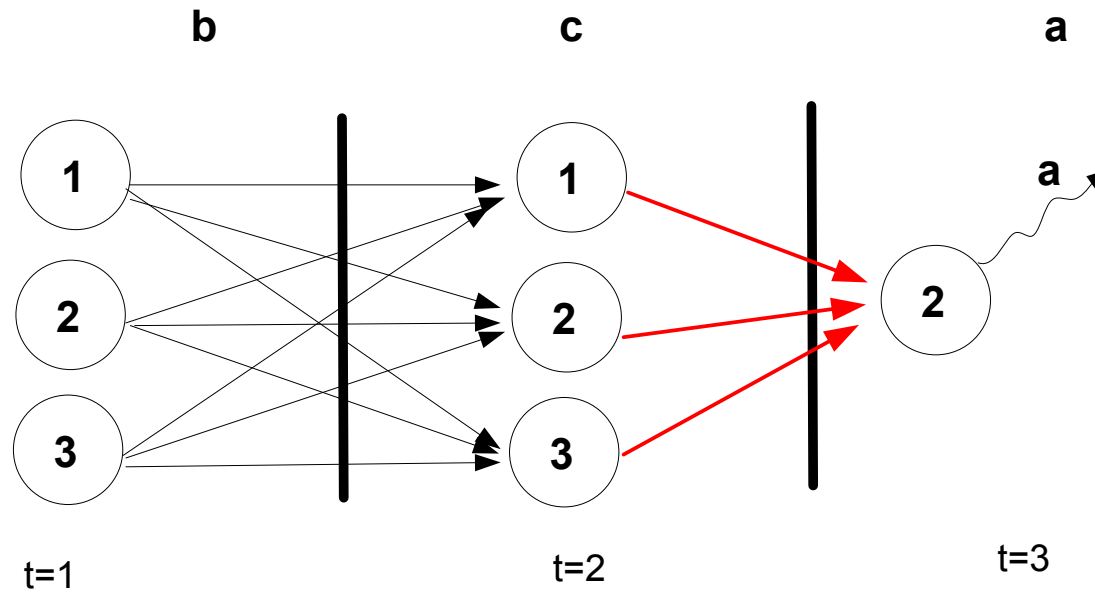
probability of reaching state 1 at t=2 having emitted before “b”

$$\alpha(1, bc) = P(c | 1) * (\alpha(1, b) * P(1 | 1) + \alpha(2, b) * P(1 | 2) + \alpha(3, b) * P(1 | 3))$$

$\alpha(1, bc)$:

1. probability that state 1 emits symbol c (**$P(c | 1)$**) *
2. probability of reaching state 1 having emitted before symbol b; i.e., the probability that when state 1 is reached, the symbol ‘b’ has already been emitted (**$\alpha(1, b) * P(1 | 1) + \alpha(2, b) * P(1 | 2) + \alpha(3, b) * P(1 | 3)$**)

$\alpha(2, bca)$: probability that M reaches state 2 at t=3 and emits symbol ‘a’ having emitted before symbols ‘bc’ with any other states.



$$\alpha(2, bca) = P(a | 2) * (\alpha(1, bc) * P(2 | 1) + \alpha(2, bc) * P(2 | 2) + \alpha(3, bc) * P(2 | 3))$$

probability of reaching state 2 at $t=3$ having emitted before "bc"

$$P(y|M) = \sum_{q \in Q} \alpha(q, |y|) P(F|q)$$

$$P(\mathbf{bcba}|\mathbf{M}) = \sum_{q \in Q} \alpha(q, |\mathbf{bcba}|) P(\mathbf{F}|q) = \sum_{q' \in Q} \alpha(q', |\mathbf{bcb}|) \mathbf{P(a|q)P(q|q')} P(\mathbf{F}|q) = \dots$$

Forward

	b t=1	c t=2	b t=3	a t=4	
1	$\alpha(1, b)$ $= P(b 1)P(1)$ $= B_{1b}\pi_1$ $= \mathbf{0.3}$	$\alpha(\mathbf{1}, bc) =$ $P(c 1) \sum_q P(1 q)\alpha(q, b) =$ $P(c 1)(P(1 1)\alpha(1, b) +$ $P(1 2)\alpha(2, b) +$ $P(1 3)\alpha(3, b)) = \mathbf{0.042}$	$\alpha(\mathbf{1}, bcb) =$ $P(b 1) \sum_q P(1 q)\alpha(q, bc) =$ $P(b 1)(P(1 1)\alpha(1, bc) +$ $P(1 2)\alpha(2, bc) +$ $P(1 3)\alpha(3, bc)) = \mathbf{0.00297}$	$\alpha(\mathbf{1}, bcb\mathbf{a}) =$ $P(a 1) \sum_q P(1 q)\alpha(q, bcb) =$ $P(a 1)(P(1 1)\alpha(1, bcb) +$ $P(1 2)\alpha(2, bcb) +$ $P(1 3)\alpha(3, bcb)) = 0$	
2	$\alpha(2, b)$ $= P(b 2)P(2)$ $= B_{2b}\pi_2 = 0$	$\alpha(\mathbf{2}, bc) =$ $P(c 2) \sum_q P(2 q)\alpha(q, b) =$ $P(c 2)(P(2 1)\alpha(1, b) +$ $P(2 2)\alpha(2, b) +$ $P(2 3)\alpha(3, b)) = \mathbf{0.015}$	$\alpha(\mathbf{2}, bcb) =$ $P(b 2) \sum_q P(2 q)\alpha(q, bc) =$ $P(b 2)(P(2 1)\alpha(1, bc) +$ $P(2 2)\alpha(2, bc) +$ $P(2 3)\alpha(3, bc)) = \mathbf{0.0126}$	$\alpha(\mathbf{2}, bcb\mathbf{a}) =$ $P(a 2) \sum_q P(2 q)\alpha(q, bcb) =$ $P(a 2)(P(2 1)\alpha(1, bcb) +$ $P(2 2)\alpha(2, bcb) +$ $P(2 3)\alpha(3, bcb)) = 0.0004455$	
3	$\alpha(3, b)$ $= P(b 3)P(3)$ $= B_{3b}\pi_3 = 0$	$\alpha(\mathbf{3}, bc) =$ $P(c 3) \sum_q P(3 q)\alpha(q, b) =$ $P(c 3)(P(3 1)\alpha(1, b) +$ $P(3 2)\alpha(2, b) +$ $P(3 3)\alpha(3, b)) = 0$	$\alpha(\mathbf{3}, bcb) =$ $P(b 3) \sum_q P(3 q)\alpha(q, bc) =$ $P(b 3)(P(3 1)\alpha(1, bc) +$ $P(3 2)\alpha(2, bc) +$ $P(3 3)\alpha(3, bc)) = 0$	$\alpha(\mathbf{3}, bcb\mathbf{a}) =$ $P(a 3) \sum_q P(3 q)\alpha(q, bcb) =$ $P(a 3)(P(3 1)\alpha(1, bcb) +$ $P(3 2)\alpha(2, bcb) +$ $P(3 3)\alpha(3, bcb)) = \mathbf{0.012231}$	
F					$P(bcba M) =$ $\sum_q \alpha(q, bcba)A_{qF} =$ $\sum_q \alpha(q, bcba)P(F q) =$ $\alpha(1, bcba)P(F 1) +$ $\alpha(2, bcba)P(F 2) +$ $\alpha(\mathbf{3}, bcba)P(\mathbf{F} 3)$ $\mathbf{0.0073386}$

Viterbi

V	b $t=1$	c $t=2$	b $t=3$	a $t=4$	
1	$V(1, b) =$ $P(b 1)P(1) =$ $B_{1b}\pi_1 = \mathbf{0.3}$	$V(\mathbf{1}, bc) =$ $P(c 1) \max_q P(1 q)V(q, b) =$ $P(c 1)P(1 1)V(1, b) =$ $\mathbf{0.042}$	$V(\mathbf{1}, bcb) =$ $P(b 1) \max_q P(1 q)V(q, bc) =$ $P(b 1)P(1 1)V(1, bc) =$ $\mathbf{0.0025}$	$V(1, bcba) =$ $P(a 1) \max_q P(1 q)V(q, bcb) =$ $P(a 1)P(1 2)V(2, bcb) =$ 0	
2	$V(2, b) =$ $P(b 2)P(2) =$ $B_{2b}\pi_2 = 0$	$V(\mathbf{2}, bc) =$ $P(c 2) \max_q P(2 q)V(q, b) =$ $P(c 2)P(2 1)V(1, b) =$ $\mathbf{0.015}$	$V(\mathbf{2}, bcb) =$ $P(b 2) \max_q P(2 q)V(q, bc) =$ $P(b 2)P(2 1)V(1, bc) =$ $\mathbf{0.0126}$	$V(2, bcba) =$ $P(a 2) \max_q P(2 q)V(q, bcb) =$ $P(a 2)P(2 1)V(1, bcb) =$ $\mathbf{0.0004}$	
3	$V(3, b) =$ $P(b 3)P(3) =$ $B_{3b}\pi_3 = 0$	$V(\mathbf{3}, bc) =$ $P(c 3) \max_q P(3 q)V(q, b) =$ $P(c 3)P(3 1)V(1, b) =$ 0	$V(\mathbf{3}, bcb) =$ $P(b 3) \max_q P(3 q)V(q, bc) =$ $P(b 3)P(3 2)V(2, bc) =$ 0	$V(3, bcba) =$ $P(a 3) \max_q P(3 q)V(q, bcb) =$ $P(a 3)P(3 2)V(2, bcb) =$ $\mathbf{0.0113}$	
F					$P(bcba M) =$ $\max_q V(q, bcba)A_{qF} =$ $\max_q V(q, bcba)P(F q) =$ $\mathbf{\max_q V(3, bcba)P(F 3)} =$ $0.0113 * 0.6 =$ $\mathbf{0.00678}$

The most probable sequence of states for generating string "bcba" is {1123F}