Sessions 9 and 10: Mathematical induction

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Mathematical proofs: axiomatic method

The standard procedure for establishing truth in mathematics was invented by Euclid, a mathematician working in Alexandria, Egypt around 300 BC. His idea was to begin with five assumptions about geometry, which seemed undeniable based on direct experience. For example, one of the assumptions was "There is a straight line segment between every pair of points". Propositions like these that are simply accepted as true are called axioms. Starting from these axioms, Euclid established the truth of many additional propositions by providing "proofs". A (mathematical) **proof** is a sequence of logical deductions (**inference process**) whose hypotheses are the axioms and previously-proved statements, and whose conclusion is the proposition in question. Although a mathematical proof is, actually, a standard process of inference (as we have studied it), the exposition of mathematical proofs are given in a non-formal way. Of course it is possible to be "formal" but the price to pay is high: an excessively long list of hypotheses and inference rules impossible to remember. You probably wrote many proofs in high school geometry class, and you'll see more in this course.

There are several common terms for a proposition that has been proved. The different terms hint at the role of the proposition within a larger body of work:

- Important propositions are called **theorems**.
- A **lemma** is a preliminary proposition useful for proving later propositions.
- A **corollary** is a proposition that follows in just a few logical steps from a lemma or a theorem.

Any proposition that is not of one of the above types is called simply **proposition**.

The definitions are not precise. In fact, sometimes a good lemma turns out to be far more important than the theorem it was originally used to prove.

A **proposition**, **lemma**, **corollary** or **theorem** is, many times, of the form $P \to Q$ (that is, an implication). P is usually called **hypothesis** and the conclusion Q is called **thesis**. Then, the **proof** is the inference process which let us deduce Q from P. Usually, $P \to Q$ is called the **direct implication**, and when we change the place of P and Q, we called the implication $Q \to P$ as the **converse implication** (which is not necessarily a theorem because it may be false!).

2 An example of set of axioms: Peano axioms

The Peano axioms, also known as the Dedekind-Peano axioms or the Peano postulates, are a set of axioms for the natural numbers presented by the 19th century Italian mathematician Giuseppe

Peano.

The Peano axioms define the arithmetical properties of the set of natural numbers, usually represented as $\mathbb{N}$, in such a way that all the known arithmetic results are derived from them. They are the following:

Peano axioms:

1. There exists a natural number called 1. (Some people also include the number 0, but not here)
2. Every natural number n has a **successor** that we will represent by $n + 1$.
3. Whenever two natural numbers are different, then their successors are different.
4. Every natural number, except 1, is the successor of another natural number.
5. **Induction principle:** Let $P(n)$ be a predicate whose universe is the set of natural numbers and assume that the following conditions are satisfied:
   - The proposition $P(1)$ is true, and
   - if the predicate P is satisfied for a natural number k (that is, if $P(k)$ is true), then its successor $k + 1$ also satisfies P (that is, $P(k + 1)$ is also true). In other words: the proposition $P(k) \to P(k + 1)$ is true.

   Then the proposition $\forall n \; P(n)$ is true (that is, $P(n)$ is true for every natural number n).

I'm sure that the first four axioms are totally evident to your mind but, perhaps, this is not the case for the last axiom (induction principle). The purpose of the next section is to make it evident to you.

3 Induction principle (or mathematical induction)

Many mathematical conjectures concern properties of the natural numbers. Consider, for example, the following problem: find a formula for the sum of the first n odd integers. A useful starting point might be to write down the sums for some small values of n and see if this gives us any idea as to what might be a possible conjecture:

- For $n = 1$, the sum is 1.
- For $n = 2$, the sum is $1 + 3 = 4$.
- For $n = 3$, the sum is $1 + 3 + 5 = 9$.
- For $n = 4$, the sum is $1 + 3 + 5 + 7 = 16$.

At this stage we notice that, so far, for each value of n, the sum is n^2. We try a few more to see if our conjecture is well founded:

- For $n = 5$, the sum is $16 + 9 = 25$.
- For $n = 6$, the sum is $25 + 11 = 36$.

If you want, you can do a computer program in order to compute the sum of the n first odd numbers, with n taking values from 1 to 100, for example. You will see that the obtained sum is always equal to n^2. This leads us to the following **conjecture**:

For every natural number n, the sum of the first n odd natural numbers is equal to n^2.

In other words, given the predicate $P(n) =$"The sum of the first n natural numbers is equal to n^2" defined in the universe of natural numbers ($\mathbb{N}$):

The proposition $\forall n\ P(n)$ is true.

But, how can we try to prove this conjecture? We can check $P(n)$ for some natural numbers n (maybe 100, or 1000, or 100000,...) but **not for ALL of them**. Surprisingly, there is an extremely useful tool that will allow us to prove the conjecture: the **induction principle**.

Suppose that we are able to prove these two conditions:

(1) $P(1)$ is true.

(2) For every natural number k, if $P(k)$ is true then $P(k+1)$ is true.

Assuming this, we have:

- $P(1)$ is true by (1).
- By (2) applied to $k = 1$, $P(2)$ **is true** because $P(1)$ is true.
- By (2) applied to $k = 2$, $P(3)$ **is true** because $P(2)$ is true.
- By (2) applied to $k = 3$, $P(4)$ **is true** because $P(3)$ is true.
- By (2) applied to $k = 4$, $P(5)$ **is true** because $P(4)$ is true.

And... we can continue indefinitely! So, $P(n)$ is true for any natural number n!

Summarizing: we have to prove that the property is true for $n = 1$ (condition (1)) and, also, assuming that the result holds for $n = k$ (this assumption is usually known as the **inductive hypothesis**) we have to deduce that it also holds for $n = k + 1$ (condition (2)). This is sufficient to assure that the result holds for every natural number. This is what **induction principle** says. In the classroom exercises we will prove (1) and (2) for this specific example.

An analogy to the process of mathematical induction is an infinite line of fireworks[1] connected together so that each is set off by the previous one in the line. Although it has been arranged that the kth firework will ignite the $(k+1)$st, nothing happens until we light the first firework in the line. This sets off the second, which sets off the third and so on to the end of the (infinite) line.

[1]For Valencian people: you may also think about an infinite "mascletà".