Sessions 17 and 18: Order relations

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Introduction

Many sets have a natural ordering of their elements. Probably the most familiar example is the set of real numbers ordered by "magnitude". We are used to statements such that $3 \leq \pi$, $-4 \leq -3$, $2 \leq \sqrt{8}$, and so on. Similarly, any family of sets is ordered by "inclusion": if $A \subseteq B$ we may regard A as being "smaller" than B. Also a set of people could be ordered by age or by height.

In this session we are going to define the concept of *order relation* with the purpose of capturing the essence of ordering and providing one to say when something is "less than" or "preceeds" another.

2 Order relation

A binary relation on a set A is an **order relation** if it is reflexive, antisymmetric and transitive.

Example 1. (a) Let $\mathbb{R}$ be the set of real numbers and consider, in it, the binary relation $\leq$.

- $\leq$ is reflexive because $a \leq a$ for every real number a.
- $\leq$ is antisymmetric because, if a and b are real numbers such that $a \leq b$ and $b \leq a$, then $a = b$
- $\leq$ is transitive because, for all real numbers a, b, c, if $a \leq b$ and $b \leq c$ then $a \leq c$.

Therefore, we conclude that $\leq$ is an order relation.

(b) Let U be any set. Consider $\mathcal{P}(U)$ (the power set of U) and the following relation R in $\mathcal{P}(U)$: for every $A, B \in \mathcal{P}(U)$, $A\ R\ B$ if and only if $A \subseteq B$. This is the "inclusion relation". Usually we do not use the notation "R" for this relation; instead, we write directly $A \subseteq B$ to denote that A is related to B. This is an order relation:

- $\subseteq$ is reflexive because $A \subseteq A$ for all $A \in \mathcal{P}(U)$.
- $\subseteq$ is antisymmetric because, if A and B are two elements of $\mathcal{P}(U)$ such that $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- $\subseteq$ is transitive because, if A, B and C are three elements of $\mathcal{P}(U)$ such that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(c) Consider the set of natural numbers $\mathbb{N}$ and, on it, the relation R defined as follows: given two natural numbers a and b, aRb if and only if a divides b. This is called the "divisibility relation". Usually we do not use the notation "R" for this relation; instead, we write $a \mid b$ to denote "a divides b" and we use the symbol $\mid$ to denote the divisibility relation. This is an order relation:

- $|$ is reflexive because a divides a for every natural number a.
- $|$ is antisymmetric because, if a and b are two natural numbers such that $a \mid b$ and $b \mid a$, then $a = b$.
- $|$ is transitive because, if a, b and c are three natural numbers such that $a \mid b$ and $b \mid c$, then $a \mid c$. Indeed, since a divides b one has that there exists a natural number k_1 such that $b = k_1 a$ and, similarly, since b divides c, there exists a natural number k_2 such that $c = k_2 b$; therefore $c = k_2 b = k_2 k_1 a$ and we conclude that a divides c.

(d) Consider the relation on the set of English words defined by "the word w_1 is related to the word w_2 if $w_1 = w_2$ or w_1 comes before w_2 in a dictionary". It can be checked easily that this relation is reflexive, antisymmetric and transitive. Hence, it is an order relation. It is called *alphabetical ordering*.

(e) Consider the set $A = \{a, b, c, d\}$ and the following relation:

$$R = \{(a,a), (b,b), (c,c), (d,d), (a,b), (b,d), (a,d), (a,c)\}.$$

Using, for example, the characterization of the properties using the matrices of the relation, one can easily check that this is an order relation.

3 Total order and partial order

An order relation R on A is a **total order relation** if $\forall\, x, y \in A$, $(xRy) \ \vee \ (yRx)$ (or, in other words, if every pair of elements of A can be "compared" by R).

If R is an order relation on A that is not a total order relation, we will say that R is a **partial order relation**.

Example 2. (a) The order relation $\leq$ on $\mathbb{R}$ is a total order relation because, for every couple of real numbers x, y, it holds that either $x \leq y$ or $y \leq x$ (we can compare every pair of real numbers).

(b) Let U be the set $\{a, b\}$. Then $\mathcal{P}(U) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. The inclusion relation $\subseteq$ on $\mathcal{P}(U)$ is not a total order relation because $\{a\} \nsubseteq \{b\}$ and $\{b\} \nsubseteq \{a\}$ (that is, the elements $\{a\}$ and $\{b\}$ are not related at all). Then, $\subseteq$ is a partial order relation, in this case.

(c) The divisibility relation $|$ on the set of natural numbers $\mathbb{N}$ is not a total order relation because, for example, 2 does not divide 3 and 3 does not divide 2 (that is, 2 and 3 are not related, they cannot be compared). Therefore $|$ is a partial order relation.

(d) The alphabetical ordering on the set of English words is a total order relation (notice that every pair of English words can be compared using this ordering).

From now on, when we will deal with an order relation, many times we will denote it by $\leq$ instead of by R, S, etc. The reason is that the symbol $\leq$ is "more suggestive" in the sense that it helps you to remind the "order flavour" of the relation. In some specific cases (as, for example, the case of the inclusion relation $\subseteq$ or the divisibility relation $|$) we will use other notations.

4 Hasse diagram of an order relation

4.1 Definition

If $\leq$ is an **order relation** on a **finite** set A, we can represent graphically $\leq$ using the usual directed graph. However, in this case, we can use a different (and simpler) diagram to represent $\leq$ that is called **Hasse diagram**. To define it, we need the following notion:

Let a be an element of A. We say that an element $b \in A$ **covers** a if $a \neq b$, $a \leq b$ and there is no element $c \in A$, different from a and b, such that $a \leq c \leq b$.

Perhaps, it is useful to introduce here some "informal definitions" because they will help you to understand the above definition and others:

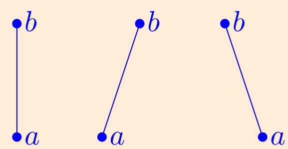
When we have x, y of A such that $x \leq y$ and $x \neq y$ we will say, **informally**, that "x is below y" (or that "y is above x"). Also, if there is no element $z \in A$, different from x and y, such that $x \leq z$ and $z \leq y$, we will say that "there is no element in de middle of x and y".

With this informal notation we can say that:

b **covers** a if and only if "b is above a" and 'there is no element in the middle" of a and b.

Now, we are prepared to define the Hasse diagram of an order relation:

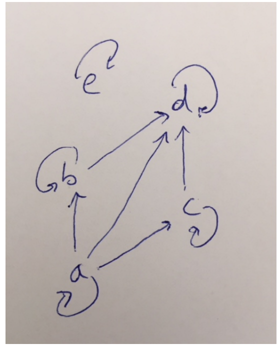
Let $\leq$ be an order relation on a finite set A. The **Hasse diagram** of $\leq$ is the graph whose vertices are the elements of A and such two of them, a and b, are connected by an edge if and only if b covers a; moreover this edge will be represented by an **ascendent** segment (with a at the bottom and b at the top).



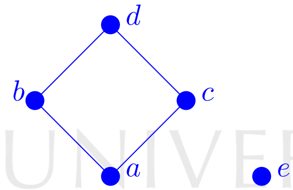
Example 3. Let us consider the set $A = \{a, b, c, d, e\}$ and the relation

$$\leq = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(b,d),(a,d),(a,c),(c,d),(a,d)\}.$$

This is an order relation (prove it!). Its associated directed graph is



Notice that b covers a, c covers a, d covers b and d covers c. And there are not more "coverings"! (Notice that, for example, d does not cover a because b and c "are in the middle"). Therefore, the Hasse diagram of this relation is the following one:

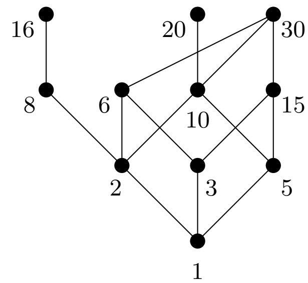


Notice that, from the Hasse diagram of an order relation, we can recover easily the elements of the relation:

- Since the relation is an order relation, all the pairs of the type (x, x) belong to it.
- Given two different elements x and y, we have that $x \leq x$ if and only if there is an **ascending path** from x to y in the Hasse diagram. (For instance, in the previous example, one can deduce that $a \leq d$ from the Hasse diagram because there is an ascending path from a to d (the path $a - b - d$).)

The computation of the Hasse diagram is specially easy when the involved relation is the divisibility relation. Look at the following example:

Example 4. Let $A = \{1, 2, 3, 5, 6, 8, 10, 15, 16, 20, 30\}$ and consider the divisibility relation on A (denoted by $|$). The Hasse diagram of this relation is represented with this figure:



4.2 An algorithm to construct Hasse diagrams

When the graph of an order relation is "complicated", it may not be easy to construct the associated Hasse diagram. The purpose of this section is to explain an easy algorithm to do that. But we need, firstly, the definition of *minimal element*, which we include below.

If $\leq$ is an order relation on a set A, it is said that an element $m \in A$ is a **minimal** element if

$$\forall\, x \in A, \quad x \leq m \to x = m.$$

In other words:
m is a minimal element if **there is no element of A below m.**

An algorithm that can be applied to construct the Hasse diagram of an order relation $\leq$ on a finite set A is the following one:

1. Compute the minimal elements of $\leq$ on A and draw a vertex for each one of them.
2. Consider the set A' obtained by removing, from A, the computed minimal elements. Compute then the minimal elements of $\leq$ **but considered as a relation defined on A'**. Draw a vertex for each one of them. Connect, with ascending segments, the vertices x drawn at the previous steps with the ones y drawn at this step whenever $x \leq y$.
3. Consider the set A'' obtained by removing, from A', the above computed minimal elements and repeat successively the process explained at Step 2 until the obtained set be empty.

In the classroom exercises you will practice with this algorithm.

5 Distinguished elements associated with an order relation

In this section we are going to define some distinguished elements that we can associate to an order relation. Some of them will be associated to the order structure of the whole set A and, others, to the subsets of A. So, let us fix a set A with an order relation $\leq$.

5.1 Distinguished elements associated with the set A

- An element $m \in A$ is a **maximum** if $\forall x \in A$ we have $x \leq m$. That is, a maximum of A is an element **of** A such that all the remaining elements of A "are below" it.
- An element $m \in A$ is a **minimum** if $\forall x \in A$, we have $m \leq x$. That is, a minimum of A is an element **of** A such that all the remaining elements of A "are above" it.
- An element $m \in A$ is **maximal** if

  $$\forall x \in A \quad (m \leq x \rightarrow m = x),$$

  that is, if there is no element of A that "is above" m.
- An element $m \in A$ is **minimal** if

  $$\forall x \in A \quad (x \leq m \rightarrow m = x),$$

  that is, if there is no element of A that "is below" m.

Example 5. Consider the ordered set A of Example 4.

- The maximal elements of A are 16, 20 and 30.
- 1 is the unique minimal element of A.
- There is no maximum.
- 1 is the minimum of A.

Theorem. The following properties are satisfied:

(a) If A has a maximum, then it is unique.

(b) If A has a minimum, then it is unique.

Proof. (a) We must prove that, if A admits two maximums m_1 and m_2, then they must be equal.

- Since m_1 is a maximum and m_2 is an element of A, we get that $m_2 \leq m_1$, and
- since m_2 is a maximum and m_1 is an element of A, we get that $m_1 \leq m_2$.

But $\leq$ is an order relation and, therefore, it is antisymmetric. We conclude, then, that $m_1 = m_2$.

(b) The proof is similar.

□

- The maximum of an ordered set A (if it exists) will be denoted by $\max(A)$.
- The minimum of an ordered set A (if it exists) will be denoted by $\min(A)$.

The following proposition is very easy to deduce from the definitions:

Proposition. The following properties are satisfied:

(a) If A has maximum, then it is unique maximal element.

(b) If A has minimum, then it is unique minimal element.

5.2 Distinguished elements associated with a subset of A

Let B be any subset of A.

- It is said that $a \in A$ is an **upper bound** of B if $\forall x \in B$, $x \leq a$, that is, if a "is above" **all** the elements of B. If B has upper bounds, then it is said that B is **upper bounded**.
- It is said that $a \in A$ is a **lower bound** of B if $\forall x \in B$, $a \leq x$, that is, if a "is below" **all** the elements of B. If B has lower bounds, then it is said that B is **lower bounded**.
- It is said that $a \in A$ is the **supremum** of B if a is the smallest upper bound of B (that is, it is the minimum of the set of upper bounds of B).
- It is said that $a \in A$ is the **infimum** of B if a is the biggest lower bound of B (that is, it is the maximum of the set of lower bounds of B).

Remark. The maximum and the minimum of a subset of A are the maximum and the minimum with respect to the same order relation, but restricted to the subset.

Also,

A subset B of A is **bounded** if it is upper bounded and lower bounded.

Proposition. The following properties are satisfied for any subset B of A:

(a) If B has a supremum, then it is unique.

(b) If B has an infimum, then it is unique.

Proof. To prove (a), let us assume that s_1 and s_2 are two supremums of B and let us prove that they must be equal.

- Since s_1 is supremum of B and s_2 is an upper bound, it holds that $s_1 \leq s_2$ (recall that the supremum is the minimum of the set of upper bounds of B), and
- since s_2 is supremum of B and s_1 is an upper bound, it holds that $s_2 \leq s_1$.

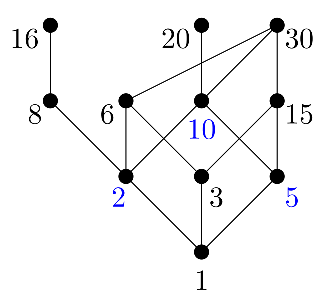
Then $s_1 = s_2$ because the relation $\leq$ is antisymmetric.

□

Let B be any subset of A.

- If B has supremum, then it will be denoted by $\sup(B)$.
- If B has infimum, then it will be denoted by $\inf(B)$.

Example 6. Let us consider the ordered set A of Example 4:



- If we consider the subset $B = \{2, 10, 5\}$, then the upper bounds of B are 10, 20, and 30 and $\sup(B) = 10$.

- The only lower bound of B is 1, and therefore it is its infimum.
- On the other hand, the maximum of B is 10, B has no minimum, 10 is a maximal element, and the minimal elements of B are 2 and 5.