MATHEMATICAL ANALYSIS

L4 - Sequences **List 1**

1. Determine the values of a_2, a_4, b_1, b_4, c_3 and c_5 for the sequences defined by:

$$a_n = \frac{(-1)^n}{(n+1)!} \quad , \quad b_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+n} \quad , \quad \begin{cases} c_n = 4n + c_{n-1} \quad , \quad n \geq 2 \\ c_1 = 1 \end{cases}$$

2. Using the two subsequences obtained with odd and even terms of the sequence sequences

$$a_n = [(-1)^n + 1] \cos(n\pi),$$

calculate if $\{a_n\}$ is convergent or divergent.

3. Calculate the limits of the sequences::

   a) $\frac{4-2n-3n^2}{2n^2+n}$
   b) $\frac{\sqrt{3n^2-5n+4}}{2n-7}$
   c) $\sqrt[3]{\frac{(3-\sqrt{n})(\sqrt{n}+2)}{8n+4}}$
   d) $\sqrt{2n^2+3} - \sqrt{n^2-n}$
   e) $\sqrt{n^2+3n} - \sqrt{n^2+3}$
   f) $\sqrt{n^2+n} - n$
   g) $\frac{4\cdot 10^n - 3\cdot 10^{2n}}{3\cdot 10^{n-1} + 2\cdot 10^{2n-1}}$
   h) $\frac{2\cdot 3^{n+1} - 3\cdot 4^{n-1}}{3^n + 2^{2n}}$

4. Find the limit for the following sequences. Euler's formula can help you:

   a) $\left(\frac{n+2}{n}\right)^n$
   b) $\left(\frac{1+3n}{5+3n}\right)^{\frac{n^2}{4n-2}}$
   c) $\left(\frac{n+1}{n}\right)^{\frac{\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}}$

5. Using Stolz criterium, find the limits of the following sequences:

   a) $\frac{1+4+\cdots+n^2}{5+8+\cdots+(n^2+4)}$
   b) $\left(\frac{1^2+2^2+\cdots+n^2}{n^3}\right)^n$
   c) $\frac{1+2+\cdots+n+(n+1)\cdots+2n}{n^2}$

6. Arrange according to magnitude order:

   a) $a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$ and $b_n = log(n)$
   b) $a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$ and $b_n = \sqrt{n}$
   c) $a_n = \sqrt{n}$ and $b_n = log(n)$
   d) $a_n = 2^n$ and $b_n = 3 + 3^2 + \cdots + 3^n$
   e) $a_n = n^2 + \log(n)$ and $b_n = 1 + 2 + 3 + \cdots + n$
   f) $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ and $b_n = n^2$
   g) $a_n = 1 + 2^2 + 3^2 + \cdots + n^2$ and $b_n = n^3$
   h) $a_n = n!$ and $b_n = 1! + 2! + \cdots + n!$

7. Arrange according to magnitude order and justifying the result the sequences: $\sqrt{n}$, n, $\log(n)$, n^2, e^n, n^3 and $n!$. Group round the following ones according to the magnitude order.

   a) $n^2 + \sqrt{n+1}$

   b) $\frac{1}{\sqrt{n+1} - \sqrt{n}}$

   c) $\frac{\sqrt{n^7} - \sqrt{n^3+1}}{5+2\sqrt{n}}$

   d) $\log\left(n^5 + e^{2n}\right)$

8. Arrange according to magnitude order:

$$3\sqrt{n^5+n} - n^2 \quad , \quad \log(n) \quad , \quad 1 + \frac{1}{2} + \frac{1}{3} + \; ... \; + \frac{1}{n}$$

Additional exercises

1. Check if the following sequences are increasing / decreasing or bounded:

   a) $a_1 = 1$, $a_{n+1} = \frac{3+2a_n}{4}$

   b) $b_1 = \sqrt{2}$, $b_{n+1} = \sqrt{2b_n}$

   c) Find the explicit form and calculate the limit for the defined above sequences.

2. Find the general term of:

   a) -1 , $+2$, -3 , $+4$, -5 , $+6$, ...

   b) $\frac{2}{3}$, $\frac{1}{3}$, $\frac{4}{27}$, $\frac{5}{81}$, ...

   c) An arithmetic sequence of difference d and first term $a_1 = a$

   d) A geometric sequence of common ratio r and first term $a_1 = a$ and the sum of the first "n" terms $\sum_{k=1}^{n} a_k$.

3. Verify that $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$ is strictly increasing and is upper bounded.

4. Check if the following sequences are increasing / decreasing or bounded:

   a) $\begin{cases} 2a_{n+1} = 2 + a_n \\ a_1 = 0 \end{cases}$

   b) $\begin{cases} a_{n+2} = n + a_{n+1} \\ a_1 = 10 \end{cases}$

5. Verify that:

   a) $a_n = \frac{10-n^2}{n+2}$ is decreasing and is upper bounded by 3

   b) $a_n = \frac{\sqrt{n}}{n+1}$ is decreasing and $0 < a_n \leq \frac{1}{2}$

   c) $a_{n+1} = 4a_n$ is increasing if and only if $a_1 > 0$.

   d) $a_{n+1} = \frac{n \cdot a_n}{n+7}$ with $a_1 = 7$ is decreasing and bounded.