Unit 1

Electrostatics of point charges

Objectives

- Know how calculate electric field and electric potential created by point charges and symmetrical distributions of charge.
- Get the concepts of electrostatic potential energy and electric potential.

1.1 Introduction

The electromagnetic interaction, the electromagnetism, is present in a wide range of phenomena of distinct fields that cover from electronic interactions in the atoms and molecules, that is to say of the same constitution of the matter, to others in which is founded the main part of the current technology: lighting, engines, broadcasting, computer science, etc...

In the past, electromagnetism has been split in two different parts, associated to their electric and magnetic effects. In fact, Hans Christian Oersted, one of the discoverers of the relations between both effects, was professor of "Electricity, Galvanism and Magnetism", and until the discovery of this relation, in the second half of the 19th century, such effects were studied in separate way. Nowadays, we know that both phenomena are a consequence of the same feature of the matter, that we call **electric charge**.

This unit aims to study electric charges at rest, the part of the electromagnetism that we call **electrostatics**. On further units will be studied the magnetic effects of moving electric charges.

1.2 Electric charge

We will begin this unit analyzing a simple experience in which are put of self-evident some characteristics of the physical quantity electric charge. Let's consider a ball of plastic hanged with an insulating thread and rubbed with skin; now, we rub with skin another piece of plastic bar shaped. If we approach the bar to the ball, ball and bar are rejected. But if we change the ball of plastic by another ball of glass and we rub it with silk, when approaching the bar of plastic rubbed with skin, the ball and the bar are attracted.

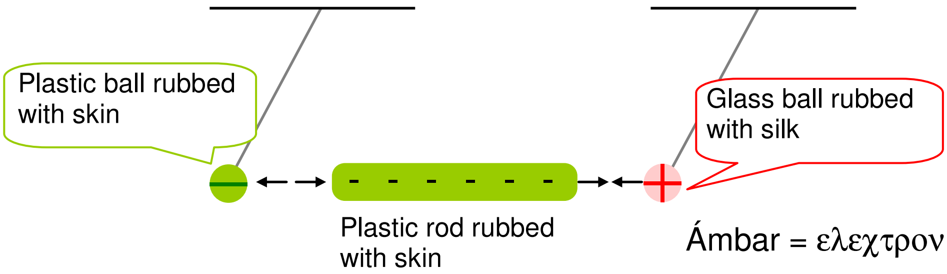


Figure 1-1. Electric forces attracting and rejecting

In the past, Greek people did similar experiences but with pieces of amber, skins and fabrics instead of plastic. From here that the origin of word "electricity" is the word "amber" written in Greek language ελεχτρον (electron).

Justification: Atomic model. Negative and positive charges

The explanation of these forces or interactions appeared in the shown systems is the existence of a feature of the matter called electric charge. There is enough to take in account two types of charge, we call positive and negative charges, to understand the phenomena of attraction and repulsion shown in Figure 1-1.

We know today that the matter is made up by atoms. The atoms are made up by a nucleus, what contains protons with positive charge and neutrons without electric charge. This nucleus is surrounded by a distribution of electrons with negative charge. The quantity of charge of an electron and a proton is the same, but of different type. Between charges of the same type they appear repulsive forces and between charges of different type, attractive forces.

The feature or characteristic of the matter that is in electrons and protons is called electric charge, Q, and it's a basic property of the matter, in the same way than it's the mass. A body is positively charged if it has lower number of electrons than of protons, and negatively charged if the number of electrons is higher than protons.

At first, the atoms have so many protons like electrons (are neutral), having the same quantity of charge of both types, and being zero their net charge (Figure 1-2).

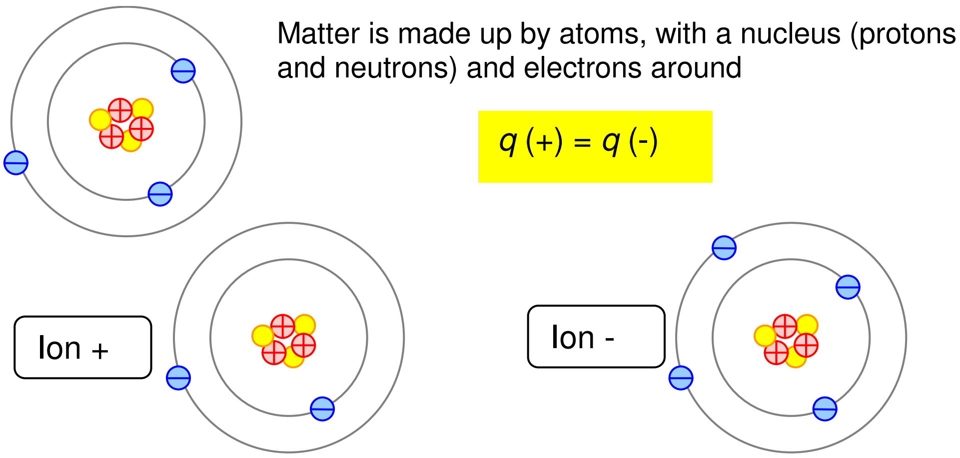


Figure 1-2. Atomic model and positively and negatively charged ions

When plastic is rubbed with skin, electrons are transferred from skin to plastic, remaining the plastic with an excess of electrons (negatively charged), and the skin with lack of electrons (positively charged). Whereas when glass is rubbed with silk, some electrons pass from glass to silk, remaining the glass positively charged and the silk negatively charged. In these charge transfers from a body to another, the electrons are always those that transfer from a body to another, never the protons, since they are into the nucleus and it's very difficult to move them out.

An atom with lack of electrons is a positive ion, and with excess of electrons, a negative ion.

The minimum electric charge can have a body is the charge of an electron (in modulus), called elementary charge, and it's written by e.

Properties: quantization and conservation

The electric charge of a body is always a whole multiple of the elementary charge *e*, that is the charge is quantized, not being possible to obtain smaller parts of this quantity.

On the other hand, the charge don't create neither destroy; it can flux, it can change its position, but can't disappear. This fact is known as the principle of conservation of the electric charge: the total charge in an isolated system remains constant.

The unit of electric charge in S.I. is the **coulomb** (C), whose definition is related to the ampere, base unit of the base quantity intensity of electric current. A coulomb is the quantity of charge fluxing along 1 second through a cross section of a conductor carrying an intensity of current of 1 ampere: 1 C = 1 A*1 s, (intensity of current will be defined in unit 3, and ampere in unit 6). Equation of dimensions of electric charge is: $[Q] = I\ T$

The elementary charge is: $e = 1{,}6 \cdot 10^{-19}$ C

1.3 Electrostatic forces. Coulomb's law

The attractive and repulsive forces dues to electric charges between charged bodies are generally much higher than the attractive gravitational forces between them.

The first one in measuring electrostatic forces was Coulomb (1785), using a torsion balance.

Two electric point charges q_1 and q_2, at rest, separated a distance r in the vacuum, exert between themself a force whose magnitude is proportional to the product of the charges and inversely related to the square of the distance. The direction is the line joining both charges, being the force repulsive if they are of the same sign and attractive if they have opposite sign.

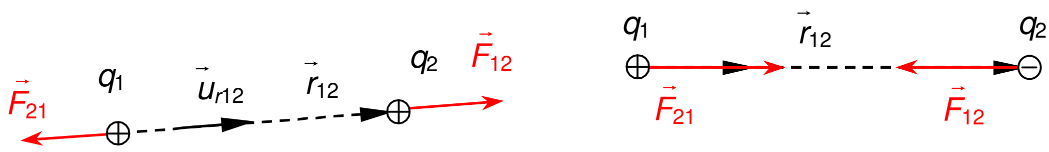


Figure 1-3. Forces between charges of the same sign and between charges of different sign

$$\vec{F}_{12} = k\frac{q_1 \cdot q_2}{r_{12}^2}\vec{u}_{r12} \qquad \text{Equation 1-1}$$

In this equation, $\vec{F}_{12}$ is the force that the charge q_1 exerts on q_2, $\vec{F}_{21}$ is the force that the charge q_2 exerts on q_1, $\vec{r}_{12}$ is the vector going q_1 from to q_2, and its unit vector $\vec{u}_{r_{12}} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|}$. Constant k on vacuum is k = 8,99·10^9 ≈ 9·10^9 N·m^2/C^2, although it's more usual to find it as a function of another constan ε_0,

$$k = 1/4\pi\varepsilon_0$$

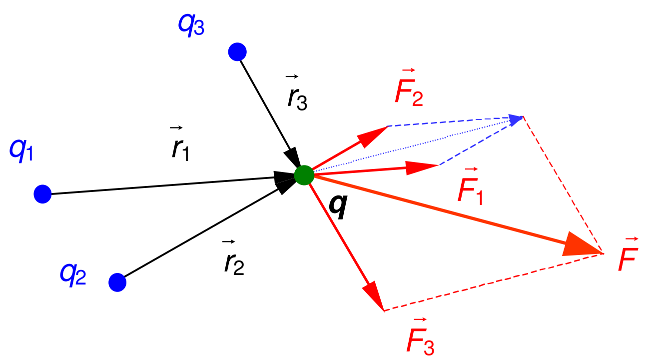
where ε_0 = 8,85·10^{-12} C^2/N·m^2 is the **permittivity or dielectric constant of the vacuum**.

Exercise:
Two positive point charges of 1 μC are placed on the *OX* axis of a cartesian coordinates system, at a distance r =1 cm. Compute the forces appear between them.

Principle of superposition of electrostatic forces

The net force produced by several charges at the same time on another, is the vector sum of the forces that would appear if they acted separately. This fact is called principle of superposition of forces, and can be also applied to other cases of forces superposition.

Therefore, if it has a distribution of charges q_i acting on a charge *q*, the net force acting on *q* is the vector sum of the forces that exerts each one of them on *q*.



$$\vec{F} = \sum \vec{F}_i = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q \cdot q_i}{r_i^2} \vec{u}_{r_i}$$

Figure 1-4. Principle of superposition

Equation 1-2

1.4 Electric field

Given an electric charge q, the space that surrounds it is modified by its presence, and it's said that in such space there is an **electric field**. If we place a second charge q_0 on a point of such field, the electric field in such point is defined as the force that would act on q_0 divided into the charge q_0. That is, the electric field $\vec{E}$ on a point of the space is the *electric force that would act on the unit of positive charge placed on such point*.

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \vec{u}_r$$

Equation 1-3

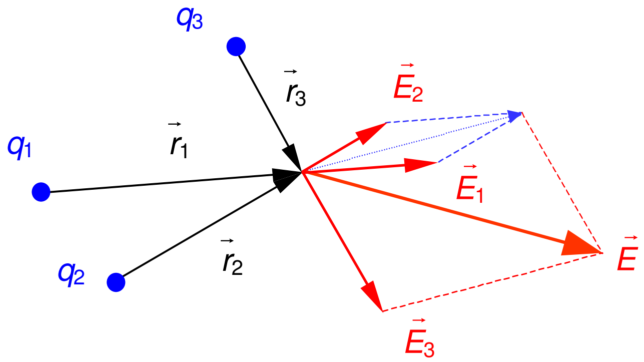
The unit of electric field is the N/C, although we will see that N/C is equivalent to V/m (volt/meter). Dimensions of electric field are $[E] = M\ L\ T^{-3}\ I^{-1}$

Electric field has been described like an effect produced by the electric charges at rest, but further of a context purely electrostatic, electric field is a quantity present in many other systems or entities more complex, like electric currents, stormy clouds, molecules, electromagnetic waves, etc., where it also exists an effect quantified by this quantity.

Electric fields in the nature (estimation)	***E* (N/C)**
In the domestic wires	10^{-2}
In the waves of radio	10^{-1}
In the atmosphere	10^{2}
In the sunlight	10^{3}
Under a stormy cloud	10^{4}
In a tube of X-rays	10^{6}
In the atom of hydrogen	$6 \cdot 10^{11}$

Electric field created by *n* point charges

In a similar way to the superposition of electrostatic forces, net electric field created by many point charges on a point is equal to the vector sum of the electric fields created by each one of the charges in such point.



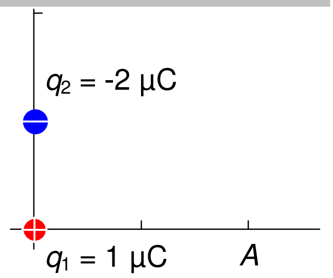
$$\vec{E} = \sum \vec{E}_i = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i^2} \vec{u}_{r_i}$$

Figure 1-5. Principle of superposition

Equation 1-4

Example 1-1

Given the point charges of the figure, compute:
a) The net electric field on point $A(2,0)$ m. Apply the principle of superposition, drawing in the chart the fields exerted by each charge separately.
b) The force would act on a negative point charge of -3 nC placed on A.



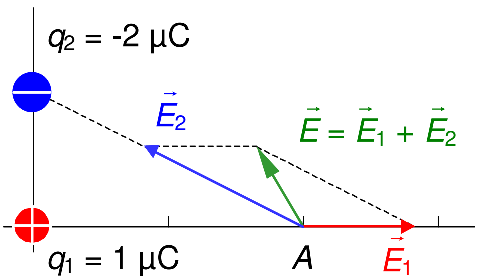
Solution

a) Electric field due to a point charge q placed on a point given by the vector $\vec{r}$ is $\vec{E} = k\dfrac{q}{r^2}\vec{u}_r$, being $\vec{u}_r$ the unit vector of $\vec{r}$:

$$\vec{E}_1 = k\frac{q_1}{r_1^2}\vec{u}_{r_1} = k\frac{q_1}{r_1^2}\vec{i} = \frac{9000}{4}\vec{i}\ \frac{\text{N}}{\text{C}}$$

$$\vec{E}_2 = k\frac{q_2}{r_2^2}\vec{u}_{r_2} = \frac{-18000}{5}\left(\frac{2\vec{i} - \vec{j}}{\sqrt{5}}\right) = \frac{3600}{\sqrt{5}}\left(-2\vec{i} + \vec{j}\right)\frac{\text{N}}{\text{C}}$$

Adding both vectors, $\vec{E} = -970\vec{i} + 1610\vec{j}$ N



b) If a charge of -3 nC is placed on A, force acting on charge is:
$\vec{F} = q\vec{E} = -3(-970\vec{i} + 1610\vec{j}) = 2910\vec{i} - 4830\vec{j}$ nN

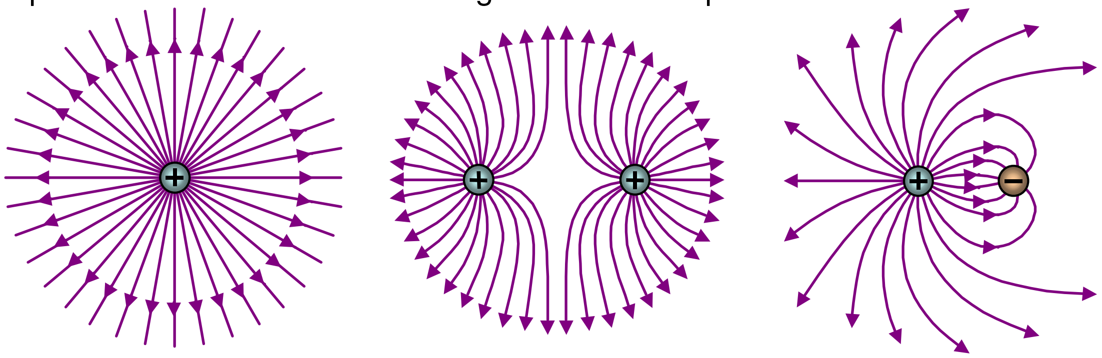
Observe that being the charge negative, vector force is opposite to the field.

Field Lines

Lines being tangents to electric field vector on each point on the space are called electric field lines. Electric field can therefore be represented by means of these lines that show the direction of electric field on any point.

The electric field created by a point charge is a central field (field vector has always radial direction), being its field lines straight lines crossing in the point where the charge is placed.

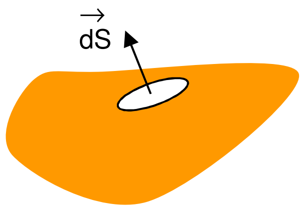
Field lines dues to several point charges are curve lines, whose shape depends on the values of the charges and of their positions.



To) b) c)

Figure 1-6. a) Electric field lines near a positive point charge. b) Electric field lines near two equal positive point charges. c) Electric field lines near a positive point charge $2q$ and a negative $-q$

1.5 Flux of the electric field. Gauss's law



Given an enclosed surface S, on any point of such surface can be defined the surface vector. To do it, we must take a little surface around such point (dS). Surface vector is a vector perpendicular to the surface, pointing outside of the volume enclosed by the surface, and magnitude the taken surface, dS. In order this definition was consistent, the taken element of surface must be a small and flat surface, and thus, a surface with any shape must be split in small infinitesimal surfaces.

Elementary flux (dϕ) of an electric field through an elementary surface $d\vec{S}$ is defined as the inner product of vectors $\vec{E}$ and $d\vec{S}$:

$$d\Phi = \vec{E} \cdot d\vec{S}$$

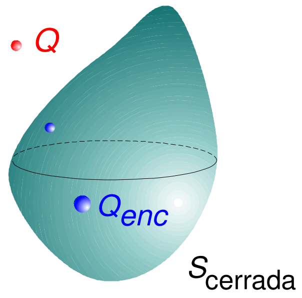
If we extend computation of elementary flux to the whole enclosed surface, we'll have the flux of the electric field through surface S:

$$\Phi = \int_S d\Phi = \int_S \vec{E} \cdot d\vec{S}$$

Dimensions of electric flux are

$[\phi] = [E][S] = ML^3T^{-3}I^{-1}$ being measured in·V m or N·m^2/C

Gauss's law says that:



$$\Phi_{closedsurf} = \frac{Q_{enclosed}}{\varepsilon_0}$$

$$\int_s \vec{E} \cdot d\vec{S} = \frac{\sum Q}{\varepsilon_0}$$

The flux of the electric field through an enclosed surface S is equal to the net charge enclosed inside S divided into ε_0.

It is important to underline that the flux through the enclosed surface only depends on the charges inside the surface, <u>does not depend on charges outside</u> the surface.

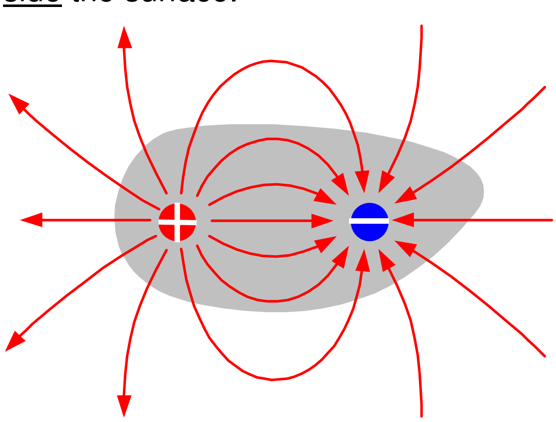


Figure 1-7. An enclosed surface enclosing an electric dipole is crossed by a net flux zero. Graphically, the number of lines going out of the surface is the same that those going in.

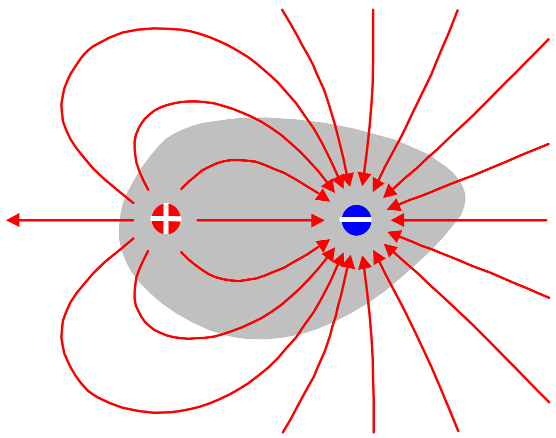


Figure 1-8. An enclosed surface enclosing a system of two charges q and $-2q$ is crossed by a negative net flux. Graphically, the number of lines going out of the surface is lower than those going in.

Gauss's law is useful to find the electric field of some distributions of charge that, in general, present a special symmetry in the distribution of the charge (spheres and infinite cylinders uniformly charged, infinite charged planes, etc...). In these cases is easier to find the flux and solve the electric field that find it directly from Coulomb's law. To do it, it is basic to look for an imaginary surface in such way that the electric field was parallel or perpendicular to the surface vector on each point, and also that the magnitude of the electric field was constant on all the points of such imaginary surface. If we do this, computation of flux can be easily done, since

- If $\vec{E}$ and $d\vec{S}$ are <u>perpendicular</u> on all the points of S

$$\Phi = \int_S d\Phi = \int_S \vec{E} \cdot d\vec{S} = 0$$

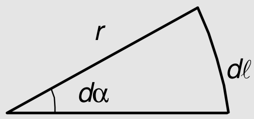
- If $\vec{E}$ and $d\vec{S}$ are parallel in all the points of S (remember magnitude of E is constant)

$$\Phi = \int_S d\Phi = \int_S \vec{E} \cdot d\vec{S} = \int_S EdS = E\int_S dS = ES$$

Gauss's law. Proof

To proof Gauss's law, a suitable magnitude for measuring angles in three dimensions must be defined; it's the **solid angle**. When we observe the Moon, for example, our sight has to cover a three-dimensional angle; we call this angle solid angle, to distinguish it of the flat angle. In the same way that the flat angle is the rate between the length of an arch and its radius, and it's measured in radians,

$$d\alpha = \frac{d\ell}{r}$$



Solid angle is the rate between a piece of spherical surface and its squared radius,

$$d\Omega = \frac{dS}{r^2}$$

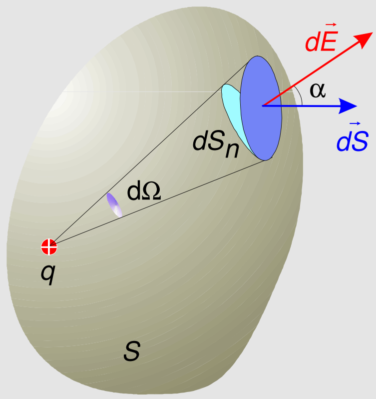
It's measured in estereoradians.

As well as the whole flat angle of a circumference is 2π radians, the whole solid angle of a sphere is 4π estereoradians.

In the figure, there is a positive point charge inside of an enclosed surface, and a elementary surface $d\vec{S}$ that musn't be perpendicular or parallel to the electric field. The electric flux crossing this surface is:

$$d\Phi = \vec{E} \cdot d\vec{S} = EdS\cos\alpha = EdS_n$$

being dS_n the component of $d\vec{S}$ parallel to the electric field.



$$d\Phi = EdS_n = \frac{q}{4\pi\varepsilon_0 r^2} d\Omega r^2 = \frac{q}{4\pi\varepsilon_0} d\Omega$$

The flux crossing a finite surface will be, then:

$$\Phi = \frac{q}{4\pi\varepsilon_0}\int d\Omega = \frac{q}{4\pi\varepsilon_0}\Delta\Omega$$

Observe that term r^2 is cancelled , and so ¡the flux does not depend on the distance!, that is to say, the flux doesn't depend on the size of the surface. This fact is related with the fact that the field is radial and diminish with the square of the distance (called gaussian field)

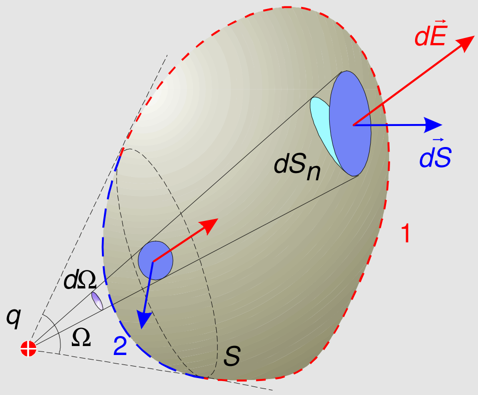
As the charge is inside S surface and enclosed by it, solid angle must be extended to the whole surface, being 4π. Substituting, we obtain:

$$\Phi = \frac{q}{4\pi\varepsilon_0} 4\pi = \frac{q}{\varepsilon_0}$$

That is Guss's law.

If the charge had been outside of the surface, this surface can be split in two areas: area 1 where the flux is positive (going out), and another area 2, where the flux is negative (going in). Net flux will be obtained adding both fluxes:

$$\Phi = \int_1 E dS_n + \int_2 E dS_n = \frac{q}{4\pi\varepsilon_0}\Omega - \frac{q}{4\pi\varepsilon_0}\Omega = 0$$



Net flux is zero because (when charge remains outside), both fluxes have opposite sign and they are cancelled. If the charge had been inside, only there would be going out flux.

If instead of a point charge we had a distributed charge, we would get to the same conclusion, taking in account the particles make up the distribution and applying the principle of superposition.

Applications of Gauss's law

Some cases are going to be considered following that, due to its symmetry, they enable using Gauss's law to compute electric field.

<u>a) Electric field created by a charged infinite plane with surface density of charge σ</u>

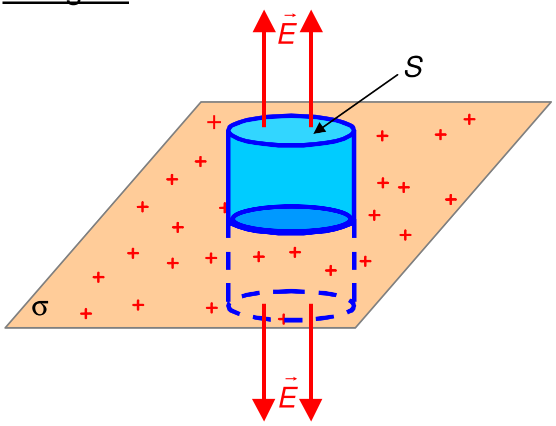


Figure 1-9. Charged infinite plane

Let's have an infinite plane charged with a surface density of charge σ across the plane. Let's take a closed surface (we'll call gaussian surface), and we going to apply Gauss's law to this surface. Due to symmetry of this problem, lines of $\vec{E}$ will be perpendicular to the plane, and being positive charges, outside of the plane. For being an infinite plane, the field lines will be all parallel. These features carry us to choose a cylindrical surface with its parallel basis as are shown in Figure 1-9. In this way, field lines are perpendicular to the basis and tangent to the side surface. Applying Gauss's law to this cylinder:

$$\Phi_{net} = \int_{S_{closed}} \vec{E}\cdot d\vec{S} = \frac{Q_{int}}{\varepsilon_0} \qquad ES + ES = \frac{\sigma S}{\varepsilon_0} \qquad E = \frac{\sigma}{2\varepsilon_0}$$

being S the area of the base of the cylinder. Note that electric field is not depending on the distance to the plane. It only depends on σ.

b) Electric field created by a charged infinite line with linear density of charge λ

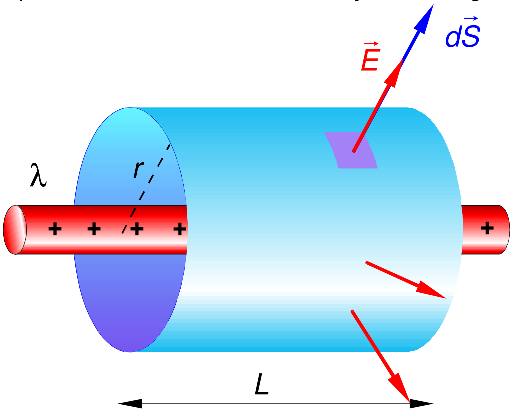


Figure 1-10. Gaussian cylindrical surface of radius *r* coaxial with a linear distribution of charge

Let's have a charged infinite line with linear density of charge λ. Due to the symmetry of problem, electric field will have a radial direction around the charged line. Therefore, let's take as gaussian surface a cylinder of radius *r* coaxial with the linear distribution of charge. In this way, field lines will cross perpendicularly the side surface of the cylinder, and will be tangent to the two basis of the cylinder. So:

$$\Phi = \int_S \vec{E} \cdot d\vec{S} = \int_{Side} EdS = ES_{Side}$$

being magnitude of electric field constant along the side surface of the cylinder. In this way, the flux is equal to the product of the magnitude of electric field multiplied by the side surface of a length L of the cylinder: $ES_{Side} = E \cdot 2\pi rL$.

This is only true if the linear distribution is supposed to be infinite; on the other way it would be necessary to consider the edge effect, being more difficult the computation of electric flux.

Applying Gauss's law:

$$\Phi = E \cdot 2\pi rL = \frac{Q_{enclosed}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}$$

Solving for E:

$$E = \frac{\lambda}{2\pi\varepsilon_0 r}$$

c) Electric field created by a spherical skin with charge density σ

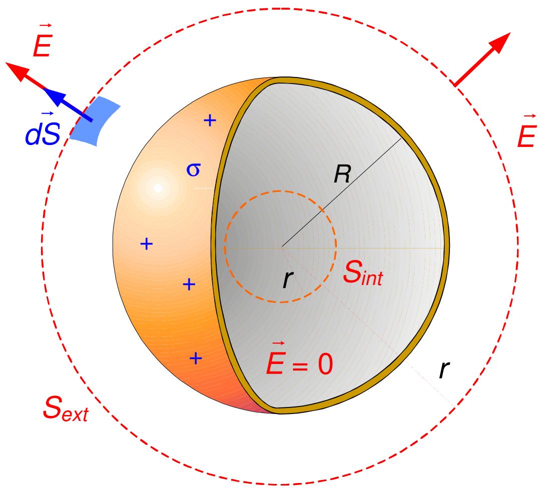


Figure 1-11. Charged spherical skin

Let's suppose a surface charge density σ on the surface of a sphere of radius R. We are going to compute the electric field in two different areas: inside and outside of the skin:

a) Inside. We consider a spherical surface, S_{int} with radius $r<R$. By Gauss's law, the flux that crosses this surface is zero (there isn't charge inside this sphere). So, as the area of the surface is not zero, it will be zero the electric field on any point belonging to the surface.

b) <u>Outside</u>. Due to symmetry of problem, the electric field will have a radial direction around the skin. If we take a gaussian spherical surface S_{ext} of radius $r>R$, the flux through this surface is:

$$\Phi = \int_{S_{ext}} \vec{E} \cdot d\vec{S} = \int_{S_{ext}} EdS = ES_{ext}$$

The simple resolution of this integral is possible because electric field is always parallel to the surface on any point of the spherical surface ($\vec{E} \cdot d\vec{S} = EdS$), and also to be constant the magnitude of E in all the points of the surface. In this way, the flux is also very simple to compute, being $ES_{ext}=E \cdot 4\pi r^2$.

On the other hand, applying Gauss's law:

$$\Phi = \frac{Q_{encerrada}}{\varepsilon_0} = \frac{\sigma \cdot 4\pi R^2}{\varepsilon_0}$$

Solving E from these equations:

$$E = \frac{\sigma R^2}{\varepsilon_0 r^2}$$

As the net charge on the surface is $Q = \sigma 4\pi R^2$, the electric field outside will be:

$$E = \frac{\frac{Q}{4\pi R^2} R^2}{\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}$$

That it is equivalent to suppose that all the charge of the spherical skin is concentrated in its center.

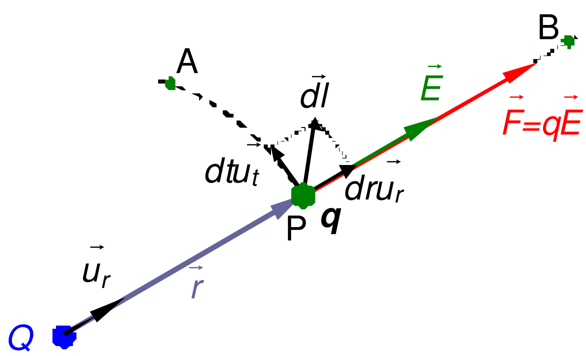
1.6 Work of the forces of the electric field.

We consider the electric field created by a point charge Q. In any point P given its position vector $\vec{r}$, the electric field created by Q will be $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{u}_r$, being $\vec{u}_r$ unit vector of $\vec{r}$. Now, we place one second point charge q in P, and we apply a small trip $d\vec{l}$ in any direction from the q charge.

The work done by the force of the electric field to move the charge $d\vec{l}$ will come from $dW = \vec{F} \cdot d\vec{l}$

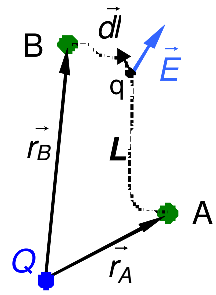
This trip $d\vec{l}$ can be split in a parallel component to the electric field and another tangent component to an arch of circumference centered in Q and passing through P, in such way that $d\vec{l} = dr\vec{u}_r + dt\vec{u}_t$

As $\vec{E}$ is always perpendicular to this arch of circumference, $\vec{u}_r \cdot \vec{u}_t = 0$, and:



$$dW = \vec{F} \cdot d\vec{l} = q\frac{Q}{4\pi\varepsilon_0 r^2}\vec{u}_r(dr\vec{u}_r + dt\vec{u}_t) = \frac{qQdr}{4\pi\varepsilon_0 r^2}$$

To compute the work done by electric force acting on q to carry it from point A to point B along any L line, as is shown in the picture, we will have to add all the works done by the force in small trips along L from A to B. In this way, the net work will be the addition of infinite terms, turning this sum into an integral, called integral of line:



$$W_{AB}^{L} = \int_{A}^{B} \vec{F} d\vec{l} = \int_{r_A}^{r_B} \frac{qQ}{4\pi\varepsilon_0 r^2} dr = - \frac{qQ}{4\pi\varepsilon_0 r}\Bigg|_{r_A}^{r_B} = \frac{qQ}{4\pi\varepsilon_0 r_A} - \frac{qQ}{4\pi\varepsilon_0 r_B}$$

$$W_{AB}^{L} = \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) \qquad \text{Equation 1-5}$$

If both charges (q and Q) are of the same sign, the force between them is repulsive; if, besides, point B is furthest than A from Q ($r_B > r_A$), then the computed work W_{AB}^{L} is positive; a positive work says that the work is done by the forces of the electric field in a spontaneous way. But if the charges had opposite sign, or the final point B was closer to Q than A ($r_B < r_A$), then W_{AB}^{L} would be negative, being necessary an external force that, winning the force of the electric field, does such work. A negative work says that the work is done against the forces of the electric field.

Obviously, the work done by the field between two points placed to the same distance of Q is zero.

1.7 Electrostatic potential energy. Electric potential. Equipotential surfaces.

It results very interesting to check that the work computed only depends on the charges q and Q and on the distances of the initial and final points (r_A and r_B) to the charge creating the electric field. This means that, although we had chosen another different way L' to go from A to B, the work done by the forces of electric field had been the same; or that if we had moved the charge q in opposite sense, from B to A, the work would have been the same but with opposite sign; or that if we had moved the charge q along a closed line, starting on a point and finishing in the same point, the net work would have been zero.

$$W_{AB}^{L} = W_{AB}^{L'} = W_{AB} \qquad W_{AB} = -W_{BA} \qquad W_{AA} = 0$$

Fields having this feature (for example, gravitational field), are called conservative fields or fields deriving from potential; the name of conservative field is related to the ability of field to “conserve” (keep) the work, to “give back it” when we move us in opposite sense; they do it, for example, gravitational field or a spring. Friction force between two surfaces, instead, is not conservative, since the work always must be done by someone external to the field. And the name of fields deriving from potential comes from the consideration of a function, called potential energy, indicating us the ability to do some work. In this type of fields, as it is known from previous courses, the work done by the field to move a particle between two points can be expressed as the difference of potential energies between both points.

In the case of the electric field, the work to carry a charge from A until B can be written as the difference of potential energy of the q charge between points A and B: $W_{AB} = U_A - U_B$. Function U is called electrostatic potential energy of a charge q in a point of the field created by the charge Q.

But previously we have seen that the work done by an electric field was

$$W_{AB} = \frac{qQ}{4\pi\varepsilon_0 r_A} - \frac{qQ}{4\pi\varepsilon_0 r_B}$$

Ans so, function

$$U = \frac{qQ}{4\pi\varepsilon_0 r} + C$$

give us the electrostatic potential energy on a point placed at a distance r from the charge creating the electric field. The constant C indicates us that infinite functions differing in a constant can be taken as potential energy on a point. This fact allows us to take the origin of energies in an arbitrary way. Usually, a charge placed very far from the creating electric field charge, is taken as having electrostatic potential energy zero (if r=∞, then U=0), and so C=0, resulting

$$U = \frac{qQ}{4\pi\varepsilon_0 r} \qquad \text{Equation 1-6}$$

It is convenient to notice that the electrostatic potential energy of a charge on a point give us the work done by the forces of the field to carry this charge from this point until the infinite, since:

$$W = \int_P^\infty \vec{F}d\vec{l} = \int_P^\infty q\vec{E}d\vec{l} = \int_P^\infty qEdr = \int_P^\infty q\frac{Q}{4\pi\varepsilon_0 r^2}dr = -\left.\frac{qQ}{4\pi\varepsilon_0 r}\right|_r^\infty = \frac{qQ}{4\pi\varepsilon_0 r}$$

If electric field was produced by a set of point charges, the computation of the work and of the electrostatic potential energy could be done by applying the principle of superposition.

Electric potential

The electrostatic potential energy, as we have seen, depends both on charge or charges creating the field, as on the charge inside the field and its position; it become useful to get a function that only depends on the electric field, but not on the charge placed on it. Therefore, electrostatic potential on a point of an electric field (V) is defined as the electrostatic potential energy that would have a positive charge of 1 C placed in such point; that is to say, it is the electrostatic potential energy by unit of electric charge:

$$V = \frac{U}{q} = \frac{Q}{4\pi\varepsilon_0 r}$$

In this expression we have supposed the origin of potentials in the infinite.

In case that we had several point charges, the potential on any point could be obtained by the addition of the potentials created by each charge separately (principle of superposition), being the potential due to *n* point charges on a point:

$$V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i}{r_i} \qquad \text{Equation 1-7}$$

being r_i the distance from the charge Q_i to the problem point.

On some particular cases is possible that we want to compute electric potential from the electric field on a point P, and not from charges creating the field; for this purpose we can compute the needed work to carry 1 C from point P to infinite along any line:

$$V_P = \int_P^\infty \vec{E}d\vec{l}$$

And the difference of potential (d.d.p.) between two points A and B of the field:

$$\Delta V = V_A - V_B = \int_A^B \vec{E}d\vec{l} \qquad \text{Equation 1-8}$$

In both cases, obviously, since we can integrate along any line, the simplest way to do it will be to choose a line of field, since the inner product turns into product of magnitudes:

$$\vec{E}d\vec{l} = Edl$$

dimensions of the electric potential are:

$$[V] = [E][\ell] = ML^2T^{-3}I^{-1} \qquad \text{V is measured in Volts}$$

From the potential on a point or from the d.d.p. between two points of an electric field, is immediate to compute as the potential energy of a q charge on a point P as the needed work to carry a q charge from a point A to another B:

$$U_P = qV_P \qquad \Delta U = U_A - U_B = q(V_A - V_B)$$

Equipotential surfaces

Electrostatic potential is a scalar quantity featuring the energy level of a point on the space, in the same way that the height of a point features the energy level of this point inside the gravitational field, or the temperature features the heat level. Observe that it’s said potential of a point, not of a charge, because a charge takes energy when it’s placed on a point with some electric potential. In this way, as we say that a kilogram of mass at 8000 m height has more energy than the same mass at 10 m, we’ll say that a charge of 1 C has more energy if it’s placed on a point with higher electrostatic potential that another.

In the same way than in the height (level curves) or the pressure (isobars), level surfaces or equipotential surfaces can be built as the set of points in the space with the same electrostatic potential.

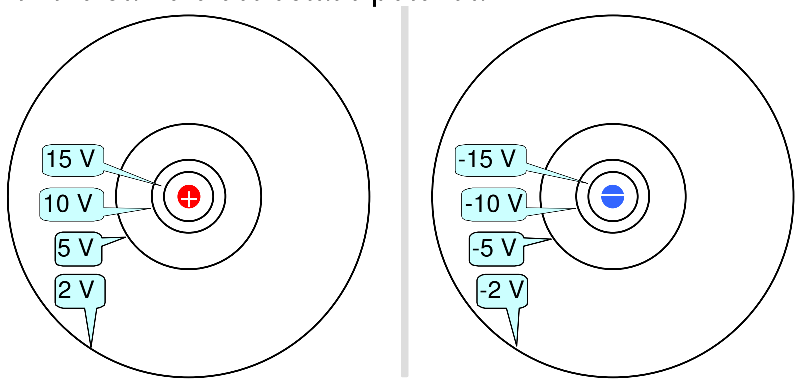


Figure 1-12. Equipotential surfaces (curves in the plane) in the vicinities of distant point charges

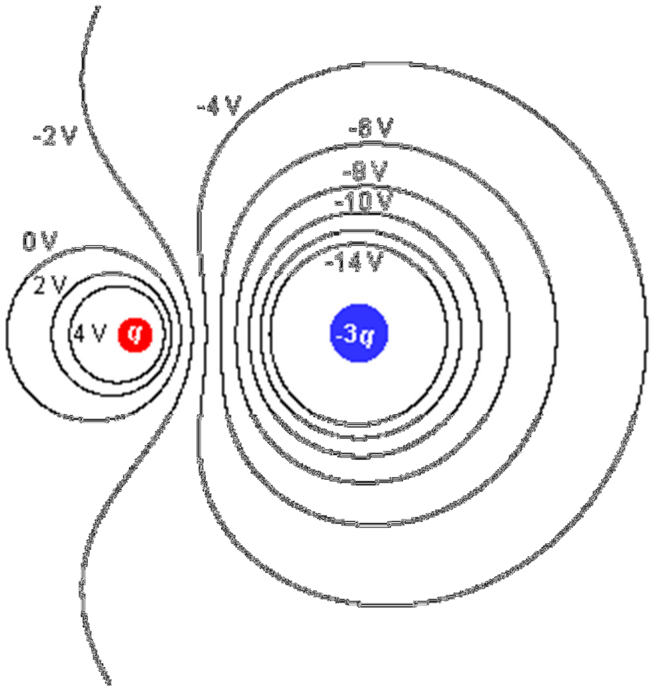


Figure 1-13. Equipotential surfaces (curves in the plane) in the vicinities of a system of charges q and $-3q$

Obviously, the work to move a charge between any two points of a equipotential surface is zero (the potential is the same on all the points of the surface), and so the electric field must be perpendicular to such surface on any point of the surface. If it wasn't, we could always find two points that carrying a charge from one to another, the work done by the field wasn't zero, and so both points wouldn't have the same potential, not belonging to the same equipotential surface. So, on any point of the field, field lines and equipotential surfaces are perpendicular.

Besides, if we let a positive charge in the field, it would move spontaneously in the same sense than the electric field (positive work), decreasing potential energy of the charge. So, the electric field says us the sense in which the electrostatic potential decreases.

The relation between electric field and electric potential can be seen in Figure 1.-14, where equipotential surfaces and electric field lines superimposed are shown for the case of an electric dipole (two equal charges of opposite sign at a distance). Observe that areas where equipotential surfaces are closer, areas where electric field is higher.

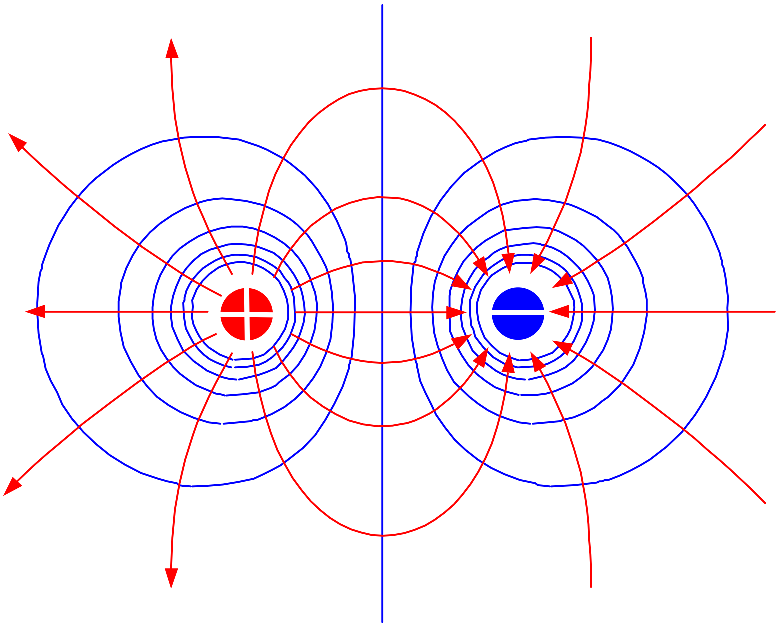


Figure 1-14. Field lines (arrows) and equipotential surfaces (closed lines are their crossing with the plane) in a system of two equal and opposite sign charges. Field lines and equipotential surfaces are perpendicular on each point.

Example 1-2

Given the point charges of **Example 1-1**, compute:

a) Electric potential on point $A(2,0)$ m and on point $B(4,2)$ m

b) The difference of potential between points A and B, $V_A - V_B$.

Solution

a) Electric potential due to a point charge q on a point at a distance r from the charge is $V = k\dfrac{q}{r}$. Therefore, substituting and applying the principle of superposition:

$$V_A = k\frac{q_1}{r_{1A}} + k\frac{q_2}{r_{2A}} = 9000\left(\frac{1}{2} + \frac{-2}{\sqrt{5}}\right) \text{ V} = \text{-3550 V}$$

$$V_B = k\frac{q_1}{r_{1B}} + k\frac{q_2}{r_{2B}} = 9000\left(\frac{1}{\sqrt{20}} + \frac{-2}{\sqrt{17}}\right) \text{ V} = \text{-2353 V}$$

c) The difference of potential between points A and B is:

$$V_A - V_B = \text{-1197 V}$$

Example 1-3

Given the point charges of **Example 1-2**, compute the work must be done by an external force to carry a negative point charge of -3 nC from A to B.

Solution

The work done by the forces of the field is:

$$W_{AB} = -q(V_B - V_A)$$

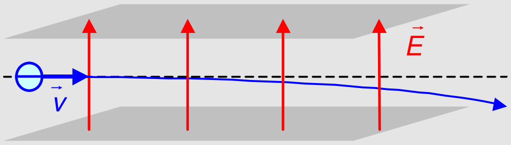
And so, external forces do a work $-W_{AB}$

$$W_{AB}\ \text{Fext} = q(V_B - V_A) = -3\ (1197) = -3{,}59\ \mu\text{J}$$

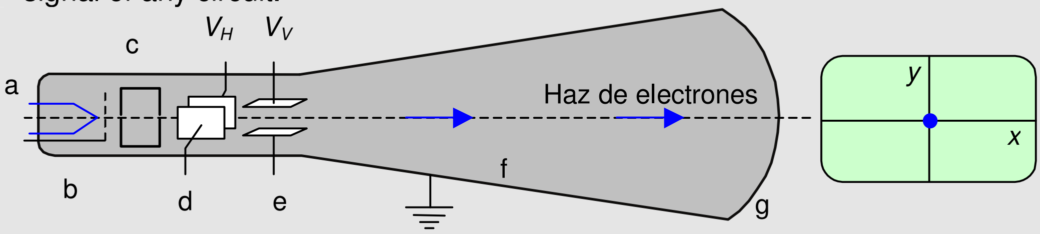
1.8 Applications

The oscilloscope of cathode ray tube

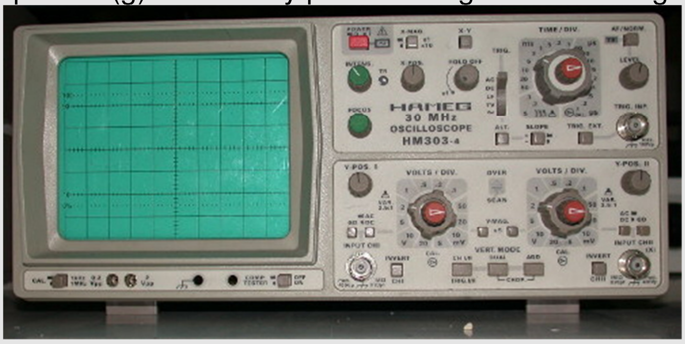
When directing a electron beam between two conductive plates having a difference of potential, and therefore an electric field, between them, the electrons divert depending of its initial speed and the magnitude of the electric field. Based on this fact, it's possible to direct a electron beam in a given direction of the space by means of two set of crossed plates. The first set will control a direction, and the second one, another perpendicular direction. In this procedure is based the television and a measurement device with multiple applications: the oscilloscope of cathodic ray.



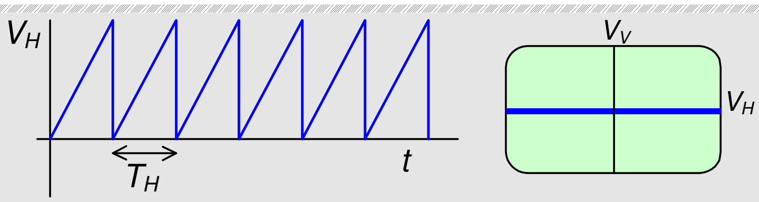
The oscilloscope of cathodic ray allows the visual representation and measurent of electric signals seen as differences of potencial (audio sources, video, medical instruments, data of computer…) and in general, any electric signal of any circuit.



It consists in a vacuum tube (f) where electrons are generated in a cathode (b) heated by a electric resistor (a); electrons are accelerated to the anode (c). Electrons cross two set of plates that produce electric fields crossed (d and e). In this way, electron beam divert in the direction *x* according the electric field produced by the plates (d), and in the direction *y* according the electric field in the plates (e). Finally, they achieve a screen of fluorescent phosphorus (g) where they produce a light of short length.

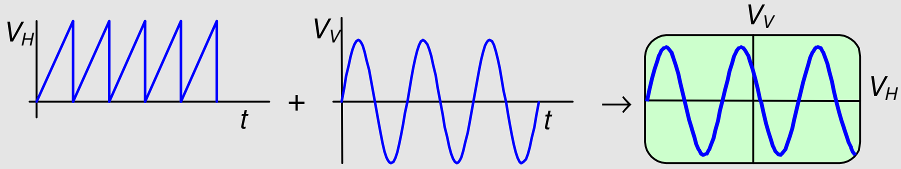


To represent an electric signal variable on time, as for example a sinusoidal signal, it is necessary to enter a triangular signal of horizontal scanning V_H (in the axis *x*, and therefore between the plates d). In this way, on electrons act an increasing electric field, that does they divert to right. If we don't enter any signal in the vertical axis, we will observe a horizontal line that sweeps the screen from left to right.

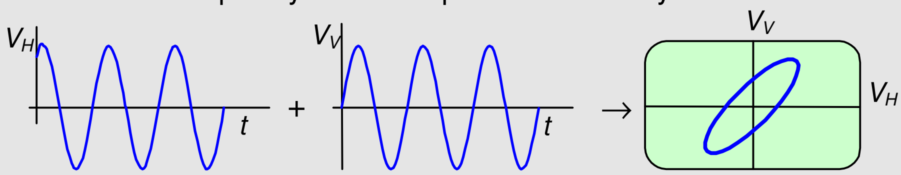


The period of triangular signal T_H, is the time the signal takes going from left to right on the screen.

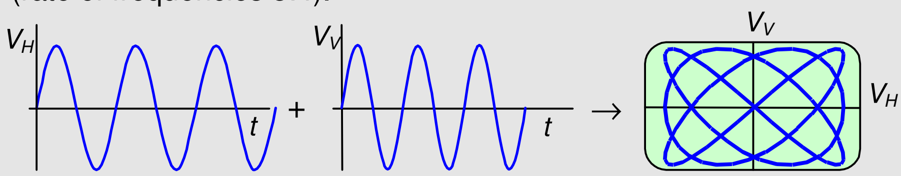
If we enter in the plates (e) a sinusoidal alternating signal at the same time that in (d) a signal of horizontal scanning like the described, the electrons will be diverted rightwards by action of the increasing horizontal field, and upwards or downwards by the action of the field entered in (e). In this way, we will obtain the superposition of both signals. With this technique, we can represent any variable signal on plates (e). To regulate the speed of the horizontal scanning, and therefore the horizontal tension, the period of the triangular signal is comprised between the seconds and the nanoseconds.

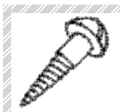


It is also possible to enter two independent signals in d and e, with a representation *XY* of both signals, obtaining the called Lissajous's figures. In the following example appear the Lissajous's figure corresponding to two signals of the same frequency but out of phase in 1/8 of cycle:



In this another example, two sinusoidal signals with different frequencies (rate of frequencies 3:4).





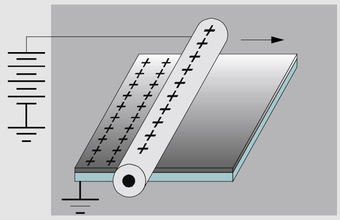
Xerography

The xerography (writing in dry) is a widely used technique, that allows obtain copies in paper from original. This technique was invented in 1938 by Chester Carlson.

The process is based in a property of some substances, called photoconductivity, consistent in the varying of electric conductivity with lighting. Like example of photoconductors materials can quote the oxide of zinc and diverse compound of selenium. In presence of light the substance acts like conductor, whereas it remains like insulator in the darkness.

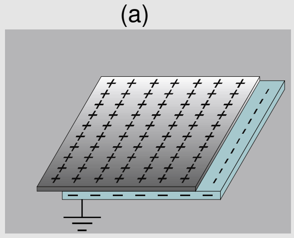
Although the machines of reproduction of xerography are very sophisticated, the process of reproduction of an original can be summarized in 5 stages:

1) In the beginning (a), a sheet covered of photoconductor material, is electrically charged with positive charge in the darkness, by means of a difference of potential of the order of 1000 V between the sheet and earth.



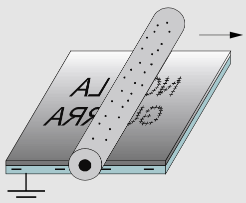
(a)

The sheet is supported on a conductor metallic plate connected to earth, so a negative charge is induced on it. The charges of different sign remain separated because the sheet is insulator on darkness.

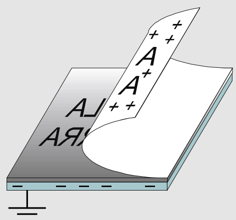


2) Later (b), light is projected over the image that goes to be copied, and by means of an optical system of lens and mirrors (not shown in the figure, this image is projected on the photoconductor surface. In the areas of the surface with intense light, the negative charge of the conductive plate neutralizes the positive charge, and the area is discharged, whereas the areas that have not received light (black zones), remain charged with positive charge. Where arrive tenuous light, the charge reduces slightly. In this way, we obtain a surface distribution of charge that reproduces the image.

(b)



(c)



(d)

3) To convert this “electric image”, not visible, in visible image, a dust (toner) constituted by colored particles negatively charged are sprinkled on the photoconductor plate (c). It will fix the dust on the “virtual electric” image positively charged, going back this “electric image” in visible image.

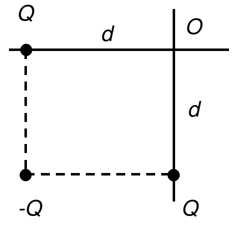
4) Last step is to transfer the toner to the paper (d). The paper has been charged positively, so that it can attract the particles of toner, fixing these in permanent way by means of heat, melting the toner.

5) Finally, the process can be repeated, cleaning the plate of any excess of toner and discharging it of any rest of electric charge with an excess of light.

1.9 Problems

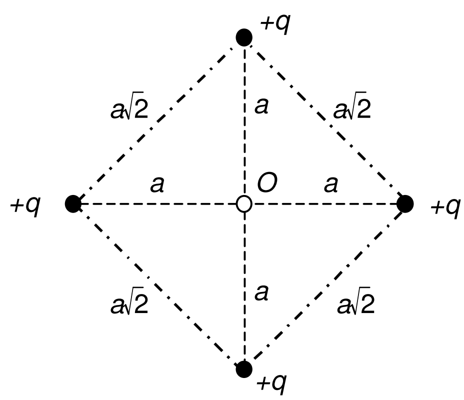
1. Given the three point charges placed as it's shown in the figure, compute the electric force $\vec{F}$ done by these forces on a charge $Q/2$ placed on point O.

Sol: $\vec{F} = \dfrac{KQ^2}{2d^2}\left(1 - \dfrac{\sqrt{2}}{4}\right)(\vec{i} + \vec{j})$



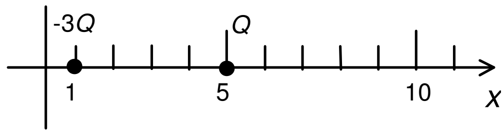
2. Given four equal point charges $+q$, at rest, placed on the vertex of a square with side $a2^{1/2}$, find the net electric force that the four charges would exert on a charge q' placed on O, and the electrostatic potential energy of q' in O.

Sol: $\vec{F} = 0 \quad U = q' \dfrac{q}{\pi \varepsilon_0 a}$

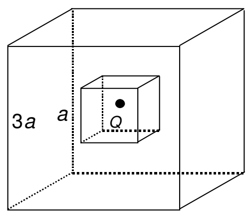


3. In the figure, two point charges, $-3Q$ in $x = 1$ and Q in $x = 5$ are shown.

¿On which points of the x axis
a) it's cancelled the electric potential
b) it's cancelled the electric field?
Sol: a) $x = 4$, $x = 7$
b) $x = 10{,}46$

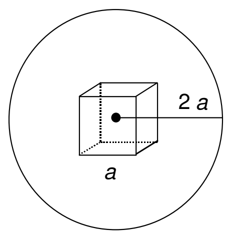


4. Let's take the point charge Q and the two cubic surfaces, parallel, centered in Q, and with sides a and $3a$, shown in the figure. Compute the rate between fluxes of the electric field through both surfaces (Φ_a/Φ_{3a}). Justify the answer.

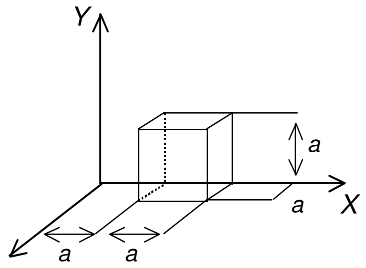


5. A cube of edge a and uniform volumetric density of charge, ρ, is placed on vacuum. It's surrounded by a spherical surface of radius $2a$. Compute the flux of the electric field through the spherical surface.

Sol: $\Phi = \dfrac{\rho a^3}{\varepsilon_0}$



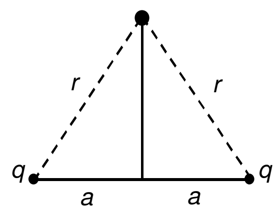
6. Given the electric field defined as $\vec{E} = (ax,0,0)$, compute:
a) Flux through the surface of the cube shown in the figure.
b) Electric charge enclosed in the cube.
Sol: to) $\Phi = a^4$
b) $Q = \varepsilon_0 a^4$



7. Apply Gauss's law to <u>deduce</u> the expression of the electric field created by an infinite plane charged with uniform surface density of charge σ.

8. A positive point charge q_1, is placed in the origin of an orthogonal coordinates system on the plane. Another negative point charge q_2 is placed on the ordinates axis at a distance of 1 m from the origin. Compute:
a) Electric field created by each one of the charges on a point *A* placed on *OX* axis at a distance of 2 m from the origin.
b) Needed work to move a charge *q* from point *A* to point *B*(4,2) m.
Apply it to the case where $q_1 = 10^{-9}$ C, $q_2 = -2 \cdot 10^{-9}$ C, $q = 3$ C.
Sol: a) $\vec{E}_1 = 2{,}25\vec{i}$ N/C $\vec{E}_2 = 1{,}61(-2\vec{i} + \vec{j})$ N/C b) $W_{AB} = 3{,}59$ J

9. Two positives and equal point charges, *q*, are placed at a distance 2*a*. A positive unit charge is placed at the same distance from each charge, *r*, as it's shown in the figure. Which must be the distance *r* for the electric force acting on unit charge gets its higher value?
Sol: $r = a(3/2)^{1/2}$

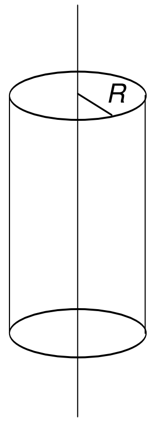


10. A straight line very long (infinite) is charged with linear uniform density of charge λ. Compute the magnitude of electric field created by this line on a point P placed at a distance *y* from the line.
Sol: $E = \lambda/(2\pi\varepsilon_0 y)$

11. Let's have a spherical and homogeneous volume density of charge ρ and radius *R.* Compute the magnitude of electric field and electrostatic potential created by such charge distribution on a point placed at a distance *r* from the centre of the sphere:
a) $r>R$; b) $r = R$; c) $r<R$

Sol: a) $E = (1/3\varepsilon_0)(\rho R^3/r^2)$; $V = (\rho/3\varepsilon_0)R^3/r$
b) $E = (1/3\varepsilon_0)\rho R$; $V = (\rho/3\varepsilon_0)R^2$

c) $E = (1/3\varepsilon_0)\rho r$; $V = (\rho/2\varepsilon_0)(R^2 - r^2/3)$



12. The figure shows a piece of a cylinder of infinite length and radius R, charged with a uniform volumetric density of charge ρ.
Compute:
a) Electric field inside and outside the cylinder.
b) Difference of potential between the axis of the cylinder and its surface.
Sol: to) $E_{in} = \rho r/2\varepsilon_0$, $E_{out} = \rho R^2/2\varepsilon_0 r$
b) $V = \rho R^2/4\varepsilon_0$

GLOSSARY

Elementary charge: Minimum value of electric charge, equivalent to the charge of the electron, $1{,}6\cdot 10^{-19}$ C.

Coulomb's law: Two electric point charges q_1 and q_2, at rest, at a distance r in vacuum, exert a force between them whose magnitude is proportional to the product of the charges and proportional inversely to the square of the distance between them, being attractive if they are of opposite sign and rejecting if they are of the same sign; its direction is the one of the straight line joining them.

$$\vec{F}_{12} = k\frac{q_1 \cdot q_2}{r_{12}^2}\vec{u}_{r12}$$

Electric field on a point of the space is the electric force exerted on a charge q_0 in such point by unit of charge. (q_0 is a small charge of probe)

$$\vec{E} = \frac{\vec{F}}{q_0}$$

Electric field created by a point charge q on a point at a distance r from it

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\vec{u}_r$$

Electric dipole: system of two equal charges and opposite sign distant a low distance.

Electric permittivity of vacuum: universal constant with value $\varepsilon_0 = 8{,}85\cdot 10^{-12}$ $C^2/N\cdot m^2$

Gauss's law. The flux of the electric field through an enclosed surface S is equal to the total charge inside S divided into ε_0

Electrostatic potential of a point of the space is the energy having any charge q_0 placed in this point, divided into such charge.

Electrostatic potential produced by a point charge q on a point at a distance r of it.

Principle of superposition: The total effect (force, field or potential) produced by a set of charges is the addition (of the force vectors, field vector or potential) of the effects produced by each one on independent way.

Equipotential surfaces. Set of points in the space having the same electrostatic potential. A given point can only belong to one equipotential surface.