Session 6: Predicates

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Introduction

Most of the examples of propositions that we have considered thus far have been straightforward in the sense that it has been relatively easy to determine if they are true or false. At worse, there were only a few cases to check in a truth table. Unfortunately, not all propositions are so easy to check. That is because some propositions may involve a large or infinite number of possible cases. For example, consider the following proposition involving prime numbers: "For every natural number n, the value of $n^2 + n + 41$ is prime".

It is not immediately clear whether this proposition is true or false. In such circumstances, it is tempting to try to determine its veracity by computing the value of the function

$$f(n) := n^2 + n + 41$$

for several values of n and then checking to see if they are prime. If any of the computed values is not prime, then we will know that the proposition is false. If all the computed values are indeed prime, then we might be tempted to conclude that the proposition is true (although we are not completely sure because we would need to check infinitely many values of n!).

We begin the checking by evaluating $f(1) = 43$, which is prime. So is $f(2) = 47$, $f(3) = 53$, ..., and $f(20) = 461$, all of which are prime. Hmmm... It is starting to look like $f(n)$ is a prime for every nonnegative integer n. In fact, continued checking reveals that $f(n)$ is prime for all $n \leq 39$. The proposition certainly does seem to be true. But $f(40) = 40^2 + 40 + 41 = 41^2$,which is not prime. So it's not true that the expression is prime for all natural numbers, and thus the proposition is false!

Although surprising, this example is not as rare as you might suspect. There are many examples of propositions that seem to be true when you check a few cases (or even many), but which turn out to be false. The key to remember is that you can't check a claim about an infinite set by checking a finite set of its elements, no matter how large the finite set.

Propositions that involve all numbers are so common that there is a special notation for them. For example, the proposition of the previous example can also be written as

$$\forall\, n \in \mathbb{N} \quad f(n) \text{ is prime.}$$

Here the symbol "$\forall$" is read"for all". The symbol "$\mathbb{N}$" stands for the set of natural numbers, namely, $1, 2, 3, \ldots$. The symbol "$\in$" is read as "is a member of" or "belongs to". Although when we write mathematics it is common to use "$\in \mathbb{N}$", in this lesson we will omit it and we will write:

$$\forall\, n \quad f(n) \text{ is prime} \tag{1}$$

or

$$\forall\, n \quad (f(n) \text{ is prime}), \tag{2}$$

stating previously that the family of objects we are talking about (that is, the set to which all the considered values of n belong) is the set of natural numbers $\mathbb{N}$. This set will be called the *universe*. Hence, once a universe is stated, when we read the symbol "$\forall\, n$" we understand "for all values of n in the universe".

In this example there are two new aspects:

- The use of the symbol $\forall$ (that is called *universal quantifier*).
- The use of a "variable" n that takes values in the universe (the set of natural numbers $\mathbb{N}$).

Then, the proposition (2) is given by "$\forall\, n$" followed by a *property* that **depends on** n ("$f(n)$ is prime"). This property that depends on the variable will be called *predicate*. We can name this *predicate*; for example using the letter P:

$$P(n) = \text{"}f(n) \text{ is prime"}.$$

Notice that we write n between parentheses (usual notation for functions) to denote that the predicate P depends on n. In fact, P **is a function!** Indeed, if we replace n by any natural number, we get a proposition. For example, replacing n by 2 we get the proposition "$f(2)$ is prime", replacing it by 3 we get "$f(3)$ is prime", etc. Then the predicate P is, in fact, a function that assigns, to any natural number, a proposition.

Let us see another example. Consider the following proposition: "There exists a sheep that has 3 eyes". This proposition can also be formalized using a *universe*, a *predicate* and a *new symbol* that we denote by $\exists$ (read as "there exists") and we call *existential quantifier*: First, we declare the universe to be "the set of all sheeps"; then we write the above proposition as

$$\exists\, x \ \ (x \text{ has 3 eyes}) \tag{3}$$

and we read it as "there exists x in the universe such that x has 3 eyes", or "there exists x in the set of sheeps such that x has 3 eyes", or "there exists a sheep x such that x has 3 eyes". Also we can name the predicate $Q(x) =$"x has 3 eyes" and we can write

$$\exists\, x \ \ Q(x).$$

The variable x represents any element in the universe (that is, any sheep) and the use of the existential quantifier expresses that *some* or *at least one* of the elements in the universe (the sheeps) satisfies the predicate Q (that is, has 3 eyes). The proposition (3) will be true if and only if there is at least one sheep with 3 eyes (may be one, or more than one).

2 Predicates

A *predicate* is an expression that depends on the value of one or more variables and such that, when the variables are replaced by elements of a certain set (called *universe*), the result is a proposition.

Along this course we will use predicates with only one variable. In the previous section we have seen a couple of examples of predicates. Let us see more of them:

Examples:

The following expressions express properties

(a) ... is red.

(b) ... has a long nose.

(c) ... is a multiple of 4.

(d) ... is divisible by 2.

The gaps (...) can be **filled** with names of suitable objects in order to build a proposition. This new proposition can be either true or false in the sense of propositional calculus. For example:

(a) **This car** is red.

(b) **John** has a long nose.

(c) **24** is a multiple of 4.

(d) **64** is divible by 2.

We will usually use capital letters to refer to the properties:

(a) $R(\ldots) =$ "... is red"

(b) $N(\ldots) =$ "... has a long nose".

(c) $M(\ldots) =$ "... is a multiple of 4"

(d) $D(\ldots) =$ "... is divisible by 2"

We will use *variables* (x, y, n, etc.) instead of gaps:

(a) $R(x) =$ "x is red"

(b) $N(x) =$ "x has a long nose".

(c) $M(n) =$ "n is a multiple of 4"

(d) $D(y) =$ "y is divisible by 2"

These expressions $R(x)$, $N(x)$, $M(n)$ and $D(y)$ are *predicates.*

We will use lowercase letters to denote specific objects or individuals. For example, we can declare $\mathbf{c}$ = "This car" and $\mathbf{j}$ = "John". Replacing in R and N the variable by these objects we have the propositions:

(a) $R(\mathbf{c})$="**c** is red"="**This car** is red"

(b) $N(\mathbf{j})$="**j** has a long nose"="**John** has a long nose"

Remark: Notice that a predicate $P(x)$ is not, itself, a proposition because it cannot be declared as true or false. However, this can be done if we replace the variable $\mathbf{x}$ by a specific object.

We call **universe** to the class of objects where the variables of certain/s predicate/s take values.

For example, the universe of the above predicate R may be the set of all cars, the universe of N the set of all persons, and the universe of M and N may be the set of natural numbers.

3 Quantifiers

The symbol $\forall$ is called the **universal quantifier** and $\forall x$ means **"for all** x**"**. For a predicate P, the expression "$\forall x\ P(x)$" (read "for all x, $P(x)$") will be a **proposition** that is true (or has logical value 1) if and only if all the propositions obtained replacing, in P, x by **all** the elements of the considered universe are true.

For example, in the universe of natural numbers, the proposition "$\forall n\quad (6n$ is a multiple of 3)" is true. However, in the same universe, the proposition "$\forall n\quad (2^n - 1$ is a prime number)" is false because, for example, $2^{11} - 1 = 2047 = 23 \cdot 89$ and, therefore, the proposition obtained replacing (in the predicate) n by 11 is false.

The symbol $\exists$ is called the **existential quantifier** and $\exists x$ means **"there is some** x**"** or **"there exists some** x**"** or **"there exists** x**"**. For a predicate P, the expression "$\exists x\ P(x)$" (read "there exists x such that $P(x)$") is a **proposition** that is true (or has logical value 1) if and only if the proposition obtained replacing, in P, x by **at least one** element of the considered universe is true.

For example, in the universe of the set of natural numbers, the proposition "$\exists n\quad (2^n - 1$ is a prime number)" is true because, for example, $2^2 - 1 = 3$ is prime and, therefore, the predicate becomes true when we replace the variable n by 2.

4 Two-place predicates

Consider the expression: "... is heavier than ...". In order to convert this expression into a proposition, the names of two objects of individuals are necessary. For instance, using this expression, we may form the proposition "**A brick** is heavier than **a hamster**". The expression "... is heavier than ..." is an example of two-place predicate and we represent it using two variables (x and y) instead of one:

$$H(x, y) = \text{“}x \text{ is heavier than } y''.$$

Two-place predicates can be quantified using the universal and exitential quantifier. However, two quantifiers are necessary to produce a proposition. The quantified expressions $\forall x\ H(x, y)$ and $\exists x\ H(x, y)$ **are not propositions** (but propositional forms depending on a single variable).

Suppose we have the two-place predicate $P(x, y) =$"$x + y = 7$" where the universe of each variable is the set of real numbers. Then, using quantifiers, the following propositions are possible:

1. $\forall x\ \exists y\ P(x, y)$
2. $\exists y\ \forall x\ P(x, y)$
3. $\forall y\ \exists x\ P(x, y)$
4. $\exists x\ \forall y\ P(x, y)$
5. $\forall y\ \forall x\ P(x, y)$
6. $\forall x\ \forall y\ P(x, y)$
7. $\exists y\ \exists x\ P(x, y)$
8. $\exists x\ \exists y\ P(x, y)$

Note that the propositions are read from left to right and the order of quantified variables is important. Consider for instance propositions 1 and 2. The first states that, for every x, there exists at least one y such that $x + y = 7$. This is clearly true. On the other hand, proposition 2 states that there exists at least one y such that, for every x, $x + y = 7$. This is not true since a single y value cannot be found for every x. Each value of x needs a different value of y to balance the equation $x + y = 7$. Therefore propositions 1 and 2 are not the same. For similar reasons, propositions 3 and 4 are also not the same.

However notice that propositions 5 and 6 are the same (they have the same meaning) and false. Similarly 7 and 8 are equal (true) propositions.

In a similar way we can consider three-place predicates, four-place predicates and, in general, n-place predicates for every natural number n.

5 Propositional forms (with predicates)

Since we have generalized the concept of proposition (allowing the use of predicates and quantifiers), we can also generalize the notion of propositional form:

A **propositional form** is an expression involving symbols $P_1, P_2, \ldots$ (representing arbitrary propositions), symbols $Q_1(x_1, \ldots), Q_2(y_1, \ldots), \ldots$ (representing arbitrary predicates), parentheses, connectives and quantifiers such that, replacing $P_1, P_2, \ldots$ by propositions and $Q_1(x_1, x_2, \ldots), Q_2(y_1, \ldots), \ldots$ by predicates (for suitable universes for the variables), the obtained expression is a **proposition**.

For example, the expression $P \wedge \forall (x\ Q(x))$ is a propositional form. We can replace P by the proposition "The president of Spain is Spiderman" and $Q(x)$ by the predicate "x is odd" (considering the set of natural numbers as the universe of x) and we obtain the proposition "The president of Spain is Spiderman and every natural number is odd".

We notice that, along this course, we will use, mainly, propositional forms involving predicates **with a single variable**.