Session 1: Propositions, connectives, truth tables, propositional forms and equivalences

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Propositions and logical values

Definition 1. A **proposition** is a sentence or statement that is either true or false, **exclusively**. The **logical value** of a proposition is 1 if it is true and 0 if it is false.

Remark 1. Sometimes the logical values are written as T and F instead of 1 and 0.

For example:

(1) *"Aristotle was a woman"* is a false proposition (logical value equal to 0).

(2) *"Where are you tonight?"* is not a proposition (it is a question).

(3) *"If a number is a power of 10, then it is even"* is a true proposition (logical value equal to 1).

(4) "$x + 4 = 0$" is not a proposition (it depends on which is the value of "x").

Unfortunately, it is not always easy to decide if a proposition is true or false, or even what the proposition means. In part, this is because the English language is riddled with ambiguities. For example, consider the following statements:

- "You may have cake, or you may have ice cream".
- "If pigs can fly, then you can understand the Chebyshev bound".
- "Every American has a dream".

What precisely do these sentences mean? Can you have both cake and ice cream or must you choose just one dessert? If the second sentence is true, then is the Chebyshev bound incomprehensible? Does the last sentence imply that all Americans have the same dream or might some of them have different dreams?

Some uncertainty is tolerable in normal conversation. But when we need to formulate ideas precisely (as in mathematics and **programming**) the ambiguities inherent in everyday language can be a real problem. We cannot hope to make an exact argument if we are not sure exactly what the statements mean. So before we start into mathematics, we need to investigate the problem of how to talk about mathematics. To get around the ambiguity of English, mathematicians have devised a special mini-language for talking about logical relationships. This language mostly uses ordinary English words and phrases such as “or”, “implies”, and “for all”. But mathematicians endow these words with definitions which are **more precise** than those found in an ordinary dictionary. Throughout this lesson we are going to deal with this precise meaning.

2 Compound propositions: connectives

In English, we can modify and combine propositions with words such as “not”, “and”, “or” and “if-then”. For example, we can combine three propositions into one like this: “If all humans are mortal and all Greeks are human, then all Greeks are mortal”.

Usually we will name propositions using capital or small letters. For example, we can name P to the proposition “all humans are mortal”, Q to the proposition “all Greeks are human” and R to “all Greeks are mortal”. With these “names”, the above compound proposition can be written as:

$$\text{If } P \text{ and } Q \text{ then } R.$$

These words (“not”, “and”, “or” and “if-then”) will be called *connectives* and, in the rest of these notes, we are going to define **precisely** what they mean in mathematical logic.

We are not going to use the words “not”, “and”, “or” and “if-then” to represent these connectives. Instead, we are going to use the symbols $\neg$, $\wedge$, $\vee$ and $\rightarrow$. Using them, the above proposition can be written more formally as:

$$P \wedge Q \rightarrow R,$$

(although we read it as “If P and Q then R”).

2.1 Connective “not” (negation)

We can precisely define these special words using truth tables. For example, if P denotes an arbitrary proposition, then the logical value of the proposition $\neg P$ is defined by the following *truth table*:

P	$\neg P$
1	0
0	1

(You can write T instead of 1 and F instead of 0 if you prefer it).

The first row of the table indicates that when proposition P is true, the proposition $\neg P$ (read “not P”) is false. The second line indicates that when P is false, $\neg P$ is true. This is probably what you would expect.

For example, if P is the proposition "The president of Spain is Spiderman", then $\neg P$ is the proposition "The president of Spain is not Spiderman".

2.2 Connective "and" (conjunction)

In general, a truth table indicates the logical value of a proposition for each possible setting of the variables. If P and Q denote arbitrary propositions, then the logical value of the proposition $P \wedge Q$ (read as "P and Q") is defined by the following truth table:

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

That is, $P \wedge Q$ is true when P and Q are simultaneously true, and it is false in the other cases.

For example, if P is as above and Q is the proposition "Gauss was a mathematician" then $P \wedge Q$ is the proposition "The president of Spain is Spiderman and Gauss was a mathematician". Its logical value is 0 because P is false and Q is true (see the third row).

2.3 Connective "or" (disjunction)

This connective, read as "or", is defined by the truth table

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

That is, $P \vee Q$ is true when one at least one of the propositions P or Q is true, and false if both P and Q are false.

For example, $P \vee Q$ is the proposition "The president of Spain is Spiderman or Gauss was a mathematician". Its logical value is 1 because P is false and Q is true (third row).

2.4 Connective "if ... then" (conditional)

The least intuitive connective is the conditional (denoted by $\to$). Here is its truth table:

P	Q	$P \to Q$
1	1	1
1	0	0
0	1	1
0	0	1

That is, $P \to Q$ is **only false** when P is true and Q is false, and it is true in the rest of the cases.

For example, $P \to Q$ means "If the president of Spain is Spiderman then Gauss was a mathematician". Observe that... it is true! (look at the third row).

Usually, given a conditional $A \to B$, the proposition A is called **antecedent** and B is called **consequent**.

We insist:

A conditional **is false exactly when** the antecedent is true and the consequent is false.

This sentence is worth remembering; a large fraction of all mathematical statements are of the if-then form!

2.5 Connective "if and only if" (biconditional)

We are going to consider a new connective, called *biconditional* and denoted by $\leftrightarrow$, that is, actually, the combination of two conditionals (we will see it later). Its truth table is

P	Q	$P \leftrightarrow Q$
1	1	1
1	0	0
0	1	0
0	0	1

That is, $P \leftrightarrow Q$ is true if both, P and Q, have the same logical value; it is false in the other cases.

For example, $P \leftrightarrow Q$ (read "P if and only if Q") means "The president of Spain is Spiderman if and only if Gauss was a mathematician". This proposition is false (see the third row of the truth table).

3 Hierarchy in connectives

The ambiguities are avoid if we establish hierarchy by using connectives. The following table divides the connectives into levels:

Hierarchy of connectives:

Level	Connectives
1	$\leftrightarrow$
2	$\rightarrow$
3	$\wedge$, $\vee$
4	$\neg$

The priority of a connective is higher than the other means that, in the absence of parentheses, the connective with **higher** level must be considered first.

For example:

- The proposition $P \wedge Q \rightarrow R$ is right since it has no misunderstanding. According to the connectives hierarchy, it is equivalent to $(P \wedge Q) \rightarrow R$.
- $P \wedge Q \vee R$ is not properly written, since the connectives $\wedge$ and $\vee$ have the same hierarchy category. Parentheses are needed to avoid vagueness: either $(P \wedge Q) \vee R$ or $P \wedge (Q \vee R)$ are valid, depending on the meaning.
- $(P \rightarrow Q) \leftrightarrow (\neg P)$ is correct, but we can erase some parentheses if we consider the hierarchy: $P \rightarrow Q \leftrightarrow \neg P$ is equivalent to it since the biconditional is the last considered connective (it has level 1).

4 Propositional forms

Definition 2. A *propositional form* is an expression involving *variables* and connectives such that, if all the variables are replaced by propositions, then the form becomes a proposition.

For example, if we consider p, q and r as variables, an example of propositional form is

$$p \wedge q \rightarrow \neg r.$$

Notice that, if we replace the variables by specific propositions, we obtain a proposition. For instance, if A is the proposition "Every multiple of 10 is even", B is the proposition "Spiderman is a woman" and C is "The number 4 is even", replacing p by A, q by B and r by C we obtain the **proposition**

$$A \wedge B \rightarrow \neg C,$$

that is, "If every multiple of 10 is even and Spiderman is a woman then the number 4 is not even".

Then, in some sense, a propositional form "represents" infinitely many propositions: all those obtained replacing the variables by specific propositions.

5 Truth tables of propositional forms

Given any propositional form, we can construct a table showing the logical value of the propositional form according with the logical values of the variables. It is the *truth table* of the propositional form.

For example, if P and Q are considered as variables, in the previous subsections we have written the truth tables of the propositional forms $\neg P$, $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$ and $P \leftrightarrow Q$.

In the following link it is explained how to compute the truth tables of more complicated propositional forms (T and F are used instead of 1 and 0):

`https://www.youtube.com/watch?v=wRMC-ttjhwM`

6 Tautologies, contradictions and contingencies

Definition 3. A *tautology* is a propositional form whose logical value is always 1, independently of the logical values of the variables. (That is, the final column of its truth table has only ones).

For example, the propositional form $P \vee \neg P$ is a tautology because it is always true, independently of the proposition that we write instead of P. Let us see it looking at the truth table:

P	$\neg P$	$P \vee \neg P$
1	0	1
0	1	1

Definition 4. A *contradiction* is a propositional form whose logical value is always 0, independently of the logical values of the variables. (That is, the final column of its truth table has only zeros).

For example, the propositional form $P \wedge \neg P$ is a contradiction: it is always false, independently of the proposition that we write instead of P. Let us see it using the truth table:

P	$\neg P$	$P \wedge \neg P$
1	0	0
0	1	0

Definition 5. A *contingency* is a propositional form that is neither a tautology nor a contradiction. (That is, the final column of its truth table has zeroes and ones).

You will find more examples of tautologies, contradictions and contingencies in the exercises.

7 Equivalence of propositional forms

Definition 6. Two propositional forms (with the same variables) are *equivalent* if they have the same truth table.

Notation: If two propositional forms A and B are *equivalent*, we will denote it by $A \equiv B$ or $A \Leftrightarrow B$.

Notice that, since all tautologies have the same truth table, all of them are equivalent. The same happens with the contradictions.

If a propositional form A is a tautology, we will write $A \equiv \tau$ or $A \Leftrightarrow \tau$.

For example, $P \vee \neg P \equiv \tau$.

If a propositional form A is a contradiction, we will write $A \equiv \phi$ or $A \Leftrightarrow \phi$.

For example, $P \wedge \neg P \equiv \phi$.

More examples of equivalent propositional forms will be shown in the classroom exercises.

8 Exercises

Solve the following exercises **before** the next class and check your solutions.

1. Represent with logical symbols the following propositions[1]:

   (a) Yesterday we went to the cinema and we didn't study.

   (b) If we are not in a hurry, then we will be saved.

   (c) I won't buy it, if you don't like it.

   (d) I phoned you on Tuesday, but you weren't at home.

   (e) Either you print the work this morning, or we won't deliver it on time and we will fail.

   (f) Neither Juan nor Pepe will come tonight.

   (g) If the road has building work, then we will take a secondary road and we will not take the motorway.

   (h) Animals, like plants, are living beings.

   (i) Propositions P and Q are equivalent if and only if the expression $P \leftrightarrow Q$ is a tautology.

2. Make the truth tables of the following propositional forms, showing if they are tautologies, contradictions or contingencies. The symbol τ represents a tautology (that is, a propositional form that is always true) and ϕ a contradiction (that is, a propositional form that is always false).

   (a) $\neg P \wedge \neg Q$

   (b) $P \wedge Q \to R$

   (c) $\neg(P \to Q) \wedge \neg(Q \to P)$

   (d) $P \to (Q \to R)$

   (e) $\neg P \wedge Q \to \neg Q \wedge R$

   (f) $(P \to Q) \wedge (R \to Q) \leftrightarrow (P \vee R) \to Q$

   (g) $\tau \to P$

   (h) $\tau \to \phi$

   (i) $\phi \to \tau$

   (j) $P \wedge Q \to \phi \vee \tau$

   Hint 1: τ represents a tautology. Since its logic value is always 1, it is not a variable. The same happens with ϕ, that represents a contradiction.

   Hint 2: *How can you check your solution?* Use the truth table generator in

   `https://web.stanford.edu/class/cs103/tools/truth-table-tool/`

   The syntax is the following: $\neg p$ is written as $!p$, $p \wedge q$ as p && q, $p \vee q$ as $p \mid\mid q$, $p \to q$ as $p - > q$ and $p \leftrightarrow q$ as $p < - > q$. Moreover, a tautology (τ) is represented by T and a contradiction (ϕ) by F.

3. Show which ones of the following expressions have logical sense and which ones not. Try to represent the correct ones using as less number of parentheses as you could.

[1] It means that you must define propositions (as "small as you can") and combine them using connectives to obtain the given one. For example, we can formalize "If John has not a car then Petra will buy one for him" by $\neg P \to Q$, where P is the proposition "John has a car" and Q is "Petra will buy a car for John".

(a) $P \vee (P \wedge Q) \wedge S$.

(b) $(P \vee \neg Q) \rightarrow R \vee P$

(c) $P \rightarrow \neg R \rightarrow S \wedge P$

(d) $((\neg Q \wedge P) \rightarrow (S \vee P)) \rightarrow Q$

(e) $[(P \rightarrow Q) \wedge (R \rightarrow Q)] \leftrightarrow (P \vee R) \rightarrow Q$

9 Solutions

(1) (a) "Yesterday we went to the cinema and we didn't study"$= C \wedge \neg S$, where
$C =$"Yesterday we went to the cinema"
$S =$"we studied yesterday"

(b) "If we are not in a hurry, then we will be saved"$= \neg H \to S$, where
$H =$"We are not in a hurry"
$S =$"we will be saved"

(c) "I won't buy it, if you don't like it"$= \neg L \to \neg B$, where
$L =$"You like it"
$B =$"I'll buy it"

(d) "I phoned you on Tuesday, but you weren't at home"$= P \wedge \neg H$, where
$P =$"I phoned you on Tuesday"
$B =$"You were at home"

(e) "Either you print the work this morning, or we won't deliver it on time and we will fail"$= P \vee (\neg D \wedge F)$, where
$P =$"You print the work this morning"
$D =$"We will deliver the work on time"
$D =$"We will fail"

(f) "Neither Juan nor Pepe will come tonight"$= \neg J \wedge \neg P$, where
$J =$"Juan will come tonight"
$P =$"Pepe will come tonight"

(g) "If the road has building work, then we will take a secondary road and we will not take the motorway"$= W \to S \wedge \neg M$, where
$W =$"The road has building work"
$S =$"We will take a secondary road"
$M =$"We will take the motorway"

(h) "Animals, like plants, are living beings"$= A \wedge P$, where
$A =$"Animals are living beings"
$P =$"Plants are living beings"

(i) "Propositions P and Q are equivalent if and only if the expression $P \leftrightarrow Q$ is a tautology"$=$ $E \leftrightarrow R$, where
$E =$"Propositions P and Q are equivalent"
$R =$"the expression $P \leftrightarrow Q$ is a tautology"

(3)

(a) $P \vee (P \wedge Q) \wedge S$. It is incorrect because $\vee$ and $\wedge$ have the same level in the hierarchy.

(b) $(P \vee \neg Q) \to R \vee P$. It is correct, but it is equivalent to $P \vee \neg Q \to R \vee P$.

(c) $P \to \neg R \to S \wedge P$. It is not correct because we must specify which conditional must be applied first.

(d) $((\neg Q \wedge P) \rightarrow (S \vee P)) \rightarrow Q$. It is correct but it is equivalent to $(\neg Q \wedge P \rightarrow S \vee P) \rightarrow Q$.

(e) $[(P \rightarrow Q) \wedge (R \rightarrow Q)] \leftrightarrow (P \vee R) \rightarrow Q$. It is correct, but it is equivalent to $(P \rightarrow Q) \wedge (R \rightarrow Q) \leftrightarrow P \vee R \rightarrow Q$