Session 19: Equivalence relations

Discrete Mathematics

Escuela Técnica Superior de Ingeniería Informática (UPV)

1 Introduction

In this Session we are going to define and study an extremely important type of binary relations: the **equivalence relations**. The purpose of an equivalence relation is to classify objects. We will see that any equivalence relation gives rise to a **partition** of the set, that is, to a **classification** of the elements of the set.

2 Definition and examples

A relation R on a set A is said to be an **equivalence relation** if it is reflexive, symmetric and transitive.

Example 1. (a) The "equality relation", $=$, is an equivalence relation on every set.

(b) Consider the set of books on a certain library and the following relation R on this set: "we say that a book a is related to another book b (that is, aRb) if a and b have the same author. (In order to simplify things, we are going to assume that every book has a unique author).

- R is, obviously, reflexive.
- R is symmetric because a and b have the same authors if and only if b and a have the same author.
- R is transitive because, if aRb (that is, a and b have the same author) and bRc (that is, b and c have the same author) then a and c have the same author, that is, aRc.

Therefore, we conclude that this is an equivalence relation.

(c) Let V the set of vertices of a non-directed graph G and consider de following relation R on V: two vertices v_1 and v_2 in V are related (that is, v_1Rv_2) if there is a path whose initial vertex is v_1 and whose final vertex is v_2.

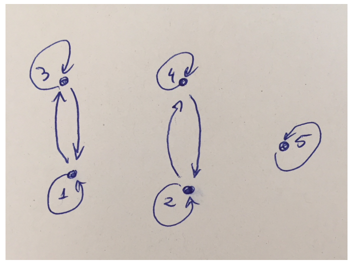
- R is reflexive because, for any vertex v, the path v (only one vertex) is a path from v to v.
- R is symmetric because there exists a path from v_1 to v_2 if and only if there exists a path from v_2 to v_1 (we only need to change the direction of the path).
- R is transitive because if there is a path from v_1 and v_2, and there is a path from v_2 to v_3, then there is a path from v_1 from v_3 (by concatenation of paths).

Therefore R is an equivalence relation.

(d) Let us consider the set $A = \{1, 2, 3, 4, 5\}$ with the following relation:

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (2, 4), (4, 2)\}.$$

It is very easy to prove (using, for example, the characterization involving boolean matrices) that R is an equivalence relation. Its associated directed graph is



3 "Classification into boxes": Equivalence classes and quotient set

Consider the equivalence relation R considered in the above example (b): that defined on the set of books of a certain library. Do, mentally, the following:

- Take a book of the library, call it b_1, and put inside all the books which are related to b_1, that is, those having the same author than b_1. Notice that, in particular, b_1 is in the box because b_1 is related to b_1 (**the relation R is reflexive**).
- When all the books with this author have been put inside the box, take another book b_2 and another box, and do the same.
- Repeat the process until finishing the books.

Finally you will have a number of boxes:

$$[b_1], [b_2], \ldots, [b_m],$$

where $[b_i]$ means "the box of b_i".

Now, let us see some properties of this distribution of books into boxes. Although you will see all the properties as evident, we will see that the essential reason of these properties is not the relation itself, but the fact that it is an equivalence relation.

- All the boxes are non-empty because b_i is in the box of $[b_i]$ for all i. This is because b_i is related with itself (**R is reflexive**).
- It is clear that every book is in one of the boxes. This means that the 'union" of the books in all the boxes is the total set of books in the library.

- Two different boxes have no book in common, that is, every book is **inside a unique** box. Let's prove this fact by contradiction: suppose that there are two boxes, say $[b_i]$ and $[b_j]$, having some book in common (call it x).
  - Since x is in the box $[b_i]$, we have that x is related to b_i, that is, xRb_i. But the relation R is **symmetric** and, therefore, b_iRx.
  - Since x is in the box $[b_j]$, we have that x is related to b_j, that is, xRb_j.
  - So, b_iRx and xRb_j. But R is **transitive** and, therefore, b_iRb_j. This means that b_i is in the box $[b_j]$, which is a contradiction (because b_i and b_j are in different boxes).

The above properties mean that, if we identify the boxes with "sets", the family of sets

$$\{[b_1], [b_2], \dots, [b_m]\}$$

is a **PARTITION** of the set of books of the library: they are non-empty, the union is exactly the set of books, and two different "boxes" are disjoint. Notice that the relevant properties that have allowed us to obtain the partition into boxes are: *reflexive*, *symmetric* and *transitive*. Therefore, this fact is not only valid for this specific equivalence relation. This is a general fact that holds for any equivalence relation. Hence, we can state the following:

Let R be an equivalence relation defined on a set A. For each $a \in A$ consider the set

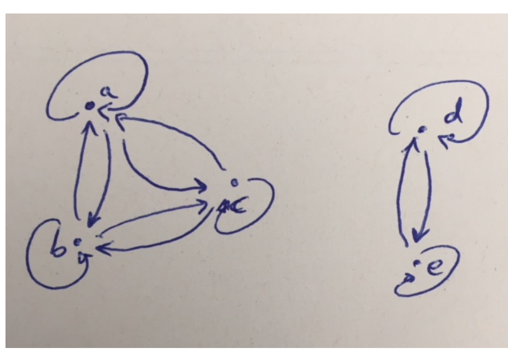
$$[a] = \{x \in A \mid xRa\},$$

that is called **the equivalence class of** a. Then, the family of sets $\{[a] \mid a \in A\}$ is a partition of A that is called **quotient set** and is denoted by A/R.

Example 2. In the set $A = \{a, b, c, d, e\}$, consider the following relation:

$$R = \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (d,e), (e,d)\}.$$

It is easy to check that this relation is reflexive, symmetric and transitive. Therefore, it is an equivalence relation. Its associated graph is the following one:



In this case:

$$[a] = \{x \in A \mid xRa\} = \{a, b, c\}.$$

$$[d] = \{x \in A \mid xRd\} = \{d, e\}.$$

Since a is related to b, one has that $a \in [b]$. But **the equivalence classes form a partition of** A **and, therefore, two different classes are disjoint**. This means that $[b] = [a]$. By the same reason, $[c] = [a]$ and $[e] = [d]$.

The **quotient set of** R is the set of equivalence classes, that is:

$$A/R = \{\{a, b, c\}, \{c, d\}\}.$$