Mathematical Analysis

Numerical Series

Contains

- General concepts
  - Partial sums. Convergence and divergence
  - Harmonic series. Generalization
- Numerical series. Exact sum.
  - Geometric, (infinite y finite sums)
  - Telescoping and reducible to telescoping
- Convergence criteria
  - Remainder test
  - Leibniz criteria for alternating series
- Approximated sums

Introduction

Everyone knows how to add two numbers together…but

How do you add infinitely many numbers?

Introduction

Problem: Given the sequence $\{a_n\}_{n\geq 1}$

$$a_1 + a_2 + a_3 + \cdots + a_{100} + a_{101} + \cdots = ¿?$$ sum of the terms?

Solution isn't evident: s = 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + (-1) +…=?

$$s = (1+(-1)) + (1+(-1)) + (1+(-1)) + \cdots = 0+0+0+\cdots = 0$$

$$s = 1 + ((-1)+1) + ((-1)+1) + ((-1)+1) + \cdots = 1+0+0+\cdots = 1$$

$$s - 1 = (-1)+1+(-1)+1+(-1)+1+\cdots = -s \;\Rightarrow\; s = \frac{1}{2}$$

Associative property, commutative property, etc …aren't (in general) valid

- When is allowed to sum?
- How and when can we sum?

$1+2+3+\cdots = ?$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = ?$

Notation: summation symbol

The index k ends at $k = n$.

The summation symbol (Greek letter sigma) $\displaystyle\sum_{k=1}^{n} a_k$ — a_k is a formula for the kth term.

The index k starts at $k = 1$.

$$\sum_{k=1}^{4} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$$

Partial sums: Convergence and divergence

Given the sequence $\{a_n\}_{n\geq 1}$ we define the partial sum as:

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$s_4 = a_1 + a_2 + a_3 + a_4$

$\vdots$

$s_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n \ , \ \forall n \in \mathbb{N}$

Recurrence formula: $\begin{cases} s_{n+1} = s_n + a_{n+1} \\ s_1 = a_1 \end{cases}$

The numerical series of general term $\{s_n\}$ is defined as:

$$\sum_{n=1}^{+\infty} a_n = \sum_{n\geq 1} a_n = \lim s_n = \sum a_n$$

The series converges when the sequence $\{s_n\}$ is convergent

The sum of the series is $s=\lim s_n$, when this limit exists and is real

An interesting case of divergence $\sum_{n\geq 1} a_n = \pm\infty$, when $s_n \to \pm\infty$

Examples

Example: $\displaystyle\sum_{n=1}^{+\infty} (-1)^{n+1} = 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots$

$\{s_n\} = \{1, 0, 1, 0, 1, 0, 1, 0, \ldots\}$ (divergent)

The series $\displaystyle\sum_{n=1}^{+\infty} (-1)^{n+1}$ is divergent (oscillating, we don't talk about sum)

Example: $\displaystyle\sum_{n=1}^{+\infty} (2n-1) = 1 + 3 + 5 + 7 + 9 + 11 + \cdots$

$\{s_n\} = \{1, 4, 9, 16, 25, 36, \ldots\} = \{n^2\}$

The series $\displaystyle\sum_{n=1}^{+\infty} (2n-1)$ diverges to +◎ (we can say that the sum is + ◎)

Example: $\displaystyle\sum_{n=1}^{+\infty} 0 = 0 + 0 + 0 + 0 + \cdots$

$\{s_n\} = \{0, 0, 0, 0, 0, \ldots\} = \{0\}$

The series $\displaystyle\sum_{n=1}^{+\infty} 0$ converges and sums 0

Examples

Example: $\displaystyle\sum_{n\geq 1} \log\left(\frac{n+1}{n}\right) = \log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \log\left(\frac{5}{4}\right) + \cdots$

$s_1 = \log\left(\frac{2}{1}\right) = \log(2) \ , \ s_2 = \log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) = \log\left(\frac{2}{1}\frac{3}{2}\right) = \log(3) \ , \ \ldots$

$s_n = \log\left(\frac{2}{1}\right) + \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) + \cdots + \log\left(\frac{n+1}{n}\right) = \log\left(\frac{2}{1}\frac{3}{2}\frac{4}{3}\cdots\frac{n+1}{n}\right) = \log(n+1)$

$\displaystyle\sum_{n\geq 1} \log\left(\frac{n+1}{n}\right)$ diverges to + ◎

Example: $\displaystyle\sum_{n\geq 1} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$

$\{s_n\} = \left\{1 - \frac{1}{n+1}\right\} = \left\{\frac{n}{n+1}\right\} \to 1$

The series $\displaystyle\sum_{n\geq 1} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ converges and sums 1

Properties

- $\sum_{n\geq p} a_n$ has the same character than $\sum_{n\geq q} a_n$, $\forall p,q \in \mathbb{N}$
- $\sum (a_n+b_n) = \sum a_n + \sum b_n$; $\sum (\alpha \cdot a_n) = \alpha \cdot \left(\sum a_n\right)$, $\alpha \neq 0$
- We can group terms (no rearrangement) of convergent series
- $\sum |a_n|$ convergent $\Rightarrow \sum a_n$ convergent
- If $0 < a_n \leq b_n,\ \forall n \geq n_0$
  - If $\sum_{n=1}^{\infty} b_n$ is convergent $\rightarrow \sum_{n=1}^{\infty} a_n$ is convergent
  - If $\sum_{n=1}^{\infty} a_n$ is divergent $\rightarrow \sum_{n=1}^{\infty} b_n$ is divergent

Examples

- $\sum_{n\geq 1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ (harmonic series)

$$s_n = a_1 + a_2 + \cdots + a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

$$\left.\begin{array}{l} s_n \text{ is increasing} \left(s_{n+1} - s_n = \dfrac{1}{n+1} > 0\right) \\ s_n \text{ isn't upper bounded} \end{array}\right\} \Rightarrow \{s_n\} \rightarrow +\infty \Rightarrow \sum_{n\geq 1} \frac{1}{n} \text{ diverges to } + ◎$$

- $\sum \frac{1}{n^{\alpha}} \begin{cases} \text{if } \alpha < 1 \text{ diverges} \\ \text{if } \alpha > 1 \text{ converges} \end{cases}$ (generalized harmonic series)

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Numerical series with exact sum

We're going to sum (exactly) these types of series

- Geometric

$$\sum_{n\geq p} r^n = r^p + r^{p+1} + r^{p+2} + \cdots + r^{p+k} + \cdots \quad \text{(r=ratio)}$$

- Telescoping

$$\sum_{n\geq p} a_n \text{ with } a_n = b_{n+1} - b_n$$

$$\text{or} \quad a_n = b_n - b_{n+1}$$

Example

Geometric series $\sum_{n\geq p} r^n = r^p + r^{p+1} + r^{p+2} + \cdots + r^{p+k} + \cdots$ (r=ratio)

$$s_n = r^p + r^{p+1} + r^{p+2} + \cdots + r^{p+n-1}$$

$$r \cdot s_n = r^{p+1} + r^{p+2} + r^{p+3} + \cdots + r^{p+n}$$

subtracting $\left.\begin{array}{l} s_n - r\cdot s_n = r^p - r^{p+n} \\ (1-r)\cdot s_n = r^p - r^{p+n} \end{array}\right\} \Rightarrow s_n = \begin{cases} \dfrac{r^p - r^{p+n}}{1-r}, & \text{if } r \neq 1 \\ n, & \text{si } r = 1 \end{cases}$

Then, $\lim s_n = s = \dfrac{r^p}{1-r}$ if and only if $|r| < 1$

$$\boxed{\text{converges if and only if } |r|<1 \text{ y } s = \frac{r^p}{1-r}}$$

Examples

Example: $\sum_{n\geq 3} \dfrac{6^n}{2\cdot 5^{n+1}} = \dfrac{1}{10}\sum_{n\geq 3}\left(\dfrac{6}{5}\right)^n$, geometric with $r = \dfrac{6}{5} > 1$ (diverges)

Example: $\sum_{n\geq 2} \dfrac{(-2)^{n+1}}{5\cdot 3^{n-1}} = -\dfrac{6}{5}\sum_{n\geq 2}\left(-\dfrac{2}{3}\right)^n$, geometric with $r = -\dfrac{2}{3}$ (converges)

$$s = -\frac{6}{5}\cdot\frac{\left(-\frac{2}{3}\right)^2}{1-\left(-\frac{2}{3}\right)} = -\frac{8}{25}$$

Example: $\sum_{n\geq 3} \dfrac{2^{3n+1}}{5\cdot 3^{2n-1}} = \dfrac{6}{5}\sum_{n\geq 3}\left(\dfrac{2^3}{3^2}\right)^n$, geometric with $r = \dfrac{8}{9}$ (converges)

$$s = \frac{6}{5}\cdot\frac{\left(\frac{8}{9}\right)^3}{1-\frac{8}{9}} = \frac{1024}{135}$$

Exercise

Exercise: Classify and sum if it's possible $\sum_{n\geq 1}\dfrac{(-1)^n\cdot(\alpha+1)^n}{6^{n+1}}$, $(\alpha\in\mathbb{R})$

The series can be written as $\dfrac{1}{6}\sum_{n\geq 1}\left(-\dfrac{(\alpha+1)}{6}\right)^n$

Then, it's a geometric series with common ratio $r = -\dfrac{(\alpha+1)}{6}$

This series convergent if and only if $\dfrac{|\alpha+1|}{6} < 1 \Leftrightarrow \alpha \in]-7,5[$

$$\sum_{n\geq 1}\frac{(-1)^n\cdot(\alpha+1)^n}{6^{n+1}} = \frac{1}{6}\left(\frac{-\frac{(\alpha+1)}{6}}{1+\frac{(\alpha+1)}{6}}\right) = -\frac{(\alpha+1)}{6(\alpha+7)}$$

Telescoping series

$$\sum_{n\geq p} a_n \quad \text{with } a_n = b_{n+1} - b_n$$

$$\text{or} \quad a_n = b_n - b_{n+1}$$

Example: $\sum_{n\geq 1}\left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right)$ convergent to $\dfrac{1}{2}$

$$s_n = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n+2}\right) = \frac{1}{2}-\frac{1}{n+2}$$

$$\{s_n\} = \left\{\frac{1}{2}-\frac{1}{n+2}\right\} = \left\{\frac{n}{2n+4}\right\} \to \frac{1}{2} = s$$

Telescoping series

Example: $\sum_{n\geq 4}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$ convergent to

$$\{s_n\}=\left\{\frac{1}{5}-\frac{1}{n+2}\right\}\to\frac{1}{5}=s$$

Reducible to telescoping series

Series like $\sum_{n\geq p}\frac{P(n)}{Q(n)}$ with $grad(Q(n))\geq grad(P(n))+2$

may be reformatted to telescoping series making simple fraction decomposition

Example: Show that $\sum_{n\geq 1}\left(\frac{4}{4n^2-1}\right)$ converges and sum 2

$$\frac{4}{4n^2-1}=\frac{4}{(2n-1)(2n+1)}=\frac{2}{2n-1}-\frac{2}{2n+1}$$

$$s_n=\left(2-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{2}{5}\right)+\cdots+\left(\frac{2}{2n-1}-\frac{2}{2n+1}\right)=2-\frac{2}{2n+1}$$

$$\{s_n\}=\left\{2-\frac{2}{2n+1}\right\}\to 2=s$$

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Convergence criteria

1) Remainder test: $\sum a_n$ convergent $\Rightarrow$ $\lim a_n=0$

$\lim a_n\neq 0 \Rightarrow \sum a_n$ divergent $\sum\frac{3^n}{2^n+1}$ diverges, because $\lim\frac{3^n}{2^n+1}=+\infty$
$\lim a_n=0 \not\Rightarrow \sum a_n$ convergent $\sum_{n\geq 1}\frac{1}{n}$ diverges to $+\infty$ $\left\{\frac{1}{n}\right\}\to 0$
$\lim a_n=0 \not\Rightarrow \sum a_n$ divergent $\sum_{n\geq 1}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1$ and $\left\{\frac{1}{n}-\frac{1}{n+1}\right\}\to 0$

This is a necessary condition, not a sufficient condition. Thus means that if the condition isn't true, the series doesn't converges.

If the condition is true, the series may be or not be convergent

Convergence criteria

2) Harmonic series:

$$\sum_{n\geq p}\frac{1}{n^{\alpha}}\begin{cases}\text{divergent if } \alpha\leq 1\\ \text{convergent if } \alpha>1\end{cases}$$

3) Geometric:

$$\sum_{n\geq p} r^{n}\begin{cases}\text{convergent if } |r|<1\\ \text{divergent if } |r|\geq 1\end{cases}$$

Leibniz criteria for alternating series

4) Leibniz criteria:

$\{a_n(>0)\}$ decreases and tends to zero $\Rightarrow \sum(-1)^{n+1}\cdot a_n$ converges

Example: $\sum\frac{(-1)^{n+1}}{n}$ converges $\left(\text{alternate}\ ,\ a_n=\frac{1}{n}\right)$

$$\left(0<s_2<s_4<s_6<s_8<\cdots<s<\cdots<s_7<s_5<s_3<s_1\right)$$

Example: $\sum\frac{(-1)^{n}}{n^2}$ converges $\left(\text{alternate}\ ,\ a_n=\frac{1}{n^2}\right)$

Example: $\sum\frac{(-1)^{n}\cdot\sqrt{n}}{2n+5}$ converges $\left(\text{alternate}\ ,\ a_n=\frac{\sqrt{n}}{2n+5}\right)$

How is an alternating series ?

An alternating series is an infinite series of the form:

$$\sum_{n\geq 1}(-1)^{n+1}a_n \quad\text{or}\quad \sum_{n\geq 1}(-1)^{n}a_n;\ a_n>0$$

A series in which the terms are alternately positive and negative is an alternating series.

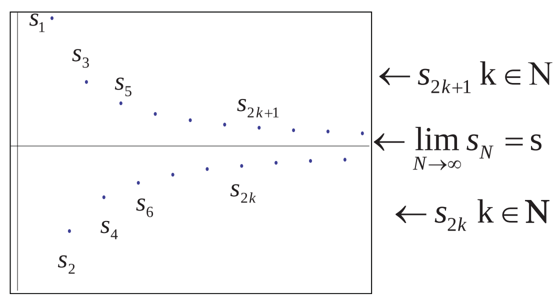
Example:

$$\sum_{n\geq 1}\frac{(-1)^{n+1}}{n}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}+\ldots+\frac{(-1)^{k+1}}{k}+\ldots$$

When can we add an alternating series?

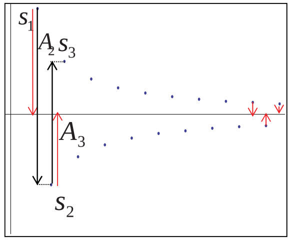
If $\{a_n\}$ is decreasing and tends to zero, then is convergent and tends to $s=\sum_{n\geq 1}\underbrace{(-1)^{n+1}a_n}_{A_n}$

and $0<s_2<s_4<s_6<s_8\ldots<s<s_7<s_5<s_3<s_1$



When can we add an alternating series?

$0<s_2<s_4<s_6<s_8 \ldots <s<s_7<s_5<s_3<s_1$ $\qquad s=\sum_{n\geq 1}(-1)^{n+1}a_n$



$\leftarrow s_{2k+1}\ \mathrm{k}\in\mathbf{N}$

$\leftarrow \lim_{N\to\infty} s_N = \mathrm{s}$

$\leftarrow s_{2k}\ \mathrm{k}\in\mathbf{N}$ $\qquad \lim_{N\to\infty} A_N = 0$

$s_1 = A_1$
$s_2 = s_1 + A_2$
$s_3 = s_2 + A_3$

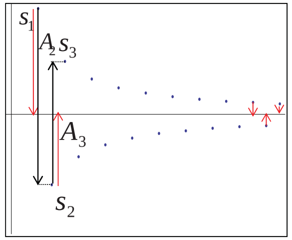
$\vdots$

$s_N = A_1 + A_2 + \ldots A_N = s_{N-1} + A_N$

Each backward or forward step is shorter than the preceding step because $|A_{N+1}| \leq |A_N|$ And since the Nth term approaches zero as N increases the size of the step we take forward or backward gets smaller and smaller.

When can we add an alternating series?

$0<s_2<s_4<s_6<s_8 \ldots <s<s_7<s_5<s_3<s_1$ $\qquad s=\sum_{n\geq 1}(-1)^{n+1}a_n$



$\leftarrow s_{2k+1}\ \mathrm{k}\in\mathbf{N}$

$\leftarrow \lim_{N\to\infty} s_N = \mathrm{s}$ $\qquad \lim_{N\to\infty} A_N = 0$

$\leftarrow s_{2k}\ \mathrm{k}\in\mathbf{N}$

$$s=\underbrace{A_1+A_2+\ldots+A_N}_{s_N}+\underbrace{A_{N+1}+\ldots}_{s-s_N\ \text{(tail)}}$$

Upper bound on the $s \cong s_N$ error

$$E_N = |s-s_N| = \left|\sum_{n=1}^{+\infty} A_n - \sum_{n=1}^{N} A_n\right| = \left|\sum_{n=N+1}^{+\infty} A_n\right| \leq |A_{N+1}|$$

Leibniz's criteria

- Leibniz's conditions :
  1. alternating series
  2. A_N decreasing with $\lim_{N\to\infty} A_N = 0$
- then $s \cong s_N$ and $E_N \leq |A_{N+1}| = a_{N+1}$

$$E_N = |s-s_N| = \left|\sum_{n=1}^{+\infty} A_n - \sum_{n=1}^{N} A_n\right| = \left|\sum_{n=N+1}^{+\infty} A_n\right| \leq |A_{N+1}|$$

Exercise 1:

Approximate $s = \sum_{n\geq 1}\frac{(-1)^{n+1}}{n}$ with at least two exact decimal digits

$A_n = (-1)^{n+1}\frac{1}{n}$ $\qquad \to \pm 0.001$

$$E_N = |s-s_N| \leq a_{N+1} = \frac{1}{N+1} < 10^{-3} \ \to N \geq 1000$$

$$s \cong s_{1000} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{1000} = 0.\underline{69}264743\ldots$$

$s = \log(2) =$
$= 0.69314718\ldots$

Exercise 2:

Approximate $s = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ using the 10th first terms

of the series. Bound the error.

$$A_n = (-1)^{n+1}\frac{1}{n}$$

$$s_{10} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{10} = \frac{1627}{2520} \cong 0.6456$$

$$E_{10} = |s - s_{10}| \leq a_{10+1} = \frac{1}{10+1} = 0.09 \cong 0.1$$

Summarizing: Leibniz's criteria

1. It is very useful for decreasing alternating series.
2. The associated error to the approximation of the sum of an alternating series with $\lim_{N\to\infty} A_N = 0$ by the finite sum of N terms, is always lower than the first term that has not been used for the summation.

$$\boxed{\lim_{N\to\infty} A_N = 0} \rightarrow \boxed{E_N = |s - s_N| \leq |a_{N+1}|}$$

Example

Example: Approximate $s = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n\cdot 2^n}$ with three exact decimal at least

The alternating series, with $A_n = (-1)^{n+1}\frac{1}{n\cdot 2^n}$

$$E_N = |s - s_N| \leq a_{N+1} = \frac{1}{(N+1)2^{N+1}} < 10^{-4} \Rightarrow N \geq 9$$

$$s \cong s_9 = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \cdots + \frac{1}{9\cdot 2^9} = 0.4055323\ldots$$

Example

Example : Approximate $s \overset{?}{=} \sum_{n\geq 1} \frac{n}{(2n+1)5^n}$ using s_4 and six exact decimals

$$|A_n| = \frac{n}{(2n+1)5^n} < \frac{1}{2}\left(\frac{1}{5}\right)^n \Rightarrow \quad E_N = |s - s_N| < \frac{1}{2}\sum_{n=N+1}^{\infty}\left(\frac{1}{5}\right)^n = \frac{1}{2}\frac{\left(\frac{1}{5}\right)^{N+1}}{1-\frac{1}{5}} = \frac{1}{8\cdot 5^N}$$

- $E_4 < \frac{1}{8\cdot 5^4} = 0.0002 \Rightarrow \quad s \cong s_4 = \sum_{n=1}^{4}\frac{n}{(2n+1)5^n} = 0.0868..$ (three exact decimals)
- $E_N < \frac{1}{8\cdot 5^N} < 10^{-7} \Rightarrow n \geq 9$ and $\quad s \cong s_9 = \sum_{n=1}^{9}\frac{n}{(2n+1)5^n} = 0.08698876\ldots$

Example

Example: Calculate $s \stackrel{?}{=} \sum_{n\geq 1} \frac{1}{n!}$ with five exact decimals

$$E_N = \cdots = \frac{1}{(N+1)!}\left(1+\frac{1}{N+2}+\frac{1}{(N+2)(N+3)}+\cdots\right) < \frac{1}{(N+1)!}\left(1+\frac{1}{N+2}+\frac{1}{(N+2)^2}+\cdots\right) < \frac{2}{(N+1)!}$$

$$E_N < \frac{2}{(N+1)!} < 10^{-6} \Rightarrow n \geq 9 \quad \text{y} \quad s \cong s_9 = 1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{9!} = 1.\underline{71828}1525\ldots$$

$$s = e-1 = 1.\underline{718281}828\ldots$$