1. Show that: $0+1+2+\ldots+n = \frac{n(n+1)}{2}$

i) $0+1+2+\ldots+n = \frac{n(n+1)}{2}$ [for n]

ii) $0+1+2+\ldots+(n+1) = \frac{(n+1)(n+1+1)}{2}$ [for (n+1)]

$\Rightarrow 0+1+2+\ldots+n+(n+1) = \frac{(n+1)(n+1+1)}{2}$

$\Rightarrow \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$

$\Rightarrow \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$

$\Rightarrow \frac{(n+2)(n+1)}{2} = \frac{(n+1)(n+2)}{2}$

$\Rightarrow \frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+2)}{2}$

Proven

2. $n \geq 2$ $\qquad$ $n+1 < n^2$

Base case:

let $n = n_0 = 2$

$n_0 \geq 2$ $\qquad$ $n_0 + 1 < n_0^2$

$\Rightarrow 2 \geq 2$ $\qquad$ $\Rightarrow 2+1 < 2^2$

$\Rightarrow 3 < 4$

for $(n+1)$ prove $(n+1)+1 < (n+1)^2$

- rewrite as: $(n+1)^2 > (n+1)+1$

$\Rightarrow (n+1)^2 > n+2$

LHS:

$(n+1)^2$

$= n^2 + 2n + 1$

$= n^2 + 2n + 1$

$= (>n+1) + 2n + 1$

$= n+1 + 2n + 1$

$= 3n + 2$ $\qquad$ $[3n > n]$

$= n+2$

proven

3 $n \geq 4$, $\cancel{n^2 > n}$ $n! > n^2$

$P(n)$: $n! > n^2$

Base case: $n_0 = 4$, $4! = 24 > 16 = 4^2$

$4! > 4^2$ Base proved

Inductive Step:

let $P(n=m)$ is true,

i.e for an $m \in \mathbb{N}$, $m! > m^2$ ——— (i)

Using (i), we have to prove that, $P(n=m+1)$ is also true.

i.e we " " " ", $(m+1)! > (m+1)^2$ ——— (*)

To prove (*),

L.H.S of (*) $= (m+1)!$

$= (m+1)\, m!$

$= m \cdot m! + m!$

$> 24m + m^2$ [from (i), $m^2 < m!$ and from base, $m \geq 4$ $\Rightarrow m! \geq 24$]

$= m^2 + 2m + 22m$

$> m^2 + 2m + 1$ [$\because 22m > 1$]

$= (m+1)^2$

[Showed]

4. $F_1 + F_3 + F_5 + F_7 + \dots + F_{2n-1} = F_{2n} - 1$ for $n \geq 1$

let $n = n_0 = 1$

$F_{2(1)-1} = F_{2(1)} - 1$

$\Rightarrow F_1 = F_2 - 1$

$\Rightarrow 1 = 2 - 1$

$\Rightarrow 1 = 1$ proven

for $(n+1)$

$F_1 + F_3 + F_5 + \dots + F_{2n-1} + F_{2(n+1)-1} = F_{2(n+1)} - 1$

$\Rightarrow (F_1 + F_3 + \dots + F_{2n-1}) + F_{2n+2-1} = F_{2n+2} - 1$

$\Rightarrow F_{2n} - 1 + F_{2n+1} = F_{2n+2} - 1$

$\Rightarrow F_{2n} + F_{2n+1} - 1 = F_{2n+2} - 1$

[in fibonacci series : $F_1 + F_2 = F_3$
$F_2 + F_3 = F_4$
$\dots\dots$]

$\Rightarrow (F_{2n} + F_{2n+1}) - 1 = F_{2n+2} - 1$

$\Rightarrow F_{2n+2} - 1 = F_{2n+2} - 1$ proven

5. $(1-x)(1+x+x^2+x^3+\dots+x^n) = 1-x^{n+1}$

for $n=0$

$(1-x)(x^0) = 1-x^{0+1}$

$\Rightarrow (1-x)(1) = 1-x$

$\Rightarrow (1-x) = 1-x$

proven

let $n \geq 0$

$(1-x)(x^0+x^1+x^2+\dots+x^n+x^{n+1}) = 1-x^{(n+1)+1}$

$\Rightarrow (1-x)(x^0+x^1+x^2+\dots+x^n)+(1-x)(x^{n+1}) = 1-x^{n+2}$

$\Rightarrow (1-x^{n+1})+(1-x)(x^{n+1})$

$\Rightarrow 1-x^{n+1}+x^{n+1}-x^{1+(n+1)}$

$\Rightarrow 1-x^{n+2}$

proven

6. $3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + \ldots + 3 \cdot 2^n$

$\Rightarrow 6 \cdot 2^0 + 6 \cdot 2^1 + 6 \cdot 2^2 + \ldots + 6 \cdot 2^{n-1}$

we know $\sum_{i=0}^{n} a \cdot r^i = a \cdot \frac{1-r^{n+1}}{1-r}$

$6 \cdot 2^0 + 6 \cdot 2^1 + 6 \cdot 2^2 + \ldots + 6 \cdot 2^{n-1} = \sum_{i=0}^{n-1} 6 \cdot 2^i = 6 \cdot \frac{1-2^{(n-1)+1}}{1-2}$

$= 6 \cdot \frac{1-2^n}{(-1)}$

$= 6(2^n - 1)$

7. $2 + 10 + 50 + \ldots + 1250$

$= 2 + 2 \cdot 5 + 2 \cdot 5^2 + \ldots + 2 \cdot 5^4$

$= 2 \cdot \frac{(1-5^{4+1})}{1-5} = 1562$

8. $a_0 = 0$, $a_1 = 2$, $a_n = 4(a_{n-1} - a_{n-2})$ for $n \geq 2$

$b_n = n.2^n = a_n$

for $n = 0$

$b_0 = 0.2^{(0)} = 0 = a_0$

for $n = 1$

$b_1 = 1 \cdot 2^{(1)} = 2 = a_1$

$a_n = 4(a_{n-1} - a_{n-2})$

$= 4((n-1)2^{n-1} - (n-2)2^{n-2})$

$= 4(n.2^{n-1} - 2^{n-1} - n.2^{n-2} + 2^{n-2+1})$

$= 4(n.2^{n-1} - n.2^{n-2} \cancel{-2^{n-1}} \cancel{+2^{n-1}})$

$= 4(n.2^{n-1} - n.2^{n-2})$

$= 4(n)(2^{n-1} - 2^{n-2})$

$= 4(n)(2^{n-2+1} - 2^{n-2})$

$= 4(n)(2^{n-2}.2 - 2^{n-2})$

$= 4(n)2^{n-2}(2-1) = 4n.2^{n-2} = 2^2.n.2^{n-2}$

$= n.2^n = b_n$

proven

$a_{n-1} = b_{n-1} = (n-1)2^{(n-1)}$

$= (n-1)2^{n-1}$

$a_{n-2} = b_{n-2} = (n-2)2^{n-2}$

9. $a_0 = 1,\ a_1 = 1,\ a_n = 2 \cdot a_{n-1} + 3 \cdot a_{n-2}$ for $n \geq 2$

$$b_n = \frac{1}{2} \cdot 3^n + \frac{1}{2} \cdot (-1)^n$$

$$a_n = b_n$$

$n = 0$

$$b_0 = \frac{1}{2}\left(3^0 + (-1)^0\right) = \frac{1}{2}(1+1) = 1 = a_0$$

$$b_1 = \frac{1}{2}\left(3^1 + (-1)^1\right) = \frac{1}{2}(3-1) = 1 = a_1$$

$$a_n = 2 \cdot a_{n-1} + 3a_{n-2}$$

$$a_{n-1} = b_{n-1} = \frac{1}{2} 3^{n-1} + \frac{1}{2}(-1)^{n-1}$$

$$= \frac{1}{2} 3^{n-1} + \frac{1}{2}(-1)^n(-1)^{-1}$$

$$= \frac{1}{2} \cdot 3^{n-1} + \frac{1}{2}(-1)^n(-1)$$

$$a_{n-2} = b_{n-2} = \frac{1}{2} 3^{n-2} + \frac{1}{2}(-1)^{n-2}$$

$$= \frac{1}{2} \cdot 3^{n-2} + \frac{1}{2}(-1)^n(-1)^{-2}$$

$$= \frac{1}{2} \cdot 3^{n-2} + \frac{1}{2}(-1)^n$$

$$a_n = 2\left(\frac{1}{2}\cdot 3^{n-1} + \frac{1}{2}(-1)^n(-1)\right) + 3\left(\frac{1}{2}\cdot 3^{n-2} + \frac{1}{2}(-1)^n\right)$$

$$= 2\left(\frac{1}{2}\cdot 3^{n-1} - \frac{1}{2}(-1)^n\right) + 3\left(\frac{1}{2}\cdot 3^{n-2} + \frac{1}{2}(-1)^n\right)$$

$$= \cancel{2}\left(\frac{1}{\cancel{2}}\left(3^{n-1} - (-1)^n\right)\right) + 3\left(\frac{1}{2}\left(3^{n-2} + (-1)^n\right)\right)$$

$$= 3^{n-1} - (-1)^n + 3\left(\frac{3^{n-2}}{2} + \frac{(-1)^n}{2}\right)$$

$$= 3^{n-1} - (-1)^n + \frac{3^{n-2+1}}{2} + \frac{3}{2}(-1)^n$$

$$= 3^{n-1} + \frac{3^{n-2+1}}{2} + \frac{3}{2}(-1)^n - (-1)^n$$

$$= 3^{n-1} + \frac{3^{n-1}}{2} + \frac{3}{2}(-1)^n - (-1)^n$$

$$= 3^{n-1}\left(1 + \frac{1}{2}\right) + (-1)^n\left(\frac{3}{2} - 1\right)$$

$$= 3^n \cdot 3^{-1}\left(\frac{3}{2}\right) + \left(\frac{1}{2}\right)(-1)^n$$

$$= 3^n \cdot \frac{1}{\cancel{3}}\left(\frac{\cancel{3}}{2}\right) + \left(\frac{1}{2}\right)(-1)^n$$

$a_n = \frac{1}{2} \cdot 3^n + \frac{1}{2} \cdot (-1)^n = b_n$

proven

10. $F(3) = 2F(2) + 3F(1) + F(0) = 10 + 9 + 2 = 21$

$F(4) = 2F(3) + 3F(2) + f(1) = 42 + 15 + 3 = 60$

$F(5) = 2F(4) + 3F(3) + F(2) = 120 + 63 + 5 = 188$

11. $T(n) = 2T(n-1) + 1$

$= 2(2T(n-2) + 1) + 1$

$= 4T(n-2) + 2 + 1$

$= 4(2T(n-3) + 1) + 2 + 1$

$= 8T(n-3) + 4 + 2 + 1$

$T(n) = 2^3 T(n-3) + 2^{3-1} + 2^{2-1} + 2^{1-1}$

$\therefore T(n) = 2^i T(n-i) + 2^{i-1} + 2^{i-2} + \dots 2 + 1$

let $i = n-2$

$$\therefore T(n) = 2^{n-2}\, T(n-(n-2)) + 2^{n-3} + 2^{n-4} + 2^{n-5} + \dots + 2 + 1$$

$$= 2^{n-2}\, T(2) + 2^{n-3} + 2^{n-4} + 2^{n-5} + \dots + 2 + 1$$

$$= 2^{n-2}\, (2T(1)+1) + 2^{n-3} + 2^{n-4} + \dots + 2 + 1$$

$$= 2^{n-1}\, T(1) + 2^{n-2} + 2^{n-3} + \dots + 2 + 1$$

$$T(n) = 2^{n-1}\cdot(1) + 2^{n-2} + 2^{n-3} + \dots + 2 + 1$$

$$\therefore T(n) = \sum_{i=0}^{n-1} 2^{i} = 1 + 2 + 4 + 8 + \dots + 2^{n-1}$$

$$2T(n) = 2 + 4 + 8 + 16 + \dots + 2^{n}$$

$$2T(n) - T(n) = T(n)$$

$$\Rightarrow (2+4+8+16+\dots+2^{n}) - (1+2+4+8+\dots+2^{n-1})$$

$$\Rightarrow (2+4+8+16+\dots+2^{n-1}+2^{n})$$

$$-(1+2+4+8+16+\dots+2^{n-1})$$

$$= 2^{n} - 1 = T(n)$$

12. $T(n) = cT(n-1) + f(n)$

$$= c(cT(n-2) + f(n-1)) + f(n)$$

$$= c^2T(n-2) + cf(n-1) + f(n)$$

$$= c^2(cT(n-3) + f(n-2)) + cf(n-1) + f(n)$$

$$= c^3T(n-3) + c^2f(n-2) + cf(n-1) + f(n)$$

$$= c^3(cT(n-4) + f(n-3)) + c^2f(n-2) + cf(n-1) + f(n)$$

$$= c^4T(n-4) + c^3f(n-3) + c^2f(n-2) + cf(n-1) + f(n)$$

$$T(n) = c^i T(n-i) + \sum_{j=0}^{i-1} c^j f(n-j)$$

let $i = n-k \Rightarrow k = n-i$

we get

$$T(n) = c^{n-k} T(k) + \sum_{j=0}^{n-k-1} c^j f(n-j)$$

$$= \sum_{j=0}^{n-k-1} c^j f(n-j) + c^{n-k} T(k)$$

$$= f(n) + cf(n-1) + c^2 f(n-2) + \dots + c^{n-k-1} f(k+1) + c^{n-k} + f(k)$$

$$T(n) = \sum_{j=0}^{n-k} c^j f(n-j) = \sum_{j=k}^{n} c^{n-j} f(j)$$

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13. $T(n) = 2T\left(\frac{n}{2}\right) + n$ $\quad T(1) = 1$

$$T(n) = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

$$= 4T\left(\frac{n}{4}\right) + n + n$$

$$= 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n$$

$$= 8T\left(\frac{n}{8}\right) + 3n$$

$$= 8\left(2T\left(\frac{n}{16}\right) + \frac{n}{8}\right) + 3n$$

$$T(n) = 16T\left(\frac{n}{16}\right) +$$

$$T(n) = 2^k\left(\frac{n}{2^k}\right) +$$

let $n = 2^k$

$n = 2^k$

$\Rightarrow \log n = k \log 2$

$\Rightarrow \frac{\log n}{\log 2} = k$

$$\therefore T(n) = n \cdot T(1) + \frac{\log n}{\log 2} \cdot n$$

$T(1) = 1$

$$T(n) = n \cdot (1) + n \cdot \frac{\log n}{\log 2}$$

14. $T(n) = 2T\left(\frac{n}{2}\right) + 1 \quad , \quad T(1) = 1$

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$

$$= 2\left(2T\left(\frac{n}{4}\right) + 1\right) + 1$$

$$= 4T\left(\frac{n}{4}\right) + 2 + 1$$

$$= 4\cancel{8}\left(2T\left(\frac{n}{8}\right) + 1\right) + 3$$

$$= 8T\left(\frac{n}{8}\right) + 4 + 3 = 8T\left(\frac{n}{8}\right) + 7$$

$$T(n) = 8\left(2T\left(\frac{n}{16}\right) + 1\right) + 7 = 16T\left(\frac{n}{16}\right) + 15$$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + 2^k - 1$$

let $2^k = n$

$$\therefore T(n) = n\, T\left(\frac{n}{n}\right) + n - 1$$

$$= n \cdot T(1) + n - 1 \qquad \Big| \; T(1) = 1$$

$$= n + n - 1$$

$$T(n) = 2n - 1$$

15. $T(n) = T(n-1) + n \quad , T(1) = 1$

$= T(n-2) + (n-1) + n$

$= T(n-3) + (n-2) + (n-1) + n$

$= T(n-4) + (n-3) + (n-2) + (n-1) + n$

$T(n) = T(n-k) + (n-k+1) + (n-k+2) + \dots + (n-1) + n$

let $k = n-1$

$T(n) = T(n-(n-1)) + (n-(n-1)+1) + \dots + (n-1) + n$

$= T(1) + (2) + (3) + \dots (n-1) + n$

$= 1 + 2 + 3 + \dots + n$

$\therefore T(n) = \dfrac{n(n+1)}{2}$

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