Mathematics IV: Linear Algebra and Fourier Analysis

(MAT216)

Summer 2023

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MAT216; *Section:* 15

Part 1: Linear Algebra

CONCEPTS OF VECTORS & LINEAR COMBINATION

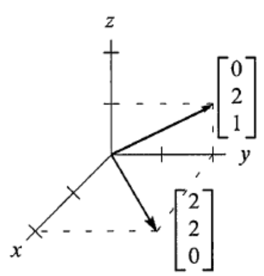
Vector Addition: $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ add to $v + w = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$

Scalar Multiplication: $2v = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$ and $-v = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$

The heart of linear algebra is in two operations-both with vectors. We add vectors to get *v* + *w*. We multiply them by numbers c and *d* to get *cv* and *d w*. Combining those two operations (adding *cv* to *d w)* gives the *linear combination cv* + *d w.*

Linear Combination: $cv + dw = c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} c + 2d \\ c + 3d \end{bmatrix}$

[Represented in class as $a\bar{A} + b\bar{B}$]



Here, if we apply linear combination on the two vectors, the vectors will **either increase or decrease in size.**

[N.B.: The points (0, 2, 1) and (2, 2, 0) can be represented using two vectors $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.]

SPACES OF VECTORS

Vector Space: Vector space is a set of vectors that immerges from the *linear combination* of vectors that must satisfy **three conditions.** They are:

1. It must include zero (0).
2. It must be closed under scalar multiplication.
3. It must be closed under vector addition.

Vector Subspace: Vector subspace is a subset of vector space that must satisfy **three conditions**. They are:

1. It must include zero (0).
2. It must be closed under scalar multiplication.
3. It must be closed under vector addition.

Extra Information

- Line passing through 0 (zero) is a subspace of two dimentional vector space.
- 0 (zero) alone can also be a subspace of two dimentional vector space.

SYSTEM OF LINEAR EQUATION

Let two linear equations are:

$$3x + 2y = 7$$
$$x + 2y = 5$$

Matrix Picture:

$$\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

Here, $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$ is a Matrix, $\begin{bmatrix} x \\ y \end{bmatrix}$ is a vector and $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ is also a vector.

$$\boxed{\text{i.e. } Matrix \times Vector = Vector}$$

Augmented Matrix:

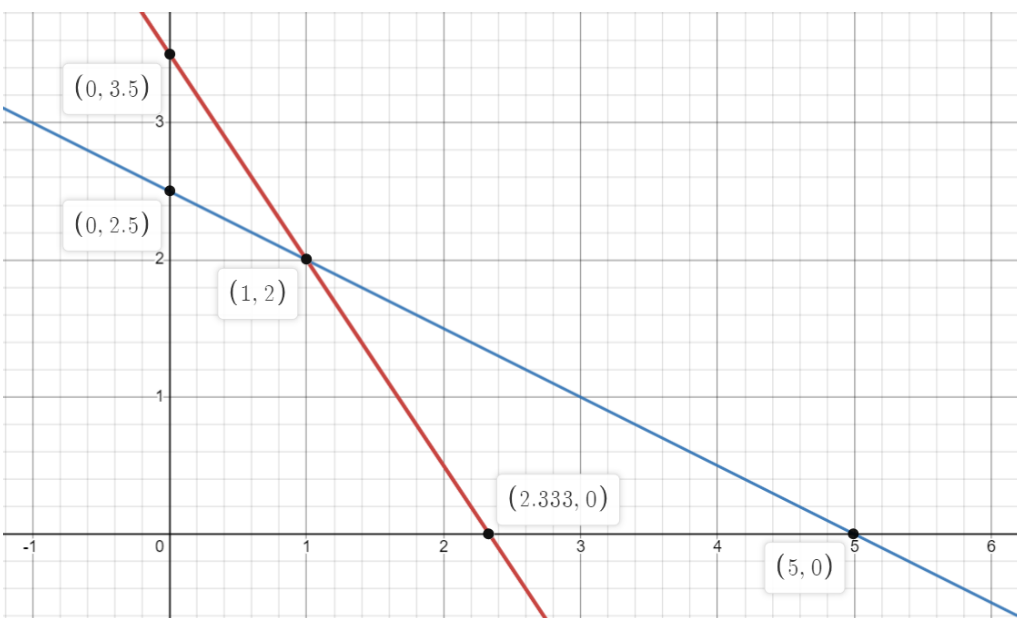
$$\left[\begin{array}{cc|c} 3 & 2 & 7 \\ 1 & 2 & 5 \end{array}\right]$$

[Required for solving linear system equations.]

Row Picture:

$$3x + 2y = 7$$
$$or, \frac{3x}{7} + \frac{2x}{7} = 1$$
$$or, \frac{x}{7/3} + \frac{y}{7/2} = 1$$

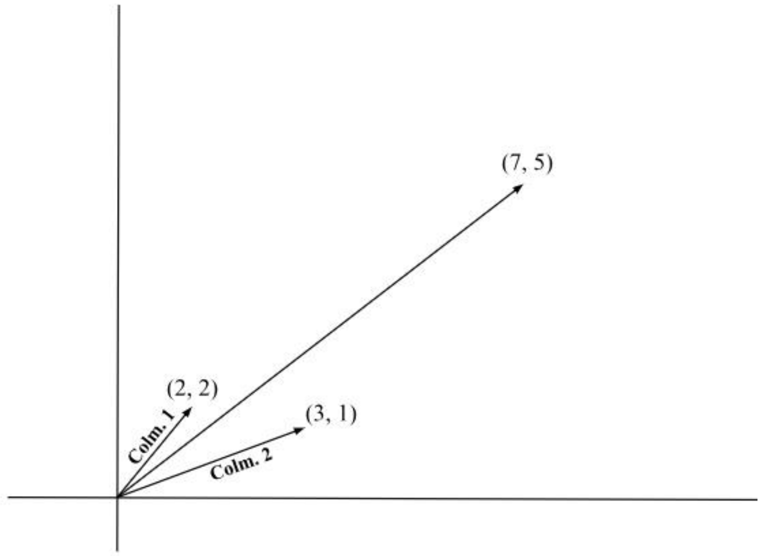
$$x + 2y = 5$$
$$or, \ \frac{x}{5} + \frac{2y}{5} = 1$$
$$or, \frac{x}{5} + \frac{y}{5/2} = 1$$



Here, the solution point is (1, 2).
The following graph is the ***Row Picture*** of the given equations.

Coloumn Picture:

$$x\begin{bmatrix}3\\1\end{bmatrix}+y\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}7\\5\end{bmatrix}$$



The following graph is the ***Column Picture*** of the given equations.

Example 1: Find the Matrix Picture, Row Picture and Column Picture of the following equations:

$$x+y+z=3$$
$$2x+3y-2z=3$$
$$4x-y+2z=5$$

Soln.:

Matrix Picture:

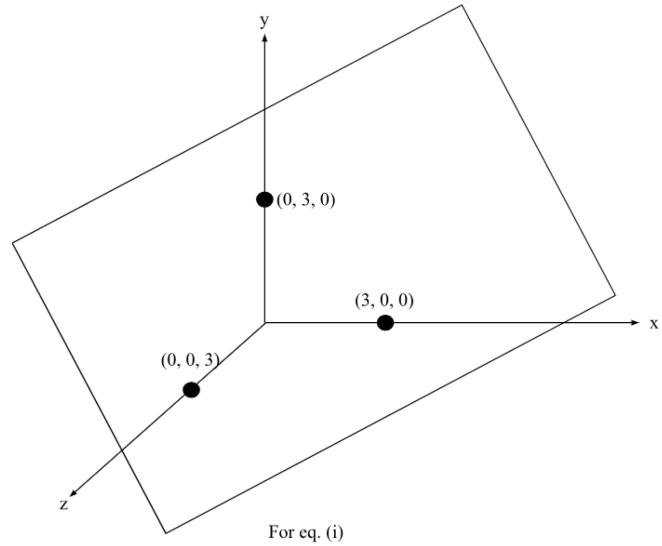
$$\begin{bmatrix}1 & 1 & 1\\2 & 3 & -2\\4 & -1 & 2\end{bmatrix}.\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\3\\5\end{bmatrix}$$

Row Pricture:

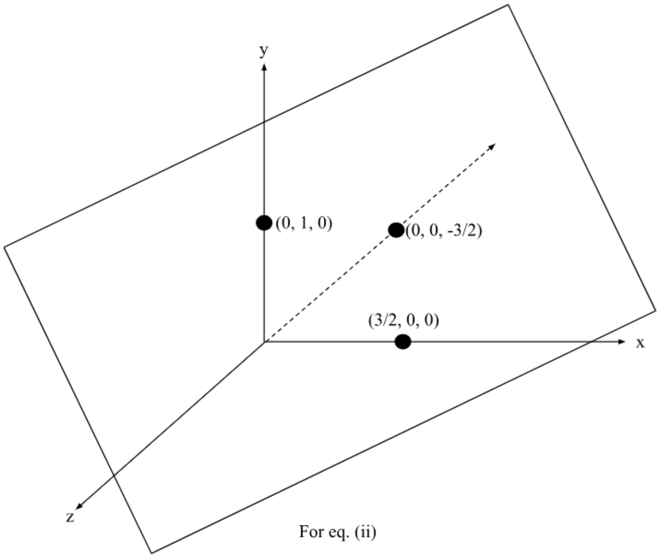
$$\frac{x}{3}+\frac{y}{3}+\frac{z}{3}=1\ \dots\dots\dots\dots\dots\dots\dots\dots (i)$$

$$\frac{x}{3/2}+\frac{y}{1}+\frac{z}{-3/2}=1\ \dots\dots\dots\dots..(ii)$$

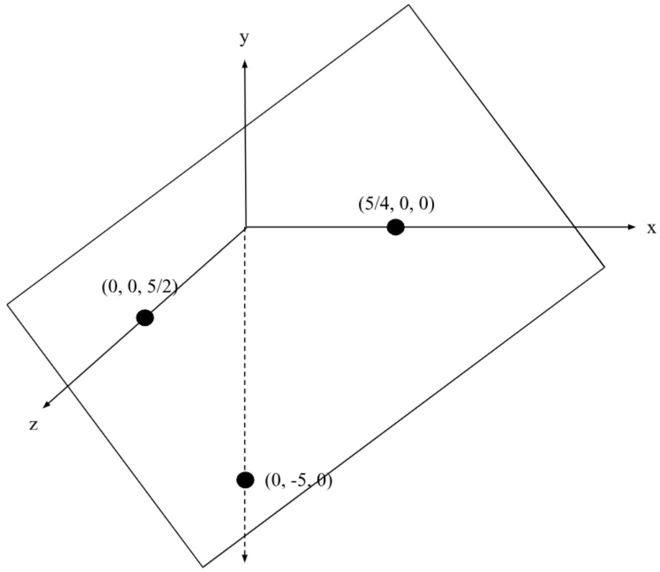
$$\frac{x}{5/4}+\frac{y}{-5}+\frac{z}{5/2}=1\ \dots\dots\dots\dots..(iii)$$



For eq. (i)



For eq. (ii)

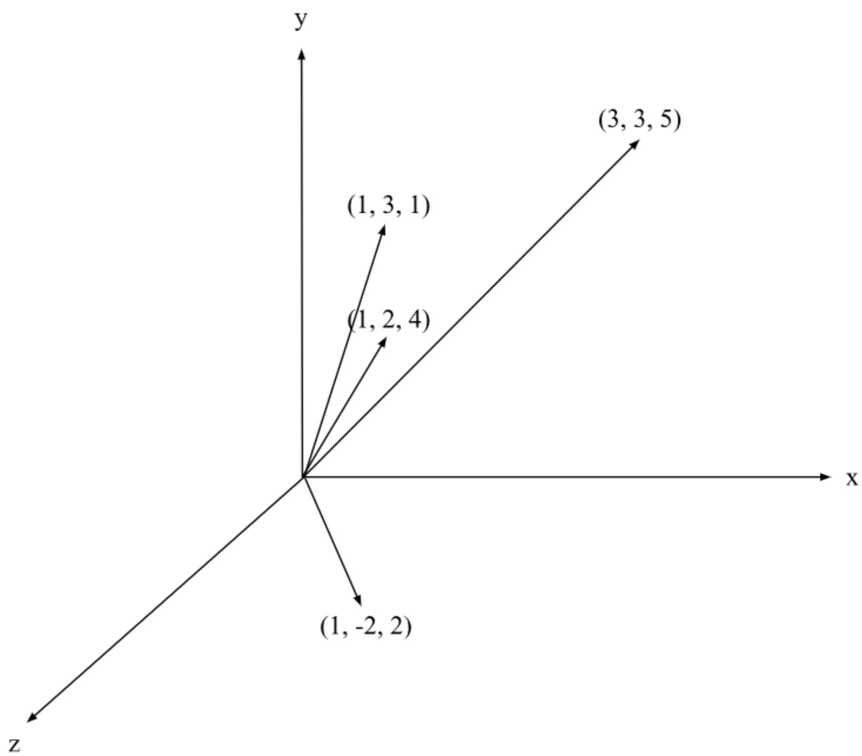


For eq. (iii)

The above three graphs are the ***Row Picture*** of the given equations.

<u>Column Picture</u>:

$$x\begin{bmatrix}1\\2\\4\end{bmatrix}+y\begin{bmatrix}1\\3\\-1\end{bmatrix}+z\begin{bmatrix}1\\-2\\2\end{bmatrix}=\begin{bmatrix}3\\3\\5\end{bmatrix}$$



The above graph is the ***Column Picture*** of the three equations.

SOLUTION OF LINEAR SYSTEM

Pivot: A **pivot position** in a matrix is the location of a leading entry in the row-echelon form of a matrix.
A **pivot column** is a column that contains a pivot position.

[We will learn about echelon later]

In a 3×3 matrix, generally the primary the diagonal is said to be its pivot.

Example: $\begin{bmatrix} \boldsymbol{a} & b & c \\ d & \boldsymbol{e} & f \\ g & h & \boldsymbol{i} \end{bmatrix}$; Here, a, e, f are the pivots of the given matrix [This is for 3×3 matrix].

While solving a linear system, we should keep in mind that, **PIVOT can never be zero.**

<u>Row/ Gaussian Elimination</u>:

Conditions:

- Pivot will never be zero.
- We have to make the given matrix into **upper triangular matrix**.
  $\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$; Here, this is a upper triangular matrix.
- If pivot is zero, we will either get **No Solution Condition** or **Infinite Solution Condition**.
  - If $0 \times Variable = Value$; we will get **No Solution Condition.**
  - If $0 \times Variable = 0$; we will get **Infinite Solution Condition.**

Example 2: Determine the solution of the linear system using Row/ Gauss Elimination.

$$x + y + z = 3$$
$$2x + 3y - 2z = 3$$
$$4x - y + 2z = 5$$

Soln.: <u>Matrix Picture</u>:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 4 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}$$

<u>Augmented Matrix</u>:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & -2 & 3 \\ 4 & -1 & 2 & 5 \end{array}\right]$$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -4 & -3 \\ 0 & -5 & -2 & -7 \end{array}\right]$; (row2 – 2×row1 = row2’); (row3 - 4×row1 = row3’)

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & -22 & -22 \end{array}\right]$; (row3’ - 5×row2 = row3”)

Using Back Elimination:

[Start from the last equation first]

$-22z = -22$
$or, z = 1$

Again, $y - 4z = -3$
$or, y = 1$

Lastly, $x + y + z = 3$
$or, x = 1$

Therefore, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
(Ans.)

Example 3: Determine the solution of the linear system using Row/ Gauss Elimination.

$$x + 2y + z = 6$$
$$x + 2y + 2z = 7$$
$$2x + 3y + z = 9$$

Soln.: Matrix Picture:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 9 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 1 & 2 & 2 & 7 \\ 2 & 3 & 1 & 9 \end{array}\right]$$

$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & -3 \end{array}\right]$; (row2 – row1 = row2’); (row3 - 2×row1 = row3’)

$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$; (row2’ ↔ row3)

Using Back Substitution:

$z = 1$

Again, $-y - z = -3$
$or, y = 2$

Finally, $x + 2y + z = 6$
$or, x = 1$

Therefore, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
(Ans.)

Example 4: Determine the solution of the linear system using Row/ Gauss Elimination.

$$x + 2y + z = 6$$
$$x + 2y + 2z = 7$$
$$x + 2y + 3z = 12$$

Soln.: Matrix Picture:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 1 & 2 & 2 & 7 \\ 1 & 2 & 3 & 12 \end{array}\right]$$

$= \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 6 \end{array}\right]$; (row2 – row1 = row2'); (row3 - row1 = row3')

Using Back Substitution:

$2z = 6$
$or, z = 2$

Again, $0.y + z = 1$
$or, 0.y = -2$

$0 \times Variable = Value$

Therefore, it is a No Solution Condition.

Gauss - Jordan Elimination:

Conditions:

- Pivot will never be zero.
- All the values over and under the Pivot values will be zero.
- We have to make the given matrix into **an Identity Matrix**.
  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; Here, this is a upper triangular matrix.
- If pivot is zero, we will either get **No Solution Condition** or **Infinite Solution Condition**.
  - ❖ If $0 \times Variable = Value$; we will get **No Solution Condition.**
  - ❖ If $0 \times Variable = 0$; we will get **Infinite Solution Condition.**

[All the steps of Gauss – Jordan Elimination will be the same upto the Back Substitution Step.]

Example 5: Determine the solution of the linear system using Gauss – Jordan Elimination.

$$x + y + z = 3$$
$$2x + 3y - 2z = 3$$
$$4x - y + 2z = 5$$

Soln.: Matrix Picture:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 4 & -1 & 2 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 3 & -2 & 3 \\ 4 & -1 & 2 & 5 \end{array}\right]$$

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -4 & -3 \\ 0 & -5 & -2 & -7 \end{array}\right]$; (row2 – 2×row1 = row2’); (row3 - 4×row1 = row3’)

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & -22 & -22 \end{array}\right]$; (row3’ - 5×row2 = row3”)

$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -4 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$; ($\frac{row3''}{-22} = row3'''$)

$= \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$; (row2’ + 4×row3’’’ = row2”)

$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$; (row1’ – row2” = row1”)

Therefore, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(Ans.)

Practice Problems

Determine the solution of the linear system using Row/ Gauss Elimination and Gauss – Jordan Elimination.

1. $2x + y + 3z = 8$
   $x + 2y + 5z = 9$
   $3x + 5y - z = 10$

2. $2x + y + 3z = 8$
   $2x + y + 5z = 9$
   $3x + 5y - z = 10$

3. $2x + y + 3z = 8$
   $2x + y + 5z = 9$
   $2x + y - z = 10$

SPECIAL TYPE OF MATRICES

Permutation Matrix: Permutation matrix emerges from identity matrix that executes *Row Exchange*.

(For Understanding) Permutation matrix is the opposite of an identity matrix. Such as: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Example:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

If we want to exchange row 2 and row 3:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b & c \\ h & i & j \\ d & e & f \end{bmatrix}$$

Elementary Matrix: Elementary matrix emerges from identity matrix that executes *Row Elimination*.
In other words, elementary matrix is a modified version of an identity matrix, that is used to eliminate or make one or more element of a matrix zero.

For example, if we want to make the A_{21} element of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 2 \\ 7 & 2 & 1 \end{bmatrix}$; we need to change the identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in such a way that it eliminates the element A_{21} (which is 4 here) when we multiply A and I. Now, f we subtract row2 of I from 4 times of its row 1, we will get the matrix $I' = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Now, multipling A and I will eliminate A_{21} which is 4.

$$A.I' = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 9 & 2 \\ 7 & 2 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 9 & 2 \\ 7 & 2 & 1 \end{bmatrix}$$

So, the elementary matrix of A_{21} is $E_{21} = I' = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 5: Find E_{31} for the matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 7 & 2 & 1 \\ 8 & 3 & 2 \end{bmatrix}$.

Soln.: Here, E_{31} means the modified identity matrix required to eliminate the A_{31} element of the matrix A. Let's consider a 3×3 identity matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Now, \qquad I' = E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{-4} & 0 & 1 \end{bmatrix}$$

(Ans.)

Verification: $A \times I' = \begin{bmatrix} 2 & 3 & 4 \\ 7 & 2 & 1 \\ \mathbf{0} & 3 & 2 \end{bmatrix}$.

REDUCED ROW ECHELON FORM (rref)

Reduced row echelon form is a type of matrix used to solve systems of linear equations. Reduced row echelon form has ***four requirements***:

- In each row, the left-most nonzero entry is 1 and the column that contains this 1 has all other entries equal to 0. This 1 is called a leading 1.
- The second row also starts with the number 1, which is further to the right than the leading entry in the first row. For every subsequent row, the number 1 must be further to the right.
- The leading entry in each row must be the only non-zero number in its column.
- Any non-zero rows are placed at the bottom of the matrix.

For example, the following matrices are **NOT** a part of RREF:

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ (The left-most nonzero entry in the second row not equal to 1, thus violating property 1 stated above.)
- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (The leading 1 in the second row is to the left of the leading 1 in the first row, thus violating property 2 stated above.)
- $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (The leading 1 in the third row is not the only nonzero in the column containg it, thus violating property 1 stated above.)

<u>Some key characteristics of pivot for solving problems using RREF</u>:

- Non zero value.
- Every row/ column (not just the primary diagonal as we learnt earlier) can contain ONLY ONE pivot.
- Will consider values for pivot from first value to last in a row/ column i.e. the first non zero value of a row/ column will be consided to be the pivot.

<u>Some key points to consider while solving RREF problems</u>:

- Figure out the pivots first.
- The elements over and under pivots should be zero. For example: $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
- If only the elements UNDER the pivots are zero, it is said to be **Echelon Form**.
- If the elements OVER and UNDER the pivots are zero, it is said to be **Reduced Echelon Form**.

Example 6: If $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & 3 & 4 & 7 \\ 2 & 5 & 6 & 9 \end{bmatrix}$; find the rref(A).

Soln.: Given,

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & 3 & 4 & 7 \\ 2 & 5 & 6 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \; ; (row2 - row1 = row2') \; ; (row3 - 2 \times row1 = row3')$$

$$= \begin{bmatrix} \mathbf{1} & 2 & 2 & 2 \\ 0 & \mathbf{1} & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; (row3' - row2' = row3") ; \text{[the red coloured elements are the pivots]}$$

$$= \begin{bmatrix} \mathbf{1} & 0 & -2 & -8 \\ 0 & \mathbf{1} & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; (row1 - 2 \times row2' = row1')$$

(Ans.)

Practice Problems

1. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 1 \\ 3 & 2 & 3 \end{bmatrix}$; Find the rref(A).

Rank: The number of pivots present in the Echelon form of a matrix is said to the Rank of that matrix.
For Example: In Example 6, we found out that the matrix A has 2 pivots. So, the rank of the matrix A will be 2.

<u>Concept of Free Variable and Fixed Variable</u>:

Free Variable: If in a linear equation one or more variable is found to have infinite solutions, they are said to be free variables.

Fixed Variable: The variables in a linear equation which have only ONE unique solution, are known as fixed variables.

Example 7: Three linear equations are given as:

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$
$$x_1 + 3x_2 + 4x_3 + 7x_4 = 0$$
$$2x_1 + 5x_2 + 6x_3 + 9x_4 = 0$$

Find the solution of the linear system.
Or, *Find the null space of the linear system (We will learn about null space later).*

Soln.: <u>Matrix Picture</u>:

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 1 & 3 & 4 & 7 \\ 2 & 5 & 6 & 9 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>Augmented Matrix</u>:

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 0 \\ 1 & 3 & 4 & 7 & 0 \\ 2 & 5 & 6 & 9 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 0 \\ 0 & 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 5 & 0 \end{array}\right] ;\ (row2 - row1 = row2') ; (row3 - 2\ \times row1 = row3')$$

$$= \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 0 \\ 0 & 1 & 2 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] ;\ (row3' - row2' = row3'')$$

[Here, we can see that the values of x_3 and x_4 are different in different rows. So, we can say them as free variables.]

So, let's assume any two values of x_3 and x_4.

Let,
$x_3 = 1$
$x_4 = 3$

[Note, we can consider any Real value for free variable except 0.]

<u>So, applying back substitution</u>:

In row2: $x_2 + 2x_3 + 5x_4 = 0$
Therefore, $x_2 = -17$

In row1: $x_1 + 2x_2 + 2x_3 + 2x_4 = 0$
Therefore, $x_1 = 26$

So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 26 \\ -17 \\ 1 \\ 3 \end{bmatrix}$

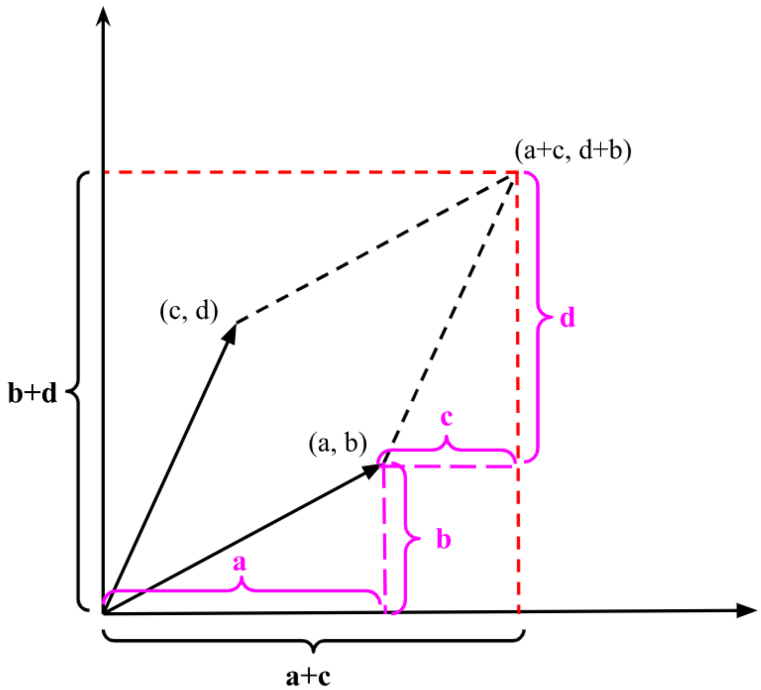
But, if the values of x_3 and x_4 change we will get different solutions. There can be infinite number of solutions. So, in order to solve this problem, we multiply a constant "c" with our solution, where c is all real numbers except 0.

Therefore, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c \begin{bmatrix} 26 \\ -17 \\ 1 \\ 3 \end{bmatrix}$

(Ans.)

DETERMINANT OF MATICES

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A, $|A| = ad - bc$.
We can prove it by the following graph:



Here,

$$\begin{aligned} \text{Area of parallelogram} &= (a+c)(b+d) - 2\left(\frac{1}{2}ab + bc + \frac{1}{2}cd\right) \\ &= ab + ad + bc + cd - ab - 2bc - cd \\ &= ad - bc \\ &= |A| \end{aligned}$$

MATRIX MULTIPLICATION

From the earlier discussion we know, $Matrix \times Vector = Vector$. So, if we multiply a 3×3 matrix with another 3×3 matrix, we will get another 3×3 matrix, i.e.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} . \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

But, actually this can be broken down into three $Matrix \times Vector = Vector$ form.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} . \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} . \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} . \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

So, solving equation (i) we will get the values of b_{11}, b_{21} and b_{31}. Similarly, solving equations (ii) and (iii) will give us the values of b_{12}, b_{22}, b_{32} and b_{13}, b_{23}, b_{33} respectively.

[We will need this basic for clearly understanding Inverse Matrix.]

INVERSE MATRIX

A matrix should follow two conditions to be invertable:

1. |A| **must not** be zero.
2. Must be a **square matrix**.

We know, if A is a matrix, then, $A^{-1} = \frac{Adj\ A}{|A|}$ and $A \times A^{-1} = I$.

Now, let $A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix}$

So, if a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 3 & 2 & 7 \end{bmatrix}$, then $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 3 & 2 & 7 \end{bmatrix} . \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So, as per our previous understanding, this equation can be expressed as three different $Matrix \times Vector = Vector$ equations, i.e.,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 3 & 2 & 7 \end{bmatrix} . \begin{bmatrix} a \\ d \\ h \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 3 & 2 & 7 \end{bmatrix} . \begin{bmatrix} b \\ e \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 3 & 2 & 7 \end{bmatrix} . \begin{bmatrix} c \\ f \\ j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now, let's try to solve the problem.

Example 8: If a matrix is given as $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 3 & 2 & 7 \end{bmatrix}$, find A^{-1} using Gauss-Jordan Elimination.

Soln.: Let, $A^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix}$

We know, $A \times A^{-1} = I$

Therefore, Matrix Picture:

$$\begin{bmatrix}1 & 2 & 3\\4 & 0 & 1\\3 & 2 & 7\end{bmatrix}.\begin{bmatrix}a & b & c\\d & e & f\\h & i & j\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Augmented Matrix:

$$\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\4 & 0 & 1 & 0 & 1 & 0\\3 & 2 & 7 & 0 & 0 & 1\end{array}\right]$$

$$=\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\0 & -8 & -11 & -4 & 1 & 0\\0 & -4 & -2 & -3 & 0 & 1\end{array}\right]\ (row2-4\times row1=row2');(row3-3\times row3=row3')$$

$$=\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\0 & -8 & -11 & -4 & 1 & 0\\0 & 0 & 7 & -2 & -1 & 2\end{array}\right]\ (2\times row3'-row2'=row3'')$$

$$=\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\0 & 1 & \frac{11}{8} & \frac{1}{2} & \frac{1}{-8} & 0\\0 & 0 & 1 & \frac{-2}{7} & \frac{-1}{7} & \frac{2}{7}\end{array}\right]\ \left(\frac{row2'}{-8}=row2''\right),\left(\frac{row3''}{7}=row3'''\right)$$

$$=\left[\begin{array}{ccc|ccc}1 & 2 & 0 & \frac{13}{7} & \frac{3}{7} & -\frac{6}{7}\\0 & 1 & 0 & \frac{25}{28} & \frac{1}{14} & -\frac{11}{28}\\0 & 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & \frac{2}{7}\end{array}\right]\ (row1-3\times row3'''=row1');(row2''-\frac{11}{8}\times row3'''=row2''')$$

$$=\left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{51}{14} & \frac{2}{7} & -\frac{1}{14}\\0 & 1 & 0 & \frac{25}{28} & \frac{1}{14} & -\frac{11}{28}\\0 & 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & \frac{2}{7}\end{array}\right]\ (row1'-2\times row2'''=row1'')$$

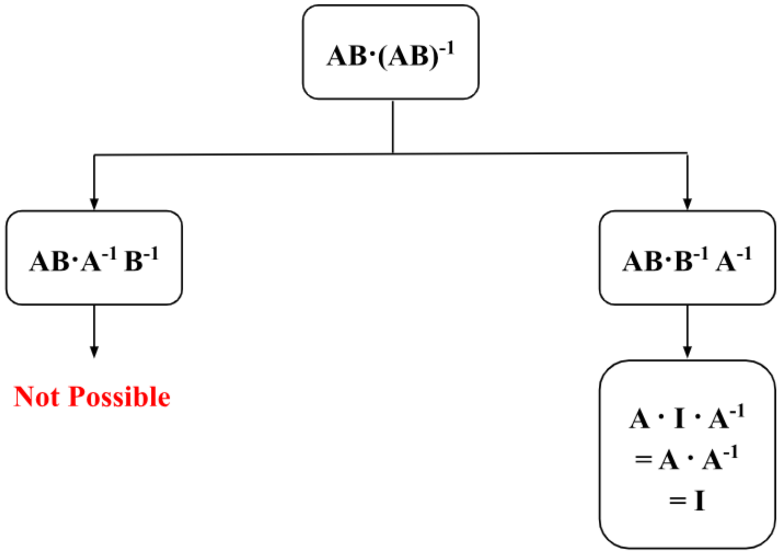
Therefore, $\begin{bmatrix}a & b & c\\d & e & f\\h & i & j\end{bmatrix}=\begin{bmatrix}\frac{51}{14} & \frac{2}{7} & -\frac{1}{14}\\\frac{25}{28} & \frac{1}{14} & -\frac{11}{28}\\-\frac{2}{7} & -\frac{1}{7} & \frac{2}{7}\end{bmatrix}$

(Ans.)

Practice Problems

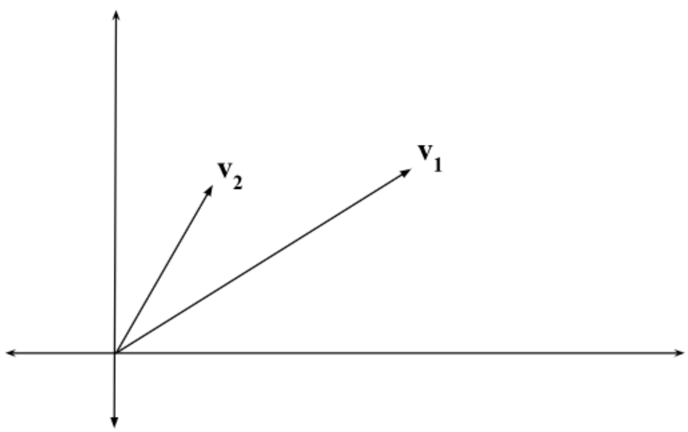
1. If $A=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$, find A^{-1} using Gauss-Jordan Elimination.
2. If $A=\begin{bmatrix}3 & 2 & 1\\0 & 5 & 7\\2 & 3 & 5\end{bmatrix}$, find A^{-1} using Gauss-Jordan Elimination.

Important Information Regarding Inverse Matrix:



Again, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$
Also, $(ABC)^{T} = C^{T}B^{T}A^{T}$

LINEAR COMBINATION



$av_1 + bv_2 =$ linear combination $(a, b \in \mathbb{R})$

Important Information
• Linear combination of one vector gives a straight line. • Linear combination of two vectors gives a plane. • Linear combination of three vectors gives a 3D space. ➢ If $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $av_1 =$ a **line** in a $\mathbb{R}^3$ (3D vector space). ➢ If $v_1 = \begin{bmatrix}1\\2\\3\end{bmatrix}, v_2 = \begin{bmatrix}3\\7\\2\end{bmatrix}$; $av_1 + bv_2 =$ a **plane** in $\mathbb{R}^3$. ➢ If $v_1 = \begin{bmatrix}1\\2\end{bmatrix}, v_2 = \begin{bmatrix}3\\2\end{bmatrix}$; $av_1 + bv_2 =$ a **plane** in $\mathbb{R}^2$.

FUNDAMENTAL SUBSPACES

There are four fundamental subspaces in linear algebra. They are:

1. Column Space: C(A)
2. Row Space: $C(A^T)$
3. Null Space: N(A)
4. Left Null Space: $N(A^T)$

1. Column Space C(A): Column space is a type of vector space which satisfies ***three conditions***:

- It must include zero (0).
- It must be closed under scalar multiplication.
- It must be closed under vector addition.

It is a linear combination of the columns of a matrix.

Let, a matrix be: $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 5 & 7 & 2 \end{bmatrix}$

Here, $c_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$; $c_2 = \begin{bmatrix} 2 \\ 0 \\ 7 \end{bmatrix}$; $c_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

If C_1, C_2, C_3, …………… are the column vectors, then column space = $a_1C_1 + a_2C_2 + a_3C_3 +$ ……………
[where, a_1, a_2, a_3, …………… $\in \mathbb{R}$].

Example 9: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, find the column space of A or C(A).

Soln: Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \ (row2 - 4 \times row1 = row2'); (row3 - 7 \times row1 = row3')$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} \ (\frac{row3'}{-2} + row2' = row3'')$$

Therefore, $C(A) = a \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + b \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}; a, b \in \mathbb{R}$

So, C(A) is a plane in $\mathbb{R}^3$.

2. Row Space C(A^T): Row space is nothing but the column space of a transpose matrix A^T. In order words, the column space of A^T is termed as the row space of A.

Example 10: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ 2 & 1 & 9 \end{bmatrix}$, find the row space of A or C(A^T).

Soln.: Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ 2 & 1 & 9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 0 & 1 \\ 3 & 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 2 \\ 0 & -8 & -3 \\ 0 & -5 & 3 \end{bmatrix} \ (row2 - 2 \times row1 = row2'); (row3 - 3 \times row1 = row3')$$

$$= \begin{bmatrix} 1 & 4 & 2 \\ 0 & -8 & -3 \\ 0 & 0 & 39 \end{bmatrix} \ (8 \times row3' - 5 \times row2' = row3'')$$

Therefore, $C(A^T) = a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix} + c \begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix}; a, b, c \in \mathbb{R}$

So, C(A^T) is a $\mathbb{R}^3$ in $\mathbb{R}^3$.

3. Null Space N(A): The set of vectors that emerge from the solution of $Ax = 0$ is termed as null space. Null space are nothing but straight lines passing through the origin.

Note: If all the given eqations have zeros on the right side (i.e. after '=' sign), there will be infinite solutions of that particular linear system.

Example 11: Three linear equations are given as:

$$x + 2y + z = 0$$
$$x + 3y + 2z = 0$$
$$2x + 5y + 3z = 0$$

Find the null space or N(A) of the linear system.

Soln.: Matrix Picture:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \ (row2 - row1 = row2'); (row3 - 2 \times row1 = row3')$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \ (row3' - row2' = row3'')$$

Therefore, z is the free variable.

Let, z = 2.

<u>Back Substitution</u>:

$y + z = 0$
$or, y = -2$

$and, x + 2y + z = 0$
$or, x = 2$

Therefore, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$

So, N(A) is a line in $\mathbb{R}^3$.

4. <u>Left Null Space $N(A^T)$</u>: Left null space is nothing but the null space of a transpose matrix A^T. In order words, the null space of A^T is termed as the left null space of A.

DO YOURSELF

Practice Problem:

1. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \end{bmatrix}$; find $N(A^T)$.

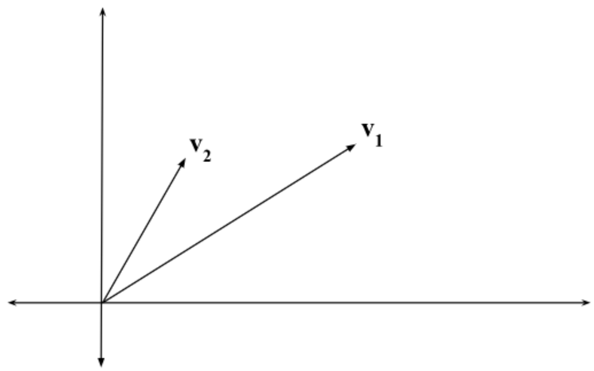
2. Three linear equations are given as:

$$x + y + z = 0$$
$$2x + 3y + 4z = 0$$
$$4x + 3y + z = 0$$

   Find the left null space of the linear system.

LINEAR INDEPENDENCE

If $v_1, v_2, v_3, v_4, \ldots \ldots \ldots \ldots \ldots$ are vectors, they can be called linearly independent if $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + \cdots \ldots \ldots \ldots \ldots = 0$; when and only when $c_1, c_2, c_3, c_4, \ldots \ldots \ldots \ldots \ldots \ldots = 0$. In other words, the sequence of vectors $v_1, v_2, v_3, v_4, \ldots \ldots \ldots \ldots \ldots$ is linearly independent if the only combination that gives the zero vector is $0v_1 + 0v_2 + 0v_3 + 0v_4 + \cdots \ldots \ldots \ldots \ldots = 0$.



Here, linear combination of the vectors v_1 and v_2 is

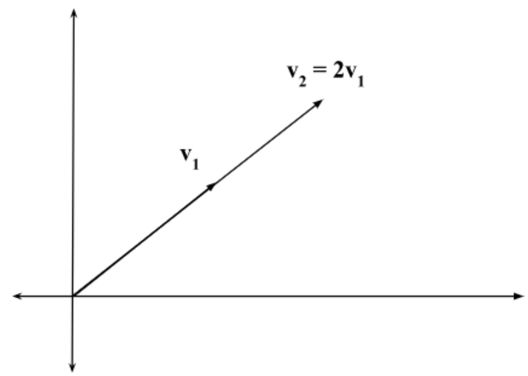
$$c_1v_1 + c_2v_2; where\ c_1, c_2 \in \mathbb{R}$$

Here, $c_1v_1 + c_2v_2 \neq 0$ if $c_1, c_2 \neq 0$.

i.e. the linear combination of v_1 and v_2 will never be zero unless the values of both c_1 and c_2 are zero.

So, the vectors v_1 and v_2 are **INDEPENDENT.**

Figure 1:



Here, linear combination of the vectors v_1 and v_2 is

$$c_1v_1 + c_2v_2; where\ c_1, c_2 \in \mathbb{R}$$

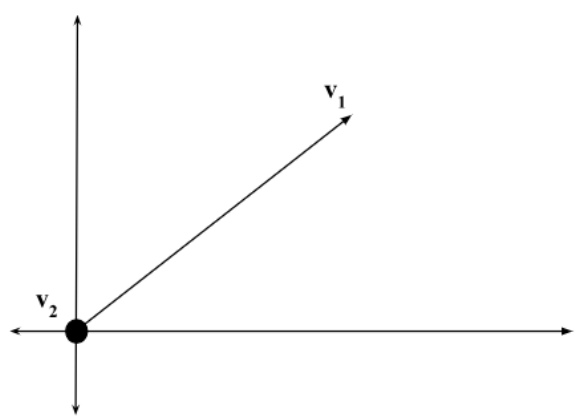
Let, $c_1 = -2$ and $c_2 = 1$,

$$\begin{aligned} \text{So, } c_1v_1 + c_2v_2 &= -2v_1 + 1 \times v_2 \\ &= -2v_1 + 2v_1 = 0 \end{aligned}$$

As, along with 0, other values c_1 and c_2 are also resulting in the linear combination of v_1 and v_2 in figure 1 to give the result 0,

So, the vectors v_1 and v_2 are **DEPENDENT.**

Figure 2:



Similarly, in figure 2, for $c_1 = 0$ and $c_2 = any\ value$, will result in the linear combination of v_1 and v_2 to become zero.

So, the vectors v_1 and v_2 are **DEPENDENT.**

Note: If the number of vectors is greater than the dimention of the vector space they are on, the vectors will always be **DEPENDENT.**

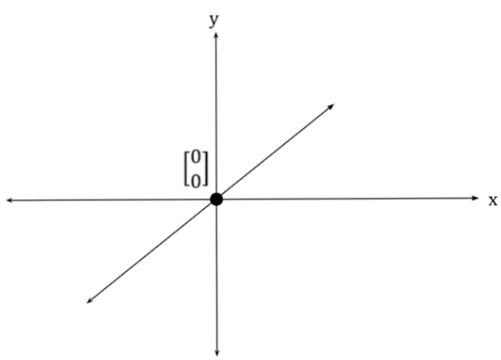
For example:
In a 2D system, if there are 3 or more vectors, they are always Dependent.
In a 3D system, if there are 4 or more vectors, they are always Dependent.
.
.
.
And so on………

VECCTOR SPACES

We know,
$\mathbb{R}^2$ is 2D real vector space over $\mathbb{R}$.
$\mathbb{R}^3$ is 3D real vector space over $\mathbb{R}$.

Subspaces of $\mathbb{R}^2$:
There are ***three types*** of subspaces of $\mathbb{R}^2$.

1. Lines through $\begin{bmatrix}0\\0\end{bmatrix}$ or origin.

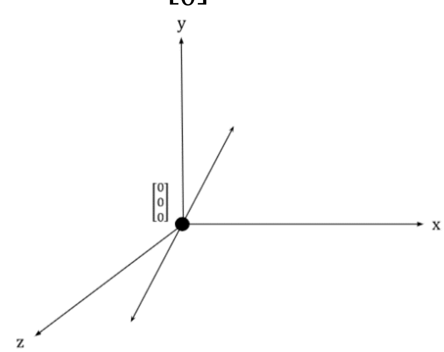


2. The point $\begin{bmatrix}0\\0\end{bmatrix}$ or origin
3. $\mathbb{R}^2$ itself.

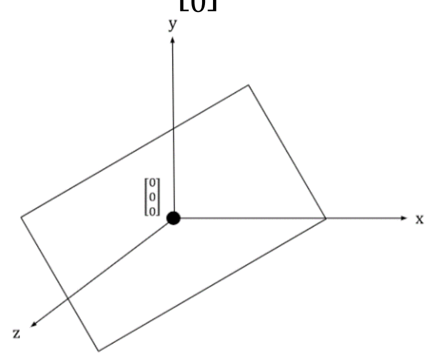
N.B.: Both $\mathbb{R}^2$ and $\mathbb{R}^3$ has different types of subspaces. But both has **infinite NUMBER** of subspaces.

Subspaces of $\mathbb{R}^3$:
There are ***four types*** of subspaces of $\mathbb{R}^3$.

1. Line through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ or origin.



2. Planes through $\begin{bmatrix}0\\0\\0\end{bmatrix}$ or origin.

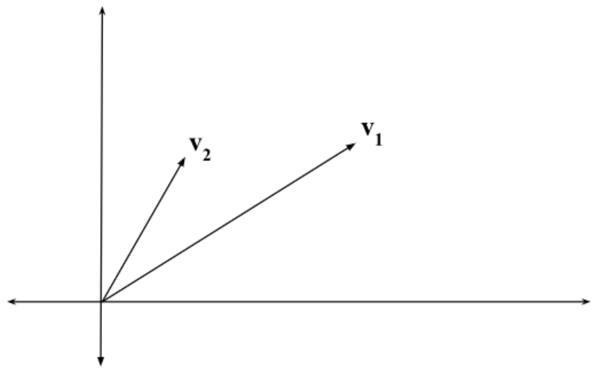


3. The point $\begin{bmatrix}0\\0\\0\end{bmatrix}$ or origin.
4. $\mathbb{R}^3$ itself.

SPANNING OF VECTORS

Spanning is nothing but rendering/creating a vector space by the linear combination of vectors, which are linearly independent.

For example:



Here, the vectors v_1 and v_2 is spanning a vector space $\mathbb{R}^2$.

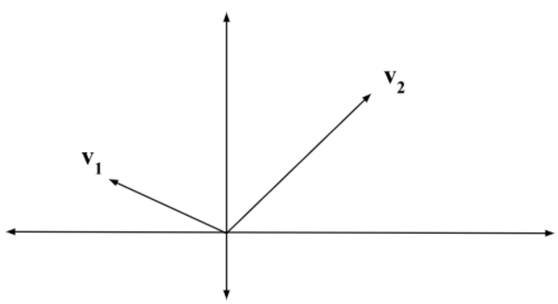
i.e. $av_1 + bv_2 \rightarrow \mathbb{R}^2$; if v_1 and v_2 are linearly independent.

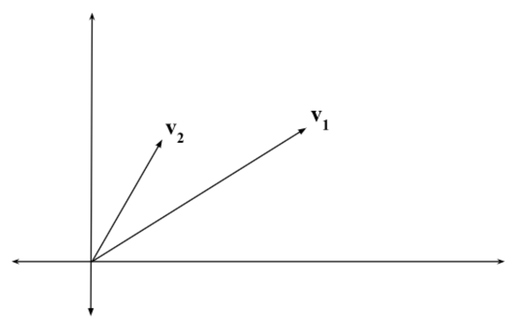
BASIS OF A VECTOR SPACE

The set of linearly independent vectors that spans the vector space is called basis of the vector space. Basis vectors are always independent.

An important feature of Basis is that "Basis is not unique."

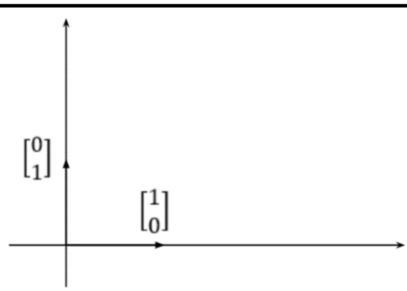
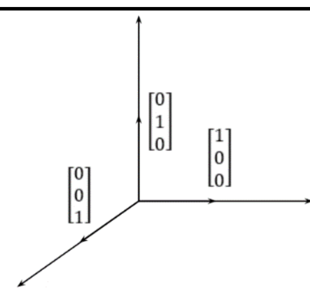
Proof:





Here, both the figures spans a $\mathbb{R}^2$ each. That is why, basis is not unique.

Important Information

 $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \rightarrow standard\ basis\ of\ \mathbb{R}^2$	 $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \rightarrow standard\ basis\ of\ \mathbb{R}^3$

Example 12: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 1 & 8 \\ 3 & 2 & 5 \end{bmatrix}$; find the basis of left null space of A or N(A^T).

Soln.: Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 1 & 8 \\ 3 & 2 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 1 & 8 \\ 3 & 2 & 5 \end{bmatrix}$$

Augmented Matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 7 & 1 & 8 \\ 3 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -13 & -4 \\ 0 & -13 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -13 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, x_3 is the free variable.
Let, $x_3 = 2$

Back Substiturtion:

$$-13x_2 - 4x_3 = 0$$
$$or, x_2 = -\frac{8}{13}$$

$$and, x_1 + 2x_2 + 3x_3 = 0$$
$$or, x_1 = -\frac{22}{13}$$

Therefore, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c \begin{bmatrix} -\frac{22}{13} \\ -\frac{8}{13} \\ 2 \end{bmatrix}$

So, the basis of N(A^T) = $\begin{bmatrix} -\frac{22}{13} \\ -\frac{8}{13} \\ 2 \end{bmatrix}$

(Ans.)

ORTHOGONALITY

Two subspaces V and W of a vector space are *orthogonal* if every vector v in V is perpendicular to every vector w in W:

Orthogonal Subspaces: $v^T w = 0$ $\boldsymbol{for\ all\ v\ in\ V\ and\ all\ w\ in\ W.}$

Let, A and B are two 3D vectors:

$$A = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$B = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$
$$\therefore A \cdot B = A_xB_x + A_yB_y + C_yD_y$$

But if $A = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$ and $B = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$, multiplication is not possible. This is because, A and B both are 3×1 matrix. So, while multiplication, their dimensions will not agree.

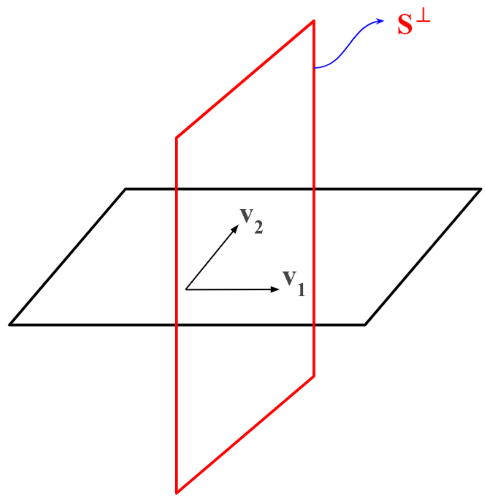
That is why, Dot multiplication of $A = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$ and $B = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$ is executed through orthogonality.

i.e., $A \cdot B = A^T \cdot B = [A_x \quad A_y \quad A_z] \cdot \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = A_xB_x + A_yB_y + C_yD_y$.

Now, let's apply this concept of orthogonality in some problems.

Example 13: If $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 8 \\ 9 \\ 0 \end{bmatrix}$ span a subspace S, then find the orthogonal subspace or $S^{\perp}$.

Soln.: $S^{\perp}$ is perpendicular on S.



Let, $\bar{X}$ is in $S^{\perp}$ and, $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

As, $S \perp S^{\perp}$, then $\bar{X} \cdot \overline{v_1} = 0$ **[Orthogonality]**

$$or, {v_1}^T \cdot X = 0$$

$$or, [1 \quad 2 \quad 3 \quad 4] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \dots\dots\dots\dots\dots\dots\dots\dots (i)$$

And, $v_2{}^T \cdot X = 0$

$$or, [7 \quad 8 \quad 9 \quad 0] \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \dots\dots\dots\dots\dots\dots\dots\dots (ii)$$

From equations (i) and (ii),

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 7 & 8 & 9 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

<u>Augmented Matrix</u>:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 7 & 8 & 9 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -6 & -12 & -28 \end{bmatrix}$$

$\therefore$ x_3 and x_4 are the free variables.

Let, $x_3 = a$ and $x_4 = b$.

<u>Back Substitution</u>:

$$-6x_2 - 12a - 28b = 0$$
$$or, x_2 = -2a - \frac{14}{3}b$$

and, $x_1 + 2x_2 + 3a + 4b = 0$

$$or, x_1 = a + \frac{16}{3}b$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a + \frac{16}{3}b \\ -2a - \frac{14}{3}b \\ a \\ b \end{bmatrix}$$

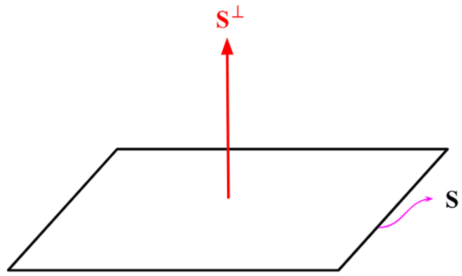
$$= \begin{bmatrix} a \\ -2a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{16}{3}b \\ -\frac{14}{3}b \\ 0 \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} \frac{16}{3} \\ -\frac{14}{3} \\ 0 \\ 1 \end{bmatrix}$$

$$= S^{\perp}$$

(Ans)

Example 13: If $\begin{bmatrix}1\\3\\2\end{bmatrix}$ and $\begin{bmatrix}7\\0\\1\end{bmatrix}$ span a subspace S, then find the orthogonal subspace or $S^{\perp}$.

Soln.: $S^{\perp}$ is perpendicular on S.



Let, $\bar{X}$ is in $S^{\perp}$ and, $\bar{X} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

As, $S \perp S^{\perp}$, then $\bar{X} \cdot \overline{v_1} = 0$ **[Orthogonality]**

$$or, {v_1}^T \cdot X = 0$$

$$or, [1 \quad 3 \quad 2] \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

And, ${v_2}^T \cdot X = 0$

$$or, [7 \quad 0 \quad 1] \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

From equations (i) and (ii),

$$\begin{bmatrix}1 & 3 & 2\\7 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

<u>Augmented Matrix</u>:

$$\begin{bmatrix}1 & 3 & 2\\7 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 3 & 2\\0 & -21 & -13\end{bmatrix}$$

$\therefore$ x_3 is the free variable.

Let, $x_3 = a$

<u>Back Substitution</u>:

$$-21x_2 - 13a = 0$$
$$or, x_2 = -\frac{13}{21}a$$

and, $x_1 + 3\left(-\frac{13}{21}a\right) + 2a = 0$

$$or, x_1 = -\frac{1}{7}a$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7}a \\ -\frac{13}{21}a \\ a \end{bmatrix}$$

$$= a\begin{bmatrix} -\frac{1}{7} \\ -\frac{13}{21} \\ 1 \end{bmatrix}$$

(Ans.)

LINEAR TRANSFORMATION

Linear transformation is a function that transforms a vector space into another vector space satisfying two conditions:

- $T(V+W) = T(V) + T(W)$
- $T(CV) = C \cdot T(V)$

N.B.: In Linear Transformation, if we give input 0, the output will be 0 i.e. the **ORIGIN would NOT CHANGE.**

Linear Transformation is performed by "MATRIX".

For example: In the equation, $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 0 & 1 \\ 2 & 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, the transformation, $T: \mathbb{R}^3 \to \mathbb{R}^3$.

Similarly, in the equation, $\begin{bmatrix} 1 & 2 & 3 \\ 7 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$, the transformation, $T: \mathbb{R}^3 \to \mathbb{R}^2$.

Example 14: Find a matrix A that converts $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ into $\begin{bmatrix} 2x+3y+z \\ x+y+z \\ y+3z \end{bmatrix}$.

Soln.: Given,

$$V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{Transforms\ to} T(V) = \begin{bmatrix} 2x+3y+z \\ x+y+z \\ y+3z \end{bmatrix}$$

Now,

$$V = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{Transforms\ to} T(V) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

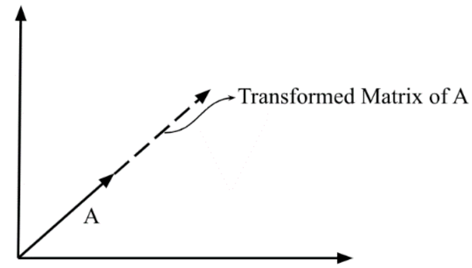
$$V = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{Transforms\ to} T(V) = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \xrightarrow{Transforms\ to} T(V) = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

(Ans.)

Note: If a matrix A undergoes linear transformation, the transformed matrix will be on the same line. We will understand it better seeing the following figure:



<u>Concept of Eigen Values and Eigen Vectors</u>:

Eigenvalue: Eigenvalues are the special set of scalars associated with the system of linear equations. It is mostly used in matrix equations. 'Eigen' is a German word that means 'proper' or 'characteristic'. Therefore, the term eigenvalue can be termed as characteristic value, characteristic root, proper values or latent roots as well. In simple words, the eigenvalue is a scalar that is used to transform the eigenvector. The basic equation is:

$$\boldsymbol{Ax} = \boldsymbol{\lambda}\mathbf{x}$$

The number or scalar value **"λ"** is an eigenvalue of A.

In Mathematics, an eigenvector corresponds to the real non zero eigenvalues which point in the direction stretched by the transformation whereas eigenvalue is considered as a factor by which it is stretched. In case, if the eigenvalue is negative, the direction of the transformation is negative.

For every real matrix, there is an eigenvalue. Sometimes it might be complex. The existence of the eigenvalue for the complex matrices is equal to the fundamental theorem of algebra.

Eigenvector: Eigenvectors are the vectors (non-zero) that do not change the direction when any linear transformation is applied. It changes by only a scalar factor. In a brief, we can say, if A is a linear transformation from a vector space V and **x** is a vector in V, which is not a zero vector, then v is an eigenvector of A if A(X) is a scalar multiple of **x**.

An **Eigenspace** of vector **x** consists of a set of all eigenvectors with the equivalent eigenvalue collectively with the zero vector. Though, the zero vector is not an eigenvector.

Let us say A is an "n × n" matrix and λ is an eigenvalue of matrix A, then **x**, a non-zero vector, is called as eigenvector if it satisfies the given below expression:

$$\boldsymbol{Ax} = \boldsymbol{\lambda}\mathbf{x}$$

"x" is an eigenvector of A corresponding to eigenvalue, λ.

Note:
- There could be infinitely many Eigenvectors, corresponding to one eigenvalue.
- For distinct eigenvalues, the eigenvectors are linearly dependent.

We will use these concepts to solve various types of maths moving forward.

Now, let's see some mathematical expressions using these concepts.

$$We\ previously\ got,$$
$$Ax = \lambda \mathrm{x}$$
$$or, Ax - \lambda \mathrm{x} = 0$$
$$or, (A - \lambda)x = 0\ [\textbf{\textit{Not Possible}}]$$

But the above equation is **not valid**. This is because, A is a MATRIX and λ is a SCALAR VALUE. So, subtraction between them is not possible. So, to make this equation valid, we need to introduce an "Identity Matrix, I".

So,

$$Ax = \lambda \cdot \mathbf{I} \cdot \mathrm{x}$$
$$or, Ax - \lambda \cdot \mathbf{I} \cdot \mathrm{x} = 0$$
$$or, (A - \lambda \mathbf{I})x = 0 \dots\dots\dots\dots\dots (i)$$

Multiplying a scalar value λ with an identity vector I, makes λI a vector as well. Now, the subtraction is possible.

Now, from equation (i), $\det(A - \lambda \mathbf{I}) = 0$.

Example 15: If $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$, find the Eigenvalue and eigenvector.

Soln.: We know,

$$Ax = \lambda \mathrm{x}$$
$$or, Ax = \lambda \cdot \mathrm{I} \cdot \mathrm{x}$$
$$or, (A - \lambda \cdot \mathrm{I})x = 0 \dots\dots\dots\dots\dots (i)$$

$\therefore \det(A - \lambda \cdot \mathrm{I}) = 0$

$or, \det\left(\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$or, \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 3 & -\lambda \end{bmatrix}\right) = 0$

$or, -\lambda(2-\lambda) - 3 = 0$

$or, -2\lambda + \lambda^2 - 3 = 0$

$or, \lambda^2 - 3\lambda + \lambda - 3 = 0$

$or, \lambda(\lambda - 3) + 1(\lambda - 3) = 0$

$or, (\lambda - 3)(\lambda + 1) = 0$

$\therefore\ \lambda = 3, -1$

When $\lambda = 3$,

$(A - 3 \cdot \mathrm{I})x = 0$ $[From\ equation\ (i)]$

$or, \left(\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$or, \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

<u>Augmented Matrix</u>:

$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}$

$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Here, x_2 is the free variable.
Let, $x_2 = a$.

<u>Back Substitution</u>:
$x_1 - a = 0$
$\therefore x_1 = a$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Again, when $\lambda = -1$,

$(A + \mathrm{I})x = 0$ $[From\ equation\ (i)]$

$or, \left(\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$or, \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

<u>Augmented Matrix</u>:

$\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$

Here, x_2 is the free variable.
Let, $x_2 = b$.

<u>Back Substitution</u>:
$3x_1 + b = 0$

$\therefore x_1 = -\frac{b}{3}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$

So,

When $\lambda = 3$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

When $\lambda = -1$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$

(Ans.)

Example 16: If $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 0 & 3 \end{bmatrix}$, find the eigenvalues of A.

Soln.: We know,

$$Ax = \lambda x$$

$$or, Ax = \lambda \cdot I \cdot x$$

$$or, (A - \lambda \cdot I)x = 0 \ldots\ldots\ldots\ldots\ldots (i)$$

$\therefore \det(A - \lambda \cdot I) = 0$

$or, \det\left(\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$

$or, \begin{vmatrix} 1-\lambda & 3 & 2 \\ 0 & 5-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$or, (1-\lambda)(5-\lambda)(3-\lambda) = 0$

$\therefore \lambda = 1, 3, 5$

Note: From Example 16 we can see that, for any **upper or lower triangular matrix**, the values of the **diagonal** are the eigenvalues.

Similarly, if a diagonal matrix is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, its eigenvalues, λ will also be the values of the diagonal, i.e., $\lambda = 1, 3, 5$. This is because, diagonal matrix is also a form of triangular matrix.

Practice Problem

1. If $A = \begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find the eigenvalue and eigenvectors.

We can check invertibility using eigenvalue and eigenvectors. The condition for invertibility is: ***A matrix is invertible if and only if there is no eigen value, $\lambda = 0$; i.e., $\lambda \neq 0$.***

Example 17: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, prove that A is not invertible.

Soln.: Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

From definition,

$\det(A - \lambda \cdot \mathrm{I}) = 0$

$or, \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$

$or, \begin{vmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{vmatrix} = 0$

$or, (1-\lambda)[(5-\lambda)(9-\lambda) - 48] - 2(36 - 4\lambda - 42) + 3(32 - 35 + 7\lambda) = 0$

$or, (1-\lambda)(45 - 5\lambda - 9\lambda + \lambda^2 - 48) + 12 + 8\lambda - 9 + 21\lambda = 0$

$or, (1-\lambda)(\lambda^2 - 14\lambda - 3) + 29\lambda + 3 = 0$

$or, \lambda^2 - 14\lambda - 3 - \lambda^3 + 14\lambda^2 + 3\lambda + 29\lambda + 3 = 0$

$or, -\lambda^3 + 15\lambda^2 + 18\lambda = 0$

$\therefore\ \lambda = 0, -1.12, 16.12$

Here, one of the values of λ is 0.

So, the matrix A is not invertible.

(Ans.)

Eigenspace: The space which is spanned by eigenvectors is called Eigenspace.

Example 18: If $A = \begin{bmatrix} 11 & -4 & -8 \\ 4 & 1 & -4 \\ 8 & -4 & -5 \end{bmatrix}$, the find the eigenspace of A corresponding $\lambda = 3$.

Soln.: From definition,

$Ax = \lambda x$

$or, Ax = \lambda \cdot \mathrm{I} \cdot x$

$or, (A - 3 \cdot \mathrm{I})x = 0$

$or, \left\{\begin{bmatrix} 11 & -4 & -8 \\ 4 & 1 & -4 \\ 8 & -4 & -5 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right\}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$or, \begin{bmatrix} 8 & -4 & -8 \\ 4 & -2 & -4 \\ 8 & -4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Augmented Matrix:

$$\begin{bmatrix} 8 & -4 & -8 \\ 4 & -2 & -4 \\ 8 & -4 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -4 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, x_2 and x_3 are free variables.
Let, x_2 = a and x_3 = b.

Back Substitution:

$$8x_1 - 4a - 8b = 0$$

$$or, x_1 = \frac{1}{2}a + b$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a + b \\ a \\ b \end{bmatrix}$$

$$= a\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen space of $A = span\left\{\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$

(Ans.)

$\therefore$ Basis of $A = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(Ans.)

DIAGONALIZATION

The conversion of a matrix into diagonal form is called **diagonalization**. The eigenvalues of a matrix are clearly represented by diagonal matrices. A Diagonal Matrix is a square matrix in which all of the elements are zero except the principal diagonal elements. 3×3

If there is an invertible n × n matrix **P** and a diagonal matrix **D** such that $\boldsymbol{A = PDP^{-1}}$, then an n × n matrix A is diagonalizable. The diagonal matrix D is generally represented by the eigenvalues of A and P is spanned by the corresponding eigenvectors.

i.e., if A is a 3×3 square matrix, then, $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{bmatrix}$ and $P = span\{v_1, v_2, v_3\}$.

Note: Any vector is diagonalizable if and only if the eigenvalues are unique and unequal. i.e., $\lambda_1 \neq \lambda_2 \neq \lambda_3$.

Example 19: Determine if matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ is diagonalizable. If so, find the matrix P that diagonalizes the given matrix A.

Soln.: From definition,

$Ax = \lambda \mathrm{x}$

$or, Ax = \lambda \cdot \mathrm{I} \cdot \mathrm{x}$

$or, (A - 3 \cdot \mathrm{I})x = 0$

$\det(A - \lambda \cdot \mathrm{I}) = 0$

$$or, \det\left(\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = 0$$

$$or, \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix} = 0$$

$or, (4-\lambda)\{(1-\lambda)(8-\lambda)+6\} + 1\{2(8-\lambda)-12\} + 6\{-2-2(1-\lambda)\} = 0$

$or, (4-\lambda)(8-\lambda-8\lambda+\lambda^2+6) + (16-2\lambda-12) + 6(-2-2+2\lambda) = 0$

$or, (4-\lambda)(14-9\lambda+\lambda^2) + 4 - 2\lambda - 24 + 12\lambda = 0$

$or, 56 - 36\lambda + 4\lambda^2 - 14\lambda + 9\lambda^2 - \lambda^3 - 20 + 10\lambda = 0$

$or, -\lambda^3 + 13\lambda^2 - 40\lambda + 36 = 0$

$or, \lambda^3 - 13\lambda^2 + 40\lambda - 36 = 0$

$or, \lambda^2(\lambda-2) - 11\lambda(\lambda-2) + 18(\lambda-2) = 0$

$or, (\lambda-2)(\lambda^2 - 11\lambda + 18) = 0$

$or, (\lambda-2)(\lambda^2 - 9\lambda - 2\lambda + 18) = 0$

$or, (\lambda-2)\{\lambda(\lambda-9) - 2(\lambda-9)\} = 0$

$or, (\lambda-2)(\lambda-2)(\lambda-9) = 0$

$\therefore\ \lambda = 2, 2, 9$

Here, all the values of λ are not equal.
So, A is **not Diagonalizable**.

(Ans.)

Example 20: Two vectors are given as $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Prove if v is an eigenvector of A. If so, find the corresponding eigenvalues.

Soln.: Given,

$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ and $v = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Now, multiplying *A* and *v*,

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12 + 0 + 6 \\ -6 + 0 + 6 \\ -6 + 0 + 8 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix}$$

$$= 2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

[**Note:** Here, 2 is the eigenvalue and the matrix multiplied with 2 is the eigenvector.]

∴ *v* is the eigenvector of *A*.
And, the eigenvalue is 2.
(Ans.)

End of **Part 1: Linear Algebra**

Part 2: Fourier Analysis

BASIC CONCEPT OF FOURIER ANALYSIS

Most of the phenomena studied in the domain of Engineering and Science are periodic in nature. For instance, current and voltage in an alternating current circuit. These periodic functions could be analyzed into their constituent components (fundamentals and harmonics) by a process called **Fourier analysis**.

FOURIER SERIES

A Fourier series is an expansion of a periodic function *f(x)* in terms of an infinite sum of sines and cosines. Fourier Series makes use of the orthogonality relationships of the sine and cosine functions.
The formula of Fourier series is given as:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin \mathrm{nx})$$

Again, if $f(x)$ is periodic from $(-L)$ to $(+L)$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)$$

Now, from the formula of Fourier series, it is clear that, in order to find the series, we need to first know the values of a_0, a_n and b_n. So, let's find their values.

For a_n:

From Fourier Series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin \mathrm{nx})$$

Taking $\cos nx$ on both sides,

$$\cos nx \;\; f(x) = \cos nx \, a_0 + \cos nx \sum_{n=1}^{\infty} (a_n \cos \mathrm{nx} + b_n \sin \mathrm{nx})$$

Integrating both sides,

$$\int_{-\pi}^{\pi} \cos nx \;\; f(x)\, dx = \int_{-\pi}^{\pi} a_0 \cos nx \, dx + \int_{-\pi}^{\pi} \cos nx \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin \mathrm{nx})\, dx$$

$$or, \int_{-\pi}^{\pi} \cos nx \;\; f(x)\, dx = 0 + a_n \pi$$

$$\boxed{\therefore \;\; \boldsymbol{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx}}$$

For b_n:

From Fourier Series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin \mathrm{nx})$$

Taking $\sin nx$ on both sides,

$$\sin nx\ f(x) = \sin nx\, a_0 + \sin nx \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin \mathrm{nx})$$

Integrating both sides,

$$\int_{-\pi}^{\pi} \sin nx\ f(x)\, dx = \int_{-\pi}^{\pi} a_0 \sin nx\, dx + \int_{-\pi}^{\pi} \sin nx \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin \mathrm{nx})\, dx$$

$$or, \int_{-\pi}^{\pi} \sin nx\ f(x)\, dx = 0 + b_n \pi$$

$$\boxed{\therefore\ \boldsymbol{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx}}$$

For a_0:

From Fourier Series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)$$

Taking $\cos 0x$ on both sides,

$$\cos 0x\ f(x) = \cos 0x\, a_0 + \cos 0x \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right)$$

Integrating both sides,

$$\int_{-\pi}^{\pi} \cos 0x\ f(x)\, dx = \int_{-\pi}^{\pi} a_0 \cos 0x\, dx + \int_{-\pi}^{\pi} \cos 0x \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}\right) dx$$

$$or, \int_{-\pi}^{\pi} f(x)\, dx = a_0 [x]_{-\pi}^{\pi} + 0$$

$$\boxed{\therefore\ \boldsymbol{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx}}$$

So, in summary, the necessary formulae for any type of Fourier series dealing with REAL numbers are:

Fourier Series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Different parameters to find Fourier Series:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Example 21: If $f(t) = \begin{cases} -1 & ; -\pi < t < 0 \\ +1 & ; \quad 0 < t < \pi \end{cases}$ & $f(t)$ is periodic over 2π, find the Fourier series of $f(t)$.

Soln.: From Euler's Formula,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,dt + \int_{0}^{\pi}(1)\,dt\right]$$

$$= \frac{1}{2\pi}\left[[t]_{-\pi}^{0} + [t]_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}(-\pi + \pi)$$

$$= 0$$

Again, from Euler's Formula,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos nt\,dt + \int_{0}^{\pi}(1)\cos nt\,dt\right]$$

$$= \frac{1}{\pi}\left[-\left[\frac{\sin nt}{n}\right]_{-\pi}^{0} + \left[\frac{\sin nt}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\times 0$$

$$= 0$$

Lastly, from Euler's Formula,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nt\,dt + \int_{0}^{\pi}(1)\sin nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{\cos nt}{n}\right]_{-\pi}^{0} + \left[\frac{\cos nt}{n}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + \frac{1}{n}\right]$$

$$= \frac{2}{n\pi}(1 - \cos n\pi)$$

$$= \frac{2}{n\pi}[1 - (-1)^n]$$

Now, from Fourier Series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

$$= 0 + \sum_{n=1}^{\infty} \left(0 + \frac{2}{n\pi}[1 - (-1)^n] \sin nt\right)$$

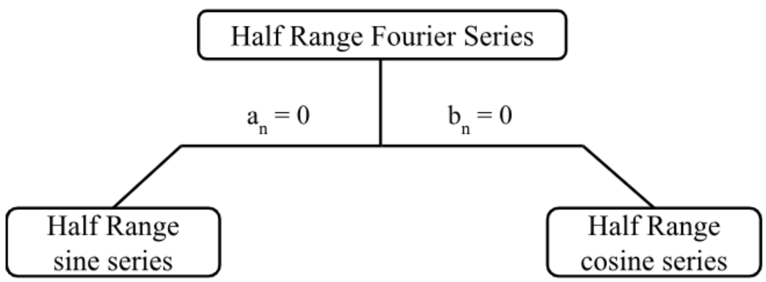
$$= \sum_{n=1}^{\infty} \frac{2}{n\pi}[1 - (-1)^n] \sin nt$$

$$= \frac{2}{\pi} \cdot 2 \sin t + 0 + \frac{2}{3\pi} \cdot 2 \sin 3t + 0 + \frac{2}{5\pi} \cdot 2 \sin 5t + \cdots \ldots \ldots \ldots \ldots \ldots$$

$$= \frac{4}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \cdots \ldots \ldots \ldots \ldots\right)$$

(Ans.)

Note: If a Fourier series contains only sin θ or cos θ, then the series is called "Half Range Fourier Series." We can understand it clearly seeing the chart below:



Example 22: If $f(t) = t - t^2$; & $f(t)$ is periodic over 2π, find the Fourier series of $f(t)$.

Soln.: From Euler's Formula,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} (t - t^2)\, dt$$

$$= \frac{1}{2\pi}\left[\left[\frac{t^2}{2}\right]_{-\pi}^{\pi} - \left[\frac{t^3}{3}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2} - \frac{\pi^3}{3} - \frac{\pi^3}{3}\right]$$

$$= -\frac{\pi^2}{3}$$

Again, from Euler's Formula,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (t - t^2)\cos nt\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Now, [Performing the integration part first]

$$\int (t - t^2)\cos nt\, dt$$

$$= (t - t^2)\int \cos nt\, dt - \int \left[(1 - 2t)\int \cos nt\, dt\right] dt$$

$$= (t - t^2)\frac{\sin nt}{n} - \int \left[(1 - 2t)\frac{\sin nt}{n}\right] dt$$

$$= (t - t^2)\frac{\sin nt}{n} - \frac{1}{n}\left[(1 - 2t)\int \sin nt\, dt - \int \left\{(-2)\int \sin nt\, dt\right\} dt\right]$$

$$= (t - t^2)\frac{\sin nt}{n} - \frac{1}{n}\left[(1 - 2t)\left(-\frac{\cos nt}{n}\right) - \int \left\{(-2)\left(-\frac{\cos nt}{n}\right)\right\} dt\right]$$

$$= (t - t^2)\frac{\sin nt}{n} - \frac{1}{n}\left[-\frac{\cos nt}{n} + \frac{2t\cos nt}{n} - \frac{2\sin nt}{n^2}\right]$$

$$= \frac{t\sin nt}{n} - \frac{t^2\sin nt}{n} + \frac{\cos nt}{n^2} - \frac{2t\cos nt}{n^2} + \frac{2\sin nt}{n^3}$$

From equation (i),

$$a_n = \frac{1}{\pi}\left[\frac{t\sin nt}{n} - \frac{t^2\sin nt}{n} + \frac{\cos nt}{n^2} - \frac{2t\cos nt}{n^2} + \frac{2\sin nt}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{2\pi(-1)^n}{n^2} - \frac{(-1)^n}{n^2} + \frac{2(-\pi)(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi(-1)^n}{n^2}\right]$$

$$= -\frac{4(-1)^n}{n^2}$$

Lastly, from Euler's Formula,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (t - t^2)\sin nt\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Now, [Performing the integration part first]

$$\int (t - t^2)\sin nt\, dt$$

$$= (t - t^2)\int \sin nt\, dt - \int \left[(1 - 2t)\int \sin nt\, dt\right] dt$$

$$= (t - t^2)\left(-\frac{\cos nt}{n}\right) - \int \left[(1-2t)\left(-\frac{\cos nt}{n}\right)\right] dt$$

$$= (t - t^2)\left(-\frac{\cos nt}{n}\right) + \frac{1}{n}\left[(1-2t)\int \cos nt\, dt - \int \left\{(-2)\int \cos nt\, dt\right\} dt\right]$$

$$= (t - t^2)\left(-\frac{\cos nt}{n}\right) + \frac{1}{n}\left[(1-2t)\left(\frac{\sin nt}{n}\right) - \int \left\{(-2)\left(\frac{\sin nt}{n}\right)\right\} dt\right]$$

$$= (t - t^2)\left(-\frac{\cos nt}{n}\right) + \frac{1}{n}\left[\frac{\sin nt}{n} - \frac{2t \sin nt}{n} + \frac{2\cos nt}{n^2}\right]$$

$$= -\frac{t\cos nt}{n} + \frac{t^2 \cos nt}{n} + \frac{\sin nt}{n^2} - \frac{2t \sin nt}{n^2} + \frac{2\cos nt}{n^3}$$

From equation (ii),

$$b_n = \frac{1}{\pi}\left[-\frac{t\cos nt}{n} + \frac{t^2 \cos nt}{n} + \frac{\sin nt}{n^2} - \frac{2t \sin nt}{n^2} + \frac{2\cos nt}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-\pi)(-1)^n}{n} + \frac{\pi^2(-1)^n}{n} + \frac{2(-1)^n}{n^3} - \frac{\pi(-1)^n}{n} - \frac{\pi^2(-1)^n}{n} - \frac{2(-1)^n}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi(-1)^n}{n}\right]$$

$$= -\frac{2(-1)^n}{n}$$

We know, the Fourier series is:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

$$= -\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[-\frac{4(-1)^n}{n^2} - \frac{2(-1)^n}{n}\right]$$

(Ans.)

COMPLEX FORM OF FOURIER SERIES

From MAT215 we know that,

$$e^{i\theta} = \cos\theta + i\sin\theta \ \ldots\ldots\ldots\ldots (i)$$
$$e^{-i\theta} = \cos\theta - i\sin\theta \ \ldots\ldots\ldots\ldots (ii)$$

Adding equation (i) and (ii),

$$e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$
$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$
$$\therefore \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Subtracting equation (i) and (ii),

$$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta$$
$$e^{i\theta} + e^{-i\theta} = 2i\sin\theta$$
$$\therefore \sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i}$$
$$= \frac{-i\left(e^{i\theta} + e^{-i\theta}\right)}{2}$$
$$\therefore \sin\theta = \frac{ie^{-i\theta} - ie^{i\theta}}{2}$$

Now, from the Fourier Series,