Step 6. Replace the column vectors of R that appear in the dependency equations by the corresponding column vectors of A.
This completes the second part of the problem.

Concept Review

- Row vectors
- Column vectors
- Row space
- Column space
- Null space
- General solution
- Particular solution
- Relationships among linear systems and row spaces, column spaces, and null spaces
- Relationships among the row space, column space, and null space of a matrix
- Dependency equations

Skills

- Determine whether a given vector is in the column space of a matrix; if it is, express it as a linear combination of the column vectors of the matrix.
- Find a basis for the null space of a matrix.
- Find a basis for the row space of a matrix.
- Find a basis for the column space of a matrix.
- Find a basis for the span of a set of vectors in R^n.

Exercise Set 4.7

1. List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ 3 & 5 & 7 & -1 \\ 1 & 4 & 2 & 7 \end{bmatrix}$$

Answer:

$\mathbf{r}_1 = (2, -1, 0, 1)$, $\mathbf{r}_2 = (3, 5, 7, -1)$, $\mathbf{r}_3 = (1, 4, 2, 7)$;

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -1 \\ 5 \\ 4 \end{bmatrix}, \mathbf{c}_3 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}, \mathbf{c}_4 = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$$

2. Express the product $A\mathbf{x}$ as a linear combination of the column vectors of A.

(a) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 0 & -1 \\ 3 & 6 & 2 \\ 0 & -1 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 6 & 2 \\ 5 & -4 & 0 \\ 2 & 3 & -1 \\ 1 & 8 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 & 5 \\ 6 & 3 & -8 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -5 \end{bmatrix}$

3. Determine whether $\mathbf{b}$ is in the column space of A, and if so, express $\mathbf{b}$ as a linear combination of the column vectors of A.

(a) $A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} -2 \\ 10 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -1 & 1 \\ 9 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

(e) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix}$

Answer:

(a) $\begin{bmatrix} -2 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \end{bmatrix}$

(b) $\mathbf{b}$ is not in the column space of A.

(c) $\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + (t-1)\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(e) $\begin{bmatrix} 4 \\ 3 \\ 5 \\ 7 \end{bmatrix} = -26\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 13\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 7\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$

4. Suppose that $x_1=-1, x_2=2, x_3=4, x_4=-3$ is a solution of a nonhomogeneous linear system $A\mathbf{x}=\mathbf{b}$ and that the solution set of the homogeneous system $A\mathbf{x}=\mathbf{0}$ is given by the formulas

$$x_1=-3r+4s, \quad x_2=r-s, \quad x_3=r, \quad x_4=s$$

(a) Find a vector form of the general solution of $A\mathbf{x}=\mathbf{0}$.

(b) Find a vector form of the general solution of $A\mathbf{x}=\mathbf{b}$.

5. In parts (a)–(d), find the vector form of the general solution of the given linear system $A\mathbf{x}=\mathbf{b}$; then use that result to find the vector form of the general solution of $A\mathbf{x}=0$.

(a)
$$\begin{aligned} x_1-3x_2&=1\\ 2x_1-6x_2&=2 \end{aligned}$$

(b)
$$\begin{aligned} x_1+x_2+2x_3&=5\\ x_1\qquad+x_3&=-2\\ 2x_1+x_2+3x_3&=3 \end{aligned}$$

(c)
$$\begin{aligned} x_1-2x_2+x_3+2x_4&=-1\\ 2x_1-4x_2+2x_3+4x_4&=-2\\ -x_1+2x_2-x_3-2x_4&=1\\ 3x_1-6x_2+3x_3+6x_4&=-3 \end{aligned}$$

(d)
$$\begin{aligned} x_1+2x_2-3x_3+x_4&=4\\ -2x_1+x_2+2x_3+x_4&=-1\\ -x_1+3x_2-x_3+2x_4&=3\\ 4x_1-7x_2\qquad-5x_4&=-5 \end{aligned}$$

Answer:

(a) $\begin{bmatrix}1\\0\end{bmatrix}+t\begin{bmatrix}3\\1\end{bmatrix};\ t\begin{bmatrix}3\\1\end{bmatrix}$

(b) $\begin{bmatrix}-2\\7\\0\end{bmatrix}+t\begin{bmatrix}-1\\-1\\1\end{bmatrix};\ t\begin{bmatrix}-1\\-1\\1\end{bmatrix}$

(c) $\begin{bmatrix}-1\\0\\0\\0\end{bmatrix}+r\begin{bmatrix}2\\1\\0\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}-2\\0\\0\\1\end{bmatrix};\ r\begin{bmatrix}2\\1\\0\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}$

(d) $\begin{bmatrix}\frac{6}{5}\\\frac{7}{5}\\0\\0\end{bmatrix}+s\begin{bmatrix}\frac{7}{5}\\\frac{4}{5}\\1\\0\end{bmatrix}+t\begin{bmatrix}\frac{1}{5}\\-\frac{3}{5}\\0\\1\end{bmatrix};\ s\begin{bmatrix}\frac{7}{5}\\\frac{4}{5}\\1\\0\end{bmatrix}+t\begin{bmatrix}\frac{1}{5}\\-\frac{3}{5}\\0\\1\end{bmatrix}$

6. Find a basis for the null space of A.

(a)
$$A=\begin{bmatrix}1&-1&3\\5&-4&-4\\7&-6&2\end{bmatrix}$$

(b)
$$A=\begin{bmatrix}2&0&-1\\4&0&-2\\0&0&0\end{bmatrix}$$

(c) $A=\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$

(d) $A=\begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$

(e) $A=\begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

7. In each part, a matrix in row echelon form is given. By inspection, find bases for the row and column spaces of A.

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Answer:

(a) $\mathbf{r}_1=[1\ 0\ 2],\ \mathbf{r}_2=[0\ 0\ 1],\ \mathbf{c}_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{c}_2=\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(b) $\mathbf{r}_1=[1\ \ -3\ 0\ 0],\ \mathbf{r}_2=[0\ 1\ 0\ 0],\ \mathbf{c}_1=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{c}_2=\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

(c) $\mathbf{r}_1=[1\ 2\ 4\ 5],\ \mathbf{r}_2=[0\ 1\ \ -3\ 0],\ \mathbf{r}_3=[0\ 0\ 1\ \ -3],\ \mathbf{r}_4=[0\ 0\ 0\ 1],$

$\mathbf{c}_1=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{c}_2=\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{c}_3=\begin{bmatrix} 4 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{c}_4=\begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

(d) $\mathbf{r}_1 = [1\,2\ -1\,5]$, $\mathbf{r}_2 = [0\,1\,4\,3]$, $\mathbf{r}_3 = [0\,0\,1\ -7]$, $\mathbf{r}_4 = [0\,0\,0\,1]$

$$\mathbf{c}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix},\ \mathbf{c}_2 = \begin{bmatrix}2\\1\\0\\0\end{bmatrix},\ \mathbf{c}_3 = \begin{bmatrix}-1\\4\\1\\0\end{bmatrix},\ \mathbf{c}_4 = \begin{bmatrix}5\\3\\-7\\1\end{bmatrix}$$

8. For the matrices in Exercise 6, find a basis for the row space of A by reducing the matrix to row echelon form.

9. By inspection, find a basis for the row space and a basis for the column space of each matrix.

(a) $\begin{bmatrix}1 & 0 & 2\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$

(b) $\begin{bmatrix}1 & -3 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix}$

(c) $\begin{bmatrix}1 & 2 & 4 & 5\\0 & 1 & -3 & 0\\0 & 0 & 1 & -3\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{bmatrix}$

(d) $\begin{bmatrix}1 & 2 & -1 & 5\\0 & 1 & 4 & 3\\0 & 0 & 1 & -7\\0 & 0 & 0 & 1\end{bmatrix}$

Answer:

(a)
$\mathbf{r}_1 = [1\ \ 0\ \ 2]$; $\mathbf{r}_2 = [0\ \ 0\ \ 1]$; $\mathbf{c}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$; $\mathbf{c}_2 = \begin{bmatrix}2\\1\\0\end{bmatrix}$

(b)
$\mathbf{r}_1 = [1\ \ -3\ \ 0\ \ 0]$; $\mathbf{r}_2 = [0\ \ 1\ \ 0\ \ 0]$; $\mathbf{c}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$; $\mathbf{c}_2 = \begin{bmatrix}-3\\1\\0\\0\end{bmatrix}$

(c) $\mathbf{r}_1 = [1\ \ 2\ \ 4\ \ 5]$; $\mathbf{r}_2 = [0\ \ 1\ \ -3\ \ 0]$; $\mathbf{r}_3 = [0\ \ 0\ \ 1\ \ -3]$;

$\mathbf{r}_4 = [0\ \ 0\ \ 0\ \ 1]$; $\mathbf{c}_1 = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$; $\mathbf{c}_2 = \begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}$; $\mathbf{c}_3 = \begin{bmatrix}4\\-3\\1\\0\\0\end{bmatrix}$; $\mathbf{c}_4 = \begin{bmatrix}5\\0\\-3\\1\\0\end{bmatrix}$

(d) $\mathbf{r}_1 = [1\ \ 2\ \ -1\ \ 5]$; $\mathbf{r}_2 = [0\ \ 1\ \ 4\ \ 3]$; $\mathbf{r}_3 = [0\ \ 0\ \ 1\ \ -7]$;

$$\mathbf{r}_4 = [0 \quad 0 \quad 0 \quad 1];\ \mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix};\ \mathbf{c}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix};\ \mathbf{c}_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \\ 0 \end{bmatrix};\ \mathbf{c}_4 = \begin{bmatrix} 5 \\ 3 \\ -7 \\ 1 \end{bmatrix}$$

10. For the matrices in Exercise 6, find a basis for the row space of *A* consisting entirely of row vectors of *A*.

11. Find a basis for the subspace of R^4 spanned by the given vectors.

(a) $(1, 1, -4, -3), (2, 0, 2, -2), (2, -1, 3, 2)$

(b) $(-1, 1, -2, 0), (3, 3, 6, 0), (9, 0, 0, 3)$

(c) $(1, 1, 0, 0),\ (0, 0, 1, 1),\ (-2, 0, 2, 2),\ (0, -3, 0, 3)$

Answer:

(a) $(1, 1, -4\ -3),\ (0, 1, -5, -2),\ \left(0, 0, 1, -\frac{1}{2}\right)$

(b) $(1, -1, 2, 0),\ (0, 1, 0, 0),\ \left(0, 0, 1, -\frac{1}{6}\right)$

(c) (1, 1, 0, 0), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)

12. Find a subset of the vectors that forms a basis for the space spanned by the vectors; then express each vector that is not in the basis as a linear combination of the basis vectors.

(a) $\mathbf{v}_1 = (1, 0, 1, 1),\ \mathbf{v}_2 = (-3, 3, 7, 1),\ \mathbf{v}_3 = (-1, 3, 9, 3),\ \mathbf{v}_4 = (-5, 3, 5, -1)$

(b) $\mathbf{v}_1 = (1, -2, 0, 3),\ \mathbf{v}_2 = (2, -4, 0, 6),\ \mathbf{v}_3 = (-1, 1, 2, 0),\ \mathbf{v}_4 = (0, -1, 2, 3)$

(c) $\mathbf{v}_1 = (1, -1, 5, 2),\ \mathbf{v}_2 = (-2, 3, 1, 0),\ \mathbf{v}_3 = (4, -5, 9, 4),\ \mathbf{v}_4 = (0, 4, 2, -3), \mathbf{v}_5 = (-7, 18, 2, -8)$

13. Prove that the row vectors of an $n \times n$ invertible matrix *A* form a basis for R^n.

14. Construct a matrix whose null space consists of all linear combinations of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 4 \end{bmatrix}$$

15. (a) Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Show that relative to an xyz-coordinate system in 3-space the null space of *A* consists of all points on the *z*-axis and that the column space consists of all points in the *xy*-plane (see the accompanying figure).

(b) Find a 3×3 matrix whose null space is the *x*-axis and whose column space is the *yz*-plane.

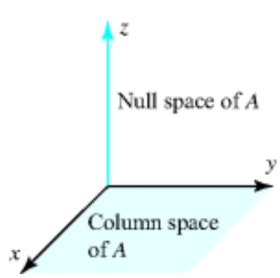

Figure Ex-15

Answer:

(b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

16. Find a 3×3 matrix whose null space is

(a) a point.

(b) a line.

(c) a plane.

17. (a) Find all 2×2 matrices whose null space is the line $3x - 5y = 0$.

(b) Sketch the null spaces of the following matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix},$$
$$C = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Answer:

(a) $\begin{bmatrix} 3a & -5a \\ 3b & -5b \end{bmatrix}$ for all real numbers *a*, *b* not both 0.

(b) Since *A* and *B* are invertible, their null spaces are the origin. The null space of *C* is the line $3x + y = 0$. The null space of *D* is the entire *xy*-plane.

18. The equation $x_1 + x_2 + x_3 = 1$ can be viewed as a linear system of one equation in three unknowns. Express its general solution as a particular solution plus the general solution of the corresponding homogeneous system. [*Suggestion:* Write the vectors in column form.]

19. Suppose that *A* and *B* are $n \times n$ matrices and *A* is invertible.Invent and prove a theorem that describes how the row spaces of AB and *B* are related.

True-False Exercises

In parts (a)–(j) determine whether the statement is true or false, and justify your answer.

(a) The span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the column space of the matrix whose column vectors are $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Answer:

True

(b) The column space of a matrix *A* is the set of solutions of $A\mathbf{x} = \mathbf{b}$.

Answer:

False

(c) If *R* is the reduced row echelon form of *A*, then those column vectors of *R* that contain the leading 1′s form a basis for the column space of *A*.

Answer:

False

(d) The set of nonzero row vectors of a matrix *A* is a basis for the row space of *A*.

Answer:

False

(e) If *A* and *B* are $n \times n$ matrices that have the same row space, then *A* and *B* have the same column space.

Answer:

False

(f) If *E* is an $m \times m$ elementary matrix and *A* is an $m \times n$ matrix, then the null space of *E A* is the same as the null space of *A*.

Answer:

True

(g) If *E* is an $m \times m$ elementary matrix and *A* is an $m \times n$ matrix, then the row space of *E A* is the same as the row space of *A*.

Answer:

True

(h) If *E* is an $m \times m$ elementary matrix and *A* is an $m \times n$ matrix, then the column space of *E A* is the same as the column space of *A*.

Answer:

False

(i) The system $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if $\mathbf{b}$ is not in the column space of *A*.

Answer:

True

(j) There is an invertible matrix *A* and a singular matrix *B* such that the row spaces of *A* and *B* are the same.

Answer:

False