Lecture note

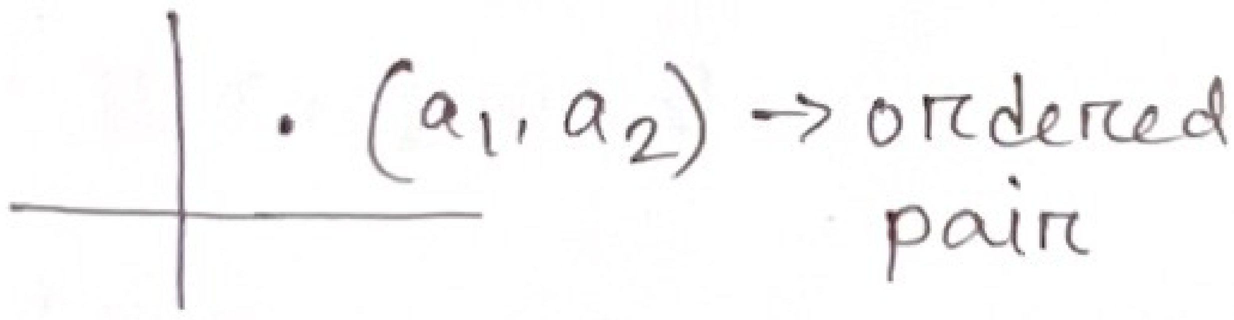
Vector space and subspace

Vectors in R^n:

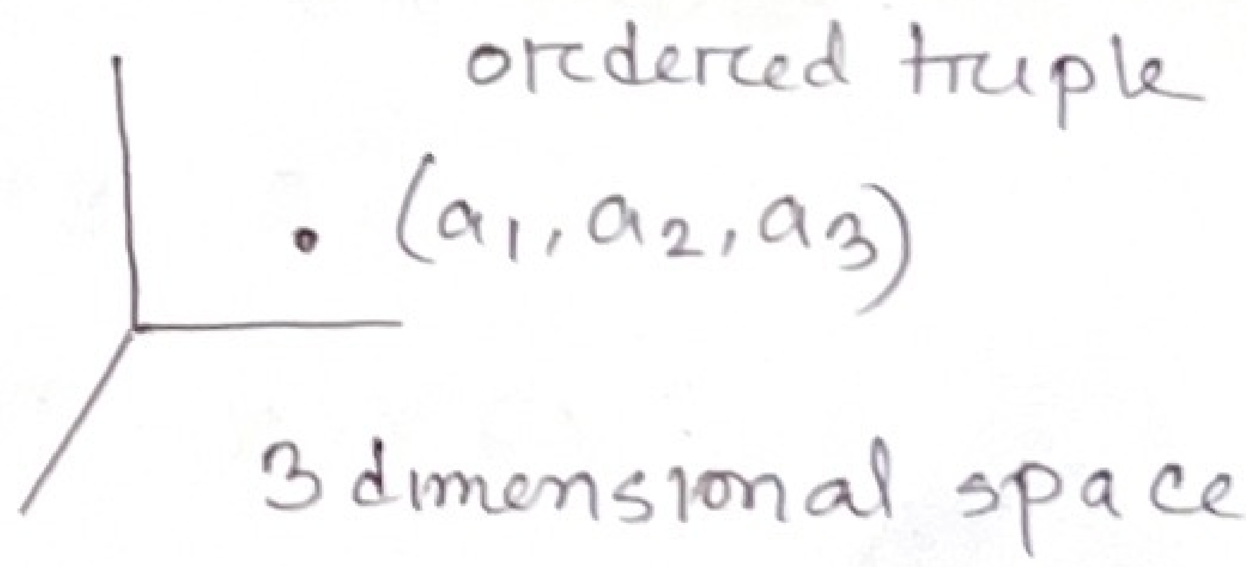
If n is a positive integer then an order n-tuple $(a_1, a_2, \dots, a_n) = \underline{a}$ is a sequence of n real numbers. Then the set of all ordered n-tuples is called n-space and is denoted R^n that

$$R^n = \{\underline{a} = (a_1, a_2, \dots \dots, a_n) ;\ a_i \in R,\ 1 \le i \le n\}$$

* The space R^n consists of all column vectors v with n-components.
* Space means the whole plane.
* $\mathbb{R}^3$ = all 3-D vectors, all vectors with three components.
* $\mathbb{R}^n$ = all vectors with n-components.



2D-space



3 dimensional space.

Example: $\begin{bmatrix} 4 \\ \pi \end{bmatrix}$ in R^2, $(1, 1, 0, 1, 1)$ is in R^5.

* We can add any vectors in R^n, and we can multiply any vector v by any scalar c.

Vector space: Let V be a non-empty set on which two operations vector addition and scalar multiplication have been defined. Then V is called a vector space over a field F (real or complex) if the following properties are true.

(i) For any $\underline{u}$ and $\underline{v}$ in V, $\underline{u}+\underline{v} \in V$ [vector addition]

(ii) $\alpha u \in V$ for any $\underline{v} \in V$ and any scalar $\alpha \in F$ [scalar multiplication]

(iii) $\underline{u}+\underline{v} = (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n)$

$$= (u_1+v_1, u_2+v_2, \dots, u_n+v_n)$$

$$= (v_1+u_1, v_2+u_2, \dots, v_n+u_n)$$

$$= \underline{v}+\underline{u} \quad \text{[commutative law]}$$

(iv) $\underline{u}+(\underline{v}+\underline{\omega}) = (\underline{u}+\underline{v})+\underline{\omega} \quad \forall \underline{u}, \underline{v}, \underline{\omega} \in V$ [Associative property]

(v) There is a vector in V denoted by $\underline{0}$, called the zero vector such that $\underline{u}+\underline{0} = \underline{0}+\underline{u} = \underline{u}$, $\forall u \in V$

(vi) For any $\underline{u} \in V$ there is a vector $-\underline{u} \in V$ called the negative of $\underline{u}$ such that $\underline{u}+(-\underline{u}) = (-\underline{u})+\underline{u} = \underline{0}$.

(vii) $\alpha(\underline{u}+\underline{v}) = \alpha\underline{u}+\alpha\underline{v}$, $\alpha \in F$ scalar, $\underline{u}, \underline{v} \in V$

(viii) $(\alpha+\beta)\underline{u} = \alpha\underline{u}+\beta\underline{u}$ for any scalar $\alpha, \beta \in F$ and $\underline{u} \in V$

(ix) $(\alpha\beta)\underline{u} = \alpha(\beta\underline{u})$

(x) $1 \cdot \underline{u} = \underline{u}$ for every vector $\underline{u} \in V$ where $1 \in F$ is called the unit scalar.

The elements of V are called vectors and the elements of F are called scalars.

* Vector spaces are also called linear spaces.

* Vector space V over an arbitrary field F is sometimes written as $V(F)$.

Example:

(i) $V = M(m, n; F)$, $F = \mathbb{R}$ and $\mathbb{C}$

$= \{m \times n \text{ real or complex matrix}\}$.

(ii) Zero vector space: $V = \{0\}$

(iii) let $F = \mathbb{R}$, set of real numbers. let P_n be the set of all polynomials of degree at most n over $\mathbb{R}$. Then $P_n(\mathbb{R})$ is a vector space.

$$P_n(\mathbb{R}) = \{P(x) = a_0x^0 + a_1x^1 + \cdots + a_nx^n\}, a_i \in \mathbb{R}, 0 \le i \le n\}$$

x is known as indeterminate.

Subspace: let V be a vector space over a field F. let W be a non-empty subset of V. Then W is called a subspace of V if W is itself a vector space over F w.r.t the operations vector addition and scalar multiplication defined on V.

Alternative: A subspace of a vector space is a set of vectors (including $\underline{0}$) that satisfies two requirements: If $\underline{v}$ and $\underline{w}$ are vectors in the subspace and c is any scalar, then

(i) $\underline{v} + \underline{w}$ is in the subspace

(ii) $c\underline{v}$ is in the subspace.

In other words, the set of vectors is "closed" under addition $\underline{v} + \underline{w}$ and multiplication $c\underline{v}$ (and $c\underline{w}$).

In short, all linear combinations stay in the subspace.

Example:

(I) $V = \{n \times n \text{ matrices}\}$

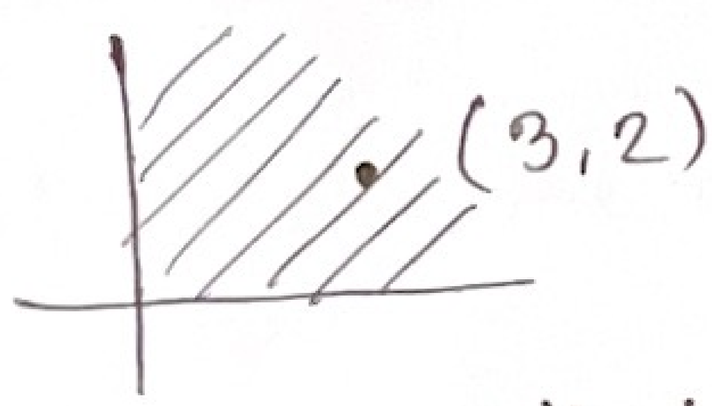
(II) $W = \{A \in M(n, R) : A = A^T\}$ = symmetric matrices.

(III) lines through the origin are also subspace.

(IV) all upper triangular matrices.

(V) all diagonal matrices.

Not a vector space: The quarter plane is not a vector space.



* multiplying by -7 will take me out of the plane.
  $\Rightarrow$ not closed under multiplication.

Not a subspace: line doesn't passes through the origin (doesn't contain zero vector).

Remark: If $A\underline{x} = \underline{0}$ is a homogeneous linear system of m equations in n unknowns, then the set of solution vectors is a subspace of R^n.

Linear combination: Let V be a vector space over a field F, and $v_1, v_2, \dots, v_n \in V$. Then a vector $\underline{w} \in V$ is called a linear combination of the vectors $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$ if $\underline{w}$ can be expressed in the form,

$$\underline{w} = k_1 \underline{v}_1 + k_2 \underline{v}_2 + \dots\dots + k_n \underline{v}_n = \sum_{i=1}^{n} k_i \underline{v}_i \,, \quad k_i \in F,\ i = \overline{1,n}$$

Example: show that $\underline{v} = (9,2,7)$ is a linear combination of $\underline{v}_1$ and $\underline{v}_2$ but $\underline{v}' = (4,-1,8)$ is not a linear combination of $\underline{v}_1$ & v_2 where, $v_1 = \{1,2,-1)$, $v_2 = (6,4,2) \in R^3$.

Solution: $\alpha_1 \alpha_2 \in R$ so,

$$\underline{v} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2$$

$$\Rightarrow (9,2,7) = \alpha_1 (1,2,-1) + \alpha_2 (6,4,2)$$

$$\Downarrow$$

solve the augmented matrix and get $\alpha_1 = -3$ and $\alpha_2 = 2$

$$\therefore \underline{v} = -3\underline{v}_1 + 2\underline{v}_2$$

Verification: $-3\underline{v}_1 + 2\underline{v}_2 = (9,2,7) = \underline{v}$.

Linear span: If $S = \{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$ is a set of vectors in a vector space $V(F)$, then the set on all linear combinations of the vectors in S is called their linear span or space spanned or generated by the vectors in S and denoted by $L(S)$.

$$L(S) = \text{span}(S) = \text{gen}(S) = \text{span}\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\}$$

Equivalent definition: If $S = \{\underline{v}_1, \dots, \underline{v}_n\}$ is a set of vectors in a vector space V, then the subspace W of V consisting of all linear combinations of the vectors in S is called the space spanned by $\underline{v}_1, \dots, \underline{v}_n$ and we say that the vectors $\underline{v}_1, \dots, \underline{v}_n$ span W. so $W = \text{span}(S) = \text{span}\{\underline{v}_1, \dots, \underline{v}_n\}$

Linear dependence and independence: A set of vectors $\{x_k, k=1,2,\dots,n\}$ in a vector space $V(F)$ is said to be linearly independent if whenever,

$$\underline{0} = c_1\underline{x}_1 + c_2\underline{x}_2 + \cdots\cdots + c_n\underline{x}_n = \sum_{i=1}^{n} c_k x_k$$

then $c_1 = c_2 = \cdots\cdots = c_n = 0$

otherwise it is called linearly dependent.

A set S with two or more vectors is linearly dependent iff at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

Basis: Let V be a vector space over a field $F.(=\mathbb{R}$ or $\mathbb{C})$. A subset B of V is called a basis for V if

(i) B is linearly independent

(ii) B spans V.

i.e. every vector in V can be expressed as a linear combination of the vectors in B.

Dimension: The dimension of a non-zero vector space V is the fewest number of linearly independent vectors which span V.

OR, The number of elements in a basis is known the dimension of a vector space $V(\mathbb{F})$.

Standard basis vector for $\mathbb{R}^n$: $\underline{e}_1 = (1,0,\dots,0)$, $\underline{e}_2 = (0,1,\dots,0)$

$e_3 = (0,0,1,\dots 0), \dots, e_n = (0,\dots,1)$.

$\therefore S = \{e_1, e_2, \dots, e_n\}$ forms a basis for $V(\mathbb{R}) = \mathbb{R}^n$.

<u>Row space, column space and null space:</u>

If A is an $m \times n$ matrix, then the subspace of F^n
spanned by the row vectors of A is called the row space
of A. And the subspace of F^m spanned by the column
vectors of A is known as the column space of A.

The solution space of the homogeneous system of linear
equations $A\underline{x} = \underline{0}$ $(x \in F^n)$, is a subspace of F^n is
called the null space of A.

<u>Rank and Nulity:</u> If A is an $m \times n$ matrix, then the
row space of A is a subspace of $\mathbb{R}^n$ and the column
space of A is a subspace of $\mathbb{R}^m$. The common dimension
of the row space and column space of A is called the
rank of A and is denoted as rank(A).

The dimension of nullspace of A is called the nulity
of A and is denoted by mulity (A).

* dim (row space) = row rank * dim(column space) = column rank.

Row space = r for an $m \times n$ matrix

$\therefore$ Column space = r, $\therefore$ Null space = n − r, null space of A^T = m − r.

The Column space

The most important subspaces are tied directly to ~~so~~ a matrix A.

Consider a system, $Ax = b$

(I) $\begin{cases} 2x+3y=5 \\ 2x+3y=7 \end{cases}$ (II) $\begin{cases} 2x+3y=5 \\ 4x+6y=10 \end{cases}$ (III) $\begin{cases} 2x+3y=5 \\ 2x-7y=-5 \end{cases}$

Ignore b first and think about A.

(I) implies $|A|=0$ → not invertible matrix → not solvable for some b

(II) implies $|A|=0$ → not invertible matrix → solvable for others b

(III) implies $|A|\neq 0$ → invertible → solvable.

So, good right sides $\underline{b}$ - the vectors that can be written as A times some vector $\underline{x}$. Those $\underline{b}$'s form the "column space" of A

To get every possible $\underline{b}$, we use every possible $\underline{x}$. So starting with the columns of A, and taking all their linear combinations. This produces the column space of A. It is a vector space made up of column vectors.

C(A) contains not just the n columns of A, but all their combinations $A\underline{x}$.

<u>Definition</u>: The column space consists of all linear combinations of the columns. The combinations are all possible vectors Ax. They fill the column space C(A).

To solve $Ax=b$ is to express b as a combination of

the columns. Equivalently, the system $Ax = b$ is solvable iff b is in the column space of A.

<u>Remember</u>: If A is an $m \times n$ matrix. Its columns have m components. The columns belong to R^m. The column space of A is a subspace of R^m.

<u>Nullspace</u>: The nullspace of A consists of all solutions to $A\underline{x} = \underline{0}$. these vectors $\underline{x}$ ~~are~~ are in R^n. the nullspace containi-ng all solutions of $A\underline{x} = \underline{0}$ is denoted by $N(A)$.

the solution vectors form a subspace. Instead of taking arbitrary value; choose special solution.

the nullspace consists of all combinations of the special solutions.

* When A is an invertible matrix, all variables are pivot variables. The simplest choices for free variables are ones and zeros. Those choices give the special solution.

* With no free variables and pivots in every column, the output from the nullbasis is an empty matrix.

If A is invertible, it RREF is the identity matrix. the nullspace is then Z.

* The true size of A is given by its rank. The rank of A is the number of pivots.