Examples: Subspaces

Let S and T be two sets. By $S \cap T$, we mean the intersection of S and T, the set of all elements common to S and T. By $S \cup T$, we mean the union of S and T, the set of all elements which are in at least one of S and T.

Theorem: The ~~the~~ intersection of two subspaces S and T of a vector space V is also a subspace of V.

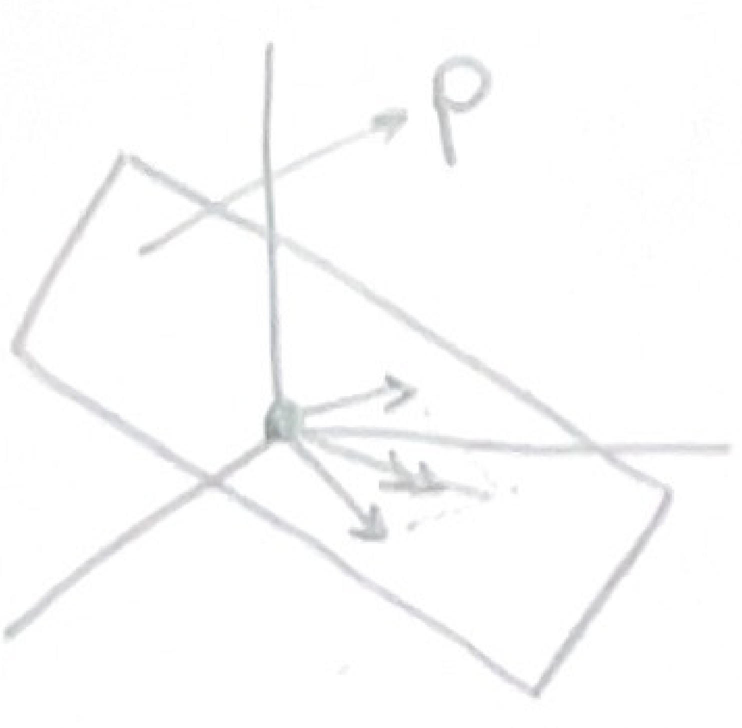
Proof: Since S and T are subspaces of V, they are non-empty and clearly $0 \in S$ and $0 \in T$. Therefore, $0 \in S \cap T$ and hence $S \cap T \neq \phi$

Now let $u, v \in S \cap T$ then $u, v \in S$ and $u, v \in T$. Since S and T are subspaces of V, $u, v \in S$ implies that $\alpha u + \beta v \in S$ where $\alpha, \beta \in F$. Similarly $u, v \in T$ implies that $\alpha u + \beta v \in T$ where $\alpha, \beta \in F$.

Hence, $u, v \in S \cap T$ implies that $\alpha u + \beta v \in S \cap T$ for $\alpha, \beta \in F$.

Therefore, $S \cap T$ is a subspace of the vector space V.

Note that the union of S and T is not a subspace of V unless $S \subset T$ or $T \subset S$.

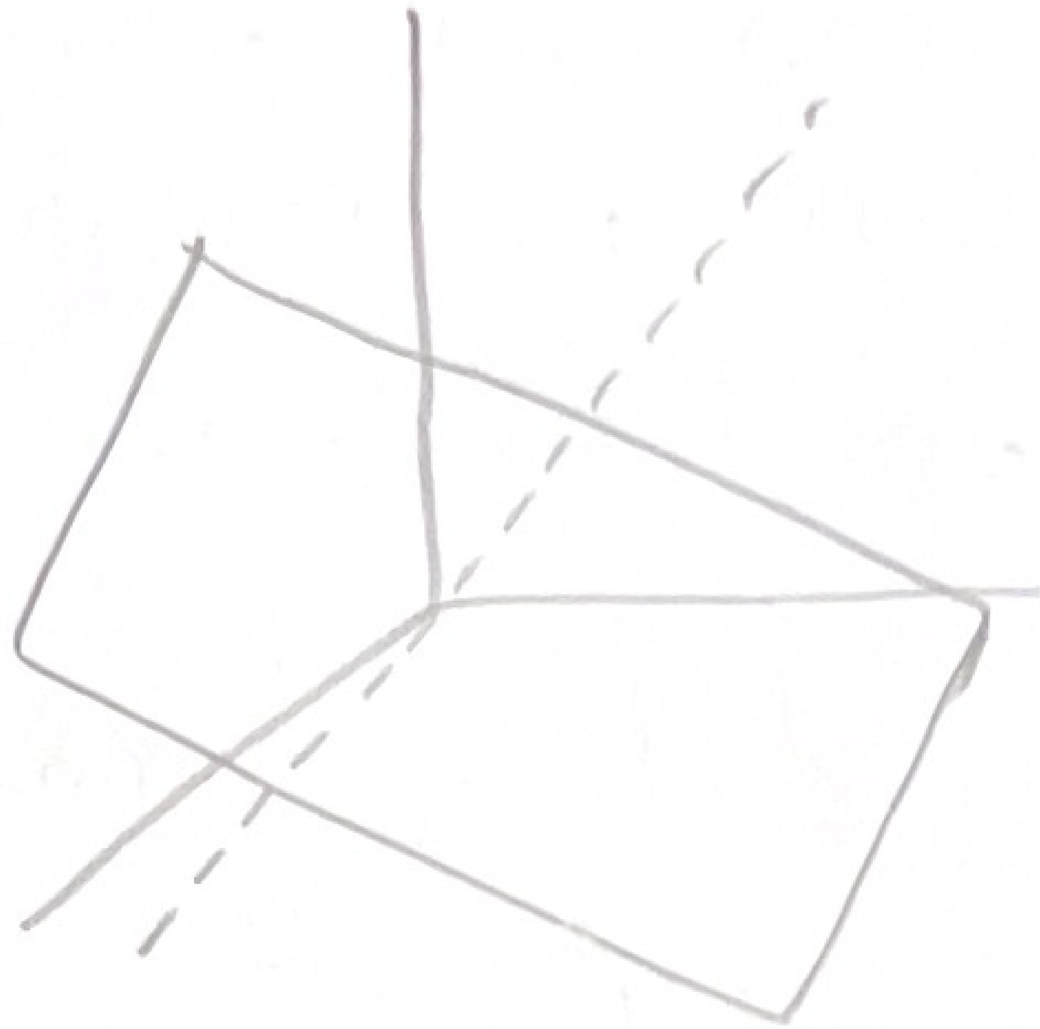


① two vectors on the plane, result is on the plane
② multiplied by a negative of a vector is on the plane

Plane passes through the origin ⇒ is a subspace of R^3

a line passes through (0,0,0)
→ Another subspace of R^3

Take two subspaces and put them together. Take union.



$P \cup L$ = all vectors in P or L or both.

Take the vectors in the plane and also the vectors on that line, put them together ⇒ addition goes somewhere else not in plane or not on the line.

(Remember most of the vectors are not on the line or in that plane

⇒ not a subspace. $P \cup L$.

$P \cap L$ = all vectors in P and L. In picture, it's only zero vector ⇒ subspace because zero vector by itself is a subspace.

Remark: Vector space requirements.

① $v+w$ and cv are in the space.

* Show that $V = \{(a, b, c, d) \in \mathbb{R}^4 : 2a - 3b + 5c - d\}$ is a subspace of $\mathbb{R}^4$.

Solution: For $\underline{0} \in \mathbb{R}^4$, $\underline{0} = (0, 0, 0, 0) \in V$ since $2 \cdot 0 - 3 \cdot 0 + 5 \cdot 0 - 0 = 0$

Hence V is non-empty.

Let $\underline{u} = (a, b, c, d)$ and $\underline{v} = (a_1, b_1, c_1, d_1) \in V$

So, $2a - 3b + 5c - d = 0$ and $2a_1 - 3b_1 + 5c_1 - d_1 = 0$

For any scalars $\alpha, \beta \in \mathbb{R}$

$$\alpha u + \beta v = \alpha(a, b, c, d) + \beta(a_1, b_1, c_1, d_1)$$
$$= (\alpha a + \beta a_1, \alpha b + \beta b_1, \alpha c + \beta c_1, \alpha d + \beta d_1)$$

Also, $2(\alpha a + \beta a_1) - 3(\alpha b + \beta b_1) + 5(\alpha c + \beta c_1) - (\alpha d + \beta d_1)$

$$= (2\alpha a - 3\alpha b + 5\alpha c - \alpha d) + (2\beta a_1 - 3\beta b_1 + 5\beta c_1 - \beta d_1)$$
$$= \alpha \cdot 0 + \beta \cdot 0$$
$$= 0$$

Therefore, $\alpha u + \beta v \in V$ and so V is a subspace of $\mathbb{R}^4$.

* $V = \{(a, b, c) \mid a, b, c \in \mathbb{R}$ and $a - 2b + 3c = 5\}$ is not a subspace of $\mathbb{R}^3$.

Solution: $0 \in \mathbb{R}^3$, $\underline{0} = (0, 0, 0) \notin V$, since $0 - 2 \cdot 0 + 3 \cdot 0 = 0 \neq 5$.

So, V is not a subspace of $\mathbb{R}^3$.

* Let V be a vector space of all 2×2 matrices over the real field $\mathbb{R}$. Show that W is not a subspace of V where

(i) W consists of all matrices with zero determinant

(ii) W consists of all matrices A for which $A^2 = A$.

Solution: Take $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Since, $|A| = |B| = 0$, $\cancel{\in}$ so $A, B \in W$.

Now, $A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $|A+B| = -1 \neq 0 \notin W$.

Hence, W is not a subspace of V.

(ii) Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ where $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \cancel{0}1 \end{pmatrix}$

$\therefore A^2 = A$, $A \in W$.

Now, $2A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ so, $(2A)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

$\neq 2A$.

Hence, W is not a subspace of V.

Example: The vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$ because $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$.

The vectors in $\mathbb{R}^3$ all have three entries, whereas the vectors in $\mathbb{R}^2$ have only two.

* The set $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$ is a subset of $\mathbb{R}^3$. Show that H is a subspace of $\mathbb{R}^3$.

Solution: The zero vector is in H, H is closed under vector addition and scalar multiplication because these operations on vectors in H always produce vectors whose third entries are zero (and so belong to H). Thus H is a subspace of $\mathbb{R}^3$.

Linear combination

* Let $\underline{u} = (1, 2, -1)$ and $\underline{v} = (6, 4, 2)$. Show that $w = (9, 2, 7)$ is a linear combination of $\underline{u}$ and $\underline{v}$.

Solution: For any two scalars k_1 and k_2, we get

$$(9, 2, 7) = k_1 \underline{u} + k_2 \underline{v} = k_1 (1, 2, -1) + k_2 (6, 4, 2)$$

$$\Rightarrow (9, 2, 7) = (k_1, 2k_1, -k_1) + (6k_2, 4k_2, 2k_2)$$

$$\Rightarrow (9, 2, 7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$$\therefore \quad k_1 + 6k_2 = 9$$
$$2k_2 + 4k_2 = 2$$
$$-k_1 + 2k_2 = 7.$$

The augmented matrix is,

$$\left[\begin{array}{cc:c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 6 & 9 \\ 0 & -8 & -16 \\ 0 & 8 & 16 \end{array}\right] \quad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_1 + R_3 \end{array}$$

$$\begin{array}{l} R_2' = -\frac{1}{8} \times R_2 \\ \sim \\ R_3' = \frac{1}{8} \times R_3 \end{array} \left[\begin{array}{cc:c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 6 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \quad R_3' = R_3 - R_2$$

So we get, $k_1 + 6k_2 = 9 \Rightarrow k_1 = 9 - 6 \times 2 = \cancel{10} \; -3.$

$k_2 = 2$

So, $(9, 2, 7) = -3\underline{u} + 2\underline{v}$. (showed)

Check: $-3\underline{u} + 2\underline{v} = -3(1, 2, -1) + 2(6, 4, 2)$

$= (-3, -6, 3) + (12, 8, 4) = (-3+12, -6+8, 3+4)$

$= (9, 2, 7)$

Linear span

* Given $\underline{v}_1$ and $\underline{v}_2$ in a vector space V, let $H = \text{Span}\{\underline{v}_1, \underline{v}_2\}$. Show that H is a subspace of V.

<u>Solution:</u> The zero vector is in H. since $\underline{0} = 0.\underline{v}_1 + 0.\underline{v}_2$.

To show that H is closed under vector addition, take two arbitrary vectors in H, say

$$\underline{u} = s_1\underline{v}_1 + s_2\underline{v}_2 \text{ and } \underline{w} = t_1\underline{v}_1 + t_2\underline{v}_2$$

By axioms, for the vector space V

$$\underline{u} + \underline{w} = (s_1\underline{v}_1 + s_2\underline{v}_2) + (t_1\underline{v}_1 + t_2\underline{v}_2)$$

$$= (s_1 + t_1)\underline{v}_1 + (s_2 + t_2)\underline{v}_2$$

So, $\underline{u} + \underline{w}$ is in H. Furthermore, if c is any scalar, then

$$c\underline{u} = c(s_1\underline{v}_1 + t_1\underline{v}_2) = (cs_1)\underline{v}_1 + (cs_2)\underline{v}_2.$$

which shows that $c\underline{u}$ is in H is closed under scalar multiplication. Thus H is a subspace of V.

<u>Theorem:</u> If $\underline{v}_1, \underline{v}_2, \ldots\ldots, \underline{v}_p$ are in a vector space V, then span $\{\underline{v}_1, \underline{v}_2, \ldots \underline{v}_p\}$ is a subspace of V.

*We call span$\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p\}$ the subspace generated or spanned by $\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_p\}$

<u>Example:</u> Let H be the set of all vectors of the form $(a-3b, b-a, a, b)$ where a and b are arbitrary scalars. That is, let $H = \{(a-3b, b-a, a, b) : a, b \in \mathbb{R}\}$. Show that H is a subspace of $\mathbb{R}^4$.

Solution: Write the vectors in H as column vectors. Then an arbitrary vector in H has the form

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\underline{v}_1} + b\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\underline{v}_2}$$

This calculation shows that $H = \text{span}\{\underline{v}_1, \underline{v}_2\}$ where $\underline{v}_1$ & $\underline{v}_2$ are the vectors indicated above. Thus H is a subspace of R^4 by the theorem.

* For what values of h will $\underline{y}$ be in the subspace of R^3 spanned by $\underline{v}_1, \underline{v}_2, \underline{v}_3$ if

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \text{ and } \underline{y} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix}$$

Ans: $h = 5$

Hint: The solution shows that $\underline{y}$ is in span$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ iff $h = 5$.

* Determine whether vectors $\underline{v}_1 = (2, -1, 3)$, $\underline{v}_2 = (4, 1, 2)$ and $\underline{v}_3 = (8, -1, 8)$ span R^3.

Solution: We must determine whether an arbitrary vector $\underline{v} = (a, b, c)$ in R^3 can be expressed as a linear combination.

so, $(a, b, c) = k_1 \underline{v}_1 + k_2 \underline{v}_2 + k_3 \underline{v}_3$.

$\Rightarrow (a, b, c) = k_1(2, -1, 3) + k_2(4, 1, 2) + k_3(8, -1, 8)$

which implies,
$$\begin{aligned} 2k_1 + 4k_2 + 8k_3 &= a \\ -k_1 + k_2 - 8k_3 &= b \\ 3k_1 + 2k_2 + 8k_3 &= c \end{aligned}$$

This problem thus reduces to determining whether or not the system is consistent for all values of a, b, and c. And the system will be consistent if the coefficient matrix is invertible or the determinant is non-zero.

Here, $A = \begin{pmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{pmatrix}$

$|A| = 0$, so, $\underline{v}_1, \underline{v}_2, \underline{v}_3$ do not span R^3.

* Show that the vectors $\underline{u} = (1,2,3)$, $\underline{v} = (0,1,2)$ and $\underline{w} = (0,0,1)$ generate R^3.

Solution: We must determine whether an arbitrary vector $\underline{v}' = (a, b, c)$ in R^3 can be expressed as a linear combination

So, $(a, b, c) = k_1 \underline{u} + k_2 \underline{v} + k_3 \underline{w}$.

$\Rightarrow (a, b, c) = k_1(1,2,3) + k_2(0,1,2) + k_3(0,0,1)$

$\therefore \quad k_1 + 0 \cdot k_2 + 0 \cdot k_3 = a \Rightarrow k_1 = 3$

and $2k_1 + k_2 = b$

and $3k_1 + 2k_2 + k_3 = c$.

Now, $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k_3 \\ k_2 \\ k_1 \end{pmatrix} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$

which is in echelon form. We have, $k_1 = a$, $k_2 = b - 2a$,

$k_3 = \cancel{b - b + 2a}\ c - 3a - 2(b - 2a) = c - 2b + a$.

Thus $\underline{u}, \underline{v}, \underline{w}$ generate (span) R^3.

Linear dependence and Independence.

* Show that the vectors $(2,-1,4)$, $(3,6,2)$ and $(2,10,-4)$ are linearly independent.

Solution: Form the matrix whose rows are the given vectors.

$$\begin{pmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{pmatrix} \sim \begin{pmatrix} 2.0 & 10 & -4 \\ 3 & 6 & 2 \\ 2 & -1 & 4 \end{pmatrix} \xrightarrow{R_1' = \frac{1}{2}R_1} \begin{pmatrix} 1 & 5 & -2 \\ 3 & 6 & 2 \\ 2 & -1 & 4 \end{pmatrix}$$

$$\xrightarrow[R_3' = R_3 - 2R_1]{R_2' = R_3 - 3R_1} \begin{pmatrix} 1 & 5 & -2 \\ 0 & -9 & 8 \\ 0 & -11 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 5 & -2 \\ 0 & -9 & 8 \\ 0 & 0 & -\frac{16}{9} \end{pmatrix}$$ which is in row echelon form.

Since the echelon form has no zero rows, the vectors are linearly independent.

* Show that the set of vectors $\{(3,0,1,-1), (2,-1,0,1), (1,1,1,-2)\}$ is linearly dependent.

Solution: Form the matrix whose rows are the given vectors.

$$\begin{pmatrix} 3 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \\ 1 & 1 & 1 & -2 \end{pmatrix} \xrightarrow{\text{Reduce it}} \begin{pmatrix} 1 & 1 & 1 & -2 \\ 3 & 0 & 1 & -1 \\ 2 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & -2 \\ 0 & -3 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This matrix is in row echelon form and has a zero row, hence the given vectors are linearly dependent.

Alternative process: set the linear combination of the given vectors equal to the zero vectors using unknown scalars. If the homogeneous system gives non-zero solution to the system, then linearly dependent, if not then linearly independent.

Basis and Dimension

* If V is a vector space of dimension n (equal to the max^m number of linearly independent vectors), every basis of V contains exactly n linearly independent vectors.

* Prove that the vectors (1, 2, 0), (0, 5, 7) and (-1, 1, 3) form a basis for R^3.

Solution: The given vectors will be a basis of R^3 iff they are linearly independent and every vector in R^3 can be written as a linear combination of the given vectors.

For any scalars k_1, k_2, k_3, we have

$$k_1(1,2,0) + k_2(0,5,7) + k_3(-1,1,3) = (0,0,0)$$

$$\Rightarrow \quad k_1 - k_3 = 0$$

$$2k_1 + 5k_2 + k_3 = 0$$

$$7k_2 + 3k_3 = 0$$

We get,

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 5 & 1 \\ 0 & 7 & 3 \end{pmatrix} \xrightarrow[R_2' = \cancel{2R_2 - R_1}]{R_2 - 2R_1} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 5 & 3 \\ 0 & 7 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3/5 \\ 0 & 1 & 3/7 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3/5 \\ 0 & 0 & \frac{3}{7} - \frac{3}{5} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3/5 \\ 0 & 0 & -6/35 \end{pmatrix}$$

This matrix is in row echelon form and there is no zero row. Hence the given vectors are linearly independent.

Now, to show the three vectors span R^3, consider an arbitrary vector $v' = (a, b, c)$ in R^3 can be expressed as a linear combination of the given vectors.

$$(a, b, c) = \alpha(1,2,0) + \beta(0,5,7) + \gamma(-1,1,3) \text{ ; for scalars } \alpha, \beta, \gamma$$

We have, $\alpha - \gamma = a$

$2\alpha + 5\beta + \gamma = b$

$7\beta + 3\gamma = c$

The augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 2 & 5 & 1 & b \\ 0 & 7 & 3 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 0 & 5 & 3 & b-2a \\ 0 & 7 & 3 & c \end{array}\right)$$

$$\xrightarrow{R_3' = R_3 - \frac{7}{5}R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & a \\ 0 & 5 & 3 & b-2a \\ 0 & 0 & -\frac{6}{5} & c - \frac{7}{5}(b-2a) \end{array}\right)$$

Therefore,

$\alpha - \gamma = a$

$5\beta + 3\gamma = b - 2a$

$-\frac{6}{5}\gamma = c - \frac{7}{5}(b-2a)$

$\therefore \gamma = -\frac{5}{6}\left(c - \frac{7}{5}(b-2a)\right) = \frac{1}{6}\left(-5c + 7(b-2a)\right)$

$\therefore \gamma = \frac{1}{6}\left(7b - 5c + (-14a)\right)$

Hence, $\beta = \frac{1}{2}(c - b + 2a)$ and $\alpha = \frac{1}{6}(7b - 5c - 8a)$.

Therefore, $(a, b, c) = \alpha(1,2,0) + \beta(0,5,7) + \gamma(-1,1,3)$.

Thus every vector in R^3 can be expressed as a linear combination of the vectors $(1,2,0)$, $(0,5,7)$, $(-1,1,3)$. Hence the given vectors form a basis of R^3.

* Determine a basis and the dimension for the solution space of the following homogeneous system.

$$x_1 - 3x_2 + x_3 = 0$$
$$2x_1 - 6x_2 + 2x_3 = 0$$
$$3x_1 - 9x_2 + 3x_3 = 0$$

Solution: The system becomes,

$$\begin{pmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now,
$$\begin{pmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have,
$$x_1 - 3x_2 + x_3 = 0$$
$$0 = 0$$
$$0 = 0$$

This system is in echelon form and has only one non-zero rows in three unknowns. So we have $3-1=2$ free variables. which are x_2 and x_3. Here x_1 is the basic variable.

$\therefore$ The dimension of the solution space is 2.

Setting $x_3 = 0$ and $x_2 = 1$, we get the special solution. and $x_1 = 3x_2 - x_3 = 3 - 0 = 3$.

Again, setting $x_3 = 1$ and $x_2 = 0$, we get $x_1 = 3x_2 - x_3 = -1$.

Hence, the respective solutions are $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \underline{v}_1$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \underline{v}_2$

so, $\{\underline{v}_1, \underline{v}_2\}$ is a basis of the solution space.

Bases for row space, column space & null space

* Find bases for the row space, column space and the null space of the matrix

(REF) (REF) (RREF)

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B$$

The three non-zero rows form a basis for the row space of A.

Basis for Row A: $\{(1,3,-5,1,5), (0,1,-2,2,7), (0,0,0,-4,20)\}$.

For the column space, observe from B that the pivots are in columns 1, 2 and 4. Hence columns 1, 2 and 4 of A (not B) form a basis for Col A.

Basis for Col A: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 3 \end{bmatrix} \right\}$

$$B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C$$

The equation $A\underline{x} = \underline{0}$ is equivalent to $C\underline{x} = \underline{0}$

$$x_1 + x_3 + x_5 = 0$$
$$x_2 - 2x_3 + 3x_5 = 0$$
$$x_4 - 5x_5 = 0$$

So, $x_1 = -x_3 - x_5$, $x_2 = 2x_3 - 3x_5$, $x_4 = 5x_5$, with x_3 and x_5 free variables. The usual calculations are, setting $x_3 = 0$ and $x_5 = 1$,

we get, $x_4 = 5x_5 = 5$, $x_2 = 2x_3 - 3x_5 = -3$, $x_1 = -x_3 - x_5 = -1$

Again, $x_3 = 1$, and $x_5 = 0$, $x_4 = 5x_5 = 0$, $x_2 = 2x_3 - 3x_5 = 2$,

$x_1 = -x_3 - x_5 = -1$.

Hence basis for Null A: $\left\{ \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

* the rank of A is the dimension of the column space of A.

<u>Rank theorem</u>: rank (A) + dim Nul (A) = n (no. of column vectors)

$\therefore$ Rank of A = 3, [A has three pivot columns]

Nulity of A = 2 [rank (A) + Nul (A) = 5 (no. of total column in A)]