# An Introduction to Quantum Mechanics 1

Dr. Hugo García Tecocoatzi

Instituto Tecnológico y de Estudios Superiores de Monterrey

August 25 2025

# Quantum Mechanics?

- Quantum mechanics is the most accurate description of nature known.
- It forms the foundation of **quantum computation** and **quantum information**.
- Our goal: understand just enough quantum mechanics to work with quantum algorithms and information.
- No prior physics knowledge is required only **elementary linear algebra**.

# Quantum Mechanics: Easier Than You Think

- Reputation: "Quantum mechanics is mysterious and difficult."
- Reality:
  - The hard part is applying it to **complex molecules** or advanced physics.
  - The **foundations** are straightforward, once you know linear algebra.
- We focus only on concepts essential for quantum computing.

# How to Learn This Chapter

- Readers with prior QM knowledge: skim for notational conventions.
- Beginners: work carefully through the material and exercises.
- Tip: If an exercise feels difficult, move on—return later.
- Key prerequisite: **linear algebra** (vectors, matrices, inner products).

1. Review of linear algebra (vectors, inner products, operators).
2. Physicists' notation (Dirac's bra-ket notation).
3. Postulates of quantum mechanics.
4. Simple systems: qubits and quantum states.

# Linear Algebra for Quantum Mechanics

- Quantum mechanics is expressed using the language of **linear algebra**.
- Core objects:
  - **Vector spaces** (e.g. $\mathbb{C}^n$).
  - **Linear operations** on these spaces.
- **Dirac notation**, which looks unfamiliar at first.
- Main challenge for beginners: not the postulates of QM, but the **linear algebraic formalism**.

# Vector Spaces

- The vector space of primary interest: $\mathbb{C}^n$.
- Elements (vectors) are $n$-tuples of complex numbers:

$$(z_1, z_2, \ldots, z_n) \quad \in \mathbb{C}^n$$

- Often written in column form:

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

- Vectors are the building blocks of quantum states.

# Vector Addition

- In $\mathbb{C}^n$, vectors can be added component-wise:

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} z_1' \\ \vdots \\ z_n' \end{bmatrix} = \begin{bmatrix} z_1 + z_1' \\ \vdots \\ z_n + z_n' \end{bmatrix}$$

- This is the familiar vector addition you already know, extended to complex entries.

# Scalar Multiplication

- A scalar (complex number $z$) can multiply a vector:

$$z \cdot \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} zz_1 \\ \vdots \\ zz_n \end{bmatrix}$$

- Scalars "scale" every component of the vector.
- Together, vector addition and scalar multiplication define the structure of a vector space.

# Dirac Notation: Vectors as Kets

- In quantum mechanics, vectors are written using **Dirac notation**.
- A vector is denoted as:

$$|\psi\rangle$$

- $\psi$ is just a label (any symbol can be used).
- The vertical bar "$|$" and angle "$\rangle$" indicate that this object is a vector.
- $|\psi\rangle$ is called a **ket**.

# The Zero Vector

- Every vector space has a special element: the **zero vector**.
- Denoted simply by:

$$0$$

- Property:

$$|v\rangle + 0 = |v\rangle \quad \text{for any vector } |v\rangle$$

- Exception: we do not write $|0\rangle$ for the zero vector.
- In quantum mechanics, $|0\rangle$ means something else (a specific quantum state).

# Scalar Multiplication in Dirac Notation

- For a scalar $z \in \mathbb{C}$ and vector $|\psi\rangle$:

$$z\,|\psi\rangle$$

  is also a vector in the same space.

- For the zero vector:

$$z \cdot 0 = 0 \quad \text{for any scalar } z$$

- Example in $\mathbb{C}^n$:

$$z \cdot (z_1, \ldots, z_n) = (zz_1, \ldots, zz_n)$$

$$2 \cdot (3, 4, i) = (6, 8, 2i)$$

# Dirac Notation: Vectors and Inner Products

- $z^*$: complex conjugate of $z$.
  Example: $(1+i)^* = 1 - i$.
- $|\psi\rangle$: vector (ket).
- $\langle\psi|$: dual vector (bra).
- $\langle\phi|\psi\rangle$: inner product between $|\phi\rangle$ and $|\psi\rangle$.

### Example

For $|\psi\rangle = \begin{bmatrix} 1 \\ i \end{bmatrix}$, $\langle\psi| = \begin{bmatrix} 1 & -i \end{bmatrix}$.

# Tensor Products

- $|\phi\rangle \otimes |\psi\rangle$: tensor product of two vectors.
- Often abbreviated as $|\phi\rangle|\psi\rangle$.
- Forms a larger vector space: $\mathbb{C}^m \otimes \mathbb{C}^n$.

## Example

$$|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

# Spanning Sets

- A set of vectors $\{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\}$ is a **spanning set** if any vector $|v\rangle$ can be written as:

$$|v\rangle = \sum_i a_i |v_i\rangle$$

- Example: In $\mathbb{C}^2$, the standard spanning set is:

$$|v_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Any vector

$$|v\rangle = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

can be written as:

$$|v\rangle = a_1|v_1\rangle + a_2|v_2\rangle$$

# Alternative Spanning Sets

- A vector space may have many different spanning sets.
- Example: Another spanning set for $\mathbb{C}^2$ is

$$|v_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad |v_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

- Any vector $|v\rangle = (a_1, a_2)$ can be written as:

$$|v\rangle = \frac{a_1+a_2}{\sqrt{2}}|v_1\rangle + \frac{a_1-a_2}{\sqrt{2}}|v_2\rangle$$

# Linear Dependence and Independence

- A set of nonzero vectors $\{|v_1\rangle, \ldots, |v_n\rangle\}$ is **linearly dependent** if there exist complex numbers $a_1, \ldots, a_n$, not all zero, such that:

$$a_1|v_1\rangle + a_2|v_2\rangle + \cdots + a_n|v_n\rangle = 0$$

- Otherwise, the vectors are **linearly independent**.
- Independence $\Rightarrow$ no vector in the set can be written as a linear combination of the others.

# Basis and Dimension

- A **basis** is a spanning set of linearly independent vectors.
- Any two bases of the same vector space contain the same number of elements.
- This number is the **dimension** of the vector space.
- Example: $\mathbb{C}^2$ has dimension 2.
- Standard basis:

$$\{|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$$

- Alternative basis:

$$\{|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ |-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$$

# Basis and Qubits

- A **basis** is a spanning set of linearly independent vectors.
- Any vector in the space can be written uniquely as a linear combination of basis vectors.
- For a single qubit, the vector space is $\mathbb{C}^2$.

## Computational Basis

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Any qubit state can be written as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 + |\beta|^2 = 1$$

- Other useful bases exist, e.g. the **Hadamard basis**:

$$|+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big), \quad |-\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big)$$

# What is an Operator?

- Mathematically: A linear operator $A$ maps vectors to vectors

$$A : |v\rangle \mapsto A|v\rangle = |w\rangle$$

- Physically: An operator represents an **action** or a **transformation**.
  - Example: Translation, rotation, reflection
  - In quantum mechanics: observables (measurable quantities) are represented by operators
- Operators encode how a system evolves or how measurements are made.

# Operators as Physical Actions

- Think of a vector $|v\rangle$ as a **state of a system**.
- An operator $A$ acts like a machine: it takes one state and produces another.

$$|v'\rangle = A|v\rangle$$

- Examples:
  - **Translation Operator**: shifts a particle in space.
  - **Rotation Operator**: rotates spin or spatial orientation.
  - **Hamiltonian Operator**: generates time evolution.

# Operators and Observables

- In quantum mechanics, every measurable property (**observable**) corresponds to an operator.

$$\text{Observable} \longleftrightarrow \text{Hermitian operator}$$

- Examples:
  - Position $\hat{x}$ operator
  - Momentum $\hat{p}$ operator
  - Energy (Hamiltonian) $\hat{H}$
- Measurement: applying the operator corresponds to asking "what value does this physical property have?"

# Why Operators Matter

- Operators connect the abstract math (linear algebra) to the real physics.
- They tell us:
  - How states evolve (dynamics)
  - What outcomes measurements can produce (eigenvalues)
  - Probabilities of outcomes (through inner products)
- **Keep in mind:** Operators are the bridge between quantum states and physical reality.

# Matrix Representation of Operators

- Let $A : V \to W$, with bases $\{|v_1\rangle, \ldots, |v_m\rangle\}$ for $V$ and $\{|w_1\rangle, \ldots, |w_n\rangle\}$ for $W$.
- Then for each $j$:

$$A|v_j\rangle = \sum_i A_{ij}|w_i\rangle$$

- The matrix with entries $A_{ij}$ is the matrix representation of $A$.
- Thus:
  - Operators $\Leftrightarrow$ Matrices
  - Vectors $\Leftrightarrow$ Column vectors
  - Composition $\Leftrightarrow$ Matrix multiplication

# Operators as Matrices

- Any $m \times n$ matrix $A$ with entries $A_{ij}$ acts as a linear operator:

$$A\left(\sum_i a_i|v_i\rangle\right) = \sum_i a_i A|v_i\rangle$$

- Action $=$ matrix multiplication on column vectors.
- Example: $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad |v\rangle = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A|v\rangle = \begin{bmatrix} x + 2y \\ y \end{bmatrix}$

# Linear Operators

- A linear operator $A : V \to W$ is a function satisfying:

$$A\left(\sum_i a_i|v_i\rangle\right) = \sum_i a_i A|v_i\rangle$$

- Acts linearly on vectors in $V$.
- If $A : V \to V$, we simply say "$A$ is defined on $V$."
- Important operators:
  - Identity operator: $I|v\rangle = |v\rangle$
  - Zero operator: $0|v\rangle = 0$

# Matrices and Hermitian Conjugates

- $A^*$: element-wise complex conjugate of matrix $A$.
- $A^T$: transpose of $A$.
- $A^\dagger$: Hermitian conjugate (adjoint): $A^\dagger = (A^T)^*$.

**Example**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}$$

- $\langle \phi | A | \psi \rangle$: inner product between $|\phi\rangle$ and $A|\psi\rangle$. Equivalently $\langle A^\dagger \phi | \psi \rangle$.

# Composition of Operators

- If $A : V \to W$ and $B : W \to X$, then their composition is:

$$(BA)(|v\rangle) = B(A(|v\rangle))$$

- Abbreviated as: $BA|v\rangle$
- Operators can be chained just like functions.
  The matrix multiplication.

# Composition ⇒ Matrix Multiplication 1

**Setup**

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad |v\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$$

**Step 1: Apply** $A$

$$A|v\rangle = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ y \end{bmatrix}$$

**Step 2: Apply** $B$ **to the result**

$$B\left(A|v\rangle\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x + 2y \\ y \end{bmatrix} = \begin{bmatrix} y \\ x + 2y \end{bmatrix}$$

# Composition $\Rightarrow$ Matrix Multiplication 2

**Compute the composite matrix** $BA$

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 & 0 \cdot 2 + 1 \cdot 1 \\ 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 2 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

**Check (matrix–vector in one shot)**

$$(BA)|v\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x + 2y \end{bmatrix}$$

$$\Rightarrow \; B(A|v\rangle) = (BA)|v\rangle$$

# Exercise: Composition = Matrix Multiplication

- Show that applying two operators in sequence is the same as multiplying their matrices.
- Method:
  1. Apply the first matrix to the vector, then the second.
  2. Multiply the matrices first, then apply the result to the vector.
  3. Verify that both give the same answer.

**Operators:**

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

**Vector:**

$$|v\rangle = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

**Evalute:**

$$BA|v\rangle$$