

Homework 3

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1 Problem 4.34

1. Apply \hat{S}_- to $|10\rangle$ and confirm that you get $\sqrt{2}\hbar|1-1\rangle$
2. Apply \hat{S}_+ to $|00\rangle$ and confirm that you get zero.
3. Show that $|11\rangle$ and $|1-1\rangle$ are eigenstates of \hat{S}^2 , with the appropriate eigenvalue.

Solution 1: $\hat{S}_- |10\rangle$

Recalling that $|10\rangle = 1/\sqrt{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ we can compute as follows,

$$\begin{aligned}
 \hat{S}_-^{(T)} |10\rangle &= (\hat{S}_-^{(1)} \oplus \hat{S}_-^{(2)}) \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
 &= \frac{1}{\sqrt{2}} \left[(\hat{S}_-^{(1)} |\uparrow\rangle_{(1)}) |\downarrow\rangle_{(2)} \oplus |\uparrow\rangle_{(1)} (\hat{S}_-^{(2)} |\downarrow\rangle_{(2)}) + (\hat{S}_-^{(1)} |\downarrow\rangle_{(1)}) |\uparrow\rangle_{(2)} \oplus |\downarrow\rangle_{(1)} (\hat{S}_-^{(2)} |\uparrow\rangle_{(2)}) \right] \\
 &= \frac{1}{\sqrt{2}} [\hbar |\downarrow\rangle_{(1)} |\downarrow\rangle_{(2)} + \hbar |\downarrow\rangle_{(1)} |\downarrow\rangle_{(2)}] \\
 &= \frac{\hbar}{\sqrt{2}} (|\downarrow\downarrow\rangle + |\downarrow\downarrow\rangle)
 \end{aligned}$$

$$\hat{S}_-^{(T)} |10\rangle = \sqrt{2}\hbar |1-1\rangle.$$

Solution 2: $\hat{S}_+ |00\rangle$

Recalling that $|00\rangle = 1/\sqrt{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ we can compute as follows,

$$\begin{aligned}
 \hat{S}_+^{(T)} |10\rangle &= (\hat{S}_+^{(1)} \oplus \hat{S}_+^{(2)}) \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
 &= \frac{1}{\sqrt{2}} \left[\left(\hat{S}_+^{(1)} |\uparrow\rangle_{(1)} \right) |\downarrow\rangle_{(2)} \oplus |\uparrow\rangle_{(1)} \left(\hat{S}_+^{(2)} |\downarrow\rangle_{(2)} \right) - \left(\hat{S}_+^{(1)} |\downarrow\rangle_{(1)} \right) |\uparrow\rangle_{(2)} \oplus |\downarrow\rangle_{(1)} \left(\hat{S}_+^{(2)} |\uparrow\rangle_{(2)} \right) \right] \\
 &= \frac{1}{\sqrt{2}} \left[\hbar |\uparrow\rangle_{(1)} |\uparrow\rangle_{(2)} - \hbar |\uparrow\rangle_{(1)} |\uparrow\rangle_{(2)} \right] \\
 &= \frac{\hbar}{\sqrt{2}} (|\uparrow\uparrow\rangle - |\uparrow\uparrow\rangle)
 \end{aligned}$$

$$\hat{S}_+^{(T)} |00\rangle = 0.$$

Solution 3: Eigenstates of \hat{S}^2

First we compute the expression of \hat{S}^2 for a system with two Hilbert spaces,

$$\begin{aligned}
 (\hat{S}^{(T)})^2 &= (\hat{S}^{(1)} \oplus \hat{S}^{(2)}) \cdot (\hat{S}^{(1)} \oplus \hat{S}^{(2)}) \\
 &= (\hat{S}^{(1)})^2 \oplus (\hat{S}^{(2)})^2 \oplus 2\vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(2)}
 \end{aligned}$$

Starting with $|11\rangle = |\uparrow\uparrow\rangle$,

$$(\hat{S}^{(T)})^2 |11\rangle = \left((\hat{S}^{(1)})^2 \oplus (\hat{S}^{(2)})^2 \oplus 2\vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(2)} \right) |\uparrow\uparrow\rangle$$

$$\begin{aligned}
 (\hat{S}^{(T)})^2 |11\rangle &= \left[(\hat{S}^{(1)})^2 |\uparrow\rangle_{(1)} |\uparrow\rangle_{(2)} \right] \oplus \left[|\uparrow\rangle_{(1)} (\hat{S}^{(2)})^2 |\uparrow\rangle_{(2)} \right] \\
 &\quad \oplus 2 \left[\hat{S}_x^{(1)} |\uparrow\rangle_{(1)} \hat{S}_x^{(2)} |\uparrow\rangle_{(2)} \oplus \hat{S}_y^{(1)} |\uparrow\rangle_{(1)} \hat{S}_y^{(2)} |\uparrow\rangle_{(2)} \oplus \hat{S}_z^{(1)} |\uparrow\rangle_{(1)} \hat{S}_z^{(2)} |\uparrow\rangle_{(2)} \right]
 \end{aligned}$$

$$\begin{aligned}
 (\hat{S}^{(T)})^2 |11\rangle &= \frac{3}{2} \hbar^2 |\uparrow\uparrow\rangle \oplus 2 \left[\frac{\hbar^2}{4} |\downarrow\downarrow\rangle \ominus \frac{\hbar^2}{4} |\downarrow\downarrow\rangle \oplus \frac{\hbar^2}{4} |\uparrow\uparrow\rangle \right] \\
 &= \frac{3}{2} \hbar^2 |\uparrow\uparrow\rangle \oplus \frac{\hbar^2}{2} |\uparrow\uparrow\rangle \\
 &= 2\hbar^2 |\uparrow\uparrow\rangle.
 \end{aligned}$$

Now, we apply the same procedure for $|1-1\rangle$,

$$(\hat{S}^{(T)})^2 |1-1\rangle = \left((\hat{S}^{(1)})^2 \oplus (\hat{S}^{(2)})^2 \oplus 2\vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(2)} \right) |\downarrow\downarrow\rangle$$

$$\begin{aligned}
(\hat{S}^{(T)})^2 |1-1\rangle &= \left[(\hat{S}^{(1)})^2 |\downarrow_{(1)} \downarrow_{(2)}\rangle \right] \oplus \left[|\downarrow_{(1)} (\hat{S}^{(2)})^2 |\downarrow_{(2)}\rangle \right] \\
&\oplus 2 \left[\hat{S}_x^{(1)} |\downarrow_{(1)} \hat{S}_x^{(2)} |\downarrow_{(2)}\rangle \oplus \hat{S}_y^{(1)} |\downarrow_{(1)} \hat{S}_y^{(2)} |\downarrow_{(2)}\rangle \oplus \hat{S}_z^{(1)} |\downarrow_{(1)} \hat{S}_z^{(2)} |\downarrow_{(2)}\rangle \right]
\end{aligned}$$

$$\begin{aligned}
(\hat{S}^{(T)})^2 |1-1\rangle &= \frac{3}{2} \hbar^2 |\downarrow\downarrow\rangle \oplus 2 \left[\frac{\hbar^2}{4} |\uparrow\uparrow\rangle \ominus \frac{\hbar^2}{4} |\uparrow\uparrow\rangle \oplus \frac{\hbar^2}{4} |\downarrow\downarrow\rangle \right] \\
&= \frac{3}{2} \hbar^2 |\downarrow\downarrow\rangle \oplus \frac{\hbar^2}{2} |\downarrow\downarrow\rangle \\
&= 2\hbar^2 |\downarrow\downarrow\rangle
\end{aligned}$$

$$\boxed{(\hat{S}^{(T)})^2 |11\rangle = 2\hbar^2 |\uparrow\uparrow\rangle, \quad (\hat{S}^{(T)})^2 |1-1\rangle = 2\hbar^2 |\downarrow\downarrow\rangle}$$

2 Problem 4.35

Quarks carry spin 1/2. Three quarks bind together to make a baryon (such as a proton or neutron): two quarks (or more precisely a quark and an antiquark) bind together to make a meson (such as the pion or the kaon). Assume the quarks are in the ground (so the orbital angular momentum is zero).

1. What spins are possible for baryons?
2. What spins are possible for mesons?

Solution 4: Possible spins for baryons

For this case we need to analyze the spin of a three particle system with spin 1/2 each. In general we can have the following states,

| Quartet | Doublet I | Doublet II |
|--|---|---|
| $ \uparrow\uparrow\uparrow\rangle$ | | |
| $\frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow\rangle + \uparrow\downarrow\uparrow\rangle + \downarrow\uparrow\uparrow\rangle)$ | $\frac{1}{\sqrt{6}}(\uparrow\uparrow\downarrow\rangle + \uparrow\downarrow\uparrow\rangle - 2 \downarrow\uparrow\uparrow\rangle)$ | $\frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow\rangle - \downarrow\uparrow\uparrow\rangle)$ |
| $\frac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow\rangle + \downarrow\uparrow\downarrow\rangle + \uparrow\downarrow\downarrow\rangle)$ | $\frac{1}{\sqrt{6}}(\downarrow\downarrow\uparrow\rangle + \downarrow\uparrow\downarrow\rangle - 2 \uparrow\downarrow\downarrow\rangle)$ | $\frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow\rangle - \downarrow\uparrow\downarrow\rangle)$ |
| $ \downarrow\downarrow\downarrow\rangle$ | | |

We are going to compute $(\hat{S}^{(T)})^2$ for one state of each group of the doublets states, due to the

symmetric relations between them and two cases for the quartet group. In this case we have that,

$$\left(\hat{S}^{(T)}\right)^2 = \left(\hat{S}^{(1)}\right)^2 \oplus \left(\hat{S}^{(2)}\right)^2 \oplus \left(\hat{S}^{(3)}\right)^2 \oplus 2\left(\vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(2)} \oplus \vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(3)} \oplus \vec{\hat{S}}^{(2)} \cdot \vec{\hat{S}}^{(3)}\right).$$

This is because, as a general property of angular momentum, we get that $\hat{L}|f\rangle = \hbar^2 l(l+1)|f\rangle$, therefore, getting the eigenvalue of the state we can compute the value of the spin.

Quartet

Starting with $|\uparrow\uparrow\uparrow\rangle$,

$$\begin{aligned} \left(\hat{S}^{(T)}\right)^2 |\uparrow\uparrow\uparrow\rangle &= 3\frac{3}{4}\hbar^2 |\uparrow\uparrow\uparrow\rangle \oplus 2\left(\frac{\hbar^2}{4} |\uparrow\uparrow\uparrow\rangle \oplus \frac{\hbar^2}{4} |\uparrow\uparrow\uparrow\rangle \oplus \frac{\hbar^2}{4} |\uparrow\uparrow\uparrow\rangle\right) \\ &= \frac{15}{4}\hbar^2 |\uparrow\uparrow\uparrow\rangle. \end{aligned}$$

Now we equate $\hbar^2 l(l+1) = 15/4\hbar^2$ which results in $l = 3/4$. By symmetry this same result is for the state $|\downarrow\downarrow\downarrow\rangle$. Following with $(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle)$,

$$\begin{aligned} \left(\hat{S}^{(T)}\right)^2 (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) &= 3\frac{3}{4}\hbar^2 (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) \\ &\oplus 2\left(\frac{\hbar^2}{4} (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) \oplus \frac{\hbar^2}{4} (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) \oplus \frac{\hbar^2}{4} (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle)\right) \\ \left(\hat{S}^{(T)}\right)^2 (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle) &= \frac{15}{4}\hbar^2 (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle). \end{aligned}$$

As before we get $l = 3/4$, and we have the same value for the state $\frac{1}{\sqrt{3}}(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\uparrow\downarrow\downarrow\rangle)$.

Doublet I

Now we compute as follows,

$$\begin{aligned} \left(\hat{S}^{(T)}\right)^2 (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\downarrow\uparrow\uparrow\rangle) &= 3\frac{3}{4}\hbar^2 (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\downarrow\uparrow\uparrow\rangle) \\ \oplus 2\left(\frac{\hbar^2}{4} (|\uparrow\uparrow\downarrow\rangle - 5|\uparrow\downarrow\uparrow\rangle + 4|\downarrow\uparrow\uparrow\rangle) \oplus \frac{\hbar^2}{4} (5|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + 4|\downarrow\uparrow\uparrow\rangle) \oplus \frac{\hbar^2}{4} (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\downarrow\uparrow\uparrow\rangle)\right) \\ \left(\hat{S}^{(T)}\right)^2 (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\downarrow\uparrow\uparrow\rangle) &= 3\frac{3}{4}\hbar^2 (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\downarrow\uparrow\uparrow\rangle) \oplus 2\frac{3}{4}\hbar^2 (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\downarrow\uparrow\uparrow\rangle) \\ &= \frac{3}{4}\hbar^2 (|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle - 2|\downarrow\uparrow\uparrow\rangle), \end{aligned}$$

which leads to $l = 1/2$ and the same goes for $\frac{1}{\sqrt{6}}(|\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\rangle - 2|\uparrow\downarrow\downarrow\rangle)$.

Doublet II Finally we compute

$$\begin{aligned} (\hat{S}^{(T)})^2(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) &= 3\frac{3}{4}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ &\oplus 2\left(\frac{\hbar^2}{4}(-3|\uparrow\downarrow\rangle + 3|\downarrow\uparrow\rangle) \oplus \frac{\hbar^2}{4}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle - 2|\uparrow\uparrow\rangle) \oplus \frac{\hbar^2}{4}(-|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle + 2|\uparrow\uparrow\rangle)\right) \end{aligned}$$

$$\begin{aligned} (\hat{S}^{(T)})^2(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) &= 3\frac{3}{4}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \ominus 2\frac{3}{4}\hbar^2(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ &= \frac{3}{4}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \end{aligned}$$

As before we get $l = 1/2$ and the same goes for $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$.

Hence we can get spins of $3/2$ and $1/2$ in baryon systems.

$$s = \left\{ \frac{3}{2}, \frac{1}{2} \right\}.$$

Solution 5: Possible spins for mesons

For this case, we already compute the eigenvalues of $(\hat{S}^{(T)})^2$ for the states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, which results in an eigenvalue of $2\hbar^2$, that is $l = 1$. Finally, we compute the eigenvalues for the states $|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle$.

$$\begin{aligned} (\hat{S}^{(T)})^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) &= 2\frac{3}{4}\hbar^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \oplus 2\frac{\hbar^2}{4}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ &= 2\hbar^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle). \end{aligned}$$

Hence, $l = 1$.

Finally,

$$\begin{aligned} (\hat{S}^{(T)})^2(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) &= 2\frac{3}{4}\hbar^2(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \oplus 2\frac{3}{4}\hbar^2(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \\ &= 0, \end{aligned}$$

leading to $l = 0$.

Therefore we can get spin of 1 and 0 in mesons systems.

$$s = \{1, 0\}.$$

3 Problem 5.4

1. If ψ_a are orthogonal, and both normalized, what is the constant A in

$$\Psi_{\pm}(\vec{r}_1, \vec{r}_2) = A [\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)]?$$

2. If $\psi_a = \psi_b$ (and it is normalized), what is A ? (This case, of course, occurs only for bosons.)

Solution 6: Constant of normalization A

First we can re-write the Ψ as follows, $\Psi_{\pm} = A(|\psi_a\psi_b\rangle \pm |\psi_b\psi_a\rangle)$. Now we can compute the inner product of the state and apply the ortho-normal properties of $|\psi_a\rangle$ and $|\psi_b\rangle$,

$$\begin{aligned} \langle\Psi|\Psi\rangle &= |A|^2 (\langle\psi_a\psi_b| \pm \langle\psi_b\psi_a|) (|\psi_a\psi_b\rangle \pm |\psi_b\psi_a\rangle) \\ &= |A|^2 (\langle\psi_a\psi_b|\psi_a\psi_b\rangle \pm \langle\psi_a\psi_b|\psi_b\psi_a\rangle \pm \langle\psi_b\psi_a|\psi_a\psi_b\rangle + \langle\psi_b\psi_a|\psi_b\psi_a\rangle) \\ &= 2|A|^2, \end{aligned}$$

since, $\langle\Psi|\Psi\rangle$ should be 1, $A = 1/\sqrt{2}$.

$$A = \frac{1}{\sqrt{2}}.$$

Solution 7: Constant of normalization A with $\psi_a = \psi_b$

From the previous procedure, we see that the second and third terms are equal to 1, leading to the following results,

$$\langle\Psi|\Psi\rangle = 4|A|^2,$$

since, $\langle\Psi|\Psi\rangle$ should be 1, $A = 1/2$.

$$A = \frac{1}{2}.$$

4 Problem 5.5

1. Write down the Hamiltonian for two noninteracting identical particles in the infinite square well. Verify that the fermion ground state given in Example 5.1 is an eigenfunction of \hat{H} , with the appropriate eigenvalue.

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(n\pi\frac{x}{a}\right), \quad E_n = n^2 \frac{\pi^2 \hbar^2}{2ma^2}.$$

2. Find the next two excited states (beyond the ones in Example 5.1)-wave functions and energies-for each of the three cases (distinguishable, identical bosons, identical fermions).

Solution 8: Hamiltonian of non-interacting identical particles

The hamiltonian that describe this system is

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x_1^2} - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x_2^2} = E\psi, \quad 0 \leq x_1, x_2 \leq a.$$

From the Example 5.1 we have,

$$\psi(x_1, x_2) = \frac{\sqrt{2}}{a} \left[\sin\left(\pi \frac{x_1}{a}\right) \sin\left(2\pi \frac{x_2}{a}\right) - \sin\left(2\pi \frac{x_1}{a}\right) \sin\left(\pi \frac{x_2}{a}\right) \right].$$

Now we compute the partial derivatives,

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x_1^2} &= \frac{\sqrt{2}}{a} \left[-\left(\frac{\pi}{a}\right)^2 \sin\left(\pi \frac{x_1}{a}\right) \sin\left(2\pi \frac{x_2}{a}\right) + \left(\frac{2\pi}{a}\right)^2 \sin\left(2\pi \frac{x_1}{a}\right) \sin\left(\pi \frac{x_2}{a}\right) \right] \\ \frac{\partial^2 \psi}{\partial x_2^2} &= \frac{\sqrt{2}}{a} \left[-\left(\frac{2\pi}{a}\right)^2 \sin\left(\pi \frac{x_1}{a}\right) \sin\left(2\pi \frac{x_2}{a}\right) + \left(\frac{\pi}{a}\right)^2 \sin\left(2\pi \frac{x_1}{a}\right) \sin\left(\pi \frac{x_2}{a}\right) \right]. \end{aligned}$$

Substituting those results into the hamiltonian we get,

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x_1^2} - \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x_2^2} &= \frac{\hbar^2}{2m} \left(-\frac{\partial^2 \psi}{\partial x_1^2} - \frac{\partial^2 \psi}{\partial x_2^2} \right) \\ &= \frac{\hbar^2}{2m} \left(\frac{\sqrt{2}}{a} \left[\left(\frac{2\pi}{a}\right)^2 + \left(\frac{\pi}{a}\right)^2 \right] \left[\sin\left(\pi \frac{x_1}{a}\right) \sin\left(2\pi \frac{x_2}{a}\right) - \sin\left(2\pi \frac{x_1}{a}\right) \sin\left(\pi \frac{x_2}{a}\right) \right] \right) \\ &= \frac{\hbar^2}{2m} \left(\left[5 \frac{\pi^2}{a^2} \right] \psi \right) \\ &= 5 \frac{\pi^2 \hbar^2}{2a^2 m} \psi. \end{aligned}$$

therefore,

$$\hat{H}\psi(x_1, x_2) = 5 \frac{\pi^2 \hbar^2}{2a^2 m} \psi(x_1, x_2),$$

which ensures that the fermion ground state is an eigenfunction with appropriate eigenvalue.

$$\boxed{\hat{H}\psi(x_1, x_2) = 5K\psi(x_1, x_2), \quad K = \frac{\pi^2 \hbar^2}{2a^2 m}.}$$

Solution 9: Energies and states of the next two excited states**Distinguishable states**

For distinguishable particles we know that $\psi_{n_1, n_2} = \psi_{n_1}(x_1)\psi_{n_2}(x_2)$, $E_{n_1, n_2} = (n_1^2 + n_2^2)K$. So now we compute $\psi_{2,2}$, $\psi_{1,3}$ and $\psi_{3,1}$,

$$\begin{aligned}\psi_{2,2} &= \frac{2}{a} \sin\left(2\pi \frac{x_1}{a}\right) \sin\left(2\pi \frac{x_2}{a}\right), & E &= 8K \\ \psi_{1,3} &= \frac{2}{a} \sin\left(\pi \frac{x_1}{a}\right) \sin\left(3\pi \frac{x_2}{a}\right), & E &= 10K \\ \psi_{3,1} &= \frac{2}{a} \sin\left(3\pi \frac{x_1}{a}\right) \sin\left(\pi \frac{x_2}{a}\right), & E &= 10K\end{aligned}$$

Identical Bosons

For this case we know that the state is $\psi_{n_1, n_2} = [\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)] / \sqrt{2}$, $E_{n_1, n_2} = (n_1^2 + n_2^2)K$, therefore,

$$\begin{aligned}\psi_{2,2} &= \frac{2}{a} \sin\left(2\pi \frac{x_1}{a}\right) \sin\left(2\pi \frac{x_2}{a}\right), & E &= 8K \\ \psi_{1,3} &= \frac{\sqrt{2}}{a} \left[\sin\left(\pi \frac{x_1}{a}\right) \sin\left(3\pi \frac{x_2}{a}\right) + \sin\left(3\pi \frac{x_2}{a}\right) \sin\left(\pi \frac{x_1}{a}\right) \right], & E &= 10K\end{aligned}$$

Identical Fermions

Finally, the identical fermions state is $\psi_{n_1, n_2} = [\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)] / \sqrt{2}$, $E_{n_1, n_2} = (n_1^2 + n_2^2)K$, therefore,

$$\begin{aligned}\psi_{1,3} &= \frac{\sqrt{2}}{a} \left[\sin\left(\pi \frac{x_1}{a}\right) \sin\left(3\pi \frac{x_2}{a}\right) - \sin\left(3\pi \frac{x_2}{a}\right) \sin\left(\pi \frac{x_1}{a}\right) \right], & E &= 10K \\ \psi_{2,3} &= \frac{\sqrt{2}}{a} \left[\sin\left(2\pi \frac{x_1}{a}\right) \sin\left(3\pi \frac{x_2}{a}\right) - \sin\left(3\pi \frac{x_2}{a}\right) \sin\left(2\pi \frac{x_1}{a}\right) \right], & E &= 13K\end{aligned}$$