

Quantum Computation

Quantum Circuits Activity

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November 16, 2025

- 1 Describe the action of the phase shift gate $p(\gamma) = |0\rangle\langle 0| + e^{i\gamma} |1\rangle\langle 1|$ on a qubit from the geometrical point of view.

Recalling that we can express a qubit as $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$. This expression allows us to create a geometrical interpretation as a point in an unit sphere. Where θ represent the angle between the \hat{x} and \hat{y} axis, and the angle φ is the angle between the \hat{x} or \hat{y} with the \hat{z} axis. With this in mind, let's compute the resulting state from the given gate,

$$\begin{aligned} p(\gamma)|\psi\rangle &= |0\rangle\langle 0| \left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle \right) + e^{i\gamma}|1\rangle\langle 1| \left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle \right) \\ &= \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi+\gamma}\sin\left(\frac{\theta}{2}\right)|1\rangle. \end{aligned}$$

We can see that the $p(\gamma)$ gate introduces a phase shift in the angle related with the $\hat{x} - \hat{z}$ or $\hat{y} - \hat{z}$ planes. That is that introduces a displacement along the latitude of the unit sphere.

- 2 The 4-qubit W-state is defined as,

$$|W_4\rangle = \frac{1}{2}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle).$$

Design a quantum circuit that upon the initial state $|0000\rangle$ constructs $|W_4\rangle$.

- 3 Design a circuit constructing the Hardy state,

$$|H\rangle = \frac{1}{\sqrt{12}}(3|00\rangle + |01\rangle + |10\rangle + |11\rangle).$$

- 4 Show how to implement the Toffoli gate in terms of single-qubit and controlled-NOT gates.

- 5 Assume that we start with a fully separable three-qubit states. First, qubits 1 and 2 become maximally entangled through an appropriate quantum operation. Your task is to design a quantum circuit that transfers this entanglement to qubits (2,3). In other words, at the end of the circuit, qubits 2 and 3 should be maximally entangled, while qubit 1 should be disentangled from the rest. You are allowed to use elementary gates alone.

- 6 In the BB84 protocol, Alice creates an 8-qubit string (in the conventional X and Z basis):

$$|+\rangle |1\rangle |+\rangle |-\rangle |0\rangle |-\rangle |+\rangle |-\rangle .$$

Use a coin to randomly determine what basis Bob uses to measure each bit, and describe the resulting bit string that Alice and Bob keep.

- 7 a boolean function $f : \{0,1\}^n \mapsto \{0,1\}$ is said to be constant if $f(x)$ has the same value for all 2^n inputs and balanced if $f(x)$ returns 0 for exactly half of all inputs and 1 for the other half,

- Consider a generalization of the Deutsch's algorithm having two registers ($n = 2$). The correspondent circuit is essentially the same as in the one register case. Discuss the conditions that would determine if a function is either balanced or constant.
- Analyze the case when the function f is neither constant or nor balanced.