

The following problems are intended to help you prepare for the final exam. You are required to submit only Problems 5 and 7, including a detailed solution for each.

1. Describe the action of the phase shift gate $P(\gamma) = |0\rangle\langle 0| + e^{i\gamma}|1\rangle\langle 1|$ on a qubit from the geometrical point of view.
2. The 4-qubit W-state is defined as

$$|W_4\rangle = \frac{1}{2} (|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$$

Design a quantum circuit that upon the initial state $|0000\rangle$ constructs $|W_4\rangle$.

3. Design a circuit constructing the Hardy state

$$|H\rangle = \frac{1}{\sqrt{12}} (3|00\rangle + |01\rangle + |10\rangle + |11\rangle).$$

4. Show how to implement the Toffoli gate in terms of single-qubit and controlled-NOT gates.
5. Assume that we start with a fully separable three-qubit state. First, qubits 1 and 2 become maximally entangled through an appropriate quantum operation. Your task is to design a quantum circuit that transfers this entanglement to qubits (2,3). In other words, at the end of the circuit, qubits 2 and 3 should be maximally entangled, while qubit 1 should be disentangled from the rest. You are allowed to use elementary gates alone.
6. In the BB84 protocol, Alice creates an 8-qubit string (in the conventional X and Z basis):

$$|+\rangle|1\rangle|+\rangle|-\rangle|0\rangle|-\rangle|+\rangle|-\rangle$$

Use a coin to randomly determine what basis Bob uses to measure each bit, and describe the resulting bit string that Alice and Bob keep.

7. A boolean function $f : \{0,1\}^n \longrightarrow \{0,1\}$ is said to be constant if $f(x)$ has the same value for all 2^n inputs and balanced if $f(x)$ returns 0 for exactly half of all inputs and 1 for the other half.
 - (a) Consider a generalization of the Deutsch's algorithm having two registers ($n = 2$). The correspondent circuit is essentially the same as in the one register case. Discuss the conditions that would determine if a function is whether balanced or constant.
 - (b) Analyze the case when the function f is neither constant nor balanced.