

# *Quantum Computation*

## *Quantum Circuits Activity*

*Francisco Vazquez-Tavares*

*November 18, 2025*

### *Mandatory exercises*

5 Assume that we start with a fully separable three-qubit states. First, qubits 1 and 2 become maximally entangled through an appropriate quantum operation. Your task is to design a quantum circuit that transfers this entanglement to qubits (2,3). In other words, at the end of the circuit, qubits 2 and 3 should be maximally entangled, while qubit 1 should be disentangled from the rest. You are allowed to use elementary gates alone.

We start with a fully separable three-qubit states  $|000\rangle$ , which can be expressed as  $|0\rangle \otimes |0\rangle \otimes |0\rangle$ . In order to get a maximally entangled state for qubits 1 and 2 we apply a Haddamard gate in qubit 1,

$$(\hat{H} \otimes \hat{I} \otimes \hat{I}) |000\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |100\rangle), \quad (1)$$

and a controlled X gate with qubit 1 as the control,

$$(\hat{\Lambda} \otimes \hat{X} \otimes \hat{I}) \frac{1}{\sqrt{2}} (|000\rangle + |100\rangle) = \frac{1}{\sqrt{2}} (|000\rangle + |110\rangle). \quad (2)$$

This final state can be expressed in terms of the Bells states for qubits 1 and 2,

$$\frac{1}{\sqrt{2}} (|000\rangle + |110\rangle) = \frac{1}{\sqrt{2}} (|00\rangle \otimes |0\rangle + |11\rangle \otimes |0\rangle) \quad (3)$$

$$= |\Psi^+\rangle \otimes |0\rangle. \quad (4)$$

Now that we achieve the maximally entangled for qubit 1 and 2, let's show how to transfer this entanglement to qubits 2 and 3. First, let's apply a controlled X gate into qubits 2 and 3 to create a maximally entangled 3 qubit state<sup>1</sup>,

$$(\hat{I} \otimes \hat{\Lambda} \otimes \hat{X}) |\Psi^+\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle). \quad (5)$$

At this point we can interpret that applying the controlled X gate after the Haddamard gate, it allows us to entangle the control with the target qubit. Hence, in order to disentangle the first qubit we apply a controlled qubit with the second qubit as controlled and the first qubit as the target,

$$(\hat{X} \otimes \hat{\Lambda} \otimes \hat{I}) \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \quad (6)$$

$$|0\rangle \otimes |\Psi^+\rangle. \quad (7)$$

<sup>1</sup> From previous homework, this is the GHZ-state.

By comparing equations (4) with (7), we can see how the maximal entanglement has been transferred from qubits 1-2 to qubits 2-3, achieving the desired quantum circuit. The graphical representation is shown in figure 1.

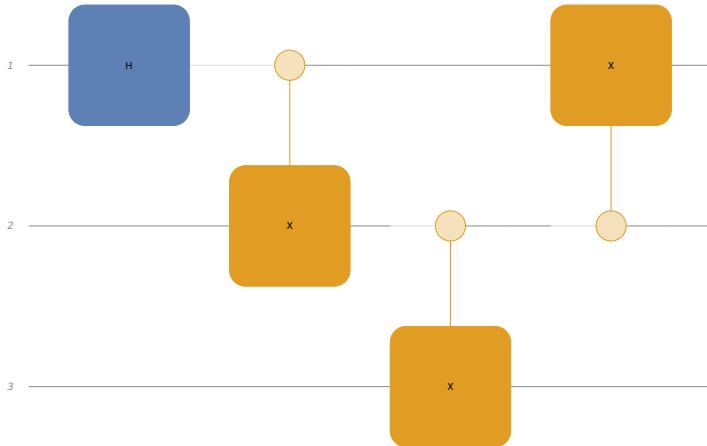


Figure 1: Transfer of entangled states circuit.

7 A boolean function  $f : \{0,1\}^n \mapsto \{0,1\}$  is said to be constant if  $f(x)$  has the same value for all  $2^n$  inputs and balanced if  $f(x)$  returns 0 for exactly half of all inputs and 1 for the other half,

- Consider a generalization of the Deutsch's algorithm having two registers ( $n = 2$ ). The correspondent circuit is essentially the same as in the one register case. Discuss the conditions that would determine if a function is wether balanced or constant.
- Analyze the case when the function  $f$  is neither constant or nor balanced.

### Optional exercises

1 Describe the action of the phase shift gate  $p(\gamma) = |0\rangle\langle 0| + e^{i\gamma}|1\rangle\langle 1|$  on a qubit from the geometrical point of view.

Recalling that we can express a qubit as  $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$ . This expression allow us to create a geometrical interpretation as a point in an unit sphere. Where  $\theta$  represent the angle between the  $\hat{x}$  and  $\hat{y}$  axis, and the angle  $\varphi$  is the angle between the  $\hat{x}$  or  $\hat{y}$  with the  $\hat{z}$  axis. With this in mind, let's compute the resulting state from the given gate,

$$\begin{aligned} p(\gamma)|\psi\rangle &= |0\rangle\langle 0| \left( \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle \right) + e^{i\gamma}|1\rangle\langle 1| \left( \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle \right) \\ &= \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi+\gamma}\sin\left(\frac{\theta}{2}\right)|1\rangle. \end{aligned}$$

We can see that the  $p(\gamma)$  gate introduces a phase shift in the angle related with the  $\hat{x} - \hat{z}$  or  $\hat{y} - \hat{z}$  planes. That is that introduces a displacement along the latitude of the unit sphere.

2 The 4-qubit W-state is defined as,

$$|W_4\rangle = \frac{1}{2} (|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle).$$

Design a quantum circuit that upon the initial state  $|0000\rangle$  constructs  $|W_4\rangle$ .

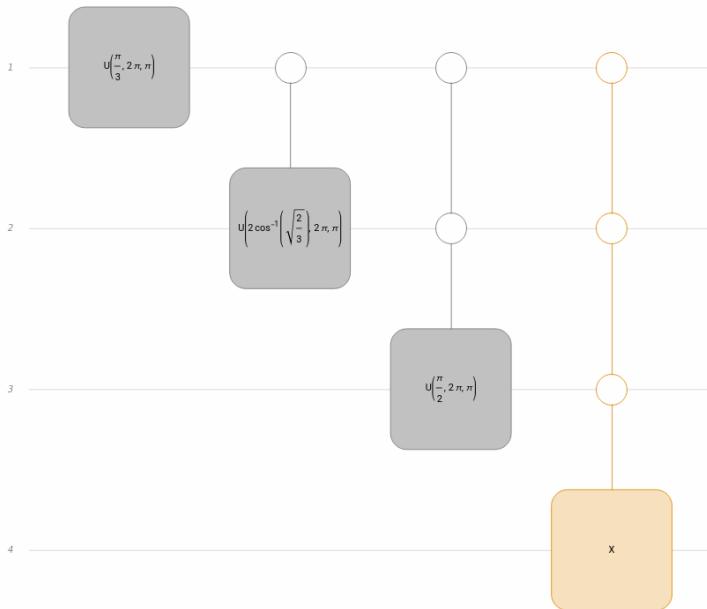


Figure 2: Quantum circuit to construct the 4-qubit W-state.

3 Design a circuit constructing the Hardy state,

$$|H\rangle = \frac{1}{\sqrt{12}} (3|00\rangle + |01\rangle + |10\rangle + |11\rangle).$$

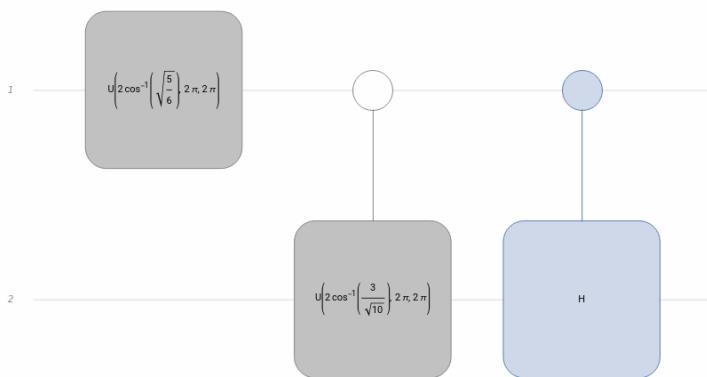


Figure 3: Quantum circuit to construct the Hardy state.

4 Show how to implement the Toffoli gate in terms of single-qubit and controlled-NOT gates.

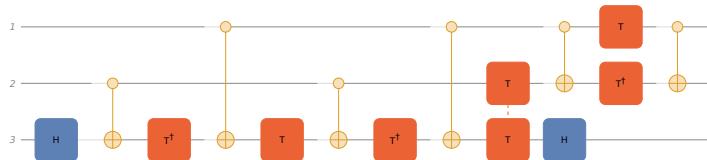


Figure 4: Toffoli gate in terms of single-qubit and controlled-NOT gates. Obtain from <https://arxiv.org/pdf/0803.2316>

6 In the BB84 protocol, Alice creates an 8-qubit string (in the conventional X and Z basis):

$$|+\rangle |1\rangle |+\rangle |- \rangle |0\rangle |- \rangle |+\rangle |- \rangle .$$

Use a coin to randomly determine what basis Bob uses to measure each bit, and describe the resulting bit string that Alice and Bob keep.