Quantum Computation
Quantum Circuits Activity

Francisco Vazquez-Tavares

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1 Describe the action of the phase shift gate $p(\gamma) = |0\rangle\langle 0| + e^{i\gamma}|1\rangle\langle 1|$ on a qubit from the geometrical point of view.

Recalling that we can express a qubti as $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$. This expression allow us to create a geometrical interpretation as a point in an unit sphere. Where θ represent the angle between the $\hat{x}$ and $\hat{y}$ axis, and the angle φ is the angle between the $\hat{x}$ or $\hat{y}$ with the $\hat{z}$ axis. With this in mind, let's compute the resulting state from the given gate,

$$
\begin{aligned}
p(\gamma)|\psi\rangle &= |0\rangle\langle 0|\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right) + e^{i\gamma}|1\rangle\langle 1|\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right) \\
&= \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi+\gamma}\sin\left(\frac{\theta}{2}\right)|1\rangle .
\end{aligned}
$$

We can se that the $p(\gamma)$ gate introduces a phase shift in the angle related with the $\hat{x}-\hat{z}$ or $\hat{y}-\hat{z}$ planes. That is that introduces a displacement along the latitude of the unit sphere.

2 The 4-qubit W-state is defined as,

$$
|W_4\rangle = \frac{1}{2}\left(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle\right).
$$

Design a quantum circuit that upon the initial state $|0000\rangle$ constructs $|W_4\rangle$.

3 Design a circuir constructing the Hardy state,

$$
|H\rangle = \frac{1}{\sqrt{12}}\left(3|00\rangle + |01\rangle + |10\rangle + |11\rangle\right).
$$

4 Show how to implement the Toffoli gate in terms of single-qubit and controlled-NOT gates.

5 Assume that we start with a fully separable three-qubit states. First, qubits 1 and 2 become maximally entangled through an appropiate quantum operation. Your task is to design a quantum circuit that transfers this entanglement to qubits (2,3). In other words, at the end of te circuit, qubits 2 and 3 should be maximally entangled, while qubit 1 should be disentangled from the rest. You are allowed to use elementary gates alone.

6 In the BB84 protocol, Alice create an 8-qubit string (in the conventional X and Z basis):

$$|+\rangle\,|1\rangle\,|+\rangle\,|-\rangle\,|0\rangle\,|-\rangle\,|+\rangle\,|-\rangle\,.$$

Use a coin to randomly determine what basis Bob uses to measure each bit, and describe the resulting bit string that Alice and Bob keep.

7 a boolean function $f : \{0,1\}^n \mapsto \{0,1\}$ is said to be constant of $f(x)$ has the same vaue for all 2^n inputs and balanced if $f(x)$ returns 0 for exactly half of all inputs and 1 for the other half,

- Consider a generalization of the Deuthsch's algorithm having two registers ($n = 2$). The correpondent circuit is essentially the same as in the one register case. Discuss the conditions that would determine if a function is wether balanced or constant.
- Analyze the case when the function f is neither constant or nor balanced.