

Quantum Optics Class-Notes and others

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Summary Class notes, post-class notes and others for the course of Quantum Optics. Semester February-June 2025

Harmonic oscillator

February 12

First Quantization

February 17

Properties of Quantum electric field

February 19

Fock States

February 19

Coherent states

February 24

According to the professor the sections ??, where the hard part of the course. The next sections we are going to describe different states and analyze their properties.

Let's remember the operator of an electric field with x component and one mode,

$$E_x = i \left(\frac{\hbar \omega}{2 \epsilon_0 V} \right)^{1/2} \left(\hat{a} \exp[-i\omega t] - \hat{a}^\dagger \exp[-i\omega t] \right),$$

which can be expressed in terms of quadratures as,

$$E_x = 2 \left(\frac{\hbar \omega}{2 \epsilon_0 V} \right)^{1/2} (\hat{X}_1 \sin(\omega t) + \hat{X}_2 \cos(\omega t))$$

So, when we want to get the expected value of the electric field of a Fock state $|n\rangle$ we get $\langle n|E|n\rangle = 0$. So we need other states to model a laser. A useful observation is that $\langle n|\hat{X}_1^2|n\rangle = 1/4(2n+1)$ and $\Delta\hat{X}_1\Delta\hat{X}_2 = 1/4(2n+1)$.

We are going to study the "Gleuber states". Which are the states that can describe the laser. For that we have 4 definitions,

Definition 1 Eigenstates of \hat{a}

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \quad \alpha \in \mathbb{C}.$$

Reminders of some properties.

$$[\hat{a}, \hat{a}^\dagger] = 1$$

$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^\dagger}{2}$$

$$\hat{X}_2 = \frac{\hat{a} - \hat{a}^\dagger}{2i}$$

$$\hat{a} = \hat{X}_1 + i\hat{X}_2$$

$$\hat{a}^\dagger = \hat{X}_1 - i\hat{X}_2$$

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$$

$$\Delta\hat{X}_1\Delta\hat{X}_2 \geq \frac{1}{4}$$

Definition 2 Displaced vacuum¹.

$$\hat{D}(\alpha) = \exp [\alpha \hat{a}^\dagger - \alpha^* \hat{a}], \quad |\alpha\rangle = \hat{D}(\alpha) |0\rangle$$

¹ It is important to use the following definition of e^x , because the argument are matrices and vectors,

$$\exp(x) = \sum \frac{x^n}{n!}.$$

Definition 3 Fock States

$$|\alpha\rangle = \exp \left[-\frac{|\alpha|^2}{2} \right] \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Definition 4

$$\Delta \hat{X}_1 \Delta \hat{X}_2 = \frac{1}{2}$$

Then we start to analyse a coherent state and a Fock state.

$$\begin{aligned} \langle \alpha | \alpha \rangle &= \exp \left[-|\alpha|^2 \right] \sum_{n=0} \sum_{m=0} \frac{\alpha^{*m} \alpha^n}{\sqrt{m!} \sqrt{n!}} \langle m | n \rangle \\ &= \exp \left[-|\alpha|^2 \right] \sum \frac{|\alpha|^{2n}}{n!} \\ &= 1 \end{aligned}$$

Later we analyse the creation and annihilation operator. Knowing that $\hat{a} |\alpha\rangle = \alpha |\alpha\rangle$ and when we compute its adjoint $\langle \alpha | \hat{a}^\dagger = \alpha^* \langle \alpha |$, we get different eigenvalues for each operator. However, if we compute the expected value of both operators in a state we get,

$$\begin{aligned} \langle \alpha | \hat{a}^\dagger \hat{a} | \alpha \rangle &= \langle \alpha | \hat{a}^\dagger \hat{a} \alpha \rangle \\ &= \langle \alpha | \hat{a}^\dagger \alpha \rangle \\ &= \alpha^* \alpha \\ &= |\alpha|^2 \end{aligned}$$

Usefull properties,

$$\begin{aligned} \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} &= 1 \\ \langle \alpha | \hat{a} \hat{a}^\dagger | \alpha \rangle &= |\alpha|^2 + 1 \\ \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle \end{aligned}$$

Now we apply the \hat{a} operator in a Fock state,

$$\begin{aligned} \hat{a} |\alpha\rangle &= \hat{a} \left(\exp \left[-\frac{|\alpha|^2}{2} \right] \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \right) \\ &= \exp \left[-\frac{|\alpha|^2}{2} \right] \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} \hat{a} |n\rangle \\ &= \exp \left[-\frac{|\alpha|^2}{2} \right] \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} (\sqrt{n} |n-1\rangle), \end{aligned}$$

When $n = 0$ is like taking photons to the vacuum, which does not make sense, hence we can translate the sum as follows,

$$\hat{a} |\alpha\rangle = \exp \left[-\frac{|\alpha|^2}{2} \right] \sum_{n=1} \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle$$

What is the probability of detecting n photons²?

² Is the Poisson distribution.

$$|\langle n | \alpha \rangle|^2 = \exp \left[-\frac{|\alpha|^2}{2} \right] \frac{|\alpha|^{2n}}{n!}$$

Now we take the expected value of $|\alpha\rangle$ with $\hat{a}, \hat{a}^\dagger, \hat{X}_1, \hat{X}_2$,

$$\begin{aligned}\langle\alpha|\hat{a}|\alpha\rangle &= \alpha \\ \langle\alpha|\hat{a}^\dagger|\alpha\rangle &= \alpha^* \\ \langle\alpha|\hat{X}_1|\alpha\rangle &= \frac{1}{2}(\alpha + \alpha^*) \\ \langle\alpha|\hat{X}_2|\alpha\rangle &= \frac{1}{2}(\alpha - \alpha^*)\end{aligned}$$

Then we checked the Quadrature noise,

$$\begin{aligned}\langle\alpha|\hat{X}_1^2|\alpha\rangle &= \langle\alpha|\frac{1}{4}(\hat{a} + \hat{a}^\dagger)^2|\alpha\rangle \\ &= \frac{1}{4}(\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1)\end{aligned}$$

therefore,

$$\langle\hat{X}_1^2|\alpha\rangle - \langle\hat{X}_1|\alpha\rangle^2 = \frac{1}{4}$$

As homework, compute

$$\langle\alpha|\hat{X}_2^2|\alpha\rangle$$

Displacement Operator

Squeezed States

February 26

Beam Splitters

March 10

In this session we start the second module of the course with analyzing the beam splitter in a quantum framework.

In the classical framework we can model the beam splitter using the Fresnel Coefficients. Assuming with that we have an power input we analyze the reflection power and a transmission coefficient. It is important to acknowledge that the reflective coefficient is complex. This can be related to a phase shift during the reflection. Also, we set that $|r|^2 + |t|^2 = 1$.

Now lets go Quantum, that is, that instead of analyzing the power input and output in the beam splitter we are going to explore the operators for each light beam. Lets assume that the operator \hat{a}_1 represent the input light beam, the operator $\hat{a}_2 = r\hat{a}_1$ represent the light beam that is reflected and finally, an operator $\hat{a}_3 = t\hat{a}_1$ that takes into account the transmitted light beam. All of these operators needs to be independent, because are representend 3 different modes. Also, we know that the creation and annihilation operators follows these commutation rules,

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \quad [\hat{a}_i, \hat{a}_j] = 0.$$

From those relations we can see that³,

$$[\hat{a}_2, \hat{a}_2^\dagger] = |r|^2, \quad [\hat{a}_3, \hat{a}_3^\dagger] = |t|^2, \quad [\hat{a}_2, \hat{a}_3^\dagger] = rt^*.$$

³

$$\begin{aligned}[\hat{a}_2, \hat{a}_2^\dagger] &= \hat{a}_2\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2 \\ &= (r\hat{a}_1)(r\hat{a}_1)^\dagger - (r\hat{a}_1)^\dagger(r\hat{a}_1) \\ &= |r|^2(\hat{a}_1\hat{a}_1^\dagger) - |r|^2(\hat{a}_1^\dagger\hat{a}_1) \\ &= |r|^2[\hat{a}_1, \hat{a}_1^\dagger],\end{aligned}$$

with the same

When we compare this results with the commutation rules that we establish before, we can see that can not be true. Therefore we need to change the definition of the operators \hat{a}_2 and \hat{a}_3 .

From previous discussions, we can consider the fluctuations of the vacuum into the system with the following transformations,

$$\begin{aligned}\hat{a}_2 &= r\hat{a}_1 + t'\hat{a}_0, \\ \hat{a}_3 &= t\hat{a}_1 + r'\hat{a}_0,\end{aligned}$$

or,

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} t' & r \\ r' & t \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix}.$$

When we compute the commutation relations⁴ we see that the commutation relations holds with the following relations,

$$\begin{aligned}r^*t' + r't^* &= 0 & r^*t + r't'^* &= 0 \\ |r'| &= |r| & |t'| &= |t| \\ |r|^2 + |t|^2 &= 1\end{aligned}$$