Homework 2

Professor: Dr. Alfonso Isaac Jaimes Nájera

Francisco Javier Vázquez Tavares A00827546

Contents

1 Problem 4.25

If the electron were a classical solid sphere, with radius,

$$r_c = \frac{e^2}{4\pi\epsilon_o mc^2}$$

(the so-called classical electron radius, obtained by assuming the electron's mass is attributable to energy sotred in its electric field, via the Einstein formula $E = mc^2$), and its angular momentum is $\hbar/2$, then how fast (in m/s) would a point on the "equator" be moving? Does this model make sense? (Actually, the radius of the electrin is known experimentally to be much less than r_c but this only makes matters worse.)

Solution 1: Classical spinning

From the classical framework the angular momentum is modeled with the following relation,

$$L = I\omega,$$

where I is the moment of interia, which in this case is $I = 2/5mr^2$ and w is the angular frequency, that can be express as $\omega = v/r$. Replacing this equivalences into the angular momentum equation we can get the following expression for v,

$$v = \frac{5}{2}\frac{L}{mr_c},$$

substituting the values of L and r_c,

$$v = \frac{5\pi\hbar\epsilon_o}{e^2}c^2.$$

Recalling the order of magnitud of the constants, $e \approx 10^{-19}$, $\hbar \approx 10^{-34}$, $\epsilon_o \approx 10^{-12}$ and $c \approx 10^8$, we get that,

$$\frac{5\pi\hbar\epsilon_o}{e^2}c \approx 90,$$

which tells us that the velocity at the ecuator is 90 times the velocity of light, which does not make sense.

$$\boxed{v \approx 90c.}$$

2 Problem 4.26

- Check that the spin matrices(1) obey the fundamental commutation relations for angular momentum(2).

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

$$\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z, \quad \left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x, \quad \left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y \tag{2}$$

- Show that the Pauli spin matrices(3) satisfy the product rule(4), where the indices stand for x, y, z and ϵ_{jkl} is the Levi-Civita symbol.

$$\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

$$\sigma_j\sigma_k = \delta_{jk} + i\sum_l \epsilon_{jkl}\sigma_l. \tag{4}$$

Solution 2: Commutation relations

We start computing $\left[\hat{S}_x, \hat{S}_y\right]$,

$$\begin{aligned}\left[\hat{S}_x, \hat{S}_y\right] &= \frac{\hbar^2}{4}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}\right] \\ &= i\hbar\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= i\hbar\hat{S}_z.\end{aligned}$$

Now $\left[\hat{S}_y, \hat{S}_z\right]$,

$$\begin{aligned}\left[\hat{S}_y, \hat{S}_z\right] &= \frac{\hbar^2}{4}\left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\right] \\ &= i\hbar\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= i\hbar\hat{S}_x.\end{aligned}$$

Finally $\left[\hat{S}_z, \hat{S}_x\right]$,

$$\begin{aligned}\left[\hat{S}_z, \hat{S}_x\right] &= \frac{\hbar^2}{4}\left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} - \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right]\\ &= (-ii)\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\\ &= i\hbar\hat{S}_y.\end{aligned}$$

Hence, it is proof that,

$$\boxed{\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z, \quad \left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x, \quad \left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y}$$

Solution 3: Pauli spin matrices properties

To proof that the product rule is satified by the Pauli matrices, we compute all the products. Strating with $\sigma_x\sigma_x$, $\sigma_y\sigma_y$, $\sigma_z\sigma_z$,

$$\begin{aligned}\sigma_x\sigma_x &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\\ \sigma_y\sigma_y &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\\ \sigma_z\sigma_z &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned}$$

Now, for the cross terms,

$$\begin{aligned}\sigma_x\sigma_y &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\\ \sigma_y\sigma_z &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = i\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\\ \sigma_x\sigma_z &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.\end{aligned}$$

Which confirms that Pauli's matrices satifies the product relation.

3 Problem 4.27

An electron is in the spin state,

$$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

1. Determine the normalization constant A.
2. Find the expectation values of S_x, S_y and S_z.
3. Find the "uncertanties" $\sigma_{S_x}, \sigma_{S_y}$ and σ_{S_z}. (Note: These sigmas are standard deviations, not Pauli matrices!)
4. Confirm that your results are consistent with all three uncertanty principles 4.100 and its cyclic permutations-only with S in place of L, of course.

Solution 4: Normalization constant

To find the normalization constant we need to get the squared modulus of the state and solve for A,

$$\begin{aligned} |\chi|^2 &= A \begin{pmatrix} -3i & 4 \end{pmatrix} A \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\ &= 25A^2, \end{aligned}$$

tacking into account that $|\chi|^2 = 1$ we get that $A = 1/5$.

$$\boxed{A = \frac{1}{5}}$$

Solution 5: Expected values of spin

To find $\left\langle \hat{S}_x \right\rangle, \left\langle \hat{S}_y \right\rangle$ and $\left\langle \hat{S}_z \right\rangle$ we compute, $\langle \chi | \hat{S}_n | \chi \rangle$,

$$\begin{aligned} \langle \chi | \hat{S}_x | \chi \rangle &= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar}{50}(-12i + 12i) = 0, \\ \langle \chi | \hat{S}_y | \chi \rangle &= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar}{50}(-12 - 12) = -\frac{12}{25}\hbar, \\ \langle \chi | \hat{S}_z | \chi \rangle &= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar}{50}(9 - 16) = -\frac{7}{50}\hbar. \end{aligned}$$

$$\boxed{\left\langle \hat{S}_x \right\rangle = 0, \quad \left\langle \hat{S}_y \right\rangle = -\frac{12}{25}\hbar, \quad \left\langle \hat{S}_z \right\rangle = -\frac{7}{50}\hbar.}$$

Solution 6: Uncertanties

To compute the uncertanty we recall that $\sigma = \sqrt{\left\langle \hat{S}^2 \right\rangle - \left\langle \hat{S} \right\rangle^2}$. From the previous task we already known $\left\langle \hat{S} \right\rangle$, hence we only need to compute $\left\langle \hat{S}^2 \right\rangle$.

$$\begin{aligned}
\langle \chi | \hat{S}_x^2 | \chi \rangle &= \frac{\hbar^2}{100} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar}{50}(9+16) = \frac{\hbar^2}{4}, \\
\langle \chi | \hat{S}_y^2 | \chi \rangle &= \frac{\hbar^2}{100} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar}{50}(9+16) = \frac{\hbar^2}{4}, \\
\langle \chi | \hat{S}_z^2 | \chi \rangle &= \frac{\hbar^2}{100} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar}{50}(9+16) = \frac{\hbar^2}{4}.
\end{aligned}$$

Now we can compute the uncertanties,

$$\begin{aligned}
\sigma_x &= \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}, \\
\sigma_y &= \sqrt{\frac{\hbar^2}{4} - \frac{144}{625}\hbar^2} = \frac{7}{50}\hbar, \\
\sigma_z &= \sqrt{\frac{\hbar^2}{4} - \frac{49}{2500}\hbar^2} = \frac{12}{25}\hbar.
\end{aligned}$$

$$\boxed{\sigma_x = \frac{\hbar}{2}, \quad \sigma_y = \frac{7}{50}\hbar, \quad \sigma_z = \frac{12}{25}\hbar.}$$

Solution 7: Checking consistency with the uncertanty principle.

With the uncertanty relation, $\sigma_A \sigma_B \geq 1/2 \left| \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle \right|$, we can test the previous results,

$$\begin{aligned}
\sigma_{\hat{S}_x} \sigma_{\hat{S}_y} &\geq \frac{1}{2} \left| i\hbar \left\langle \hat{S}_z \right\rangle \right| \\
\frac{\hbar}{2} \frac{7}{50} \hbar &\geq \frac{1}{2} \frac{7}{50} \hbar^2 \\
\frac{7}{100} \hbar^2 &\geq \frac{7}{100} \hbar^2
\end{aligned}
\qquad
\begin{aligned}
\sigma_{\hat{S}_y} \sigma_{\hat{S}_z} &\geq \frac{1}{2} \left| i\hbar \left\langle \hat{S}_x \right\rangle \right| \\
\frac{7}{50} \frac{12}{25} \hbar &\geq \frac{1}{2} 0 \\
\frac{42}{625} \hbar^2 &\geq 0
\end{aligned}
\qquad
\begin{aligned}
\sigma_{\hat{S}_x} \sigma_{\hat{S}_z} &\geq \frac{1}{2} \left| i\hbar \left\langle \hat{S}_y \right\rangle \right| \\
\frac{\hbar}{2} \frac{12}{25} \hbar &\geq \frac{1}{2} \frac{12}{25} \hbar^2 \\
\frac{12}{50} \hbar^2 &\geq \frac{12}{50} \hbar^2
\end{aligned}$$

This tells us that the previous results are consistent with the uncerntainty principle.

$$\boxed{\sigma_{\hat{S}_x}\sigma_{\hat{S}_y} \geq \frac{1}{2}\left|i\hbar\left\langle\hat{S}_z\right\rangle\right|, \quad \sigma_{\hat{S}_y}\sigma_{\hat{S}_z} \geq \frac{1}{2}\left|i\hbar\left\langle\hat{S}_x\right\rangle\right|, \quad \sigma_{\hat{S}_x}\sigma_{\hat{S}_z} \geq \frac{1}{2}\left|i\hbar\left\langle\hat{S}_y\right\rangle\right|}$$

4 Problem 4.32 a

In Example 4.3,

$$\chi(t) = a\left|\chi_+\right\rangle e^{-iE_+t/\hbar} + b\left|\chi_-\right\rangle e^{-iE_-t/\hbar}.$$

If you measure the component of spin angular momentum along the x direction, at time t, what is the probability that you would get $+\hbar/2$?

Solution 8: Probability of measuring spin angular momentum along the x direction.

First we recall that $a = \cos[\alpha/2]$ and $b = \sin[\alpha/2]$ are the initial conditions of the state and that

$$\left|\chi_+\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \left|\chi_-\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Taking into account that, we can re-write $\chi(t)$ as,

$$\left|\chi(t)\right\rangle = \begin{pmatrix}\cos[\alpha/2]e^{i\gamma B_0 t/2}\\ \sin[\alpha/2]e^{-i\gamma B_0 t/2}\end{pmatrix}.$$

Now, to compute the probability of measuring $\hbar/2$ we need remember that the expected value of $\left|\chi_+\right\rangle$ is $\hbar/2$, therefore, we need to compute the modulus square of the projection of the state $\left|\chi(t)\right\rangle$ into $\left|\chi_+\right\rangle$ to get the probability. The mathematical procedure is shown below,

$$\begin{aligned}\left\langle\chi_+|\chi(t)\right\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\cos[\alpha/2]e^{i\gamma B_0 t/2}\\ \sin[\alpha/2]e^{-i\gamma B_0 t/2}\end{pmatrix}\\ &= \frac{1}{\sqrt{2}}\left(\cos[\alpha/2]e^{i\gamma B_0 t/2} + \sin[\alpha/2]e^{-i\gamma B_0 t/2}\right).\end{aligned}$$

Now we can compute the probability,

$$\begin{aligned}\left|\left\langle\chi_+|\chi(t)\right\rangle\right|^2 = &\left[\frac{1}{\sqrt{2}}\left(\cos[\alpha/2]e^{-i\gamma B_0 t/2} + \sin[\alpha/2]e^{i\gamma B_0 t/2}\right)\right]\\ &\left[\frac{1}{\sqrt{2}}\left(\cos[\alpha/2]e^{i\gamma B_0 t/2} + \sin[\alpha/2]e^{-i\gamma B_0 t/2}\right)\right]\end{aligned}$$

$$
\begin{aligned}
|\langle\chi_+|\chi(t)\rangle|^2 &= \frac{1}{2}\Big[1+\cos[\alpha/2]\sin[\alpha/2]\Big(e^{-i\gamma B_0 t}+e^{i\gamma B_0 t}\Big)\Big] \\
&= \frac{1}{2}\Big[1+\frac{1}{2}\sin[\alpha]2\cos[\gamma B_0 t]\Big] \\
&= \frac{1}{2}[1+\sin[\alpha]\cos[\gamma B_0 t]]
\end{aligned}
$$

$$
\boxed{|\langle\chi_+|\chi(t)\rangle|^2 = \frac{1}{2}[1+\sin[\alpha]\cos[\gamma B_0 t]]}
$$