Homework 1

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1 Problem 4.2

Use separation of variable in *cartesian* coordinates to solve the infinite *cubical* well (or particle in a box):

$$V(x,y,z) = \begin{cases} 0, & \forall x,y,z \in [0,a] \\ \infty, & \forall x,y,z \notin [0,a] \end{cases}$$

1. Find the stationary states, and the corresponding energies.
2. Call the distinct energies $E_1, E_2, \dots$ in order of increasing energy. Find E_1, E_2, E_3, E_4, E_5 and E_6. Determine their degeneracies (that is, the number of different states that share the same energy).
3. What is the degeneracy of E_{14}, and why is this case interesting?

Solution 1: Stationary states

To find the stationary states of the infinite cubical well, we are going to solve the time independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\nabla\psi = E\psi,\ \forall x,y,z \in [0,a],$$

with the following boundary conditions $\psi(0,0,0) = \psi(a,a,a) = 0$. To solve the equation we are going to use the method of separation of variables, that is, that we assume that the solution of the differential equation has the following form $\psi(x,y,z) = X(x)Y(y)Z(z)$. Substituting this solution to the differential equation, we can perfom some algebraic manipulation,

$$
\begin{gathered}
-\frac{\hbar^2}{2m}\nabla\psi = E\psi \\
\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)X(x)Y(y)Z(z) = -\frac{2m}{\hbar^2}EX(x)Y(y)Z(z) \\
Y(y)Z(z)\frac{\partial^2}{\partial x^2}X(x)+X(x)Z(z)\frac{\partial^2}{\partial y^2}Y(y)+X(x)Y(y)\frac{\partial^2}{\partial z^2}Z(z) = -\frac{2m}{\hbar^2}EX(x)Y(y)Z(z) \\
\frac{1}{X(x)}\frac{\partial^2}{\partial x^2}X(x)+\frac{1}{Y(y)}\frac{\partial^2}{\partial y^2}Y(y)+\frac{1}{Z(z)}\frac{\partial^2}{\partial z^2}Z(z) = -\frac{2m}{\hbar^2}E
\end{gathered}
$$

2 Problem 4.3

Use

$$
\begin{gathered}
P_l^m(x) \equiv \left(1-x^2\right)^{|m|/2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{|m|}P_l(x) \\
P_l(x) \equiv \frac{1}{2l!}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l}\left(x^2-1\right)^{l} \\
Y_l^m(\theta,\phi) = \epsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}e^{im\phi}P_l^m(\cos[\theta])
\end{gathered}
$$

to construct Y_0^0 and Y_2^l. Check that they are normalized and orthogonal.

3 Problem 4.13

- Find $\langle r\rangle$ and $\langle r^2\rangle$ for an electron in the ground state of hydrogen. Express your answers in terms of the Bohr radius.
- Find $\langle x\rangle$ and $\langle x^2\rangle$ for an electron in the ground stat of hydrogen. *Hint:* this requires no noew integration-note that $r^2 = x^2+y^2+z^2$, and explot the symmetry of the ground state.
- Find $\langle x^2\rangle$ in the state $n=2, l=1, m=1$. *Warning:* This state is not symmetrical in x, y, z. Use $x = r\sin\theta\cos\phi$.

4 Problem 4.14

What is the *most probable* value of r, in the ground state of hydrogen? (The answer is not zero!) *Hint:* First ypu must figure out the probability that the electron would be found between r and $r + dr$.

5 Problem 4.23

In problem 4.3 you showed that

$$Y_2^l(\theta, \phi) = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi}.$$

Apply the raising operator to find $Y_2^2(\theta, \phi)$. Use equation $A_l^m = \hbar\sqrt{l(l+1) - m(m \pm 1)} = \hbar\sqrt{(l \mp m)(l \pm m + 1)}$ to get the normalization.

[heading=bibintoc,title=References]