align(center, text(17pt)[**Class Notes**])

1. February 12

1.1. Hmk solution

The eigen values and eiggen values are $\lambda = \{1, 2, -1\}$ with the following eigen vectors, $|1\rangle = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$.
Now, for the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

the eigenvectors and values are, $|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad |2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad |3\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Now we have a state β, $|\beta\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

In an observable all possible values are the eigenvalues.
So when we multiple two vectors together $\left\langle V_Q \middle| \beta \right\rangle$
Now, when we measure Q we get -1.
After the measure the state is $|-1\rangle$ in Q. Sow now we can get 1 or -1 from the eigen states of R.

1.2. Hermitian conjugate and hermitian operator

The **hermitian conjugate** of and operator $\hat{Q}$ is the operator $\hat{Q}^\dagger$ such that

$$\left\langle f \middle| \hat{Q} g \right\rangle = \left\langle \hat{Q}^\dagger f \middle| g \right\rangle \forall f \wedge g$$

An hermitian operator, then, is equal to its hermitian conjugate: $\hat{Q} = \hat{Q}^\dagger$.
Every observable is represented with an hermitian operator. A hamiltonina is an hermitian operator.
Observale are represented by hermitian operators.
Useful facts:

- Hermitian operator have **real** eigenvalues.
- Eigenfunction of an hermitian operator are orthogonal to each other. (They are a good basis)
- Hermitian conjugate on an operator in a matrix is the same as conjugate trnaspose.

1.3. Harmonic oscilator

Near a minimum of any potential a good approximation is using a quadratic potential (harmonic oscilation).
$12kx^2$
Normally in quantum we use the letter ω, where represent the resonance frequency.
$V = \frac{1}{2}m\omega^2 x^2$
Since we are dealing quantum mechanics,

$$\begin{aligned}\hat{H} &= p^2 2m + 12m\omega^2\hat{x}^2\\ &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\\ &= \frac{1}{2m}\left(\hat{p}^2 + (m\omega x)^2\right)\end{aligned}$$

So now we are going to factorize the squared terms. To do that we are going to introduce an operator $\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} + i\hat{p})$ and its hermitian conjugate, $\hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} - i\hat{p})$.
Now as an excersise we compute the multipication of the operators,

$$\hat{a}^\dagger\hat{a} = \frac{1}{2m\hbar\omega}\left[(m\omega x)^2 + \hat{p}^2 + im\omega\hat{x}\hat{p} - \operatorname{im}\omega\hat{p}\hat{x}\right]$$

So, when we change the order of the operator, the signs changes.

$$\hat{a}\hat{a}^\dagger = \frac{1}{2m\hbar\omega}\left[(m\omega x)^2 + \hat{p}^2 - im\omega\hat{x}\hat{p} + \operatorname{im}\omega\hat{p}\hat{x}\right]$$

The commutator is important $[\hat{x}, \hat{p}] = i\hbar$.
So

$$\begin{aligned}a^\dagger a &= \frac{1}{1m\hbar\omega}\left[(m\omega\hat{x})^2 + \hat{p}^2 + im\omega i\hbar\right]\\ &= \frac{1}{1m\hbar\omega}\left[(m\omega\hat{x})^2 + \hat{p}^2 + m\omega\hbar\right]\end{aligned}$$

Sow, we can we re-write the hamiltonian as,

$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{a}\hat{a}^\dagger - \frac{1}{2}\right)$$

Now we explore (how much the operator changes when the order is changed.)

$$\begin{aligned}\left[\hat{a}, \hat{a}^\dagger\right] &= \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\\ &= \frac{1}{2m\hbar\omega}\left(m\omega\hat{x}^2 + \hat{p}^2 + im\omega(\hat{x}\hat{p} - \hat{p}\hat{x})\right) - ...\end{aligned}$$

$\left[\hat{a}, \hat{a}^\dagger\right] = 1$

1.3.1. Now we solve the time independet

We are going to solve $\hat{H}\Psi = E\Psi$

If Ψ is a solution, then $\hat{a}\Psi$ is a solution and the corresponding energy $(H(\hat{a}\Psi)) = (E - \hbar\omega)(\hat{a}\Psi))$. t is also true that $H(\hat{a}\Psi)) = (E + \hbar\omega)\left(\hat{a}^{\dagger}\Psi\right)$

Now we are going to proof:

We start that ψ is a solution,

$$
\begin{aligned}
\hat{H}(\hat{a}\psi) &= \hbar\omega\left(\hat{a}\hat{a}^{\dagger} - \frac{1}{2}\right)(\hat{a}\psi) \\
&= \hbar\omega\left(\hat{a}\hat{a}^{\dagger}\hat{a} - \frac{1}{2}\hat{a}\right)\psi \\
&= \hat{a}\left(\hbar\omega\left(\hat{a}^{\dagger}\hat{a} - \frac{1}{2}\right)\right)\psi \\
&= \hat{a}\left(\hat{H} - \hbar\omega\right)\psi \\
&= \hat{a}(E - \hbar\omega)\psi \\
&= (E - \hbar\omega)\hat{a}\psi
\end{aligned}
$$

We are going to compute a lot $\hat{a}$ and $\hat{a}^{\dagger}$.

We know that the energy needs to be greater than zero. Lets name a state ψ_o that when we apply $\hat{a}$ to lower the energy at a minimum value.

$$
\begin{aligned}
\hat{a}\psi_{o(x)} &= 0 \\
(m\omega\hat{x} + i\hat{p})\psi_{o(x)} &= 0 \\
m\omega x\psi_{o(x)} + \hbar\frac{\partial}{\partial\psi_o} &= 0
\end{aligned}
$$

and the solution of that is a gaussian.

We wnat to knwo the energy of that state,

$$\hat{H}\psi_o = \hbar\frac{\omega}{2}\psi_o$$

which is $\hbar\frac{\omega}{2}$

The analytic solution can be not take into account.

We are going to deal with algebraic operators.

1.3.2. New operator

Now we are going to define a new operator: $\hat{n} = \hat{a} + \hat{a}^{\dagger}$. Which represent the exitations, The hamiltonian can be re-writed as,

$$\hat{H} = \hbar\omega(\hat{n} + 12).$$

1.3.3. Notation

It common to use the following notaiton: $|0\rangle = \psi_o$, $|1\rangle = \psi_1$.
The functions can be normilize,

$$\hat{a}|n\rangle = c_n|n-1\rangle$$

The constant c_n is to normalize the state.

$$\begin{aligned}\langle\psi|\psi\rangle &= 1\\ \left\langle n\middle|\hat{a}^\dagger\hat{a}\middle|n\right\rangle &= |c_n|^2\ \langle n-1|n-1\rangle = |c_n|^2\\ n\langle n|n\rangle &= |c_n|^2\\ n &= |c_n|^2\end{aligned}$$

Hence,

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

$\hat{a}^\dagger\hat{a}$ is hermitian?
Monday from 11 after class.

1.3.4. Examples?

We start we the ground state,

$$\begin{aligned}\hat{a}^\dagger|0\rangle &= |1\rangle\\ \hat{a}^{\dagger 2}|o\rangle &= \sqrt{2}|2\rangle\\ \hat{a}^{\dagger 3}|o\rangle &= \sqrt{2}\sqrt{3}|2\rangle\\ |n\rangle &= \frac{1}{n!}\hat{a}^{\dagger N}\hat{0}\end{aligned}$$

1.3.5. Infinity independent oscilators

We can get all the oscilators with a bif state $|(n_1, n_2)\rangle$
$\left[\hat{a}_i, \hat{a}_j\right] = 0$ and $\left[\hat{a}_1, \hat{a}_j^\dagger\right] = \delta_{12}$
Problems 2.13 and 3.5 of the Griffiths. Just the algebraic method.