

Homework 2

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Contents

1	Problem 4.25	1
2	Problem 4.26	2
3	Problem 4.27	4
4	Problem 4.32 a	6

1 Problem 4.25

If the electron were a classical solid sphere, with radius,

$$r_c = \frac{e^2}{4\pi\epsilon_0 mc^2}$$

(the so-called classical electron radius, obtained by assuming the electron's mass is attributable to energy stored in its electric field, via the Einstein formula $E = mc^2$), and its angular momentum is $\hbar/2$, then how fast (in m/s) would a point on the “equator” be moving? Does this model make sense? (Actually, the radius of the electron is known experimentally to be much less than r_c but this only makes matters worse.)

Solution 1: Classical spinning

From the classical framework the angular momentum is modeled with the following relation,

$$L = I\omega,$$

where I is the moment of inertia, which in this case is $I = 2/5 mr^2$ and ω is the angular frequency, that can be expressed as $\omega = v/r$. Replacing these equivalences into the angular momentum equation we can get the following expression for v ,

$$v = \frac{5}{2} \frac{L}{mr_c},$$

substituting the values of L and r_c ,

$$v = \frac{5\pi\hbar\epsilon_0}{e^2} c^2.$$

Recalling the order of magnitude of the constants, $e \approx 10^{-19}$, $\hbar \approx 10^{-34}$, $\epsilon_0 \approx 10^{-12}$ and $c \approx 10^8$, we get that,

$$\frac{5\pi\hbar\epsilon_0}{e^2} c \approx 90,$$

which tells us that the velocity at the ecuator is 90 times the velocity of light, which does not make sense.

$$v \approx 90c.$$

2 Problem 4.26

- Check that the spin matrices(1) obey the fundamental commutation relations for angular momentum(2).

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1)$$

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y \quad (2)$$

- Show that the Pauli spin matrices(3) satisfy the product rule(4), where the indices stand for x, y, z and ϵ_{jkl} is the Levi-Civita symbol.

$$\sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3)$$

$$\sigma_j \sigma_k = \delta_{jk} + i \sum_l \epsilon_{jkl} \sigma_l. \quad (4)$$

Solution 2: Commutation relations

We start computing $[\hat{S}_x, \hat{S}_y]$,

$$\begin{aligned} [\hat{S}_x, \hat{S}_y] &= \frac{\hbar^2}{4} \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right] \\ &= i\hbar \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= i\hbar\hat{S}_z. \end{aligned}$$

Now $[\hat{S}_y, \hat{S}_z]$,

$$\begin{aligned} [\hat{S}_y, \hat{S}_z] &= \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \\ &= i\hbar \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= i\hbar\hat{S}_x. \end{aligned}$$

Finally $[\hat{S}_z, \hat{S}_x]$,

$$\begin{aligned} [\hat{S}_z, \hat{S}_x] &= \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \\ &= (-i\hbar) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= i\hbar \hat{S}_y. \end{aligned}$$

Hence, it is proof that,

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$

Solution 3: Pauli spin matrices properties

To proof that the product rule is satisfied by the Pauli matrices, we compute all the products. Strating with $\sigma_x \sigma_x$, $\sigma_y \sigma_y$, $\sigma_z \sigma_z$,

$$\begin{aligned} \sigma_x \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \sigma_y \sigma_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \sigma_z \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Now, for the cross terms,

$$\begin{aligned} \sigma_x \sigma_y &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \sigma_y \sigma_z &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_x \sigma_z &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \end{aligned}$$

Which confirms that Pauli's matrices satisfies the product relation.

3 Problem 4.27

An electron is in the spin state,

$$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

1. Determine the normalization constant A .
2. Find the expectation values of S_x, S_y and S_z .
3. Find the “uncertainties” $\sigma_{S_x}, \sigma_{S_y}$ and σ_{S_z} . (Note: These sigmas are standard deviations, not Pauli matrices!)
4. Confirm that your results are consistent with all three uncertainty principles 4.100 and its cyclic permutations-only with S in place of L , of course.

Solution 4: Normalization constant

To find the normalization constant we need to get the squared modulus of the state and solve for A ,

$$\begin{aligned} |\chi|^2 &= A \begin{pmatrix} -3i & 4 \end{pmatrix} A \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\ &= 25A^2, \end{aligned}$$

tacking into account that $|\chi|^2 = 1$ we get that $A = 1/5$.

$$A = \frac{1}{5}$$

Solution 5: Expected values of spin

To find $\langle \hat{S}_x \rangle, \langle \hat{S}_y \rangle$ and $\langle \hat{S}_z \rangle$ we compute, $\langle \chi | \hat{S}_n | \chi \rangle$,

$$\begin{aligned} \langle \chi | \hat{S}_x | \chi \rangle &= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (-12i + 12i) = 0, \\ \langle \chi | \hat{S}_y | \chi \rangle &= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (-12 - 12) = -\frac{12}{25} \hbar, \\ \langle \chi | \hat{S}_z | \chi \rangle &= \frac{\hbar}{50} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar}{50} (9 - 16) = -\frac{7}{50} \hbar. \end{aligned}$$

$$\langle \hat{S}_x \rangle = 0, \quad \langle \hat{S}_y \rangle = -\frac{12}{25}\hbar, \quad \langle \hat{S}_z \rangle = -\frac{7}{50}\hbar.$$

Solution 6: Uncertainties

To compute the uncertainty we recall that $\sigma = \sqrt{\langle \hat{S}^2 \rangle - \langle \hat{S} \rangle^2}$. From the previous task we already know $\langle \hat{S} \rangle$, hence we only need to compute $\langle \hat{S}^2 \rangle$.

$$\begin{aligned} \langle \chi | \hat{S}_x^2 | \chi \rangle &= \frac{\hbar^2}{100} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar^2}{50} (9 + 16) = \frac{\hbar^2}{4}, \\ \langle \chi | \hat{S}_y^2 | \chi \rangle &= \frac{\hbar^2}{100} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar^2}{50} (9 + 16) = \frac{\hbar^2}{4}, \\ \langle \chi | \hat{S}_z^2 | \chi \rangle &= \frac{\hbar^2}{100} \begin{pmatrix} -3i & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3i \\ 4 \end{pmatrix} = \frac{\hbar^2}{50} (9 + 16) = \frac{\hbar^2}{4}. \end{aligned}$$

Now we can compute the uncertainties,

$$\begin{aligned} \sigma_x &= \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}, \\ \sigma_y &= \sqrt{\frac{\hbar^2}{4} - \frac{144}{625}\hbar^2} = \frac{7}{50}\hbar, \\ \sigma_z &= \sqrt{\frac{\hbar^2}{4} - \frac{49}{2500}\hbar^2} = \frac{12}{25}\hbar. \end{aligned}$$

$$\sigma_x = \frac{\hbar}{2}, \quad \sigma_y = \frac{7}{50}\hbar, \quad \sigma_z = \frac{12}{25}\hbar.$$

Solution 7: Checking consistency with the uncertainty principle.

With the uncertainty relation, $\sigma_A \sigma_B \geq 1/2 |\langle [\hat{A}, \hat{B}] \rangle|$, we can test the previous results,

$$\begin{aligned} \sigma_{\hat{S}_x} \sigma_{\hat{S}_y} &\geq \frac{1}{2} |i\hbar \langle \hat{S}_z \rangle| & \sigma_{\hat{S}_y} \sigma_{\hat{S}_z} &\geq \frac{1}{2} |i\hbar \langle \hat{S}_x \rangle| & \sigma_{\hat{S}_x} \sigma_{\hat{S}_z} &\geq \frac{1}{2} |i\hbar \langle \hat{S}_y \rangle| \\ \frac{\hbar}{2} \frac{7}{50} \hbar &\geq \frac{1}{2} \frac{7}{50} \hbar^2 & \frac{7}{50} \frac{12}{25} \hbar &\geq \frac{1}{2} 0 & \frac{\hbar}{2} \frac{12}{25} \hbar &\geq \frac{1}{2} \frac{12}{25} \hbar^2 \\ \frac{7}{100} \hbar^2 &\geq \frac{7}{100} \hbar^2 & \frac{42}{625} \hbar^2 &\geq 0 & \frac{12}{50} \hbar^2 &\geq \frac{12}{50} \hbar^2 \end{aligned}$$

This tells us that the previous results are consistent with the uncertainty principle.

$$\sigma_{\hat{S}_x} \sigma_{\hat{S}_y} \geq \frac{1}{2} |i\hbar \langle \hat{S}_z \rangle|, \quad \sigma_{\hat{S}_y} \sigma_{\hat{S}_z} \geq \frac{1}{2} |i\hbar \langle \hat{S}_x \rangle|, \quad \sigma_{\hat{S}_x} \sigma_{\hat{S}_z} \geq \frac{1}{2} |i\hbar \langle \hat{S}_y \rangle|$$

4 Problem 4.32 a

If you measure the component of spin angular momentum along the x direction, at time t , what is the probability that you would get $+\hbar/2$?

Solution 8: Practice spin state

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