

Homework 2

Professor: Dr. Alfonso Isaac Jaimes Nájera

Francisco Javier Vázquez Tavares A00827546

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1 Problem 4.25

If the electron were a classical solid sphere, with radius,

$$r_c = \frac{e^2}{4\pi\epsilon_0 mc^2}$$

(the so-called classical electron radius, obtained by assuming the electron's mass is attributable to energy stored in its electric field, via the Einstein formula $E = mc^2$), and its angular momentum is $\hbar/2$, then how fast (in m/s) would a point on the “equator” be moving? Does this model make sense? (Actually, the radius of the electron is known experimentally to be much less than r_c but this only makes matters worse.)

Solution 1: Classical spinning

From the classical framework the angular momentum is modeled with the following relation,

$$L = I\omega,$$

where I is the moment of inertia, which in this case is $I = 2/5 mr^2$ and ω is the angular frequency, that can be expressed as $\omega = v/r$. Replacing these equivalences into the angular momentum equation we can get the following expression for v ,

$$v = \frac{5}{2} \frac{L}{mr_c},$$

substituting the values of L and r_c ,

$$v = \frac{5\pi\hbar\epsilon_0}{e^2} c^2.$$

Recalling the order of magnitude of the constants, $e \approx 10^{-19}$, $\hbar \approx 10^{-34}$, $\epsilon_0 \approx 10^{-12}$ and $c \approx 10^8$, we get that,

$$\frac{5\pi\hbar\epsilon_0}{e^2} c \approx 90,$$

which tells us that the velocity at the ecuator is 90 times the velocity of light, which does not make sense.

$$v \approx 90c.$$

2 Problem 4.26

- Check that the spin matrices(1) obey the fundamental commutation relations for angular momentum(2).

$$\hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1)$$

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y \quad (2)$$

- Show that the Pauli spin matrices(3) satisfy the product rule(4), where the indices stand for x, y, z and ϵ_{jkl} is the Levi-Civita symbol.

$$\sigma_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3)$$

$$\sigma_j \sigma_k = \delta_{jk} + i \sum_l \epsilon_{jkl} \sigma_l. \quad (4)$$

Solution 2: Commutation relations

We start computing $[\hat{S}_x, \hat{S}_y]$,

$$\begin{aligned} [\hat{S}_x, \hat{S}_y] &= \frac{\hbar^2}{4} \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right] \\ &= i\hbar \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= i\hbar\hat{S}_z. \end{aligned}$$

Now $[\hat{S}_y, \hat{S}_z]$,

$$\begin{aligned} [\hat{S}_y, \hat{S}_z] &= \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right] \\ &= i\hbar \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= i\hbar\hat{S}_x. \end{aligned}$$

Finally $[\hat{S}_z, \hat{S}_x]$,

$$\begin{aligned} [\hat{S}_z, \hat{S}_x] &= \frac{\hbar^2}{4} \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \\ &= (-i\hbar) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= i\hbar \hat{S}_y. \end{aligned}$$

Hence, it is proof that,

$$[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$

Solution 3: Pauli spin matrices properties

To proof that the product rule is satisfied by the Pauli matrices, we compute all the products,

Yes, it worked

3 Problem 4.27

An electron is in the spin state,

$$\Xi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

- Determine the normalization constant A .
- Find the expectation values of S_x, S_y and S_z .
- Find the “uncertainties” $\sigma_{S_x}, \sigma_{S_y}$ and σ_{S_z} . (Note: These sigmas are standard deviations, not Pauli matrices!)
- Confirm that your results are consistent with all three uncertainty principles 4.100 and its cyclic permutations-only with S in place of L , of course.

Solution 4: Practice spin state

Sol

4 Problem 4.32 a

If you measure the component of spin angular momentum along the x direction, at time t , what is the probability that you would get $+\hbar/2$?

Solution 5: Practice spin state

Sol