Homework 2

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Contents

1 Problem 4.34

1. Aplly $\hat{S}_-$ to $|10\rangle$ and confirm that you get $\sqrt{2}\hbar\,|1-1\rangle$
2. Apply $\hat{S}_+$ to $|00\rangle$ and confirm that you get zero.
3. Show that $|11\rangle$ and $|1-1\rangle$ are eqigenstates of $\hat{S}^2$, with the appropriate eigenvalue.

Solution 1: $\hat{S}_-\,|10\rangle$

Recalling that $|10\rangle = 1/\sqrt{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ we can compute as follows,

$$\begin{aligned}
\hat{S}_-^{(T)}\,|10\rangle &= \left(\hat{S}_-^{(1)} \oplus \hat{S}_-^{(2)}\right)\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
&= \frac{1}{\sqrt{2}}\left[\left(\hat{S}_-^{(1)}\,|\uparrow\rangle_{(1)}\right)|\downarrow\rangle_{(2)} \oplus |\uparrow\rangle_{(1)} \cancelto{0}{\left(\hat{S}_-^{(2)}\,|\downarrow\rangle_{(2)}\right)} + \cancelto{0}{\left(\hat{S}_-^{(1)}\,|\downarrow\rangle_{(1)}\right)}|\uparrow\rangle_{(2)} \oplus |\downarrow\rangle_{(1)}\left(\hat{S}_-^{(2)}\,|\uparrow\rangle_{(2)}\right)\right] \\
&= \frac{1}{\sqrt{2}}\left[\hbar\,|\downarrow\rangle_{(1)}\,|\downarrow\rangle_{(2)} + \hbar\,|\downarrow\rangle_{(1)}\,|\downarrow\rangle_{(2)}\right] \\
&= \frac{\hbar}{\sqrt{2}}\left(|\downarrow\downarrow\rangle + |\downarrow\downarrow\rangle\right)
\end{aligned}$$

$$\boxed{\hat{S}_-^{(T)}\,|10\rangle = \sqrt{2}\hbar\,|1-1\rangle\,.}$$

Solution 2: $\hat{S}_+ \left|00\right\rangle$

Recalling that $\left|00\right\rangle = 1/\sqrt{2}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle)$ we can compute as follows,

$$\begin{aligned}
\hat{S}_+^{(T)} \left|10\right\rangle &= \left(\hat{S}_+^{(1)} \oplus \hat{S}_+^{(2)}\right) \frac{1}{\sqrt{2}} (\left|\uparrow\downarrow\right\rangle + \left|\downarrow\uparrow\right\rangle) \\
&= \frac{1}{\sqrt{2}} \left[\cancelto{0}{\left(\hat{S}_+^{(1)} \left|\uparrow\right\rangle_{(1)}\right)} \left|\downarrow\right\rangle_{(2)} \oplus \left|\uparrow\right\rangle_{(1)} \left(\hat{S}_+^{(2)} \left|\downarrow\right\rangle_{(2)}\right) - \left(\hat{S}_+^{(1)} \left|\downarrow\right\rangle_{(1)}\right) \left|\uparrow\right\rangle_{(2)} \oplus \left|\downarrow\right\rangle_{(1)} \cancelto{0}{\left(\hat{S}_+^{(2)} \left|\uparrow\right\rangle_{(2)}\right)} \right] \\
&= \frac{1}{\sqrt{2}} \left[\hbar \left|\uparrow\right\rangle_{(1)} \left|\uparrow\right\rangle_{(2)} - \hbar \left|\uparrow\right\rangle_{(1)} \left|\uparrow\right\rangle_{(2)} \right] \\
&= \frac{\hbar}{\sqrt{2}} (\left|\uparrow\uparrow\right\rangle - \left|\uparrow\uparrow\right\rangle)
\end{aligned}$$

$$\boxed{\hat{S}_+^{(T)} \left|00\right\rangle = 0.}$$

Solution 3: Eigenstates of $\hat{S}^2$

First we compute the expression of $\hat{S}^2$ for a system with two Hilbert spaces,

$$\begin{aligned}
\left(\hat{S}^{(T)}\right)^2 &= \left(\hat{S}^{(1)} \oplus \hat{S}^{(2)}\right) \cdot \left(\hat{S}^{(1)} \oplus \hat{S}^{(2)}\right) \\
&= \left(\hat{S}^{(1)}\right)^2 \oplus \left(\hat{S}^{(2)}\right)^2 \oplus 2\vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(2)}
\end{aligned}$$

Starting with $\left|11\right\rangle = \left|\uparrow\uparrow\right\rangle$,

$$\left(\hat{S}^{(T)}\right)^2 \left|11\right\rangle = \left(\left(\hat{S}^{(1)}\right)^2 \oplus \left(\hat{S}^{(2)}\right)^2 \oplus 2\vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(2)} \right) \left|\uparrow\uparrow\right\rangle$$

$$\begin{aligned}
\left(\hat{S}^{(T)}\right)^2 \left|11\right\rangle = &\left[\left(\hat{S}^{(1)}\right)^2 \left|\uparrow\right\rangle_{(1)} \left|\uparrow\right\rangle_{(2)} \right] \oplus \left[\left|\uparrow\right\rangle_{(1)} \left(\hat{S}^{(2)}\right)^2 \left|\uparrow\right\rangle_{(2)} \right] \\
&\oplus 2 \left[\hat{S}_x^{(1)} \left|\uparrow\right\rangle_{(1)} \hat{S}_x^{(2)} \left|\uparrow\right\rangle_{(2)} \oplus \hat{S}_y^{(1)} \left|\uparrow\right\rangle_{(1)} \hat{S}_y^{(2)} \left|\uparrow\right\rangle_{(2)} \oplus \hat{S}_z^{(1)} \left|\uparrow\right\rangle_{(1)} \hat{S}_z^{(2)} \left|\uparrow\right\rangle_{(2)} \right]
\end{aligned}$$

$$\begin{aligned}
\left(\hat{S}^{(T)}\right)^2 \left|11\right\rangle &= \frac{3}{2}\hbar^2 \left|\uparrow\uparrow\right\rangle \oplus 2 \left[\frac{\hbar^2}{4} \left|\downarrow\downarrow\right\rangle \ominus \frac{\hbar^2}{4} \left|\downarrow\downarrow\right\rangle \oplus \frac{\hbar^2}{4} \left|\uparrow\uparrow\right\rangle \right] \\
&= \frac{3}{2}\hbar^2 \left|\uparrow\uparrow\right\rangle \oplus \frac{\hbar^2}{2} \left|\uparrow\uparrow\right\rangle \\
&= 2\hbar^2 \left|\uparrow\uparrow\right\rangle .
\end{aligned}$$

Now, we apply the same procedure for $\left|1-1\right\rangle$,

$$\left(\hat{S}^{(T)}\right)^2 \left|1-1\right\rangle = \left(\left(\hat{S}^{(1)}\right)^2 \oplus \left(\hat{S}^{(2)}\right)^2 \oplus 2\vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(2)} \right) \left|\downarrow\downarrow\right\rangle$$

$$\left(\hat{S}^{(T)}\right)^2 |1-1\rangle = \left[\left(\hat{S}^{(1)}\right)^2 |\downarrow\rangle_{(1)} |\downarrow\rangle_{(2)}\right] \oplus \left[|\downarrow\rangle_{(1)} \left(\hat{S}^{(2)}\right)^2 |\downarrow\rangle_{(2)}\right] \\ \oplus 2\left[\hat{S}_x^{(1)} |\downarrow\rangle_{(1)} \hat{S}_x^{(2)} |\downarrow\rangle_{(2)} \oplus \hat{S}_y^{(1)} |\downarrow\rangle_{(1)} \hat{S}_y^{(2)} |\downarrow\rangle_{(2)} \oplus \hat{S}_z^{(1)} |\downarrow\rangle_{(1)} \hat{S}_z^{(2)} |\downarrow\rangle_{(2)}\right]$$

$$\begin{aligned} \left(\hat{S}^{(T)}\right)^2 |1-1\rangle &= \frac{3}{2}\hbar^2 |\downarrow\downarrow\rangle \oplus 2\left[\frac{\hbar^2}{4}|\uparrow\uparrow\rangle \ominus \frac{\hbar^2}{4}|\uparrow\uparrow\rangle \oplus \frac{\hbar^2}{4}|\downarrow\downarrow\rangle\right] \\ &= \frac{3}{2}\hbar^2 |\downarrow\downarrow\rangle \oplus \frac{\hbar^2}{2}|\downarrow\downarrow\rangle \\ &= 2\hbar^2 |\downarrow\downarrow\rangle \end{aligned}$$

$$\boxed{\left(\hat{S}^{(T)}\right)^2 |11\rangle = 2\hbar^2 |\uparrow\uparrow\rangle, \quad \left(\hat{S}^{(T)}\right)^2 |1-1\rangle = 2\hbar^2 |\downarrow\downarrow\rangle}$$

2 Problem 4.35

Quarks carry spin 1/2. Three quarks bind together to make a make baryon (such as a proton or neutron): two quarks (or more precisely a quark and an antiquark) bind together to make a meson (such as the pion or the kaon). Assume the quarks are in the ground (so the orbital angular momentum is zero).

- What spins are possible for baryons?
- What spins are possible for mesons?

3 Problem 5.4

- If ψ_a are orthogonal, and both normalized, what is the constant A in 5.10?
- If $\psi_a = \psi_b$ (and it is normalized), what is A? (This case, of course, occurs only for bosons.)

4 Problem 5.5

- Write down the Hamiltonian for two noninteracting identical particles in the infinite square well. Verify that the fermion ground state given in Example 5.1 is an eienfunction of H, with the appropriate eigenvalue.
- Find the next two excited states (beyond the ones in Example 5.1)-wave functions and energies-for each of the three cases (distinguishable, identical bosons, identical fermions).