Quantumm Optics Class-Notes and others

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Summary Class notes, post-class notes and others for the course of Quantum Optics. Semester February-June 2025

Harmonic oscilator

February 12

First Quantization

February 17

Properties of Quantumm electric field

Februrary 19

Fock States

February 19

Coherent states

February 24

According to the professor the sections **??**, where the hard part of the course. The next sections we are going to describe different states and analize there properties.

Lets remember the operator of an enlectric field with x component and one mode,

$$E_x = i\left(\frac{\hbar\omega}{2\epsilon_o V}\right)^{1/2}\left(\hat{a}\exp\left[-i\omega t\right] - \hat{a}^\dagger\exp\left[-i\omega t\right]\right),$$

which can be expressed in terms of quadratures as,

$$E_x = 2\left(\frac{\hbar\omega}{2\epsilon_o V}\right)^{1/2}\left(\hat{X}_1\sin\left(\omega t\right) + \hat{X}_2\cos\left(\omega t\right)\right)$$

So, when we want to get the expeted value of the electric field of a Fock state $|n\rangle$ we get $\langle n|E|n\rangle = 0$. So we need other states to model a laser. A useful observation is that $\langle n|\hat{X}_1^2|n\rangle = 1/4(2n+1)$ and $\Delta\hat{X}_1\Delta\hat{X}_2 = 1/4(2n+1)$.

We are going to study the "*Gleuber states*". Which are the states that can describe the laser. For that we have 4 definitions,

Reminders of some properties.

$$\left[\hat{a},\hat{a}^\dagger\right] = 1$$
$$\hat{X}_1 = \frac{\hat{a}+\hat{a}^\dagger}{2}$$
$$\hat{X}_2 = \frac{\hat{a}-\hat{a}^\dagger}{2i}$$
$$\hat{a} = \hat{X}_1 + i\hat{X}_2$$
$$\hat{a}^\dagger = \hat{X}_1 - i\hat{X}_2$$
$$\left[\hat{X}_1,\hat{X}_2\right] = \frac{i}{2}$$
$$\Delta\hat{X}_1\Delta\hat{X}_2 \geq \frac{1}{4}$$

Definition 1 Eigenstates of $\hat{a}$

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle, \quad \alpha\in\mathbb{C}.$$

Definition 2 Displaced vacuum[1].

$$\hat{D}(\alpha) = \exp\left[\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}\right], \quad |\alpha\rangle = \hat{D}(\alpha)|0\rangle$$

[1] It is important to use the following definition of e^x, becuase the argument are matrices and vectors,

$$\exp(x) = \sum \frac{x^n}{n!}.$$

Definition 3 Fock States

$$|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Definition 4

$$\Delta\hat{X}_1 \Delta\hat{X}_2 = \frac{1}{2}$$

Then we start to analyse a coherent state and a Fock state.

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= \exp\left[-|\alpha|^2\right] \sum_{n=0}\sum_{m=0} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!}\sqrt{n!}} \langle m|n\rangle \\
&= \exp\left[-|\alpha|^2\right] \sum \frac{|\alpha|^{2n}}{n!} \\
&= 1
\end{aligned}$$

Later we analyse the creation and anhilihation operator. Knowing that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and when we compute its adjoint $\langle\alpha|\hat{a}^{\dagger} = \alpha^{*}\langle\alpha|$, we get different eigenvalues for each operator. However, if we compute the expected value of both operators in a state we get,

Usefull properties,

$$\begin{aligned}
\hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a} &= 1 \\
\langle\alpha|\hat{a}\hat{a}^{\dagger}|\alpha\rangle &= |\alpha|^2 + 1 \\
\hat{a}|n\rangle &= \sqrt{n}|n-1\rangle
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\hat{a}^{\dagger}\hat{a}|\alpha\rangle &= \left\langle\alpha\middle|\hat{a}^{\dagger}\hat{a}\alpha\right\rangle \\
&= \langle\alpha|\hat{a}^{\dagger}\alpha \\
&= \alpha^{*}\alpha \\
&= |\alpha|^2
\end{aligned}$$

Now we apply the $\hat{a}$ operator in a Fock state,

$$\begin{aligned}
\hat{a}|\alpha\rangle &= \hat{a}\left(\exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} |\alpha\rangle\right) \\
&= \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} \hat{a}|\alpha\rangle \\
&= \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} \left(\sqrt{n}|\alpha-1\rangle\right),
\end{aligned}$$

When $n = 0$ is like taking photons to the vaccum, which does not make sense, hence we can translate the sum as follows,

$$\hat{a}|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=1} \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle$$

What is the probability of detecting n photons[2]?

[2] Is the Poisson distribution.

$$|\langle n|\alpha\rangle|^2 = \exp\left[-\frac{|\alpha|^2}{2}\right] \frac{|\alpha|^2}{n!}$$

Now wetake the expected value of $|\alpha\rangle$ with $\hat{a}, \hat{a}^{\dagger}, \hat{X}_1, \hat{X}_2$,

$$
\begin{aligned}
\langle\alpha|\hat{a}|\alpha\rangle &= \alpha \\
\langle\alpha|\hat{a}^{\dagger}|\alpha\rangle &= \alpha^* \\
\langle\alpha|\hat{X}_1|\alpha\rangle &= \frac{1}{2}(\alpha+\alpha^*) \\
\langle\alpha|\hat{X}_2|\alpha\rangle &= \frac{1}{2}(\alpha-\alpha^*)
\end{aligned}
$$

Then we checked the Qadrature noise,

$$
\begin{aligned}
\langle\alpha|\hat{X}_1^2|\alpha\rangle =& \ \langle\alpha|\frac{1}{4}(\hat{a}+\hat{a}^{\dagger})^2|\alpha\rangle \\
&= \frac{1}{4}\Big(\alpha^2+\alpha^{*2}+2|\alpha|^2+1\Big)
\end{aligned}
$$

therefore,

$$
\left\langle \hat{X}_1^2 \middle| \hat{X}_1^2 \right\rangle - \left\langle \hat{X}_1 | \hat{X}_1 \right\rangle^2 = \frac{1}{4}
$$

As homework, compute $\langle\alpha|\hat{X}_2^2|\alpha\rangle$

Displacement Operator

Squeezed States

February 26