Quantumm Optics Class-Notes and others

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Summary Class notes, post-class notes and others for the course of Quantum Optics. Semester February-June 2025

Harmonic oscilator

February 12

First Quantization

February 17

Properties of Quantumm electric field

Februrary 19

Fock States

February 19

Coherent states

February 24

According to the professor the sections,,,, where the hard part of the course. The next sections we are going to describe different states and analize there properties.

Lets remember the operator of an enlectric field with x component and one mode,

$$E_x = i\left(\frac{\hbar\omega}{2\epsilon_o V}\right)^{1/2}\left(\hat{a}\exp\left[-i\omega t\right] - \hat{a}^\dagger\exp\left[-i\omega t\right]\right),$$

which can be expressed in terms of quadratures as,

$$E_x = 2\left(\frac{\hbar\omega}{2\epsilon_o V}\right)^{1/2}\left(\hat{X}_1\sin\left(\omega t\right) + \hat{X}_2\cos\left(\omega t\right)\right)$$

So, when we want to get the expeted value of the electric field of a Fock state $|n\rangle$ we get $\langle n|E|n\rangle = 0$. So we need other states to model a laser. A useful observation is that $\langle n|\hat{X}_1^2|n\rangle = 1/4(2n+1)$ and $\Delta\hat{X}_1\Delta\hat{X}_2 = 1/4(2n+1)$.

We are going to study the "*Gleuber states*". Which are the states that can describe the laser. For that we have 4 definitions,

Reminders of some properties.

$$\begin{aligned}
\left[\hat{a},\hat{a}^\dagger\right] &= 1\\
\hat{X}_1 &= \frac{\hat{a}+\hat{a}^\dagger}{2}\\
\hat{X}_2 &= \frac{\hat{a}-\hat{a}^\dagger}{2i}\\
\hat{a} &= \hat{X}_1 + i\hat{X}_2\\
\hat{a}^\dagger &= \hat{X}_1 - i\hat{X}_2\\
\left[\hat{X}_1,\hat{X}_2\right] &= \frac{i}{2}\\
\Delta\hat{X}_1\Delta\hat{X}_2 &\geq \frac{1}{4}
\end{aligned}$$

Definition 1 Eigenstates of $\hat{a}$

$$\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle, \quad \alpha\in\mathbb{C}.$$

Definition 2 Displaced vacuum[1].

$$\hat{D}(\alpha) = \exp\left[\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right], \quad |\alpha\rangle = \hat{D}(\alpha)\,|0\rangle$$

[1] It is important to use the following definition of e^x, becuase the argument are matrices and vectors,

$$\exp(x) = \sum \frac{x^n}{n!}.$$

Definition 3 Fock States

$$|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \sum \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle$$

Definition 4

$$\Delta\hat{X}_1\Delta\hat{X}_2 = \frac{1}{2}$$

Then we start to analyse a coherent state and a Fock state.

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= \exp\left[-|\alpha|^2\right] \sum_{n=0}\sum_{m=0} \frac{\alpha^{*m}\alpha^n}{\sqrt{m!}\sqrt{n!}}\,\langle m|n\rangle \\
&= \exp\left[-|\alpha|^2\right] \sum \frac{|\alpha|^{2n}}{n!} \\
&= 1
\end{aligned}$$

Later we analyse the creation and anhilihation operator. Knowing that $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$ and when we compute its adjoint $\langle\alpha|\,\hat{a}^\dagger = \alpha^*\,\langle\alpha|$, we get different eigenvalues for each operator. However, if we compute the expected value of both operators in a state we get,

$$\begin{aligned}
\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle &= \left\langle\alpha\middle|\hat{a}^\dagger\hat{a}\alpha\right\rangle \\
&= \langle\alpha|\,\hat{a}^\dagger\alpha \\
&= \alpha^*\alpha \\
&= |\alpha|^2
\end{aligned}$$

Usefull properties,

$$\begin{aligned}
\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1 \\
\langle\alpha|\hat{a}\hat{a}^\dagger|\alpha\rangle = |\alpha|^2 + 1 \\
\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle
\end{aligned}$$

Now we apply the $\hat{a}$ operator in a Fock state,

$$\begin{aligned}
\hat{a}\,|\alpha\rangle &= \hat{a}\left(\exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}}\,|\alpha\rangle\right) \\
&= \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}}\hat{a}\,|\alpha\rangle \\
&= \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}}\left(\sqrt{n}\,|\alpha-1\rangle\right),
\end{aligned}$$

When $n = 0$ is like taking photons to the vaccum, which does not make sense, hence we can translate the sum as follows,

$$\hat{a}\,|\alpha\rangle = \exp\left[-\frac{|\alpha|^2}{2}\right] \sum_{n=1} \frac{\alpha^n}{\sqrt{(n-1)!}}\,|n-1\rangle$$

What is the probability of detecting n photons[2]?

[2] Is the Poisson distribution.

$$|\langle n|\alpha\rangle|^2 = \exp\left[-\frac{|\alpha|^2}{2}\right] \frac{|\alpha|^2}{n!}$$

Now wetake the expected value of $|\alpha\rangle$ with $\hat{a}, \hat{a}^\dagger, \hat{X}_1, \hat{X}_2$,

$$\begin{aligned}
\langle\alpha|\hat{a}|\alpha\rangle &= \alpha \\
\langle\alpha|\hat{a}^\dagger|\alpha\rangle &= \alpha^* \\
\langle\alpha|\hat{X}_1|\alpha\rangle &= \frac{1}{2}(\alpha + \alpha^*) \\
\langle\alpha|\hat{X}_2|\alpha\rangle &= \frac{1}{2}(\alpha - \alpha^*)
\end{aligned}$$

Then we checked the Qadrature noise,

$$\begin{aligned}
\langle\alpha|\hat{X}_1^2|\alpha\rangle = &\ \langle\alpha|\frac{1}{4}(\hat{a} + \hat{a}^\dagger)^2|\alpha\rangle \\
&= \frac{1}{4}\left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1\right)
\end{aligned}$$

therefore,

$$\left\langle \hat{X}_1^2 \middle| \hat{X}_1^2 \right\rangle - \langle \hat{X}_1|\hat{X}_1\rangle^2 = \frac{1}{4}$$

As homework, compute $\langle\alpha|\hat{X}_2^2|\alpha\rangle$

Displacement Operator

Squeezed States

February 26

Beam Splitters

March 10

In this session we start the second module of the course with analyzing the beam splitter in a quantum framework.

In the classical framework we can model the beam splitter using the Fresnel Coefficients. Assuming with that we have an power input we analyze the reflection power and a transmision coefficient. It is important to acknoledge that the reflective coefficient is complex. This can be relatead to a phase shift during the reflection. Also, we set that $|r|^2 + |t|^2 = 1$.

Now lets go Quantum, that is, that instead of analyzing the power input and output in the beam splitter we are going to explore the operators for each light beam. Lets assume that the operator $\hat{a}_1$ represent the input light beam, the operator $\hat{a}_2 = r\hat{a}_1$ represent the light beam that is reflected and finally, an operator $\hat{a}_3 = r\hat{a}_1$ that takes into account the transmitted light beam. All of these operators needs to be indepenedent, because are represented 3 different modes. Also, we kwnow that the creation and anhilihation operators follows these commutation rules,

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}, \quad \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0, \quad \left[\hat{a}_i, \hat{a}_j\right] = 0.$$

From those relations we can see that[3],

$$\left[\hat{a}_2, \hat{a}_2^\dagger\right] = |r|^2, \quad \left[\hat{a}_3, \hat{a}_3^\dagger\right] = |t|^2, \quad \left[\hat{a}_2, \hat{a}_3^\dagger\right] = rt^*.$$

[3]
$$\begin{aligned}
\left[\hat{a}_2, \hat{a}_2^\dagger\right] &= \hat{a}_2\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_2 \\
&= (r\hat{a}_1)(r\hat{a}_1)^\dagger - (r\hat{a}_1)^\dagger(r\hat{a}_1) \\
&= |r|^2\left(\hat{a}_1\hat{a}_1^\dagger\right) - |r|^2\left(\hat{a}_1^\dagger\hat{a}_1\right) \\
&= |r|^2\left[\hat{a}_1, \hat{a}_1^\dagger\right],
\end{aligned}$$
with the same

When we compare this results with the commutation rules that we stablish before, we can see that can not be true. Therefore we need to change the definition of the operators $\hat{a}_2$ and $\hat{a}_3$.

From previus discussions, we can consider the fluctuations of the vacuum into the system with the followinf transformations,

$$\hat{a}_2 = r\hat{a}_1 + t^{'}\hat{a}_0,$$
$$\hat{a}_3 = t\hat{a}_1 + r^{'}\hat{a}_0,$$

or,

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} t^{'} & r \\ r^{'} & t \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix}.$$

When we compute the commutation relations[4] we see that the commutation relations holds with the following relations,

$$r^* t^{'} + r^{'} t^* = 0 \quad r^* t + r^{'} t^{'*} = 0$$
$$\left|r^{'}\right| = |r| \quad \left|t^{'}\right| = |t|$$
$$|r|^2 + |t|^2 = 1.$$

Also, tese relations are known as the reciprocity relations and can be also derived on the basis of energy conservation.

Now, lets analyse a 50:50 beam splitter,[5] assuming the reflected beam suffers a $\pi/2$ phase shifts, the input and output modes are related according to,

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix}.$$

Since the trnasformation between input and output modes must be unitary[6], we can write,

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \hat{a} U^{\dagger} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix} \hat{a} U.$$

Since we are evolving the operator, this constitues a Heisenberg picture formulation of a beam spltter. For the specific transformation of 50:50 beam splitter, the operator $\hat{U}$ has the form,

$$\hat{U} = \exp\left[i\frac{\pi}{4}\left(\hat{a}_0^{\dagger}\hat{a}_1 + \hat{a}_0\hat{a}_1^{\dagger}\right)\right].$$

Now, lets do the following example, consider the single photon input state $|0\rangle_0 |1\rangle_1$ which we may write as $\hat{a}_1^{\dagger}|0\rangle_0 |0\rangle_1$. For the 50:50 beam splitter we find that[7] $\hat{a}_1^{\dagger} = \left(i\hat{a}_2^{\dagger} + \hat{a}_3^{\dagger}\right)/\sqrt{2}$.

Tacking into account all of those properties, we can write,

$$\begin{aligned} |0\rangle_0 |1\rangle_1 &\to \hat{a}_1^{\dagger}|0\rangle_2 |0\rangle_3 \\ &\to \frac{1}{\sqrt{2}}\left(i\hat{a}_2^{\dagger} + \hat{a}_3^{\dagger}\right)|0\rangle_2 |0\rangle_3 \\ &\to \frac{1}{\sqrt{2}}\left(i\hat{a}_2^{\dagger}|0\rangle_2 |0\rangle_3 + |0\rangle_2 \hat{a}_3^{\dagger}|0\rangle_3\right) \\ &\to \frac{1}{\sqrt{2}}(i|1\rangle_2 |0\rangle_3 + |0\rangle_2 |1\rangle_3). \end{aligned}$$

[4]

$$\begin{aligned}\left[\hat{a}_2, \hat{a}_2^{\dagger}\right] &= \left(r\hat{a}_1 + t^{'}\hat{a}_0\right)\left(\hat{a}_1^{\dagger}r^* + \hat{a}_0^{\dagger}t^{'*}\right) \\ &\quad - \left(\hat{a}_1^{\dagger}r^* + \hat{a}_0^{\dagger}t^{'*}\right)\left(r\hat{a}_1 + t^{'}\hat{a}_0\right)\end{aligned}$$

$$\left[\hat{a}_2, \hat{a}_2^{\dagger}\right] = |r|^2\hat{a}_1\hat{a}_1^{\dagger} + rt^{'*}\hat{a}_1\hat{a}_0^{\dagger} + t^{'}r^*\hat{a}_0\hat{a}_1^{\dagger} + |t|^2\hat{a}_0\hat{a}_0^{\dagger}$$
$$- |r|^2\hat{a}_1^{\dagger}\hat{a}_1 - rt^{'*}\hat{a}_0^{\dagger}\hat{a}_1 - t^{'}r^*\hat{a}_1^{\dagger}\hat{a}_0 - |t|^2\hat{a}_0^{\dagger}\hat{a}_0$$

This can be reduce if we impose that $rt^{'*} = r^* t^{'}$,

$$\begin{aligned}\left[\hat{a}_2, \hat{a}_2^{\dagger}\right] &= |r|^2 \cancelto{1}{\left[\hat{a}_1, \hat{a}_1^{\dagger}\right]} + rt^{'*}\cancelto{0}{\left[\hat{a}_1, \hat{a}_0^{\dagger}\right]} \\ &\quad + t^{'}r^*\cancelto{0}{\left[\hat{a}_0, \hat{a}_1^{\dagger}\right]} + |t|^2\cancelto{1}{\left[\hat{a}_0, \hat{a}_0^{\dagger}\right]}\end{aligned}$$

which holds that $\left[\hat{a}_2, \hat{a}_2^{\dagger}\right] = |r|^2 + |t|^2 = 1$

[5] If the beam splitter is constructed as a single dielectric layer, the reflected and transmitted beams will differ in phase by a factor of $\exp(\pm i\pi/2) = \pm i$.

[6] Not sure why

[7]

$$\begin{pmatrix} \hat{a}_0^{\dagger} \\ \hat{a}_1^{\dagger} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_0^{\dagger} \\ \hat{a}_1^{\dagger} \end{pmatrix}$$

Be careful, it is the same matrix due to the conjugate.

This result can be interpreted as a single-photon incident at one of the input ports of the beam splitter, the other port containing only vacuum, will be either trnasmitted or reflected with equal probability.

Little notes of entanglement. The mathematical representation of an entangle state is as follows,

$$|\Psi\rangle = \sum |\psi_1\rangle |\psi_2\rangle$$