Lecture Activity 5

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Lecture Activities 5

Apply the Gram-Schmidt procedure to construct an orthonormal basis

$$|w_1\rangle = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad |w_2\rangle = \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad |w_3\rangle = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

① Normalice $|w_1\rangle$

$$|u_1\rangle = \frac{|w_1\rangle}{\| |w_1\rangle \|} = \frac{|w_1\rangle}{\sqrt{\langle w_1|w_1\rangle}} = \frac{1}{\sqrt{2}}|w_1\rangle =$$

② Subtract projection of $|w_2\rangle$ on $|u_1\rangle$:

$$|u_2\rangle = |w_2\rangle - \langle u_1|w_2\rangle|u_1\rangle$$

$$= |w_2\rangle - \frac{1}{\sqrt{2}}|u_1\rangle = \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\0\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1\end{bmatrix}$$

③ Normalize $|u_2\rangle$

$$|u_2\rangle = \frac{|u_2\rangle}{\sqrt{\langle u_2|u_2\rangle}} = \sqrt{\frac{2}{3}}\begin{bmatrix}1/2\\-1/2\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{6}\\-1/\sqrt{6}\\\sqrt{2}/\sqrt{3}\end{bmatrix} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

④ Subtract projections

$$|u_3\rangle = |w_3\rangle - \left(\langle u_1|w_3\rangle|u_1\rangle + \langle u_2|w_3\rangle|u_2\rangle\right)$$

$$= |w_3\rangle - \left(\frac{1}{\sqrt{2}}|u_1\rangle + \frac{1}{\sqrt{6}}|u_2\rangle\right)$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/6 \\ -1/6 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

⑤ Normalize

$$|u_3\rangle = \frac{|u_3\rangle}{\sqrt{\langle u_3|u_3\rangle}} = \frac{\sqrt{3}}{2}|u_3\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

⑥ Answer and proof

$$|u_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad |u_2\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \quad |u_3\rangle = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\langle u_1|u_2\rangle = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{6}}(1 - 1 + 0) = 0$$

$$\langle u_1|u_3\rangle = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{3}}(-1 + 1 + 0) = 0$$

$$\langle u_2|u_3\rangle = \frac{1}{\sqrt{6}}\cdot\frac{1}{\sqrt{3}}(-1 - 1 + 2) = 0$$

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Eigenvalues and Eigenvectors

Compute the eigenvalues and eigenvectors of

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

① Finding characteristic equation:

$$\det(B-\lambda I)=0 \rightarrow \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right)=0 \rightarrow \lambda^2-1=0$$

② Solve for the eigenvalues λ

$$\lambda^2 - 1 = 0 \rightarrow \lambda^2 = 1 \rightarrow \lambda = \pm 1 \rightarrow \lambda \in \{-1, 1\}$$

③ Solving $(B - \lambda I)|v\rangle = 0$ to find eigenvector

3.a)

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \rightarrow \begin{cases} -a + b = 0 \\ a - b = 0 \end{cases} \Rightarrow a = b$$

$$\lambda = +1 \rightarrow |+1\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3.b)

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = 0 \rightarrow \begin{cases} a + b = 0 \\ a + b = 0 \end{cases} \Rightarrow a = -b$$

$$\lambda = -1 \rightarrow |-1\rangle = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

④ Normalizing eigenvectors

$$|+1\rangle = \frac{|+1\rangle}{\sqrt{\langle +1|+1\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|-1\rangle = \frac{|-1\rangle}{\sqrt{\langle -1|-1\rangle}} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

3 Jypter notebook

Notebook with the procedures/answers of the lecture activity.

[28]:
```
# Import the numpy library
import numpy as np

# Usefull function to print matrices (https://gist.github.com/braingineer/
↪d801735dac07ff3ac4d746e1f218ab75)
def matprint(mat, fmt="g"):
    col_maxes = [max([len(("{:"+fmt+"}").format(x)) for x in col]) for col in␣
↪mat.T]
    for x in mat:
        for i, y in enumerate(x):
            print(("{:"+str(col_maxes[i])+fmt+"}").format(y), end="  ")
        print("")
```

3.1 2 Finding eigenvalues and eigenvectors (Page 28)

Considering the following matrix

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We are going to use python to find the eigenvalues and eigen vectors.

[29]:
```
# Declare the matrix
B = np.array([[0, 1],[1, 0]]);

# Print the matrix
print("Matrix\n")
matprint(B, fmt="g")
print("\n")

# Use the command to find the eigenvalues and eigenvectors
eigenvalues, eigenvectors = np.linalg.eig(B)

# Print the answers
for i in range(len(eigenvalues)):
    print(f"Eigenvalue {i+1}: {eigenvalues[i]}")
    print(f"Corresponding eigenvector:\n{eigenvectors[:, i]}\n")
```

```
Matrix

0  1
1  0


Eigenvalue 1: 1.0
Corresponding eigenvector:
```

```
[0.70710678 0.70710678]

Eigenvalue 2: -1.0
Corresponding eigenvector:
[-0.70710678  0.70710678]
```

3.2 For each eigenvalue, write down its corresponding eigenvector (coding)

We are going to use the same command to find eigenvalues and eigenvectors of the following matrix

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{pmatrix}.$$

[35]:

```
M = np.array([[2, 0, 0],
              [0, 3, 4],
              [0, 4, 9]]);

# Print the matrix
print("Matrix\n")
matprint(M, fmt="g")
print("\n")

vals, vecs = np.linalg.eig(M)

# Print the answers
for i in range(len(vals)):
    print(f"Eigenvalue {i+1}: {vals[i]}")
    print(f"Corresponding eigenvector:\n{vecs[:, i]}\n")
```

```
Matrix

2  0  0
0  3  4
0  4  9


Eigenvalue 1: 11.0
Corresponding eigenvector:
[0.         0.4472136  0.89442719]

Eigenvalue 2: 1.0
Corresponding eigenvector:
[ 0.          0.89442719 -0.4472136 ]

Eigenvalue 3: 2.0
Corresponding eigenvector:
```

```
[1. 0. 0.]
```