Homework 2

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1 Problem 4.34

1. Aplly $\hat{S}_-$ to $|10\rangle$ and confirm that you get $\sqrt{2}\hbar\,|1-1\rangle$
2. Apply $\hat{S}_+$ to $|00\rangle$ and confirm that you get zero.
3. Show that $|11\rangle$ and $|1-1\rangle$ are eqigenstates of $\hat{S}^2$, with the appropriate eigenvalue.

Solution 1: $\hat{S}_-\,|10\rangle$

Recalling that $|10\rangle = 1/\sqrt{2}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ we can compute as follows,

$$\begin{aligned}
\hat{S}_-^{(T)}\,|10\rangle &= \left(\hat{S}_-^{(1)} \oplus \hat{S}_-^{(2)}\right)\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\\
&= \frac{1}{\sqrt{2}}\left[\left(\hat{S}_-^{(1)}\,|\uparrow\rangle_{(1)}\right)|\downarrow\rangle_{(2)} \oplus |\uparrow\rangle_{(1)}\cancelto{0}{\left(\hat{S}_-^{(2)}\,|\downarrow\rangle_{(2)}\right)} + \cancelto{0}{\left(\hat{S}_-^{(1)}\,|\downarrow\rangle_{(1)}\right)}|\uparrow\rangle_{(2)} \oplus |\downarrow\rangle_{(1)}\left(\hat{S}_-^{(2)}\,|\uparrow\rangle_{(2)}\right)\right]\\
&= \frac{1}{\sqrt{2}}\left[\hbar\,|\downarrow\rangle_{(1)}\,|\downarrow\rangle_{(2)} + \hbar\,|\downarrow\rangle_{(1)}\,|\downarrow\rangle_{(2)}\right]\\
&= \frac{\hbar}{\sqrt{2}}\left(|\downarrow\downarrow\rangle + |\downarrow\downarrow\rangle\right)
\end{aligned}$$

$$\boxed{\hat{S}_-^{(T)}\,|10\rangle = \sqrt{2}\hbar\,|1-1\rangle\,.}$$

Solution 2: $\hat{S}_+ |00\rangle$

Recalling that $|00\rangle = 1/\sqrt{2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ we can compute as follows,

$$\begin{aligned}
\hat{S}_+^{(T)} |10\rangle &= \left(\hat{S}_+^{(1)} \oplus \hat{S}_+^{(2)}\right) \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
&= \frac{1}{\sqrt{2}} \left[\cancelto{0}{\left(\hat{S}_+^{(1)} |\uparrow\rangle_{(1)}\right)} |\downarrow\rangle_{(2)} \oplus |\uparrow\rangle_{(1)} \left(\hat{S}_+^{(2)} |\downarrow\rangle_{(2)}\right) - \left(\hat{S}_+^{(1)} |\downarrow\rangle_{(1)}\right) |\uparrow\rangle_{(2)} \oplus |\downarrow\rangle_{(1)} \cancelto{0}{\left(\hat{S}_+^{(2)} |\uparrow\rangle_{(2)}\right)} \right] \\
&= \frac{1}{\sqrt{2}} \left[\hbar |\uparrow\rangle_{(1)} |\uparrow\rangle_{(2)} - \hbar |\uparrow\rangle_{(1)} |\uparrow\rangle_{(2)} \right] \\
&= \frac{\hbar}{\sqrt{2}} (|\uparrow\uparrow\rangle - |\uparrow\uparrow\rangle)
\end{aligned}$$

$$\boxed{\hat{S}_+^{(T)} |00\rangle = 0.}$$

Solution 3: Eigenstates of $\hat{S}^2$

First we compute the expression of $\hat{S}^2$ for a system with two Hilbert spaces,

$$\begin{aligned}
\left(\hat{S}^{(T)}\right)^2 &= \left(\hat{S}^{(1)} \oplus \hat{S}^{(2)}\right) \cdot \left(\hat{S}^{(1)} \oplus \hat{S}^{(2)}\right) \\
&= \left(\hat{S}^{(1)}\right)^2 \oplus \left(\hat{S}^{(2)}\right)^2 \oplus 2\vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(2)}
\end{aligned}$$

Starting with $|11\rangle = |\uparrow\uparrow\rangle$,

$$\left(\hat{S}^{(T)}\right)^2 |11\rangle = \left(\left(\hat{S}^{(1)}\right)^2 \oplus \left(\hat{S}^{(2)}\right)^2 \oplus 2\vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(2)} \right) |\uparrow\uparrow\rangle$$

$$\begin{aligned}
\left(\hat{S}^{(T)}\right)^2 |11\rangle = &\left[\left(\hat{S}^{(1)}\right)^2 |\uparrow\rangle_{(1)} |\uparrow\rangle_{(2)} \right] \oplus \left[|\uparrow\rangle_{(1)} \left(\hat{S}^{(2)}\right)^2 |\uparrow\rangle_{(2)} \right] \\
&\oplus 2 \left[\hat{S}_x^{(1)} |\uparrow\rangle_{(1)} \hat{S}_x^{(2)} |\uparrow\rangle_{(2)} \oplus \hat{S}_y^{(1)} |\uparrow\rangle_{(1)} \hat{S}_y^{(2)} |\uparrow\rangle_{(2)} \oplus \hat{S}_z^{(1)} |\uparrow\rangle_{(1)} \hat{S}_z^{(2)} |\uparrow\rangle_{(2)} \right]
\end{aligned}$$

$$\begin{aligned}
\left(\hat{S}^{(T)}\right)^2 |11\rangle &= \frac{3}{2}\hbar^2 |\uparrow\uparrow\rangle \oplus 2 \left[\frac{\hbar^2}{4} |\downarrow\downarrow\rangle \ominus \frac{\hbar^2}{4} |\downarrow\downarrow\rangle \oplus \frac{\hbar^2}{4} |\uparrow\uparrow\rangle \right] \\
&= \frac{3}{2}\hbar^2 |\uparrow\uparrow\rangle \oplus \frac{\hbar^2}{2} |\uparrow\uparrow\rangle \\
&= 2\hbar^2 |\uparrow\uparrow\rangle .
\end{aligned}$$

Now, we apply the same procedure for $|1-1\rangle$,

$$\left(\hat{S}^{(T)}\right)^2 |1-1\rangle = \left(\left(\hat{S}^{(1)}\right)^2 \oplus \left(\hat{S}^{(2)}\right)^2 \oplus 2\vec{\hat{S}}^{(1)} \cdot \vec{\hat{S}}^{(2)} \right) |\downarrow\downarrow\rangle$$

$$\left(\hat{S}^{(T)}\right)^2 |1-1\rangle = \left[\left(\hat{S}^{(1)}\right)^2 |\downarrow\rangle_{(1)} |\downarrow\rangle_{(2)}\right] \oplus \left[|\downarrow\rangle_{(1)} \left(\hat{S}^{(2)}\right)^2 |\downarrow\rangle_{(2)}\right]$$
$$\oplus\, 2\left[\hat{S}_x^{(1)} |\downarrow\rangle_{(1)} \hat{S}_x^{(2)} |\downarrow\rangle_{(2)} \oplus \hat{S}_y^{(1)} |\downarrow\rangle_{(1)} \hat{S}_y^{(2)} |\downarrow\rangle_{(2)} \oplus \hat{S}_z^{(1)} |\downarrow\rangle_{(1)} \hat{S}_z^{(2)} |\downarrow\rangle_{(2)}\right]$$

$$\begin{aligned}\left(\hat{S}^{(T)}\right)^2 |1-1\rangle &= \frac{3}{2}\hbar^2 |\downarrow\downarrow\rangle \oplus 2\left[\frac{\hbar^2}{4}|\uparrow\uparrow\rangle \ominus \frac{\hbar^2}{4}|\uparrow\uparrow\rangle \oplus \frac{\hbar^2}{4}|\downarrow\downarrow\rangle\right] \\ &= \frac{3}{2}\hbar^2 |\downarrow\downarrow\rangle \oplus \frac{\hbar^2}{2}|\downarrow\downarrow\rangle \\ &= 2\hbar^2 |\downarrow\downarrow\rangle\end{aligned}$$

$$\boxed{\left(\hat{S}^{(T)}\right)^2 |11\rangle = 2\hbar^2 |\uparrow\uparrow\rangle\,, \quad \left(\hat{S}^{(T)}\right)^2 |1-1\rangle = 2\hbar^2 |\downarrow\downarrow\rangle}$$

2 Problem 4.35

Quarks carry spin 1/2. Three quarks bind together to make a make baryon (such as a proton or neutron): two quarks (or more precisely a quark and an antiquark) bind together to make a meson (such as the pion or the kaon). Assume the quarks are in the ground (so the orbital angular momentum is zero).

1. What spins are possible for baryons?
2. What spins are possible for mesons?

3 Problem 5.4

1. If ψ_a are orthogonal, and both normalized, what is the constant A in

$$\Psi_{\pm}(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\psi_a(\vec{r}_2)\right]?$$

2. If $\psi_a = \psi_b$ (and it is normalized), what is A? (This case, of course, occurs only for bosons.)

Solution 4: Constant of normalization A

First we can re-write the Ψ as follows, $\Psi_{\pm} = A\left(|\psi_a\psi_b\rangle \pm |\psi_b\psi_a\rangle\right)$. Now we can compute the inner product of the state and apply the ortho-normal properties of $|\psi_a\rangle$ and $|\psi_b\rangle$,

$$\begin{aligned}\langle\Psi|\Psi\rangle &= |A|^2\left(\langle\psi_a\psi_b| \pm \langle\psi_b\psi_a|\right)\left(|\psi_a\psi_b\rangle \pm |\psi_b\psi_a\rangle\right)\\ &= |A|^2\left(\langle\psi_a\psi_b|\psi_a\psi_b\rangle \pm \langle\psi_a\psi_b|\psi_b\psi_a\rangle \pm \langle\psi_b\psi_a|\psi_a\psi_b\rangle + \langle\psi_b\psi_a|\psi_b\psi_a\rangle\right)\\ &= 2|A|^2,\end{aligned}$$

since, $\langle\Psi|\Psi\rangle$ should be 1, $A = 1/\sqrt{2}$.

$$\boxed{A = \frac{1}{\sqrt{2}}.}$$

Solution 5: Constant of normalization A with $\psi_a = \psi_b$

From the previous procedure, we see that the second and third terms are equal to 1, leading to hte following results,

$$\langle\Psi|\Psi\rangle = 4|A|^2,$$

since, $\langle\Psi|\Psi\rangle$ should be 1, $A = 1/2$.

$$\boxed{A = \frac{1}{2}.}$$

4 Problem 5.5

1. Write down the Hamiltonian for two noninteracting identical particles in the infinite square well. Verify that the fermion ground state given in Example 5.1 is an eienfunction of $\hat{H}$, with the appropriate eigenvalue.

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(n\pi\frac{a}{x}\right), \quad E_n = n^2\frac{\pi^2\hbar^2}{2ma^2}.$$

2. Find the next two excited states (beyond the ones in Example 5.1)-wave functions and energies-for each of the three cases (distinguishable, identical bosons, identical fermions).

Solution 6: Hamiltonian of non-interacting identical particles

The hamiltonian that describe this system is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_2^2} = E\psi, \quad 0 \le x_1, x_2 \le a.$$

From the Example 5.1 we have,

$$\psi(x_1,x_2)=\frac{\sqrt{2}}{a}\left[\sin\left(\pi\frac{x_1}{a}\right)\sin\left(2\pi\frac{x_2}{a}\right)-\sin\left(2\pi\frac{x_1}{a}\right)\sin\left(\pi\frac{x_2}{a}\right)\right].$$

Now we compute the partial derivatives,

$$\begin{aligned}\frac{\partial^2\psi}{\partial x_1^2}&=\frac{\sqrt{2}}{a}\left[-\left(\frac{\pi}{a}\right)^2\sin\left(\pi\frac{x_1}{a}\right)\sin\left(2\pi\frac{x_2}{a}\right)+\left(\frac{2\pi}{a}\right)^2\sin\left(2\pi\frac{x_1}{a}\right)\sin\left(\pi\frac{x_2}{a}\right)\right]\\ \frac{\partial^2\psi}{\partial x_2^2}&=\frac{\sqrt{2}}{a}\left[-\left(\frac{2\pi}{a}\right)^2\sin\left(\pi\frac{x_1}{a}\right)\sin\left(2\pi\frac{x_2}{a}\right)+\left(\frac{\pi}{a}\right)^2\sin\left(2\pi\frac{x_1}{a}\right)\sin\left(\pi\frac{x_2}{a}\right)\right].\end{aligned}$$

Substituing those results into the hamiltonian we get,

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_1^2}-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x_2^2}&=\frac{\hbar^2}{2m}\left(-\frac{\partial^2\psi}{\partial x_1^2}-\frac{\partial^2\psi}{\partial x_2^2}\right)\\ &=\frac{\hbar^2}{2m}\left(\frac{\sqrt{2}}{a}\left[\left(\frac{2\pi}{a}\right)^2+\left(\frac{\pi}{a}\right)^2\right]\left[\sin\left(\pi\frac{x_1}{a}\right)\sin\left(2\pi\frac{x_2}{a}\right)-\sin\left(2\pi\frac{x_1}{a}\right)\sin\left(\pi\frac{x_2}{a}\right)\right]\right)\\ &=\frac{\hbar^2}{2m}\left(\left[5\frac{\pi^2}{a^2}\right]\psi\right)\\ &=5\frac{\pi^2\hbar^2}{2a^2m}\psi.\end{aligned}$$

therefore,

$$\hat{H}\psi(x_1,x_2)=5\frac{\pi^2\hbar^2}{2a^2m}\psi(x_1,x_2),$$

which ensures that the fermion ground state is an eigenfunction with appropriate eigenvalue.

$$\boxed{\hat{H}\psi(x_1,x_2)=5K\psi(x_1,x_2),\quad K=\frac{\pi^2\hbar^2}{2a^2m}.}$$

Solution 7: Energies and states of the next two excited states

Distinguishable states

For distinguishable particles we known that $\psi_{n_1,n_2}=\psi_{n_1}(x_1)\psi_{n_2}(x_2),\quad E_{n_1,n_2}=(n_1^2+n_2^2)K$. So now we compute $\psi_{2,2}, \psi_{1,3}$ and $\psi_{3,1}$,

$$\boxed{\begin{aligned}\psi_{2,2} &= \frac{2}{a}\sin\left(2\pi\frac{x_1}{a}\right)\sin\left(2\pi\frac{x_2}{a}\right), \quad E = 8K\\ \psi_{1,3} &= \frac{2}{a}\sin\left(\pi\frac{x_1}{a}\right)\sin\left(3\pi\frac{x_2}{a}\right), \quad E = 10K\\ \psi_{3,1} &= \frac{2}{a}\sin\left(3\pi\frac{x_1}{a}\right)\sin\left(\pi\frac{x_2}{a}\right), \quad E = 10K\end{aligned}}$$

Identical Bosons

For this case we know that the state is $\psi_{n_1,n_2} = \left[\psi_{n_1}(x_1)\psi_{n_2}(x_2) + \psi_{n_2}(x_1)\psi_{n_1}(x_2)\right]/\sqrt{2}, \quad E_{n_1,n_2} = (n_1^2 + n_2^2)K$, therefore,

$$\boxed{\begin{aligned}\psi_{2,2} &= \frac{2}{a}\sin\left(2\pi\frac{x_1}{a}\right)\sin\left(2\pi\frac{x_2}{a}\right), \quad E = 8K\\ \psi_{1,3} &= \frac{\sqrt{2}}{a}\left[\sin\left(\pi\frac{x_1}{a}\right)\sin\left(3\pi\frac{x_2}{a}\right) + \sin\left(3\pi\frac{x_2}{a}\right)\sin\left(\pi\frac{x_1}{a}\right)\right], \quad E = 10K\end{aligned}}$$

Identical Fermions

Finally, the identical fermions state is $\psi_{n_1,n_2} = \left[\psi_{n_1}(x_1)\psi_{n_2}(x_2) - \psi_{n_2}(x_1)\psi_{n_1}(x_2)\right]/\sqrt{2}, \quad E_{n_1,n_2} = (n_1^2 + n_2^2)K$, therefore,

$$\boxed{\begin{aligned}\psi_{1,3} &= \frac{\sqrt{2}}{a}\left[\sin\left(\pi\frac{x_1}{a}\right)\sin\left(3\pi\frac{x_2}{a}\right) - \sin\left(3\pi\frac{x_2}{a}\right)\sin\left(\pi\frac{x_1}{a}\right)\right], \quad E = 10K\\ \psi_{2,3} &= \frac{\sqrt{2}}{a}\left[\sin\left(2\pi\frac{x_1}{a}\right)\sin\left(3\pi\frac{x_2}{a}\right) - \sin\left(3\pi\frac{x_2}{a}\right)\sin\left(2\pi\frac{x_1}{a}\right)\right], \quad E = 13K\end{aligned}}$$