Quantum Computation
Quantum Circuits Activity

Francisco Vazquez-Tavares

November 20, 2025

Mandatory exercises

5 Assume that we start with a fully separable three-qubit states. First, qubits 1 and 2 become maximally entangled through an appropiate quantum operation. Your task is to design a quantum circuit that transfers this entanglement to qubits (2,3). In other words, at the end of the circuit, qubits 2 and 3 should be maximally entangled, while qubit 1 should be disentangled from the rest. You are allowed to use elementary gates alone.

We start with a fully separable three-qubit states $|000\rangle$, which can be expressed as $|0\rangle \otimes |0\rangle \otimes |0\rangle$. In order to get a maximally entangled state for qubits 1 and 2 we apply a Haddamard gate in qubit 1,

$$\left(\hat{H} \otimes \hat{I} \otimes \hat{I}\right)|000\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |100\rangle\right), \tag{1}$$

and a controlled X gate with qubit 1 as the control,

$$\left(\hat{\Lambda} \otimes \hat{X} \otimes \hat{I}\right)\frac{1}{\sqrt{2}}\left(|000\rangle + |100\rangle\right) = \frac{1}{\sqrt{2}}\left(|000\rangle + |110\rangle\right). \tag{2}$$

This final state can be expressed in terms of the Bells states for qubits 1 and 2,

$$\frac{1}{\sqrt{2}}\left(|000\rangle + |110\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle \otimes |0\rangle + |11\rangle \otimes |0\rangle\right) \tag{3}$$

$$= \left|\Psi^{+}\right\rangle \otimes |0\rangle. \tag{4}$$

Now that we achive the maximally entangled for qubit 1 and 2, let's show con to transfer this entanglement to qubits 2 and 3. First, let's apply a controlled X gate into qubits 2 and 3 to create a maximally entangled 3 qubit state[1],

[1] From previous homework, this is the GHZ-state.

$$\left(\hat{I} \otimes \hat{\Lambda} \otimes \hat{X}\right)\left|\Psi^{+}\right\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right). \tag{5}$$

At this point we can interpert that applying the controlled X gate after the Haddamard gate, it allow us to entangle the control with the target qubit. Hence, in order to disentangle the first qubit we apply a controlled qubit with the second qubit as controlled and the first qubit as the target,

$$\left(\hat{X} \otimes \hat{\Lambda} \otimes \hat{I}\right)\frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) \tag{6}$$

$$|0\rangle \otimes \left|\Psi^{+}\right\rangle. \tag{7}$$

By comparing equations (4) with (7), we can see how the maximal entanglement has been transferred from qubits 1-2 to qubits 2-3, achieving the desired quantum circuit. The graphical representation is shown in figure 1.

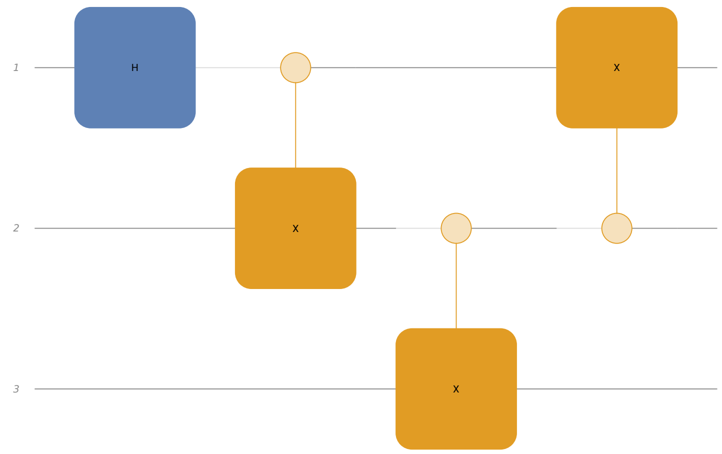

Figure 1: Transfer of entangled states cricuit.

7 A boolean function $f : \{0,1\}^n \mapsto \{0,1\}$ is said to be constant of $f(x)$ has the same vaue for all 2^n inputs and balanced if $f(x)$ returns 0 for exactly half of all inputs and 1 for the other half,

- Consider a generalization of the Deuthsch's algorithm having two registers ($n = 2$). The correpondent circuit is essentially the same as in the one register case. Discuss the conditions that would determine if a function is wether balanced or constant.

  Haddamard

$$(\hat{H} \otimes \hat{H} \otimes \hat{I}) \left(U_f \right) (\hat{H} \otimes \hat{H} \otimes \hat{I})$$

  The third register is the target. The boolean thing is $|y \oplus f(x)\rangle$

- Analyze the case when the function f is neither constant or nor balanced.

Optional exercises

$$U_3(\theta, \phi, \lambda) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -e^{i\lambda} \\ e^{i\phi} \sin\left(\frac{\theta}{2}\right) & e^{i(\lambda+\phi)} \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

1 Describe the action of the phase shift gate $p(\gamma) = |0\rangle\langle 0| + e^{i\gamma}|1\rangle\langle 1|$ on a qubit from the geometrical point of view.

Recalling that we can express a qubti as $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$. This expression allow us to create a geometrical interpretation as a point in an unit sphere. Where θ represent the angle between the $\hat{x}$ and $\hat{y}$ axis, and the angle φ is the angle between the $\hat{x}$ or $\hat{y}$ with the $\hat{z}$ axis. With this in mind, let's compute the resulting state from the given gate,

$$
\begin{aligned}
p(\gamma)|\psi\rangle &= |0\rangle\langle 0|\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right) + e^{i\gamma}|1\rangle\langle 1|\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right) \\
&= \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi+\gamma}\sin\left(\frac{\theta}{2}\right)|1\rangle .
\end{aligned}
$$

We can se that the $p(\gamma)$ gate introduces a phase shift in the angle related with the $\hat{x} - \hat{z}$ or $\hat{y} - \hat{z}$ planes. That is that introduces a displacement along the latitude of the unit sphere.

2 The 4-qubit W-state is defined as,

$$
|W_4\rangle = \frac{1}{2}\left(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle\right).
$$

Design a quantum circuit that upon the initial state $|0000\rangle$ constructs $|W_4\rangle$.

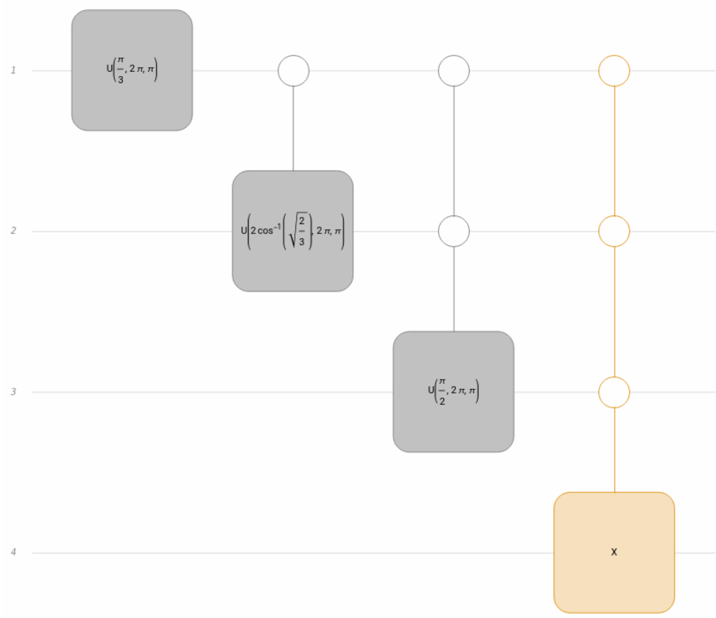

Figure 2: Quantum circuit to construct the 4-qubit W-state.

3 Design a circuir constructing the Hardy state,

$$
|H\rangle = \frac{1}{\sqrt{12}}\left(3|00\rangle + |01\rangle + |10\rangle + |11\rangle\right).
$$

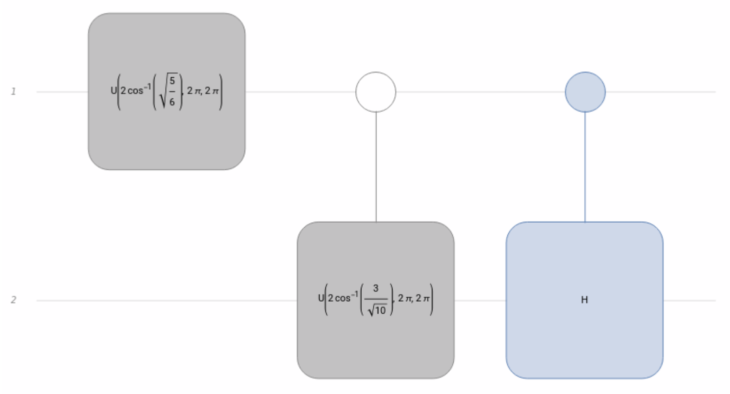

Figure 3: Quantum circuit to construct the Hardy state.

4 Show how to implement the Toffoli gate in terms of single-qubit and controlled-NOT gates.

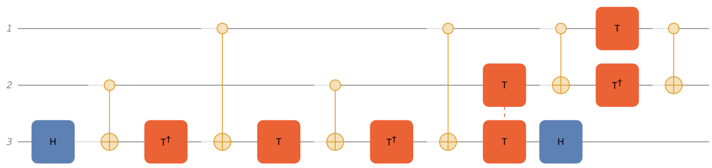

Figure 4: Toffoli gate in terms of single-qubit and controlled-NOT gates. Obtain from https://arxiv.org/pdf/0803.2316

6 In the BB84 protocol, Alice create an 8-qubit string (in the conventional X and Z basis):

$$|+\rangle\,|1\rangle\,|+\rangle\,|-\rangle\,|0\rangle\,|-\rangle\,|+\rangle\,|-\rangle\,.$$

Use a coin to randomly determine what basis Bob uses to measure each bit, and describe the resulting bit string that Alice and Bob keep.

[?]

References