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Multi-Party Quantum Teleportation with Selective Receiver

Mohammad Bolokian[1] · **Monireh Houshmand**[1] · **Mohammad-Sadegh Sadeghizadeh**[1] · **Maryam Parvaneh**[1]

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Abstract
Quantum teleportation is the protocol of transmitting quantum states without physically moving them. After the proposal for the first scheme of quantum teleportation in 1993, different schemes have been proposed, according to the application required. In this paper, a new quantum teleportation protocol is proposed for an arbitrary number of users. First, an initial entangled stated made of $\boldsymbol{n+1}$ qubits is distributed between the sender and $\boldsymbol{n}$ users. In each run, the sender decides which user is the receiver of the transmitted state. The receiver can retrieve the sent state with the collaboration of the other users. The applications of this protocol are in quantum networks, cloud computing and also quantum cryptography.

Keywords Quantum cryptography · Teleportation · Multi users · Entanglement · Measurement

1 Introduction

The process of transferring a quantum particle from one place to another without navigation of the physical distance between them is called the quantum teleportation. In quantum teleportation, an unknown quantum state is transmitted by sharing the initial entanglement between transmitter and receiver and the classical transmition between them. The most important

✉ Monireh Houshmand
m.hooshmand@imamreza.ac.ir

Mohammad Bolokian
mohammad.bolokian@yahoo.com

Mohammad-Sadegh Sadeghizadeh
m_sadegh.sadeghizadeh@yahoo.com

Maryam Parvaneh
m_Parvaneh_S@yahoo.com

[1] Imam Reza International University, Mashhad, Iran

feature of this type of information transmission is the indirect transmission of information, which makes the enemy unable to access and change information. Quantum teleportation is one of the most important protocols in quantum information and plays a key role in the implementation of telecommunications, computing and quantum networks [1].

Quantum teleportation was first proposed in 1993 by Bennett et al. using Einstein–Podolsky–Rosen (EPR) entangled states [1]. This protocol is implemented by photon polarized state [2], optical coherent state [3] and nuclear magnetic resonance [4]. After proposing the scheme of Bennett et al., many schemes in this field have been proposed up to now, which can be classified into three general categories: one-way quantum teleportation, two-way quantum teleportation, and quantum control teleportation [5–30].

In a one-way quantum teleportation, there is only one transmitter and one receiver. At first, in this protocol, an entangled state is shared between the transmitter and receiver. According to the results of sent measurement results, the qubits containing the information is transmitted from transmitter to receiver using distributed entanglement [5–9]. The most important difference in the designs presented in this type of quantum teleportation is the type of channel used and the number of sent qubits. In these designs, in addition to EPR, GHZ [5], GHZ-like [6], W [7] and Cluster [8] states are used as channels. In reference [9], more than one qubit is transferred simultaneously.

There is a transmitter, a receiver, and a controller in a quantum control teleportation. In this type of quantum teleportation, the transmitter sends its information to the receiver using channel under supervision of controller [10–15].

In two-way quantum teleportation, both users play the role of both transmitter and receiver and send their information to each other simultaneously using the primary shared channel [16–33].

In reference [34], n EPR state is shared between Alice and Bob. In this protocol, the state belonging to Alice is transmitted to one of them. Bob has n output pathway, and based on the result of Alice's measurement, Bob determines Alice state can be accessed in which pathway. In the paper [35], two protocols are proposed. In the first protocol, a single qubit state is transmitted to two receivers simultaneously by a four qubit channel. In the second protocol, an unknown single-qubit state is transmitted to Bob and another single-qubit state to Charlie through a four-qubit channel. Since in quantum mechanics an unknown quantum state cannot be copied, in the second protocol it is not possible to send an identical state to two receivers.

In this paper, for the first time, the quantum teleportation protocol is proposed with the ability to choose among the desired number of users. In this protocol, by sharing the initial entanglement between users and classical exchanges, the transmitter decides which single-qubit state having the information to be transmitted to which user and also sends the necessary information to receive the mode. This protocol is applicable in quantum information processing networks, quantum cloud computing and quantum cryptography [15].

This paper is organized in such a way that at first, the primary concepts of quantum computing is stated in Section 2. Then in Section 3, we describe the steps of the proposed protocol.. In Section 4, the protocol is evaluated and at the end, it is finished with the conclusion of the paper.

2 The Basic Concepts of Quantum Computing

This section deals with the basic concepts of quantum computing theory [36, 37], which is based on quantum mechanics. Quantum computing is established based on the concept equivalent to bit in the classical world, called quantum bit or qubit. A qubit, with a vector that is described in two-dimensional Hilbert space, and for this space, base basis represented by symbols $|0\rangle = [1\ \ 0]$ and $|1\rangle = [0\ \ 1]$ are selected. The base basis $|0\rangle$ and $|1\rangle$ are the quantum peers of 0 and 1 classical bits, respectively.

Unlike classical bits, qubits can be placed in any superposition of $|0\rangle$ and $|1\rangle$ such as $\alpha|0\rangle + \beta|1\rangle$, which α and β are complex numbers and $|\alpha|^2 + |\beta|^2 = 1$. n qubit register is a quantum state whose state space is the 2^n dimensional Hilbert space. Assuming the base basis $\{|0\rangle, |1\rangle\}$ for two-dimensional Hilbert space, the base basis of the 2^n dimensional Hilbert space are the following set:

$$\{|i_1\rangle\otimes|i_2\rangle\otimes\ldots\otimes|i_n\rangle \quad ; i_1, i_2, \ldots, i_n = 0, 1\},$$

which is also shown in this way:

$$\{|i_1 i_2\ldots i_n\rangle \ ; i_1, i_2, \ldots, i_n = 0, 1\} \tag{1}$$

The state of n qubit register can be written as a linear sum of the base basis in this form:

$$\sum_{\mathrm{i}_1,\ldots,\mathrm{i}_n=0,1} \mathrm{a}_{\mathrm{i}_1,\mathrm{i}_2,\ldots,\mathrm{i}_n}|\mathrm{i}_1\rangle\otimes|\mathrm{i}_2\rangle\otimes\ldots\otimes|\mathrm{i}_n\rangle, \tag{2}$$

The state $|i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_n\rangle$ where $i_1 = i_2 = \ldots = i_n = i$ is written as $|i\rangle^{\otimes n}$.

A very remarkable feature in quantum mechanics is the entanglement characteristic. An entangled state is a state of the system in which the quantum state of a constituent component cannot be described independently of the state of the other constituent components, even if the particles are physically apart from each other and the change in one particle is immediately transferred to another particle.

Quantum operations can be accomplished with a network of gates (operators). Each quantum gate is a linear transformation that is explained by a unitary matrix. A matrix U is unitary, if $UU^\dagger = I$, that $U^\dagger$ is the transpose conjugate of matrix U. For mentioning some examples of quantum gates, we can refer to the Hadamard matrix and the Controlled-NOT, CNOT gate. The definition of a Hadamard matrix is as follows.

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3}$$

The CNOT gate is a two-qubit gate. The first qubit is in the role of source (control) and the second qubit is in the role of target. If the source qubit is $|1\rangle$, the CNOT gate inverts the target qubit, and if the source qubit is $|0\rangle$, the target qubit exits without change.

A quantum measuring device is described with a Hermitian matrix. Quantum measurement is a random phenomenon that changes the measured state of $|\varnothing\rangle$. The measured value is one of the eigenvalues of the measurement matrix. The probability of observing λ_i eigenvalue is obtained from the following relation:

$$p(\lambda_i) = \langle\varnothing|P_i|\varnothing\rangle \tag{4}$$

where $P_i = |\lambda_i\rangle \quad \langle\lambda_i|$ is the eigenvector corresponds to λ_i eigenvalue. The state after measurement is as follows.

$$|\varnothing^+\rangle = \frac{P_i|\varnothing\rangle}{\sqrt{\langle\varnothing|P_i|\varnothing\rangle}} \tag{5}$$

The measurement matrices used in this paper are the matrices Z and X defined in the following relations.

$$Z=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, X=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \tag{6}$$

Eigenvectors Z are equal to $|0\rangle$ and $|1\rangle$ and eigenvectors of X are equal to $|+\rangle=\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ and $|-\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$.

3 The Proposed Protocol

In the proposed $n+1$ user protocol, Alice is the transmitter who wants to transmits the single qubit state of information ($|\varphi\rangle\ _a$) to one of the n users $Bob_1\ \ Bob_2\ \ ...\ \ Bob_n$. The $|\varphi\rangle\ _a$ state is as follows.

$$|\varphi\rangle_a=\alpha|0\rangle+\beta|1\rangle, |\alpha|^2+|\beta|^2=1 \tag{7}$$

The protocol is divided into five steps, which are described in the following sections: 1) channel sharing, 2) gate applying by Alice, 3) user measurement, and 4) state recovery by the intended receiver.

3.1 Channel Sharing

Alice first generates the channel used in this protocol as follows:

$$|G\rangle_{AB_1B_2\ldots B_n}=\frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes(n+1)}+|1\rangle^{\otimes(n+1)}\right) \tag{8}$$

She then holds the qubit A and sends the qubit B_i to $\mathrm{Bob_i}$. The whole state of the system (including the state $|\varphi\rangle\ _a$) is as follows.

$$\begin{aligned}|\Psi\rangle_{aAB_1B_2\ldots B_n}&=|\varphi\rangle_a\otimes|G\rangle_{AB_1B_2\ldots B_n}=\frac{1}{\sqrt{2}}(\alpha|0\rangle+\beta|1\rangle)\left(|0\rangle^{\otimes(n+1)}+|1\rangle^{\otimes(n+1)}\right)\\&=\frac{1}{\sqrt{2}}\left(\alpha|0\rangle^{\otimes(n+2)}+\alpha|0\rangle|1\rangle^{\otimes(n+1)}+\beta|1\rangle|0\rangle^{\otimes(n+1)}+\beta|1\rangle^{\otimes(n+1)}\right)\end{aligned} \tag{9}$$

3.2 Gate Applying by Alice

Alice applies the CNOT operator to qubit a as the control and qubit A as the target qubit. The state after applying the operator is as follows:

$$\Big|\Psi\rangle_{aAB_1B_2\ldots B_n}=\frac{1}{\sqrt{2}}\left(\alpha|0\rangle^{\otimes(n+2)}+\alpha|0\rangle|1\rangle^{\otimes(n+1)}+\beta|11\rangle|0\rangle^{\otimes n}+\quad\beta|10\rangle|1\rangle^{\otimes n}\right) \tag{10}$$

Alice then applies the Hadamard qubit on Gate a. The state changes as follows:

$$|\Psi\rangle_{aAB_1B_2\ldots B_n}=\frac{1}{2}\left(\alpha|0\rangle^{\otimes(n+2)}+\alpha\Big|1\rangle|0\rangle^{\otimes(n+1)}+\alpha\Big|0\rangle|1\rangle^{\otimes(n+1)}+\alpha|1\rangle^{\otimes(n+2)}+\beta|01\rangle|0\rangle^{\otimes n}-\beta|11\rangle|0\rangle^{\otimes n}+\beta|00\rangle|1\rangle^{\otimes n}-\beta|10\rangle|1\rangle^{\otimes n}\right) \tag{11}$$

3.3 User's Measurement

In the third step, Alice applies a two-qubit measurement to the qubits a and A. According to Relation (12), the results of Alice measurement and the post-measurement states of Bob_1 Bob_2 ... Bob_n can be seen in Table 1. Alice reveals the result of the measurement on the public channel.

$$|\Psi\rangle_{aAB_1B_2\dots B_n} = \left|00\right\rangle \frac{\alpha|0\rangle^{\otimes n}+\beta|1\rangle^{\otimes n}}{2} + \left|01\right\rangle \frac{\alpha|1\rangle^{\otimes n}+\beta|0\rangle^{\otimes n}}{2} + \left|10\right\rangle \frac{\alpha|0\rangle^{\otimes n}-\beta|1\rangle^{\otimes n}}{2} + \left|11\right\rangle \frac{\alpha|1\rangle^{\otimes n}-\beta|0\rangle^{\otimes n}}{2} \tag{12}$$

In this step, Alice decides which user is the state receiver and sends the classic bit equal to "0" to the desired receiver and the bit equal to "1" to the other users. Next, we assume that the B_n receiver is the desired one. The protocol steps are similar for the other users as the receiver. After receiving the classical bits, Bob_1 Bob_2 ... $\mathrm{Bob}_{\mathrm{n}-1}$ measure their qubits in the basis X and announce the measurement result. Imagine we show the result of a measurement corresponding to $|+\rangle$ and $|-\rangle$ with the symbols "+" and "–". The state belongs to B_n according to the measurement result of other users is written in Table 2. In this table, by saying $num\{-\}$, we mean the total number of observed *"–"* as a result of measuring Bob_1 Bob_2 ... $\mathrm{Bob}_{\mathrm{n}-1}$ The proof is provided in appendix.

3.4 The Receiving of Sent State

According to Table 2, after receiving the measurement results of users, B_n by applying the gates mentioned in Table 3, it achieves $|\varphi\rangle_{\ a}$ state.

4 Evaluation

In this research, by producing and distributing $n+1$ entangled qubits between the transmitter and the n user, during protocol procedure, the transmitter is able to decide that information will be transferred to which n users. In this protocol, the receiver will be able to retrieve the qubits containing the information only with the help of other users. This extended scheme of the quantum teleportation is proposed for the first time in this paper. In addition, in this protocol, such as a data multiplexor, it is possible to select one of the users as a receiver, because the recovery of information by the receiver requires the receiving of other users' measurement results. Therefor like quantum secret sharing protocols, security is established in the presence of the dishonest user [38]. Due to non cloning theorem in quantum mechanics [36, 37], it is not possible to send an unknown quantum state to several users, and each time the protocol is

Table 1 Alice Measurement

Collapsed state of qubits $B_1B_2\dots B_n$	Result of the measurement Alice	Probability of occurrence
$\alpha\|0\rangle^{\otimes n}+\beta\|1\rangle^{\otimes n}$	00	25%
$\alpha\|1\rangle^{\otimes n}+\beta\|0\rangle^{\otimes n}$	01	25%
$\alpha\|0\rangle^{\otimes n}-\beta\|1\rangle^{\otimes n}$	10	25%
$\alpha\|1\rangle^{\otimes n}-\beta\|0\rangle^{\otimes n}$	11	25%

Table 2 User's measurement

The state B_n when the $num\{-\}$ is odd	The state B_n when the $num\{-\}$ is even	Result of the measurement Alice
$\alpha\|0\rangle-\beta\|1\rangle$	$\alpha\|0\rangle+\beta\|1\rangle$	00
$\beta\|0\rangle-\alpha\|1\rangle$	$\beta\|0\rangle+\alpha\|1\rangle$	01
$\alpha\|0\rangle-\beta\|1\rangle$	$\alpha\|0\rangle+\beta\|1\rangle$	10
$\beta\|0\rangle-\alpha\|1\rangle$	$\beta\|0\rangle+\alpha\|1\rangle$	11

implemented, only one of the users is the receiver of the state. If the sender shares the entangled state with each receiver independently, then $2n$ qubites would be required and each receiver can get the teleported qubit independent of the other users. This protocol is applicable in quantum information processing networks, cloud computing and quantum cryptography.

5 Conclusion

In this paper, for the first time, a protocol with the presence of a transmitter and the possibility of transferring a single-qubit state to one of the n users is proposed. Only with the cooperation of other users, the mentioned receiver can access the state. In the future, we can add some controller (s) to the protocol. The efficiency of the protocol in noise channels can also be examined.

Appendix: Proving the correctness of the protocol

In this section, we prove that the qubit belongs to $\mathrm{Bob_n}$ according to the result of Alice's measurement and the number of even / odd values of "-" in the measurements of other users, corresponds to Table 2. Proof is done by mathematical induction. First we prove the correctness of Table 2 for the $k=2$ state (minimum number of receivers for this protocol). Then we show that if the table correctness is valid for $k=n-1$, then it is also valid for $k=n$.

The proof for $k=2$

In this section, we rewrite the protocol for dual-receiver mode. Alice first generates the channel used in this protocol as follows.

$$\left|G\right\rangle_{AB_1B_2}=\frac{1}{\sqrt{2}}\left(|000\rangle+\left|111\right\rangle\right) \tag{13}$$

Table 3 the applying gate of $\mathrm{Bob_n}$ according to users' measurement results

the applied gate is to the qubit B_n when the $num\{-\}$ is odd	the applied gate to the qubit B_n when the $num\{-\}$ is even	result of the Alice' measurement Alice
Z	I	00
ZX	X	01
Z	I	10
ZX	X	11

She then holds the qubit A, sends the qubit B_1 to Bob_1 and the qubit B_2 to Bob_2. The complete state of the system (by considering the state $|\varphi\rangle_A$) is as follows:

$$|\Psi\rangle_{aAB_1B_2} = |\varphi\rangle_a \otimes |G\rangle_{AB_1B_2} = \frac{1}{\sqrt{2}}(\alpha|0000\rangle + \alpha|0111\rangle + \beta|1000\rangle + \beta|1111\rangle) \quad (14)$$

After applying the CNOT operator on qubit a as the control and qubit A as the target and then applying the Hadamard gate on the qubit a, the state of the system changes as follows.

$$|\Psi\rangle_{aAB_1B_2} = \frac{1}{2}[\alpha|0000\rangle + \alpha|0111\rangle + \alpha|1000\rangle + \alpha|1111\rangle + \beta|0100\rangle + \beta|0011\rangle - \beta|1100\rangle - \beta|1011\rangle] \quad (15)$$

In the third step, Alice measures the two qubits according to computational basis a and A. According to Relation (16), the measurement results and post-measurement states of Bob_1 and Bob_2 can be seen in Table 4. Alice reveals the result of the measurement on the public channel.

$$|\Psi\rangle_{aAB_1B_2} = |00\rangle\frac{\alpha|00\rangle + \beta|11\rangle}{2} + |01\rangle\frac{\alpha|11\rangle + \beta|00\rangle}{2} + |10\rangle\frac{\alpha|00\rangle - \beta|11\rangle}{2} + |11\rangle\frac{\alpha|11\rangle - \beta|00\rangle}{2} \quad (16)$$

In this step, suppose Bob_2 is the receiver of Alice. Bob_1 measures its qubit based on the X and declares the result. According to Relations (17) the collapsed stateB_2 corresponds to Table 5.

$$\begin{aligned}
|00\rangle + \beta|11\rangle &= |+\rangle\left(\frac{\alpha|0\rangle + \beta|1\rangle}{2}\right) + |-\rangle\left(\frac{\alpha|0\rangle - \beta|1\rangle}{2}\right)\\
\alpha|11\rangle + \beta|00\rangle &= |+\rangle\left(\frac{\alpha|1\rangle + \beta|0\rangle}{2}\right) + |-\rangle\left(\frac{\beta|0\rangle - \alpha|1\rangle}{2}\right)\\
\alpha|00\rangle - \beta|11\rangle &= |+\rangle\left(\frac{\alpha|0\rangle - \beta|1\rangle}{2}\right) + |-\rangle\left(\frac{\alpha|0\rangle + \beta|1\rangle}{2}\right)\\
\alpha|11\rangle - \beta|00\rangle &= |+\rangle\left(\frac{\alpha|1\rangle - \beta|0\rangle}{2}\right) + |-\rangle\left(\frac{\alpha|1\rangle + \beta|0\rangle}{2}\right)
\end{aligned} \quad (17)$$

According to Table 5, when the measurement result is "+", it means that $num\{-\}$ is an even number, and when it is "-", it means that $num\{-\}$ is an odd number, the post-measurement state is according to Table 2.

Table 4 Alice measurement

Collapsed state of qubits B_1B_2	Result of the measurement Alice	Probability of occurrence
$\alpha\|00\rangle + \beta\|11\rangle$	00	25%
$\alpha\|11\rangle + \beta\|00\rangle$	01	25%
$\alpha\|00\rangle - \beta\|11\rangle$	10	25%
$\alpha\|11\rangle - \beta\|00\rangle$	11	25%

Inductive reasoning

Assume that $k = n - 1$ mode is valid for the correctness of Table 2. Now we prove that for $k = n$ mode, the correctness of Table 2 is also valid. Proof is provided for the case that the result of Alice's measurement is "00". For other results, Alice's measurement is in the same way.

According to the induction assumption, the $\alpha|0\rangle^{\otimes(n-1)} + \beta|1\rangle^{\otimes(n-1)}$ state can be written as follows.

$$\alpha|0\rangle^{\otimes(n-1)} + \beta|1\rangle^{\otimes(n-1)} = \sum_{\tau_i \in \{+,-\}, num\{-\} is\ even} |\tau_1 \ldots \tau_{n-1}\rangle \left(\frac{\alpha|0\rangle + \beta|1\rangle}{2}\right) + \sum_{\tau_i \in \{+,-\}, num\{-\} is\ odd} |\tau_1 \ldots \tau_{n-1}\rangle \left(\frac{\alpha|0\rangle - \beta|1\rangle}{2}\right) \tag{18}$$

Now by adding a user, the post-measurement state of Alice according to Table 1 is in the form of relation (19).

$$\alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n} = |0\rangle\alpha|0\rangle^{\otimes(n-1)} + |1\rangle\beta|1\rangle^{\otimes(n-1)} \tag{19}$$

According to Relation (18), The Relation (19) can be written as follows:

$$\begin{aligned}
&\alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n} = \alpha|0\rangle^{\otimes(n-1)}|0\rangle + \beta|1\rangle^{\otimes(n-1)}|1\rangle = \\
&\sum_{\tau_i \in \{+,-\}, num\{-\} is\ even} |\tau_1 \ldots \tau_{n-1}\rangle \left(\frac{|0\rangle\alpha|0\rangle + |1\rangle\beta|1\rangle}{2}\right) + \sum_{\tau_i \in \{+,-\}, num\{-\} is\ odd} |\tau_1 \ldots \tau_{n-1}\rangle \left(\frac{|0\rangle\alpha|0\rangle - |1\rangle\beta|1\rangle}{2}\right) \\
&= \sum_{\tau_i \in \{+,-\}, num\{-\} is\ even} \left|\tau_1 \ldots \tau_{n-1}\right\rangle \frac{|+\rangle + |-\rangle}{\sqrt{2}} \left(\frac{\alpha|0\rangle}{2}\right) + \sum_{\tau_i \in \{+,-\}, num\{-\} is\ even} \left|\tau_1 \ldots \tau_{n-1}\right\rangle \frac{|+\rangle - |-\rangle}{\sqrt{2}} \left(\frac{\beta|1\rangle}{2}\right) + \\
&\sum_{\tau_i \in \{+,-\}, num\{-\} is\ odd} \left|\tau_1 \ldots \tau_{n-1}\right\rangle \frac{|+\rangle + |-\rangle}{\sqrt{2}} \left(\frac{\alpha|0\rangle}{2}\right) + \sum_{\tau_i \in \{+,-\}, num\{-\} is\ odd} \left|\tau_1 \ldots \tau_{n-1}\right\rangle \frac{|+\rangle - |-\rangle}{\sqrt{2}} \left(\frac{-\beta|1\rangle}{2}\right) = \\
&\sum_{\tau_i \in \{+,-\}, num\{-\} is\ even} |\tau_1 \ldots \tau_{n-1}\rangle|+\rangle \left(\frac{\alpha|0\rangle + \beta|1\rangle}{2}\right) + \sum_{\tau_i \in \{+,-\}, num\{-\} is\ odd} |\tau_1 \ldots \tau_{n-1}\rangle|+\rangle \left(\frac{\alpha|0\rangle - \beta|1\rangle}{2}\right) + \\
&\sum_{\tau_i \in \{+,-\}, num\{-\} is\ even} |\tau_1 \ldots \tau_{n-1}\rangle|-\rangle \left(\frac{\alpha|0\rangle - \beta|1\rangle}{2}\right) + \sum_{\tau_i \in \{+,-\}, num\{-\} is\ odd} |\tau_1 \ldots \tau_{n-1}\rangle|-\rangle \left(\frac{\alpha|0\rangle + \beta|1\rangle}{2}\right) = \\
&\sum_{\tau_i \in \{+,-\}, num\{-\} is\ even} |\tau_1 \ldots \tau_n\rangle \left(\frac{\alpha|0\rangle + \beta|1\rangle}{2}\right) + \sum_{\tau_i \in \{+,-\}, num\{-\} is\ odd} |\tau_1 \ldots \tau_n\rangle \left(\frac{\alpha|0\rangle - \beta|1\rangle}{2}\right)
\end{aligned} \tag{20}$$

Given the equality of Relation (21), which results from Relation (20), the correctness of Table 2 is proved

Table 5 User's measurements

The state B_2 when the result is a "-" measurement	The state B_2 when the result is a "+" measurement	Result of the measurement Alice
$\alpha\|0\rangle - \beta\|1\rangle$	$\alpha\|0\rangle + \beta\|1\rangle$	00
$\beta\|0\rangle - \alpha\|1\rangle$	$\beta\|0\rangle + \alpha\|1\rangle$	01
$\alpha\|0\rangle - \beta\|1\rangle$	$\alpha\|0\rangle + \beta\|1\rangle$	10
$\beta\|0\rangle - \alpha\|1\rangle$	$\beta\|0\rangle + \alpha\|1\rangle$	11

$$\alpha|0\rangle^{\otimes(n-1)}+\beta|1\rangle^{\otimes(n-1)}=\sum_{\tau_i\in\{+,-\},num\{-\}is\ even}|\tau_1\ldots\tau_n\rangle\left(\frac{\alpha|0\rangle+\beta|1\rangle}{2}\right)+\sum_{\tau_i\in\{+,-\},num\{-\}is\ odd}|\tau_1\ldots\tau_n\rangle\left(\frac{\alpha|0\rangle-\beta|1\rangle}{2}\right) \tag{21}$$

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