Homeworks

1. Homework for February 12

consider the following operators on a Hilbert space $\mathbb{V}^3(C)$:

$$L_x = \frac{1}{2^{\frac{1}{2}}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_y = \frac{1}{2^{\frac{1}{2}}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad L_z = \frac{1}{2^{\frac{1}{2}}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

1. What are the possible values one can obtain if L_z is measured?
2. Take the state in which $L_z = 1$. In this state what are $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and ΔL_x?
3. Find the normalized eigenstates and the eigenvalues of L_x in the L_z basis.
4. If the particle is in the state with $L_z = -1$, and L_x is measured, what are the possible outcomes and their probabilities?

1.1.

Since the possible values from an operator are there eigenvalues, looking that the operator L_z is diagonalized, the possible values at a measurement are $0, +1, -1$, hence,

$$|L_z = 1\rangle, \quad |L_z = 0\rangle, \quad |L_z = -1\rangle.$$

Finally, tacking advantage that the operator is already diagonalized, there eigenvectors are,

$$|L_z = 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |L_z = 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |L_z = -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

1.2.

Now to compute the expected values of $\langle L_x \rangle$, $\langle L_x^2 \rangle$, and ΔL_x when $|L_z = 1\rangle$ we do as follows,

$$\begin{aligned} \langle L_x \rangle &= \langle L_z = 1 | L_x | L_z = 1 \rangle \\ &= (1 \ \ 0 \ \ 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} (1 \ \ 0 \ \ 0) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= 0. \end{aligned}$$

Now, to compute $\langle L_x^2 \rangle$,

$$\begin{aligned}
\langle L_x^2 \rangle &= \langle L_z = 1 \big| L_x^2 \big| L_z = 1 \rangle \\
&= (1 \ \ 0 \ \ 0) \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\
&= \frac{1}{2} (1 \ \ 0 \ \ 0) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{2}.
\end{aligned}$$

Finally, the previous results help us to compute ΔL_x as follows,

$$\begin{aligned}
\Delta L_x &= \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2} \\
&= \sqrt{\left(\frac{1}{2}\right)^2 - 0^2} \\
&= \sqrt{\left(\frac{1}{2}\right)^2} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

1.3.

To get the normalized eigenstates and the eigenvalues of L_x in the L_z basis.

For the eigenvalues we are going to use the determinant method,

$$\begin{aligned}
\begin{vmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{vmatrix} &= -\lambda^3 - \left(-\lambda \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot 0\right) - \left(\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot -\lambda\right) + \left(0 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\right) - (0 \cdot -\lambda \cdot 0) \\
&= -\lambda^3 - \left(-\lambda \frac{1}{2}\right) + (0) - \left(\frac{1}{2} \cdot -\lambda\right) + (0) - (0) \\
&= -\lambda^3 - \left(-\lambda \frac{1}{2}\right) - \left(\frac{1}{2} \cdot -\lambda\right) \\
&= -\lambda^3 + \frac{\lambda}{2} + \frac{\lambda}{2} \\
&= -\lambda^3 + \lambda
\end{aligned}$$

therefore, to get the eigenvalues we need to find the roots of $-\lambda^3 + \lambda = 0$ wich are $\lambda = \{0, +1, -1\}$. Once we know the eigenvalues, we start to compute the eigenvectors with the follwoing property,

$$(L_x - \lambda I)|L_x = \lambda\rangle = 0,$$

where we define $|L_x = \lambda\rangle$ as,

$$|L_x = \lambda\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

therefore,

$$(L_x - \lambda I)|L_x = \lambda\rangle = \left(\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right) \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda \cdot a + \frac{1}{\sqrt{2}} \cdot b + 0 \\ \frac{1}{\sqrt{2}} \cdot a - \lambda \cdot b + \frac{1}{\sqrt{2}} \cdot c \\ 0 \cdot a + \frac{1}{\sqrt{2}} \cdot b - \lambda \cdot c \end{pmatrix}$$

$$= \begin{pmatrix} -a\lambda + \frac{b}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} - b\lambda + \frac{c}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} - c\lambda \end{pmatrix}$$

and we equate it to $|0\rangle$,

$$(L_x - \lambda I)|L_x = \lambda\rangle = |0\rangle$$

$$\begin{pmatrix} -a\lambda + \frac{b}{\sqrt{2}} \\ \frac{a}{\sqrt{2}} - b\lambda + \frac{c}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} - c\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now we need to solve that system of equations for each eigenvalue.
(Later I will add the procedure)

$$|L_x = 1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \quad |L_x = 0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad |L_x = -1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

1.4.

Finally, to measure the operator L_x when $L_z = -1$
https://phys
icsisbeautiful.com/resources/principles-of-quantum-mechanics/problems/4.
2.1/solutions/shankar20exercises2004.02.01.pdf/9tVFTcE7Xy4WdP74UD8
m2S/