Quantum Computation
The qubit geometry

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The expectation value of an operator (or quantum gate) A over a qubit $|\psi\rangle$ is defined as

$$\langle A \rangle = \langle \psi | A | \psi \rangle . \tag{1}$$

Consider the general state

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle , \quad a, b \in \mathbb{C}, \tag{2}$$

and define the map

$$|\psi\rangle \mapsto \hat{n} = (\langle X \rangle , \langle Y \rangle , \langle Z \rangle) . \tag{3}$$

> **Problem 1:**
>
> Show that the entries of the vector $\hat{n}$ fulfill
>
> $$\begin{aligned} n_x &= \langle X \rangle = 2\,\mathrm{Re}(\bar{a}b) \\ n_y &= \langle Y \rangle = 2\,\mathrm{Im}(\bar{a}b) \\ n_z &= \langle Z \rangle = |a|^2 - |b|^2. \end{aligned}$$
>
> and its norm is equal to 1. The overline $\bar{x}$ stands for the complex conjugate of x. You might work with the conventional matrix representation.

Let's start with n_x,

$$\begin{aligned} n_x = \langle \psi | X | \psi \rangle &= \begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \\ &= \bar{a}b + \bar{b}a. \end{aligned}$$

Recalling that $\mathrm{Re}(z) = (z + \bar{z})/2$, we can re-write,

$$n_x = 2\,\mathrm{Re}(\bar{a}b).$$

Moving forward to n_y we repeat the same process,

$$\begin{aligned} n_y = \langle \psi | Y | \psi \rangle &= \begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix} \begin{pmatrix} ib \\ -ia \end{pmatrix} \\ &= i\left(\bar{a}b - \bar{b}a\right) . \end{aligned}$$

Recalling that $\mathrm{Im}(z) = (z - \bar{z})/2$, we can re-write,

$$n_y = 2\,\mathrm{Im}(\bar{a}b).$$

Finally, for n_z, we repeat one last time,

$$\begin{aligned} n_z = \langle\psi|Z|\psi\rangle &= \begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix} \begin{pmatrix} a \\ -b \end{pmatrix} \\ &= |a|^2 - |b|^2. \end{aligned}$$

Now, to compute the norm we need to sum the squared of the components and take the squared of the result. Let's start by computing the squared of each component,

$$\begin{aligned} n_x^2 &= \left(\bar{a}b + \bar{b}a\right)^2 = \bar{a}^2b^2 + 2|a|^2|b|^2 + \bar{b}^2a^2 \\ n_y^2 &= \left(i\left(\bar{a}b - \bar{b}a\right)\right)^2 = -\bar{a}^2b^2 + 2|a|^2|b|^2 - \bar{b}^2a^2 \\ n_z^2 &= \left(|a|^2 - |b|^2\right)^2 = |a|^4 - 2|a|^2|b|^2 + |b|^4. \end{aligned}$$

Now, we add the terms,

$$n_x^2 + n_y^2 + n_z^2 = |a|^4 + 2|a|^2|b|^2 + |b|^4.$$

The nex step is to assume that the state $|\psi\rangle$ is normalize, such that $\langle\psi|\psi\rangle = 1 = |a|^2 + |b|^2$. We can derive the following identities, $|a|^2 = 1 - |b|^2$ and $|a|^4 = 1 - 2|b|^2 + |b|^4$.

$$\begin{aligned} n_x^2 + n_y^2 + n_z^2 &= 1 - 2|b|^2 + |b|^4 + 2|b|^2 - 2|b|^4 + |b|^4 \\ &= 1. \end{aligned}$$

Hence the norm of $\hat{n}$ is 1.

Problem 2:

The qubit $|\psi\rangle$ can be parametrized in the following way,

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \tag{4}$$

with $\theta \in [0, \pi/2]$, $\quad \varphi \in [1, 2\pi]$. Using the results obtained in the previous part prove that the components of $\hat{n}$ are the usual spherical coordinates,

$$\begin{aligned} n_x &= \sin\theta\cos\varphi \\ n_y &= \sin\theta\sin\varphi \\ n_z &= \cos\theta. \end{aligned}$$

This procedure justifies why an arbitrary qubit is identified with a point in the Bloch sphere, which is also called the qubit projective space.

We can identify that $a = \cos(\theta/2)$ and $b = e^{i\varphi}\sin(\theta/2)$, hence,

$$\begin{aligned} n_x &= \bar{a}b + \bar{b}a \\ &= \left(e^{i\varphi} + e^{-i\varphi}\right)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ &= 2\cos(\varphi)\,\frac{1}{2}\sin(\theta) \\ &= \cos\varphi\sin\theta. \end{aligned}$$

Usefull trigonometric and complex exponentials identities for the excersices.

$$\begin{aligned} \cos(A)\sin(B) &= \frac{\sin(A+B) - \sin(A-B)}{2} \\ \cos(x) &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \\ \sin(x) &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \end{aligned}$$

Moving on to the next component,

$$\begin{aligned} n_y &= \bar{a}b - \bar{b}a \\ &= \left(e^{i\varphi} - e^{-i\varphi}\right)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ &= 2\sin(\varphi)\,\frac{1}{2}\sin(\theta) \\ &= \sin\varphi\sin\theta. \end{aligned}$$

Finally,

$$\begin{aligned} n_z &= |a|^2 - |b|^2 \\ &= \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \\ &= \cos(\theta). \end{aligned}$$