Quantum Computation
Quantum Circuits Activity

Francisco Vazquez-Tavares

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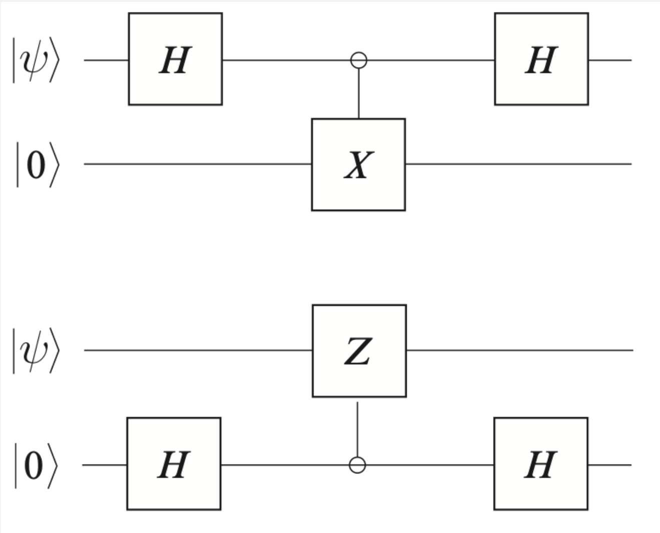
Problem 1:

Compare the effect of the following two circuits



Let's consider the following general state $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$ with $a, c \in \mathbb{C}$. The first circuit can be represented with the following algebraic expression

$$\left[\left(\hat{H}\otimes\mathbb{1}\right)\left(\Lambda\hat{X}\right)\left(\hat{H}\otimes\mathbb{1}\right)\right]\left(|\psi\rangle\otimes|0\rangle\right).$$

Where $\Lambda\hat{X}$ denotes the controlled $\hat{X}$ gate $(|0\rangle\langle 0|\otimes\mathbb{1}+|1\rangle\langle 1|\otimes\hat{X})$, $\hat{H}$ is the Haddamard gate and $\hat{X}$ is the X gate.

Starting with the first gate,

$$\begin{aligned}
\left(\hat{H}\otimes\mathbb{1}\right)\left(|\psi\rangle\otimes|0\rangle\right) &= \hat{H}\,|\psi\rangle\otimes\mathbb{1}\,|0\rangle\\
&= \left[\frac{a}{\sqrt{2}}(|0\rangle+|1\rangle)+\frac{b}{\sqrt{2}}(|0\rangle-|1\rangle)\right]\otimes\mathbb{1}\,|0\rangle\\
&= \left[\frac{(a+b)}{\sqrt{2}}\,|0\rangle+\frac{(a-b)}{\sqrt{2}}\,|1\rangle\right]\otimes\mathbb{1}\,|0\rangle\\
&= \frac{(a+b)}{\sqrt{2}}\,|00\rangle+\frac{(a-b)}{\sqrt{2}}\,|10\rangle.
\end{aligned}$$

Now we compute the controlled $\hat{X}$ gate with the new state with the following mnemonic rule, *It flips the second qubit if the first qubit is* 1 *and leaves unchanged otherwise*, therefore

$$\Lambda\hat{X}\left[\frac{(a+b)}{\sqrt{2}}\,|00\rangle+\frac{(a-b)}{\sqrt{2}}\,|10\rangle\right] = \frac{(a+b)}{\sqrt{2}}\,|00\rangle+\frac{(a-b)}{\sqrt{2}}\,|11\rangle = |\psi_2\rangle.$$

Finally, we apply the last Haddamard gate into the nwe state,

$$\begin{aligned}
(\hat{H}\otimes\mathbb{1})\,|\psi_2\rangle &= \frac{(a+b)}{\sqrt{2}}\hat{H}\,|0\rangle\otimes\mathbb{1}\,|0\rangle + \frac{(a-b)}{\sqrt{2}}\hat{H}\,|1\rangle\otimes\mathbb{1}\,|1\rangle \\
&= \frac{(a+b)}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right)\otimes\mathbb{1}\,|0\rangle + \frac{(a-b)}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right)\otimes\mathbb{1}\,|1\rangle \\
&= \frac{(a+b)}{2}\left(|00\rangle+|10\rangle\right) + \frac{(a-b)}{2}\left(|01\rangle-|11\rangle\right).
\end{aligned}$$

After expanding the expression and minor algebraic manipulations we can express the final state in terms of the of the Bell states,

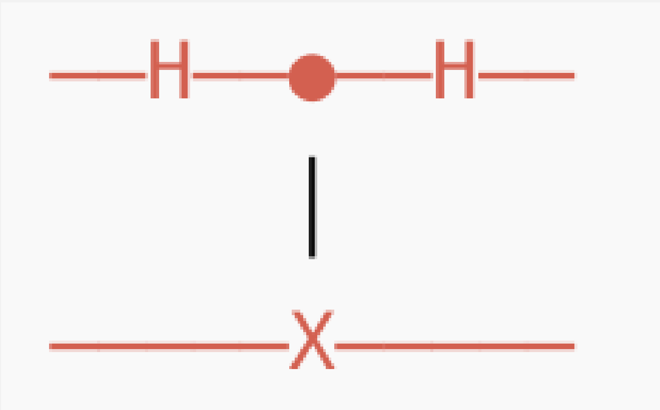
$$(\hat{H}\otimes\mathbb{1})\,|\psi_2\rangle = \frac{a}{\sqrt{2}}\left(|\Psi^+\rangle+|\Phi^+\rangle\right) + \frac{b}{\sqrt{2}}\left(|\Psi^+\rangle-|\Phi^-\rangle\right)$$

Bottom circuit.

$$\left[(\mathbb{1}\otimes\hat{H})\,(\Lambda\hat{Z})\,(\mathbb{1}\otimes\hat{H})\right]\left(|\psi\rangle\otimes|0\rangle\right)$$

Problem 2:

Show that the following quantum circuit is equivalent to a controlled Z-gate



Problem 3:

The three qubit GHZ-state is defined as

$$|GHZ\rangle = \frac{1}{2}\left(|000\rangle+|111\rangle\right).$$

Design a circuit that upon of the separable state $|000\rangle$ constructs the GHZ-state.