Algebraic and Transcendental functions

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Ellipse

An ellipse is defined as the curve formed by all the points that satisfies that the sum of the distances r_1 and r_2 from two fixed points F_1 and F_2 separated by a distance of $2c$ is constant with value $2a$. (fig.1).

Where the value a is commnly refered as *semimajor axis*, the points F_1 and F_2 as *foci* and the half point between the foci is refered as the *origin* of the coordinate system.

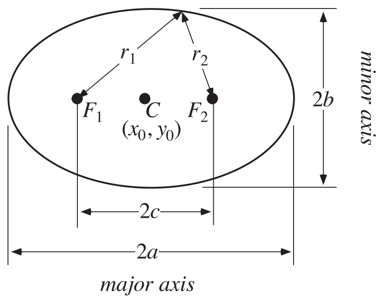

Figure 1: Ellipse. It is helpfull to acknowledge that we can create a right triangle with sides of length c and b and hypotenuse a.

Derivation of the equation

To derive the algebraic form of the ellipse from its definition, we start by translating the definition of the ellipse into an algebra representation. First, we represent the foci as $F_1 = (c,0)$ and $F_2 = (-c,0)$. Now, we recall the equation to compute the distance[1] between two points, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Since we want to find the points in the plane that satisfies the definiiton of the ellipse, we are going to find the distance between the foci and all the plane,

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2}.$$

Now we are going to apply the restricction that the sum of those distances are equal to $2a$,

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

With this equation we are going to apply several algebraic tricks[2] to simplify the equation. Let's start by isolating one distance and then squaring the equation,

$$\begin{aligned}
\sqrt{(x-c)^2 + y^2} &= 2a - \sqrt{(x+c)^2 + y^2} \\
\left[\sqrt{(x-c)^2 + y^2}\right]^2 &= \left[2a - \sqrt{(x+c)^2 + y^2}\right]^2 \\
(x-c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\
x^2 - 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\
\cancelto{0}{x^2} - 2cx + c^2 + \cancelto{0}{y^2} &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + \cancelto{0}{x^2} + 2cx + c^2 + \cancelto{0}{y^2} \\
-4cx &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2}.
\end{aligned}$$

[1] As a reminder, the distance equation resembles the pythagorean relation,

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \longleftrightarrow c^2 = a^2 + b^2.$$

This relation gives insight of the mathematical similarities between the circle and the ellipse equations.

[2] Basically we are going to get rid of the radicals ($\sqrt{x}$). It is prefered to use exponents rather than radicals, because exponents provide a unifed notation system that also reduces steps and potential errors.

$$\left[2a - \sqrt{(x+c)^2 + y^2}\right]\left[2a - \sqrt{(x+c)^2 + y^2}\right]$$
$$4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

Now, we are going to solve for the distance $\sqrt{(x+c)^2+y^2}$,

$$\begin{aligned}
-4cx &= 4a^2 - 4a\sqrt{(x+c)^2+y^2} \\
4a\sqrt{(x+c)^2+y^2} &= 4a^2 + 4cx \\
\sqrt{(x+c)^2+y^2} &= \frac{4a^2+4cx}{4a} \\
\sqrt{(x+c)^2+y^2} &= a + \frac{cx}{a}.
\end{aligned}$$

Now we square it again[3],

[3] $(x+c)^2 = x^2+2cx+c^2$

$$\begin{aligned}
\left[\sqrt{(x+c)^2+y^2}\right]^2 &= \left[a+\frac{cx}{a}\right]^2 \\
x^2 + \cancelto{0}{2cx} + c^2 + y^2 &= a^2 + \cancelto{0}{2cx} + \frac{c^2x^2}{a^2} \\
x^2 + c^2 + y^2 &= a^2 + \frac{c^2x^2}{a^2}.
\end{aligned}$$

Now we multiply the equation by a^2 to transform the fraction term into a non fraction term,

$$\begin{aligned}
\left[x^2+c^2+y^2 \right. &= \left. a^2+\frac{c^2x^2}{a^2}\right] a^2 \\
a^2x^2 + a^2c^2 + a^2y^2 &= a^4 + c^2x^2 \\
a^2x^2 - c^2x^2 + a^2y^2 &= a^4 - a^2c^2 \\
x^2\left(a^2-c^2\right) + a^2y^2 &= a^2\left(a^2-c^2\right).
\end{aligned}$$

We can simplify the expression by using the pythagorean relation $a^2 = b^2+c^2, \to b^2 = a^2-c^2$,

$$\begin{aligned}
x^2\cancelto{b^2}{\left(a^2-c^2\right)} + a^2y^2 &= a^2\cancelto{b^2}{\left(a^2-c^2\right)} \\
b^2x^2 + a^2y^2 &= a^2b^2.
\end{aligned}$$

Finally, we divide all the expression by a^2b^2,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where b is normally known as the *semiminor axis* and a as *semimajor axis*[4].

[4] This equation is similar to the *unit* circle equation ($x^2+y^2=1$), where $a=1$ and $b=1$.

General equation

From the derivation we assume that the origin of the ellipse is at $(0,0)$, so now we are going to see how to modify the equation to shift that origin point to any other point inside the plane. Using the same notation from the circle, we are going to shifts the ellipse to the coordinate (h,k)[5],

[5] Due to equating the function to 1 the domain and range is $x \in [h-a, h+a]$ and $f(x,y) \in [k-b, k+b]$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Note 1: Expanded form

Now we are going to expand the expression,

$$
\begin{aligned}
\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2} &= 1\\
\frac{1}{a^2}\left(x^2-2xh+h^2\right)+\frac{1}{b^2}\left(y^2-2yk+k^2\right) &= 1\\
\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xh}{a^2}-\frac{2yk}{b^2}+\frac{h^2}{a^2}+\frac{k^2}{b^2} &= 1
\end{aligned}
$$

As same with the circle equation, we can approch this as a two variable function $f(x,y)$ equated to 1, $f(x,y) = 1$. However, we need to take into account that this is not a bijective function, which leads to have a bounded domain and range with no inverse function.

Examples

Example 1: Factored equation of the ellipse

Find the center, axis and foci of the following ellipse,

$$\frac{(x-2)^2}{100}+\frac{(y-5)^2}{64}=1.$$

Comparing the given equation with the general equation for the ellipse we can get the following insight,

$$
\begin{gathered}
\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad \longleftrightarrow \quad \frac{(x-2)^2}{100}+\frac{(y-5)^2}{64}=1\\
-h=-2, \quad -k=-5, \quad a^2=100, \quad b^2=64\\
h=2, \quad k=5, \quad a=10, \quad b=8.
\end{gathered}
$$

Therefore, we know that the orign of the ellipse is at $(2,5)$, also that the semiminor axis is equal to 8 and the semimajor axis is equal to 10. Lastly, to compute the foci of the ellipse we need to add and substract the x component of the origin to the semimajor axis,

$$
\begin{aligned}
F_1 &= (2,5)-(10,0)=(-8,5)\\
F_2 &= (2,5)+(10,0)=(12,5),
\end{aligned}
$$

hence, $F_1=(-8,5), \quad F_2=(12,5)$.

Example 2: Expanded equation of the ellipse

Find the center, axis and foci of the following ellipse,

$$x^2+4y^2+6x+16y+3=0.$$

As before, we compare this expression with the expanded form of the ellipse,

$$
\begin{aligned}
\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xh}{a^2}-\frac{2yk}{b^2}+\frac{h^2}{a^2}+\frac{k^2}{b^2}=1 &\longleftrightarrow x^2+4y^2+6x+16y=-3 \\
\frac{1}{a^2}\left(x^2-2xh+h^2\right)+\frac{1}{b^2}\left(y^2-2yk+k^2\right)=1 &\longleftrightarrow \left(x^2+6x\right)+4\left(y^2+4y\right)=-3.
\end{aligned}
$$

As we can see, we are missing terms. If we just say that $a = 1$ and $b = 1/2$, there is no solution for the equation. Hence, we need to complete the square,

$$
\begin{aligned}
(x+\alpha)^2 &= x^2+2\alpha x+\alpha^2 \implies 6x=2\alpha x \to \alpha=3 \\
(x+3)^2 &= x^2+6x+9,
\end{aligned}
$$

and

$$
\begin{aligned}
(y+\alpha)^2 &= y^2+2\alpha y+\alpha^2 \implies 4y=2\alpha y \to \alpha=2 \\
(y+2)^2 &= y^2+4y+4.
\end{aligned}
$$

To keep the same expression we need to re-write those terms as follows,

$$
\left(x^2+6x\right)=(x+3)^2-9, \quad \left(y^2+4y\right)=(y+2)^2-4,
$$

therefore,

$$
\begin{aligned}
\cancelto{(x+3)^2-9}{\left(x^2+6x\right)}+4\cancelto{(y+2)^2-4}{\left(y^2+4y\right)} &= -3 \\
(x+3)^2-9+4\left[(y+2)^2-4\right] &= -3 \\
(x+3)^2+4(y+2)^2 &= -3+9+16 \\
(x+3)^2+4(y+2)^2 &= 22,
\end{aligned}
$$

finally, we divide by 22 the equation,

$$
\begin{gathered}
\frac{1}{22}(x+3)^2+\frac{4}{22}(y+2)^2=1 \longleftrightarrow \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \\
3=-h, \quad 2=-k, \qquad \frac{1}{22}=a^2, \quad \frac{2}{11}=b^2 \\
h=-3, \quad k=-2, \qquad a=\sqrt{22}, \quad b=\sqrt{\frac{2}{11}}.
\end{gathered}
$$

Hence, the ellipse is center at $O = (-3,-2)$ with focal points at $F_1=(-3-\sqrt{22},-2)$ and $F_2=(-3+\sqrt{22},-2)$.

Exercises

Ellipse

- Find the center, vertices and foci of the following ellipse

$$
\frac{(x-1)^2}{16}+\frac{(y+5)^2}{25}=1
$$

.

- Write the compacted equation of the ellipse $16x^2 - 48x + 9y^2 + 12y + 28 = 0$.

Review

- The hydrogen ion concentration of a sample of a mystery liquid is $\mathrm{H}^+ = 6.88 \times 10^{-3}$. Calculate the pH of the substance considering that $\mathrm{pH} = -\log[\mathrm{H}^+]$.
- Determine if the following functions are inverses by computing both compositions, $f(x) = (x-5)/2$ and $g(x) = 2x+5$.
- Express the domain and rage of the functions $f(x) = (x-5)^{1/2}$ and $g(x) = 1/(x-2)^2$.
- Find the critical point of the function $f(x) = \exp[(x-4)(x+4)]$.

References

James Stewart, L. Redlin, Saleem Watson, and Phyllis Panman. *Precalculus: mathematics for calculus*. Cengage Learning, Boston, MA, seventh edition edition, 2016. ISBN 9781305071759.

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