

# Algebraic and Transcendental functions

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**Summary** Class sketches for the subject Algebraic and Transcendental Functions. Semester January-May 2025

## Hyperbola

A hyperbola is a curve defined by all points that satisfies that the difference of the distance from a point  $P$  to a point  $F_1$  with the distance from a point  $P$  to a point  $F_2$  remains as a constant value of  $2a$ , as shown in figure1. The distance between point  $F_1$  and  $F_2$  is defined to be  $2c$ .

### Derivation of the equation

We are going to call the distance from point  $P$  to point  $F_1$  as  $r_1$ , and the distance from point  $P$  to point  $F_2$  as  $r_2$ . By definition, the subtraction of those distances needs to be equal to  $2a$ , therefore we have the following equation,

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a.$$

As before, we are going to modify that equation to express it in a more friendly way, taking into account that  $b^2 = c^2 - a^2$ ,

$$\begin{aligned} \left[ \sqrt{(x-c)^2 + y^2} \right]^2 &= \left[ 2a - \sqrt{(x+c)^2 + y^2} \right]^2 \\ (x-c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\ \cancel{x^2} - 2cx + \cancel{y^2} &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + \cancel{x^2} + 2cx + \cancel{y^2} \\ \frac{1}{4a} \left[ 4a\sqrt{(x+c)^2 + y^2} \right] &= \frac{1}{4a} [4a^2 + 4cx] \\ \left[ \sqrt{(x+c)^2 + y^2} \right]^2 &= \left[ a + \frac{cx}{a} \right]^2 \\ (x+c)^2 + y^2 &= a^2 + 2cx + \frac{c^2x^2}{a^2} \\ x^2 + 2cx + c^2 + y^2 &= a^2 + 2cx + \frac{c^2x^2}{a^2} \\ x^2 + c^2 + y^2 &= a^2 + \frac{c^2x^2}{a^2} \\ x^2 \left( 1 - \frac{c^2}{a^2} \right) + y^2 &= a^2 - \frac{b^2}{a^2} \\ \frac{1}{-b^2} \left[ \frac{-b^2}{a^2} x^2 + y^2 \right] &= \frac{-b^2}{-b^2} \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \end{aligned}$$

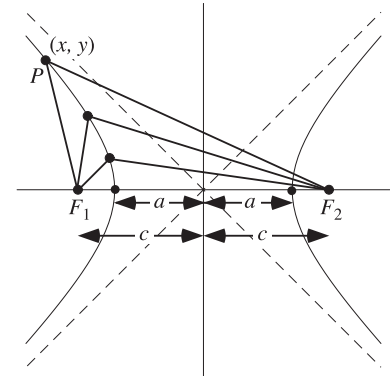


Figure 1: Skeeth of an hyperbola

*General equation*

Finally we are going to take into account that the center is at any point of the plane,

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

**Note 1: Expanded equation**

For completeness, we are going to expand the quadratic terms,

$$\begin{aligned}\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1 \\ \frac{1}{a^2} [x^2 - 2hx + h^2] - \frac{1}{b^2} [y^2 - 2ky + k^2] &= 1 \\ \frac{x^2}{a^2} - \frac{2hx}{a^2} + \frac{h^2}{a^2} - \frac{y^2}{b^2} + \frac{2ky}{b^2} - \frac{k^2}{b^2} &= 1\end{aligned}$$

This equation ( $f(x,y) = 1$ ) is not a bijective function. On the other hand, the range can go from plus infinity to minus infinity  $f(x,y) \in (-\infty, \infty)$ , however, the domain is not defined between the distance of the two vertex  $x \in (-\infty, h-a] \cup [h+a, \infty)$ .

**Note 2: Asymptotes of the hyperbola**

As mentioned in the note before, the domain is not defined in the interval  $x \in (h-a, h+a)$ , indicating the existence of asymptotes. Here we are going to see the derivation of the asymptotes. To do that, we are going to equate the function to zero  $f(x,y) = 0$ ,

$$\begin{aligned}\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 0 \rightarrow \frac{(x-h)^2}{a^2} = \frac{(y-k)^2}{b^2} \\ \left[ \frac{(x-h)^2}{a^2} \right]^{1/2} &= \left[ \frac{(y-k)^2}{b^2} \right]^{1/2} \\ x-h &= \pm \frac{a}{b}(y-k)\end{aligned}$$

*Examples***Example 1: Factored form of a vertical hyperbola**

We are going to find the center, vertices, foci and asymptotes of the following hyperbola,

$$\frac{(y-8)^2}{4} - \frac{(x+1)^2}{9} = 1.$$

Let's start with the center, which is  $C = (-1, 8)$ . Now that we know that and also the values of  $a = 2$  and  $b = 3$ , we can compute  $c$  using the relation  $b^2 \equiv c^2 - a^2 \rightarrow c = \sqrt{9+4} = \sqrt{13}$ . An important observation is that the term with  $x$  is negative, which tells us that is a vertical hyperbola.

With that information we can compute the vertex of the hyperbola as follows,

$$\begin{aligned}v_1 &= (-1, 8 - 2) & v_2 &= (-1, 8 + 2) \\v_1 &= (-1, 6) & v_2 &= (-1, 10)\end{aligned}$$

Then, we do something similar for the foci,

$$\begin{aligned}F_1 &= (-1, 8 - \sqrt{13}) & F_2 &= (-1, 8 + \sqrt{13}) \\F_1 &= (-1, 4.39) & F_2 &= (-1, 11.60)\end{aligned}$$

Finally, to compute the asymptotes of the hyperbola, we modify the general equation for hyperbola asymptotes since is a vertical hyperbola,  $y - k = \pm \frac{a}{b}(x - h)$ ,

$$\begin{aligned}y - k &= \pm \frac{a}{b}(x - h) \rightarrow y - 8 = \pm \frac{2}{3}(x + 1) \\y &= \pm \frac{2}{3}(x + 1) + 8\end{aligned}$$

### Example 2: Factored form of a horizontal hyperbola

We are going to find the center, vertices, foci and asymptotes of the following hyperbola,

$$\frac{(x + 8)^2}{25} - \frac{(y - 9)^2}{4} = 1.$$

Let's start with the center, which is  $C = (-8, 9)$ . Now that we know that and also the values of  $a = 5$  and  $b = 2$ , we can compute  $c$  using the relation  $b^2 \equiv c^2 - a^2$ ,  $\rightarrow c = \sqrt{25 + 4} = \sqrt{29}$ . An important observation is that the term with  $y$  is negative, which tells us that is a horizontal hyperbola.

With that information we can compute the vertex of the hyperbola as follows,

$$\begin{aligned}v_1 &= (-8 - 5, 9) & v_2 &= (-8 + 5, 9) \\v_1 &= (-13, 9) & v_2 &= (-3, 9)\end{aligned}$$

Then, we do something similar for the foci,

$$\begin{aligned}F_1 &= (-8 - \sqrt{29}, 9) & F_2 &= (-8 + \sqrt{29}, 9) \\F_1 &= (-13.38, 9) & F_2 &= (-2.61, 9)\end{aligned}$$

Finally, to compute the asymptotes of the hyperbola, we use the general equation for hyperbola,  $x - h = \pm \frac{a}{b}(y - k)$ ,

$$\begin{aligned}x - h &= \pm \frac{a}{b}(y - k) \rightarrow x + 8 = \pm \frac{5}{2}(y - 9) \\x &= \pm \frac{5}{2}(y - 9) - 8\end{aligned}$$

## Exercises

### Hyperbola

- Construct the hyperbola equation from its asymptotes,  $y = \pm 1/2(x + 4) - 3$ .
- Find the center, vertices, foci and asymptotes of the following hyperbola  $(x - 3)^2/9 - (y - 6)^2/4 = 1$ .

### Review

- Solve for  $x$ ,  $\log_5(4x) - 3 = -8$
- Determine the horizontal asymptote for the following function  $f(x) = 1/3(e^x - 4)^4$
- Determine the vertical and horizontal asymptotes of the following function  $f(x) = (7x - 15)/(x - 5)$ .
- Find the inverse function of  $f(x) = 2x + 4$ .

## References

James Stewart, L. Redlin, Saleem Watson, and Phyllis Panman. *Precalculus: mathematics for calculus*. Cengage Learning, Boston, MA, seventh edition, 2016. ISBN 9781305071759.

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