Algebraic and Transcendental functions

Francisco Vazquez-Tavares

April 7, 2025

Summary Class sketches for the subject Algebraic and Transcendental Functions. Semester January-May 2025

Parabola

A parabola is the set of all points in the plane that are equidistant from a given line L, called directrix, and a given point F, called focus, which is not on le line L. The distance between the directrix and the focus is $p = 2a$, where a is the distance from the vertex to the directrix or focus. (fig.1)

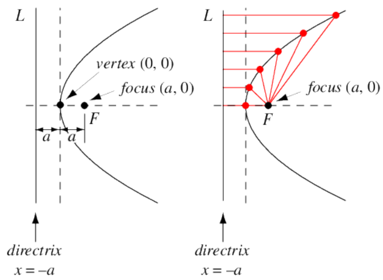

Figure 1: The figure from the left are the gemetrical representation of the directrix, focus and vertex. On the right it is shown the distances from the focus to the parabola are equal from the point of the parabola to the directrix.

Derivation of the equation

we are going to use the same tools from the last sessions. Since the parabola is the set of all points in the plane that are equidistance from a given point and a given line, we use the distance function between two points and a distance between a point and a line. The distance between the focus ($(0,a)$) and any other point in the plane is $d = \sqrt{(x-a)^2 + y^2}$. Tacking into account the fig.1, the directrix is the vertical line $x = -a$, hence, the distance any point[1] of the line to the vertex is $x + a$.

Now that we know the distances from any point of the plane to the focus and the distance from the directrix to the vertex, we can apply the equidistant restriction,

$$\sqrt{(x-a)^2 + y^2} = x + a.$$

Now we are going to make some algebraic manipulations to get a more friendly expression,

$$\begin{aligned}
(x-a)^2 + y^2 &= (x+a)^2 \\
\cancelto{0}{x^2} - 2ax + \cancelto{0}{a^2} + y^2 &= \cancelto{0}{x^2} + 2ax + \cancelto{0}{a^2} \\
y^2 &= 4ax.
\end{aligned}$$

[1] In this distance we can "ignore" the y component of the point, because we are only considereing perpendicular points from the directrix.

General equation

As the other conic sections, we are going to consider the situation in which the vertex is not at the origin, as shown in figure1, instead is at an arbitrary point in the plane (h,k). Therefore[2],

$$(y-k)^2 = 4a(x-h).$$

[2] Please be **very** carefull with the parathensis, they are really important.

What if...

Before moving on with the examples and exercises, it is important to consider the case in which the directrix is not a vertical line, but rather an horizontal one, and the focus is at $F=(0,a)$, not $F=(a,0)$. Following the same procedure as before[3] we get,

$$x^2 = 4ay.$$

Finally, to consider a vertex at any point in the plane,

$$(x-h)^2 = 4a(y-k).$$

> **Note 1: Expanded form**
>
> $$\begin{aligned} (x-h)^2 &= 4a(y-k) \\ x^2 - 2hx + h^2 &= 4ay - 4ak \\ \frac{1}{4a}[4ay] &= \frac{1}{4a}\left[x^2 - 2hx + h^2 + 4ak\right] \\ y &= \frac{x^2}{4a} - \frac{hx}{2a} + \frac{h^2}{4a} + k \end{aligned}$$
>
> In this case this function (a parabola with horizontal directrix) is bijective and has the following domain and range $x \in (-\infty,\infty)$ and $f(x,y) \in (-\infty,\infty)$. However, we need to be careful, because the parabola with a vertical directrix is not a bijective function, because for each value of x it corresponds two values of y.

3

$$\begin{aligned} \sqrt{x^2+(y-a)^2} &= y+a \\ x^2+(y-a)^2 &= (y+a)^2 \\ x^2 + \cancelto{0}{y^2} - 2ax + \cancelto{0}{a^2} &= \cancelto{0}{y^2} + 2ay + \cancelto{0}{a^2} \\ x^2 &= 4ay. \end{aligned}$$

Examples

> **Example 1: Factored form of the parabola**
>
> We are going to write the factored form of the parabola with the following information, the focus is at $F=(-3,-7)$ and the directrix is at $y=3$.
> Since the directrix is an horizontal line we are going to use the folowing expression for the parabola, $(x-h)^2 = 4a(y-k)$. We know that (h,k) is the coordinate for the vertex of the parabola and the distance from the vertex to the focus is the same distance from the vertex to the directrix and that distance is the parameter a. Therefore we need to compute the distance in the y direction from the focus to the directrixand divided by two,
>
> $$\begin{aligned} a &= \frac{F_y - L}{2} \\ &= \frac{-7-3}{2} \\ &= -5. \end{aligned}$$
>
> Now, we add 5 to the y component of the focus, giving us that the vertex is at $V=(-3,-2)$. Finally we substitute those values into the factored equation,
>
> $$(x+3)^2 = -20(y+2).$$

Example 2: Expanded form of the parabola

We are going to find the vertex, focus and directrix of the following parabola $3x^2+24x+y+47=0$.
As we did we the ellipse, we are going to use the same methodology to find the vertex, focus and directrix of the parabola in its expanded expression. Looking at the equation we can identify that the squared term is in the x, telling us that the parabola has an horizontal directrix, which lead us to use the following equation as a reference,

$$y=\frac{x^2}{4a}-\frac{hx}{2a}+\frac{h^2}{4a}+k\longleftrightarrow y=-3x^2-24x-47$$
$$\frac{x^2}{4a}=-3x^2,\quad -\frac{hx}{2a}=-24x,\quad \frac{h^2}{4a}+k=-47.$$

First we are going to solve the equation with x^2 to find a, then we are going to solve the equation with x to solve h and finally, we are going to use the values of h and a to find k in the last equation.

$$\frac{x^2}{4a}=-3x^2\rightarrow a=-\frac{1}{12}$$
$$-\frac{hx}{2a}\bigg|_{a=-1/12}=-24x\rightarrow h=-4$$
$$\frac{h^2}{4a}\bigg|_{a=-1/12,h=4}+k=-47\rightarrow k=1,$$

Therefore, the vertex of the parabola is $v=(-4,1)$, the focus is at $F=(-4,1-1/12)$ and the directrix is at $y=1+1/12$.

Exercises

Parabola

- Find the vertex, focus and directrix of the parabola $-y^2+2y-x+1=0$.
- find the equation of the parabola with vertex $(2,1)$ and directrix $y=-5$
- Find the intersection points (2 points) of between these two parabolas, $x^2=4(3/2)y$ and $y^2=4(3/2)x$.

Review

- Consider the following function $f(x)=3/2x^2$ and $g(x)=4/3x$. Compute $f(x)/g(x)$ and determine the domain and range of the new function.
- Define the coordinate of the empty hole $f(x)=(x^3(x-3))/(4(x-3))$.
- Use the change base formula to compute $\log_{(1-5/3)}(2000)$.
- An unkwon element has a half-life of 800 days. Suppose that you have a sample of 400 mg. Which is the decay rate of this element?

References

James Stewart, L. Redlin, Saleem Watson, and Phyllis Panman. *Precalculus: mathematics for calculus*. Cengage Learning, Boston, MA, seventh edition edition, 2016. ISBN 9781305071759.

Eric W. Weisstein. Ellipse, a. URL `https://mathworld.wolfram.com/Ellipse.html`.

Eric W. Weisstein. Pythagorean theorem, b. URL `https://mathworld.wolfram.com/PythagoreanTheorem.html`.

Eric W. Weisstein. Circle, c. URL `https://mathworld.wolfram.com/Circle.html`.