Algebric and Transcendental functions

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Summary Class sketches for the subject Algebraic and Transcendental Functions. Semester January-May 2025

Circle

A circle is the set of points in a plane that are equidistant from a given point O. The point O is called the center, while the distance r from the center is called the radius. Twice the radius is known as the diameter $d = 2r$ (fig.1).

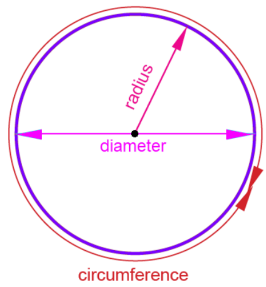

Figure 1: Sketch for the board

Derivation of the equation

The main characteristic of the circle is "*equidistant*", that is, that the distance between all those points and the point O are equal. A usefull theorem that translates that characteristic into an equation is the **Pythagorean theorem**. Which describes the relationship between the sides of a right triangle (fig.2) with the equation,

$$a^2 + b^2 = c^2.$$

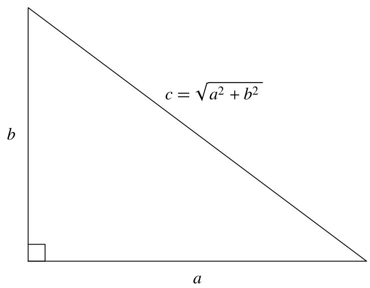

Figure 2: Sketch for the board for the pythagoream theorem.

We can show that any point on the circle forms a right triangle with the horizontal and vertical distances from the point (x, y) to the point O (fig.3).

In that sense, we can relate a with x, b with y and c with the radius. In the special case when the radius is equal to one, we have a *unit circle*,

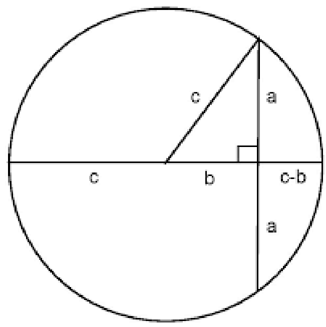

Figure 3: Relation between pythagorean theorem and the circle.

> **Definition 1: Unit circle**
>
> The unit circle is the circle of radius 1 centered at the origin in the xy-plane. It is represented with the following equation,
>
> $$x^2 + y^2 = 1.$$

General equation

Now, let's give the mathematical representation of the point O as (h, k). Using the knowledge of function translations, we can move the origin of the circle (fig.4) as follows,

$$(x - h)^2 + (y - k)^2 = r^2. \tag{1}$$

Now, lets get some mathematical exercises.

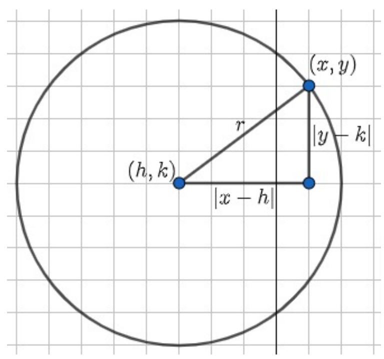

Figure 4: Diplaced circle to the point (h, k).

> **Note 1: Some algebra stuff.**
>
> We are going to expand the equation(1) to get familiarize ourselves

with different expressions of the same mathematical object.

$$\begin{aligned}(x-h)^2+(y-k)^2&=r^2\\ \left(x^2-2hx+h^2\right)+\left(y^2-2ky+k^2\right)&=r^2\\ x^2+y^2-2hx-2ky+h^2+k^2&=r^2\end{aligned}$$

As we can see, this can be approach as a function of two independent variables x and y ($f(x,y)$) equated to a constant value r^2, $f(x,y)=r^2$. This function is not bijective, indicating that there is no inverse function and the domain and range are bounded.

Examples

Example 1: Factored equation of the circle.

Identify the radius and center of the following circle,

$$(x+3)^2+(y+1)^2=9.$$

Recalling the definition of the circle, $(x-h)^2+(y-k)^2=r^2$, and comparing both expressions, we can make the following relations,

$$x+3\longleftrightarrow x-h,\quad y+1\longleftrightarrow y-k,\quad 9\longleftrightarrow r^2.$$

Therefore,

$$\begin{aligned}3&=-h,\quad 1=-k,\quad 9=r^2\\ h&=-3,\quad k=-1,\quad r=9^{1/2}=3.\end{aligned}$$

This tell's us that the circle origin is at $(-3,-1)$ with a radius of $r=3$.

Example 2: Expanded equation of the circle.

Identify the radius and center of the following circle,

$$x^2+y^2-8x+4y+4=0.$$

Recalling the note, we can make the following comparsion as before,

$$x^2+y^2-8x+4y+4=0\quad\longleftrightarrow\quad x^2+y^2-2hx-2ky+h^2+k^2-r^2=0.$$

giving us the following relations,

$$\begin{aligned}-8x=-2hx,&\quad +4y=-2ky,\\ h=\frac{-8}{-2},&\quad k=\frac{4}{-2},\\ h=4,&\quad k=-2.\end{aligned}$$

Hence, the origin of the circle is at $(4,-2)$. To find the radius, we start by substituting the values of h and k into the equation and comparing the final terms,

$$\begin{aligned} x^2+y^2-8x+4y+4=0 &\quad\longleftrightarrow\quad x^2+y^2-2(4)x-2(-2)y+(4)^2+(-2)^2-r^2=0. \\ x^2+y^2-8x+4y+4=0 &\quad\longleftrightarrow\quad x^2+y^2-8x+4y+16+4-r^2=0, \end{aligned}$$

therefore

$$\begin{aligned} 4 &= 16+4-r^2 \\ 4-20 &= -r^2 \\ r &= 16^{1/2} \\ r &= 4. \end{aligned}$$

Finally, the radius of the circle is 4.

Exercises

Circle

1. With help of the pythagorean theorem find the function of a circle with origin at $O=(3/2,2)$ and the point $(2,3/2)$.
2. Write the equation of the circle with center at $(6,-9)$ and radius $1/4$.
3. Find the origing, radius of the following circle equation $x^2-8x+y^2-10y+5=0$, then write the factored equation.

Review

1. Taking into account the following functions $f(x)=x+c$ and $g(x)=x+h$, compute $(f\circ g)(x)-f(x)$.
2. Determine the vertical and horizontal asymptotes of the following function $f(x)=(7x-15)/(x-5)$.
3. Use the properties of logarithms to condes the following expression $\log\left[\ln\left(x^6\right)/6\ln(x)\right]-\log[8\ln(x)]$.
4. Determine the critical point, asymptote, domain and range of the following function $f(x)=\ln(x+6)-4$.

   Derivation of the equation/functionnote 1.
   Example/Exercises.

References

James Stewart, L. Redlin, Saleem Watson, and Phyllis Panman. *Precalculus: mathematics for calculus*. Cengage Learning, Boston, MA, seventh edition edition, 2016. ISBN 9781305071759.

Eric W. Weisstein. Pythagorean theorem, a. URL `https://mathworld.wolfram.com/PythagoreanTheorem.html`.

Eric W. Weisstein. Circle, b. URL `https://mathworld.wolfram.com/Circle.html`.