Algebraic and Transcendental functions

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Summary Class sketches for the subject Algebraic and Transcendental Functions. Semester January-May 2025

Ellipse

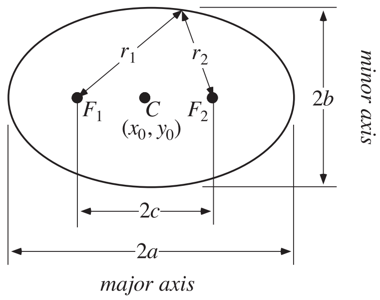

Figure 1: Ellipse

An ellipse is defined as the curve formed by all the points that satisfies that the sum of the distances r_1 and r_2 from two fixed points F_1 and F_2 separated by a distance of $2c$ is constant with value $2a$. (fig.1).

Where the value a is commnly refered as *semimajor axis*, the points F_1 and F_2 as *foci* and the half point between the foci is refered as the *origin* of the coordinate system.

Derivation of the equation

To derive the a;gebraic form of the ellipse from its definition, we start by translating the definition of the ellipse into an algebra representation. First, we represent the foci as $F_1 = (c, 0)$ and $F_2 = (-c, 0)$. Now, we recall the equation to compute the distance between two points, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Since we want to find the points in the plane that satisfies the definiiton of the ellipse, we are going to find the distance between the foci and all the plane,

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2}.$$

Now we are going to apply the restricction that the sum of those distances are equal to $2a$,

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

With this equation we are going to apply several algebraic tricks[1] to simplify the equation. Let's start by isolating one distance and then squaring the equation,

[1] Basically we are going to get rid of the radicals ($\sqrt{x}$) and fractions.

$$\begin{aligned}
\sqrt{(x-c)^2 + y^2} &= 2a - \sqrt{(x+c)^2 + y^2} \\
\left[\sqrt{(x-c)^2 + y^2}\right]^2 &= \left[2a - \sqrt{(x+c)^2 + y^2}\right]^2 \\
(x-c)^2 + y^2 &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2 \\
x^2 - 2cx + c^2 + y^2 &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \\
\cancelto{0}{x^2} - 2cx + c^2 + \cancelto{0}{y^2} &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + \cancelto{0}{x^2} + 2cx + c^2 + \cancelto{0}{y^2} \\
-4cx &= 4a^2 - 4a\sqrt{(x+c)^2 + y^2}.
\end{aligned}$$

$$\left[2a - \sqrt{(x+c)^2 + y^2}\right]\left[2a - \sqrt{(x+c)^2 + y^2}\right]$$
$$4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

Now, we are going to solve for the distance $\sqrt{(x+c)^2+y^2}$,

$$
\begin{aligned}
-4cx &= 4a^2 - 4a\sqrt{(x+c)^2+y^2} \\
4a\sqrt{(x+c)^2+y^2} &= 4a^2 + 4cx \\
\sqrt{(x+c)^2+y^2} &= \frac{4a^2+4cx}{4a} \\
\sqrt{(x+c)^2+y^2} &= a + \frac{cx}{a}.
\end{aligned}
$$

Now we square it again[2],

[2] $(x+c)^2 = x^2+2cx+c^2$

$$
\begin{aligned}
\left[\sqrt{(x+c)^2+y^2}\right]^2 &= \left[a+\frac{cx}{a}\right]^2 \\
x^2 + \cancelto{0}{2cx} + c^2 + y^2 &= a^2 + \cancelto{0}{2cx} + \frac{c^2x^2}{a^2} \\
x^2 + c^2 + y^2 &= a^2 + \frac{c^2x^2}{a^2}.
\end{aligned}
$$

Now we multiply the equation by a^2 to transform the fraction term into a non fraction term,

$$
\begin{aligned}
\left[x^2 + c^2 + y^2 \right.&= \left. a^2 + \frac{c^2x^2}{a^2}\right] a^2 \\
a^2x^2 + a^2c^2 + a^2y^2 &= a^4 + c^2x^2 \\
a^2x^2 - c^2x^2 + a^2y^2 &= a^4 - a^2c^2 \\
x^2\left(a^2-c^2\right) + a^2y^2 &= a^2\left(a^2-c^2\right).
\end{aligned}
$$

We can simplify the expression by using the pythagorean relation $a^2 = b^2+c^2, \to b^2 = a^2-c^2$,

$$
\begin{aligned}
x^2\cancelto{b^2}{\left(a^2-c^2\right)} + a^2y^2 &= a^2\cancelto{b^2}{\left(a^2-c^2\right)} \\
b^2x^2 + a^2y^2 &= a^2b^2.
\end{aligned}
$$

Finally, we divide all the expression by a^2b^2,

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,
$$

where b is normally known as the *semiminor axis* and a as *semimajor axis*[3].

[3] This equation is similar to the *unit* circle equation $(x^2+y^2=1)$, where $a=1$ and $b=1$.

General equation

From the derivation we assume that the origin of the ellipse is at $(0,0)$, so now we are going to see how to modify the equation to shift that origin point to any other point inside the plane. Using the same notation from the circle, we are going to shifts the ellipse to the coordinate (h,k)[4],

[4] Due to equating the function to 1 the domain and range is $x \in [h-a, h+a]$ and $f(x,y) \in [k-b, k+b]$

$$
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.
$$

Note 1: Expanded form

Now we are going to expand the expression,

$$
\begin{aligned}
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} &= 1 \\
\frac{1}{a^2}\left(x^2 - 2xh + h^2\right) + \frac{1}{b^2}\left(y^2 - 2yk + k^2\right) &= 1 \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xh}{a^2} - \frac{2yk}{b^2} + \frac{h^2}{a^2} + \frac{k^2}{b^2} &= 1
\end{aligned}
$$

As same with the circle equation, we can approch this as a two variable function $f(x,y)$ equated to 1, $f(x,y) = 1$. However, we need to take into account that this is not a bijective function, which leads to have a bounded domain and range with no inverse function.

Examples

Example 1: Factored equation of the ellipse

Find the center, vertices and foci of the following ellipse,

$$\frac{(x-2)^2}{100} + \frac{(y-5)^2}{64} = 1.$$

Example 2: Expanded equation of the ellipse

Exercises

Ellipse

- Find the center, vertices and foci of the following ellipse $(x-1)^2/16 + (y+5)^2/25 = 1$.

Review

References

James Stewart, L. Redlin, Saleem Watson, and Phyllis Panman. *Precalculus: mathematics for calculus*. Cengage Learning, Boston, MA, seventh edition edition, 2016. ISBN 9781305071759.

Eric W. Weisstein. Pythagorean theorem, a. URL `https://mathworld.wolfram.com/PythagoreanTheorem.html`.

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