Algebraic and Transcendental functions

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Summary Class sketches for the subject Algebraic and Transcendental Functions. Semester January-May 2025

Recap

Parabola

$$\sqrt{(x-a)^2+y^2} = x+a \to (y-k)^2 = 4a(x-h)$$
$$\sqrt{x^2+(y-a)^2} = y+a \to (x-h)^2 = 4a(y-k)$$

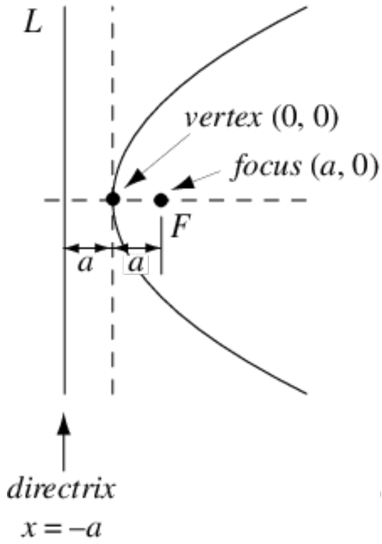

Figure 1: Parabola

Note 1: Expanded form

$$\begin{aligned}(x-h)^2 &= 4a(y-k)\\ x^2-2hx+h^2 &= 4ay-4ak\\ \frac{1}{4a}[4ay] &= \frac{1}{4a}\left[x^2-2hx+h^2+4ak\right]\\ y &= \frac{x^2}{4a}-\frac{hx}{2a}+\frac{h^2}{4a}+k\end{aligned}$$

In this case this function (a parabola with horizontal directrix) is bijective and has the following domain and range $x \in (-\infty,\infty)$ and $f(x,y) \in (-\infty,\infty)$. However, we need to be careful, because the parabola with a vertical directrix is not a bijective function, because for each value of x it corresponds two values of y.

Circle

$$a^2+b^2 = c^2 \to (x-h)^2+(y-k)^2 = r^2 \tag{1}$$

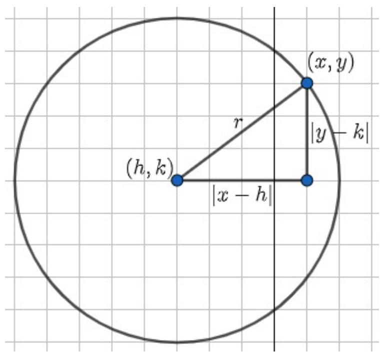

Figure 2: Circle

Note 2: Expanded form

We are going to expand the equation(**??**) to get familiarize ourselves with different expressions of the same mathematical object.

$$\begin{aligned}(x-h)^2+(y-k)^2 &= r^2\\ \left(x^2-2hx+h^2\right)+\left(y^2-2ky+k^2\right) &= r^2\\ x^2+y^2-2hx-2ky+h^2+k^2 &= r^2\end{aligned}$$

As we can see, this can be approach as a function of two independent variables x and y ($f(x,y)$) equated to a constant value r^2, $f(x,y) = r^2$. This function is not bijective, indicating that there is no inverse function and the domain and range are bounded.

Ellipse

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a \rightarrow \frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

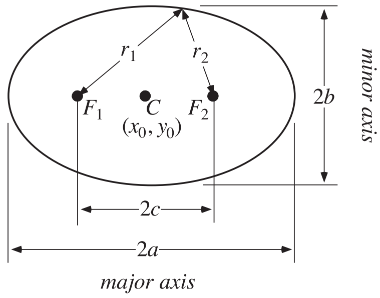

Figure 3: Ellipse

Note 3: Expanded form

Now we are going to expand the expression,

$$\begin{aligned}\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}&=1\\ \frac{1}{a^2}\left(x^2-2xh+h^2\right)+\frac{1}{b^2}\left(y^2-2yk+k^2\right)&=1\\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xh}{a^2}-\frac{2yk}{b^2}+\frac{h^2}{a^2}+\frac{k^2}{b^2}&=1\end{aligned}$$

As same with the circle equation, we can approch this as a two variable function $f(x,y)$ equated to 1, $f(x,y) = 1$. However, we need to take into account that this is not a bijective function, which leads to have a bounded domain and range with no inverse function.

Hyperbola

$$\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}=2a \rightarrow \frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

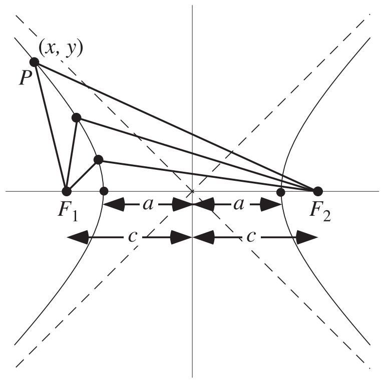

Figure 4: Hyperbola

Note 4: Expanded equation

For completeness, we are going to expand the quadratic terms,

$$\begin{aligned}\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}&=1\\ \frac{1}{a^2}\left[x^2-2hx+h^2\right]-\frac{1}{b^2}\left[y^2-2ky+k^2\right]&=1\\ \frac{x^2}{a^2}-\frac{2hx}{a^2}+\frac{h^2}{a^2}-\frac{y^2}{b^2}+\frac{2ky}{b^2}-\frac{k^2}{b^2}&=1\end{aligned}$$

This equation ($f(x,y) = 1$) is not a bijective function. On the other hand, the range can go from plus infinity to minus infinity $f(x,y) \in (-\infty,\infty)$, however, the domain is not defined between the distance of the two vertex $x \in (-\infty, h-a] \cup [h+a,\infty)$.

Note 5: Asymptotes of the hyperbola

As mentioned in the note before, the domain is not defined in the inverval $x \in (h-a, h+a)$, indicating the existance of asymptotes. Here we are going to see the derivation of the asymptotes. To do

> that, we are going to equate the function to zero $f(x,y)=0$,
>
> $$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \to \frac{(x-h)^2}{a^2}=\frac{(y-k)^2}{b^2}$$
> $$\left[(x-h)^2\right]^{1/2}=\left[\frac{a^2}{b^2}(y-k)^2\right]^{1/2}$$
> $$x-h=\pm\frac{a}{b}(y-k)$$

Exercises

Conic sections

1. With help of the pythagorean theorem find the function of a circle with origin at $O=(3/2,2)$ and the point $(2,3/2)$.
2. Write the equation of the circle with center at $(6,-9)$ and radius $1/4$.
3. Find the origing, radius of the following circle equation $x^2-8x+y^2-10y+5=0$, then write the factored equation.
4. Find the vertex, focus and directrix of the parabola $-y^2+2y-x+1=0$.
5. find the equation of the parabola with vertex $(2,1)$ and directrix $y=-5$
6. Find the intersection points (2 points) of between these two parabolas, $x^2=4(3/2)y$ and $y^2=4(3/2)x$.
7. Construct the hyperbola equation from it's asymptotes, $y=\pm1/2(x+4)-3$.
8. Find the center, vertices, foci and asymptotes of the following hyperbola $(x-3)^2/9-(y-6)^2/4=1$.
9. Find the center, vertices and foci of the following ellipse
$$\frac{(x-1)^2}{16}+\frac{(y+5)^2}{25}=1$$
.
10. Write the compacted equation of the ellipse $16x^2-48x+9y^2+12y+28=0$.

References

James Stewart, L. Redlin, Saleem Watson, and Phyllis Panman. *Precalculus: mathematics for calculus*. Cengage Learning, Boston, MA, seventh edition edition, 2016. ISBN 9781305071759.

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Eric W. Weisstein. Circle, c. URL `https://mathworld.wolfram.com/Circle.html`.