Algebraic and Transcendental functions

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Summary Class sketches for the subject Algebraic and Transcendental Functions. Semester January-May 2025

Hyperbola

A hyperbola is a curve defined by all points that satifies that the difference of the distance from a point P to a point F_1 with the distance from a point P to a point F_2 remains as a constant value of $2a$, as shown in figure1. The distance between point F_1 and F_2 is defined to be $2c$.

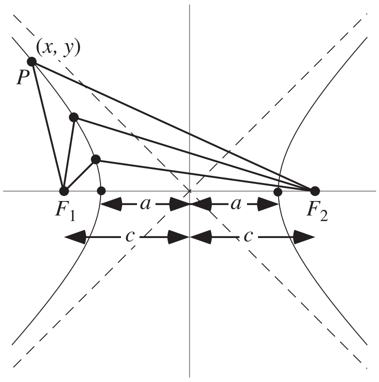

Figure 1: Skecth of an hyperbola

Derivation of the equation

We are going to call the distance from point P to point F_1 as r_1, and the distance from point P to point F_2 as r_2. By definition, the substraction of those distances needs to be equal to $2a$, therefore we have the following equation,

$$\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}=2a.$$

As before, we are going to modify that equation to express it in a more friendly way, taking into account that $b^2=c^2-a^2$,

$$\begin{aligned}
\left[\sqrt{(x-c)^2+y^2}\right]^2 &= \left[2a-\sqrt{(x+c)^2+y^2}\right]^2\\
(x-c)^2+y^2 &= 4a^2-4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2\\
\cancelto{0}{x^2}-2cx+\cancelto{0}{c^2}+\cancelto{0}{y^2} &= 4a^2-4a\sqrt{(x+c)^2+y^2}+\cancelto{0}{x^2}+2cx+\cancelto{0}{c^2}+\cancelto{0}{y^2}\\
\frac{1}{4a}\left[4a\sqrt{(x+c)^2+y^2}\right] &= \frac{1}{4a}\left[4a^2+4cx\right]\\
\left[\sqrt{(x+c)^2+y^2}\right]^2 &= \left[a+\frac{cx}{a}\right]^2\\
(x+c)^2+y^2 &= a^2+2cx+\frac{c^2x^2}{a^2}\\
x^2+\cancelto{0}{2cx}+c^2+y^2 &= a^2+\cancelto{0}{2cx}+\frac{c^2x^2}{a^2}\\
x^2+c^2+y^2 &= a^2+\frac{c^2x^2}{a^2}\\
x^2\cancelto{-b^2/a^2}{\left(1-\frac{c^2}{a^2}\right)}+y^2 &= \cancelto{-b^2}{a^2-c^2}\\
\frac{1}{-b^2}\left[\frac{-b^2}{a^2}x^2+y^2\right] &= \frac{\cancelto{1}{-b^2}}{\cancel{-b^2}}\\
\frac{x^2}{a^2}-\frac{y^2}{b^2} &= 1
\end{aligned}$$

General equation

Finally we are going to take into account that the center is at any point of the plane,

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Note 1: Expanded equation

For completeness, we are going to expand the quadratic terms,

$$\begin{aligned} \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1 \\ \frac{1}{a^2}\left[x^2 - 2hx + h^2\right] - \frac{1}{b^2}\left[y^2 - 2ky + k^2\right] &= 1 \\ \frac{x^2}{a^2} - \frac{2hx}{a^2} + \frac{h^2}{a^2} - \frac{y^2}{b^2} + \frac{2ky}{b^2} - \frac{k^2}{b^2} &= 1 \end{aligned}$$

This equation ($f(x,y) = 1$) is not a bijective function. On the other hand, the range can go from plus infinity to minus infinity $f(x,y) \in (-\infty,\infty)$, however, the domain is not defined between the distance of the two vertex $x \in (-\infty, h-a] \cup [h+a, \infty)$.

Note 2: Asymptotes of the hyperbola

As mentioned in the note before, the domain is not defined in the inverval $x \in (h-a, h+a)$, indicating the existance of asymptotes. Here we are going to see the derivation of the asymptotes. To do that, we are going to equate the function to zero $f(x,y) = 0$,

$$\begin{aligned} \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \rightarrow \frac{(x-h)^2}{a^2} &= \frac{(y-k)^2}{b^2} \\ \left[(x-h)^2\right]^{1/2} &= \left[\frac{a^2}{b^2}(y-k)^2\right]^{1/2} \\ x - h &= \pm\frac{a}{b}(y-k) \end{aligned}$$

Examples

Example 1: Factored form of a vertical hyperbola

We are going to find the center, vertices, foci and asymptotes of the following hyperbola,

$$\frac{(y-8)^2}{4} - \frac{(x+1)^2}{9} = 1.$$

Let's start with the center, which is $C = (-1,8)$. Now that we know that and also the values of $a = 2$ and $b = 3$, we can compute c using the relation $b^2 \equiv c^2 - a^2, \rightarrow c = \sqrt{9+4} = \sqrt{13}$. An important observation is that the term with x is negative, which tells us that is a vertical hyperbola.

With that information we can compute the vertex of the hyperbola as follows,

$$v_1 = (-1, 8-2) \qquad v_2 = (-1, 8+2)$$
$$v_1 = (-1, 6) \qquad v_2 = (-1, 10)$$

Then, we do something similar for the foci,

$$F_1 = (-1, 8-\sqrt{13}) \qquad F_2 = (-1, 8+\sqrt{13})$$
$$F_1 = (-1, 4.39) \qquad F_2 = (-1, 11.60)$$

Finally, to compute the asymptotes of the hyperbola, we modify the general equation for hyperbola asymptotes since is a vertical hyperbola, $y-k = \pm\frac{a}{b}(x-h)$,

$$y-k = \pm\frac{a}{b}(x-h) \to y-8 = \pm\frac{2}{3}(x+1)$$
$$y = \pm\frac{2}{3}(x+1)+8$$

Example 2: Factored form of a horizontal hyperbola

We are going to find the center, vertices, foci and asymptotes of the following hyperbola,

$$\frac{(x+8)^2}{25} - \frac{(y-9)^2}{4} = 1.$$

Let's start with the center, which is $C = (-8, 9)$. Now that we know that and also the values of $a = 5$ and $b = 2$, we can compute c using the relation $b^2 \equiv c^2 - a^2, \to c = \sqrt{25+4} = \sqrt{29}$. An important observation is that the term with y is negative, which tells us that is a horizontal hyperbola.

With that information we can compute the vertex of the hyperbola as follows,

$$v_1 = (-8-5, 9) \qquad v_2 = (-8+5, 9)$$
$$v_1 = (-13, 9) \qquad v_2 = (-3, 9)$$

Then, we do something similar for the foci,

$$F_1 = (-8-\sqrt{29}, 9) \qquad F_2 = (-8+\sqrt{29}, 9)$$
$$F_1 = (-13.38, 9) \qquad F_2 = (-2.61, 9)$$

Finally, to compute the asymptotes of the hyperbola, we use the general equation for hyperbola, $x-h = \pm\frac{a}{b}(y-k)$,

$$x-h = \pm\frac{a}{b}(y-k) \to x+8 = \pm\frac{5}{2}(y-9)$$
$$x = \pm\frac{5}{2}(y-9)-8$$

Exercises

Hyperbola

- Construct the hyperbola equation from it's asymptotes, $y = \pm 1/2(x+4) - 3$.
- Find the center, vertices, foci and asymptotes of the following hyperbola $(x-3)^2/9 - (y-6)^2/4 = 1$.

Review

- Solve for x, $\log_5(4x) - 3 = -8$
- Determine the horizontal asymptote for the following function $f(x) = 1/3(e^x - 4)^4$
- Determine the vertical and horizontal asymptotes of the following function $f(x) = (7x - 15)/(x - 5)$.
- Find the inverse function of $f(x) = 2x + 4$.

References

James Stewart, L. Redlin, Saleem Watson, and Phyllis Panman. *Precalculus: mathematics for calculus*. Cengage Learning, Boston, MA, seventh edition edition, 2016. ISBN 9781305071759.

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