Homework 1

Quantum Optics

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> **Problem 1**
>
> Plot the photon number distribution $p(n) = |\langle n|\psi\rangle|^2$ for a coherent state $|\alpha\rangle$, a Fock sate, and a single mode thermal state, for $\langle n\rangle = 1, 10, \text{and}\, 100$ (9 plots in total). For each plot additionally provide Δn and the most probable outcome of the photon number measurement.

(a) A coherent state.

A coherent state is $|\alpha\rangle$ is defined as the eigenstate of the annihilation operator $\hat{a}$ with eigenvalue α. They can be written in the Fock state eigenbasis as

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{1}$$

Let us start by computing the following inner product

$$\begin{aligned}
\langle n|\alpha\rangle &= \mathrm{e}^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \langle n|m\rangle\,, \\
&= \mathrm{e}^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \delta_{nm}, \\
&= \mathrm{e}^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}
\end{aligned} \tag{2}$$

Then the photon number distribution can be formed from the squared absolute value of th previous computation. This results in a Poisson distribution with $\lambda = \langle n\rangle$.

$$\begin{aligned}
P(n) &= |\langle n|\alpha\rangle|^2, \\
&= \frac{|\alpha|^{2n}}{n!} \mathrm{e}^{-|\alpha|^2}, \\
&= \left\{ |\alpha|^2 = \langle n\rangle \right\}, \\
&= \frac{\langle n\rangle^n}{n!} \mathrm{e}^{-\langle n\rangle}.
\end{aligned} \tag{3}$$

Furthermore, the uncertainty can be computed from

$$\begin{aligned}
\Delta n^2 &= \left\langle n^2 \right\rangle - \langle n\rangle^2\,, \\
&= \langle\alpha| a^\dagger a a^\dagger a |\alpha\rangle - |\alpha|^4, \\
&= |\alpha|^2 \langle\alpha| a a^\dagger |\alpha\rangle - |\alpha|^4, \\
&= |\alpha|^2 \langle\alpha| 1 + a^\dagger a |\alpha\rangle - |\alpha|^4, \\
&= |\alpha|^2
\end{aligned} \tag{4}$$

Therefore $\Delta n = |\alpha| = \sqrt{\langle n \rangle}$. Figure 1 shows the photon number distribution for the coherent state and their standard deviations. We find that the most probable number of photons is anticlimatically $n = \langle n \rangle$, but notably $n = \langle n \rangle - 1$ is also a maximum in the studied cases.

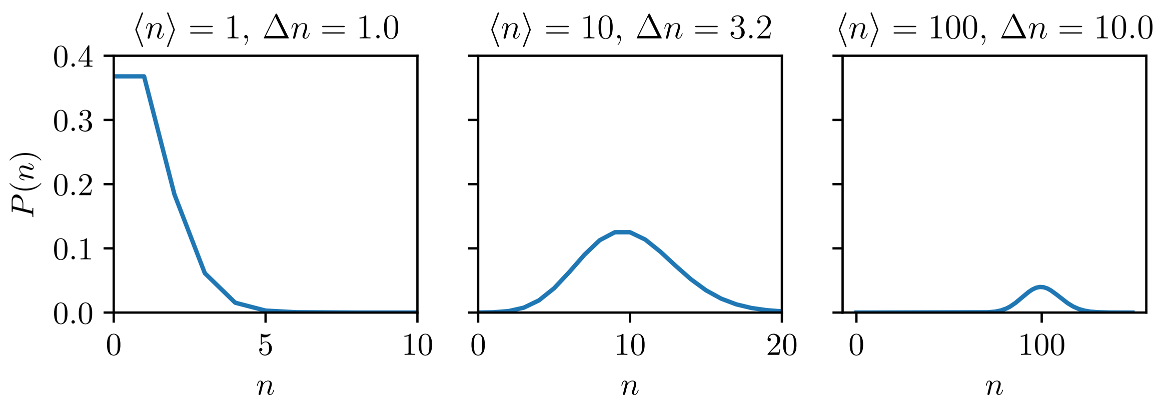

Figure 1: Photon number distribution for a coherent state.

(b) A Fock state.

For a Fock state $|n\rangle$, the situation is much simpler, here

$$P(n) = |\langle n|m\rangle|^2 = \delta_{nm}. \tag{5}$$

Note that

$$\langle n \rangle = \langle m| \hat{n} |m\rangle = m. \tag{6}$$

So the photon number distribution becomes

$$P(n) = \begin{cases} 1 & n = \langle n \rangle, \\ 0 & \text{otherwise}. \end{cases} \tag{7}$$

With zero uncertainty $\Delta n = 0$. Figure 2 shows this ditribution wich is a simple peak at the average number of photons, making them the most probable with zero uncertainty.

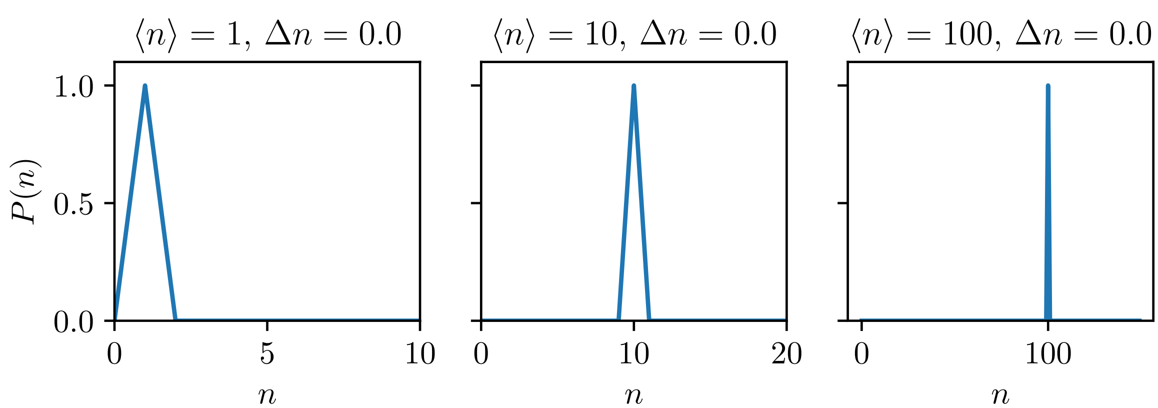

Figure 2: Photon number distribution for a Fock state.

(c) A single mode thermal state. A thermal state can be described by the density matrix,

$$\hat{\rho}_{\mathrm{TH}} = \frac{1}{Z} e^{-\beta\hbar\omega(\hat{n}+1/2)}, \tag{8}$$

where $\beta = 1/k_B T$, and Z is the partition function (computed in class)

$$Z = \mathrm{tr}\left\{e^{-\beta\hat{H}}\right\} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} \tag{9}$$

Note that, a function of any hermitian operator $f(\hat{A})$ with eigenvalues a and eigenkets $|a\rangle$ can be decomposed in its eigenbasis as

$$f(\hat{A}) = \sum_a f(a)\,|a\rangle\langle a| \tag{10}$$

Thus we can decompose the density matrix as

$$\begin{aligned} \hat{\rho} &= \frac{1 - e^{\beta\hbar\omega}}{e^{\beta\hbar\omega/2}} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega(n+1/2)}\,|n\rangle\langle n|\,, \\ &= (1 - e^{\beta\hbar\omega}) \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}\,|n\rangle\langle n| \end{aligned} \tag{11}$$

Here, note that from the average number of photons (computed in class)

$$\langle n\rangle = \frac{1}{e^{\beta\hbar\omega} - 1}, \tag{12}$$

$$\implies e^{-\beta\hbar\omega} = \frac{\langle n\rangle}{1 + \langle n\rangle} \tag{13}$$

Putting back into the density matrix, we can obtain

$$\hat{\rho}_{\mathrm{TH}} = \frac{1}{1 + \langle n\rangle} \sum_{n=0}^{\infty} \left(\frac{\langle n\rangle}{1 + \langle n\rangle}\right)^n |n\rangle\langle n| \tag{14}$$

The probability distribution of finding n photons is then

$$\begin{aligned} P(n) &= \langle n|\,\rho_{\mathrm{TH}}\,|n\rangle\,, \\ &= \langle n|\,\frac{1}{1 + \langle n\rangle} \sum_{m=0}^{\infty} \left(\frac{\langle n\rangle}{1 + \langle n\rangle}\right)^m |m\rangle\langle m|\,|n\rangle\,, \\ &= \frac{1}{1 + \langle n\rangle} \sum_{m=0}^{\infty} \left(\frac{\langle n\rangle}{1 + \langle n\rangle}\right)^m \delta_{nm}, \\ &= \frac{\langle n\rangle^n}{(\langle n\rangle + 1)^{n+1}} \end{aligned} \tag{15}$$

Before computing the uncertainty, it will be helpful to get first

$$
\begin{aligned}
\langle n^2 \rangle &= \mathrm{tr}\{n^2 \rho_{\mathrm{TH}}\}, \\
&= \frac{e^{-\beta\hbar\omega/2}}{Z} \sum_{n=0}^{\infty} n^2 e^{-\beta\hbar\omega n}, \\
&= \frac{e^{-\beta\hbar\omega/2}}{Z} \sum_{n=0}^{\infty} \frac{\partial^2}{\partial(-\beta\hbar\omega)^2} e^{-\beta\hbar\omega n}, \\
&= \frac{e^{-\beta\hbar\omega/2}}{Z} \frac{\partial^2}{\partial(-\beta\hbar\omega)^2} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n}, \\
&= (1 - e^{-\beta\hbar\omega}) \frac{\partial^2}{\partial(-\beta\hbar\omega)^2} \frac{1}{1 - e^{-\beta\hbar\omega}},
\end{aligned}
\tag{16}
$$

From here, note that

$$
\frac{\mathrm{d}^2}{\mathrm{d}x^2} \frac{1}{1 - e^x} = \frac{e^x}{(1 - e^x)^2} + \frac{2e^{2x}}{(1 - e^x)^3}
\tag{17}
$$

Then, going back we get

$$
\begin{aligned}
\langle n^2 \rangle &= (1 - e^{-\beta\hbar\omega}) \left[\frac{e^{-\beta\hbar\omega}}{(1 - e^{-\beta\hbar\omega})^2} + \frac{2e^{-2\beta\hbar\omega}}{(1 - e^{-\beta\hbar\omega})^3} \right], \\
&= \frac{e^{-\beta\hbar\omega}}{(1 - e^{-\beta\hbar\omega})} + \frac{2e^{-2\beta\hbar\omega}}{(1 - e^{-\beta\hbar\omega})^2}, \\
&= \langle n \rangle + 2 \langle n \rangle^2
\end{aligned}
\tag{18}
$$

The uncertainty in this state can then be computed directly

$$
\begin{aligned}
\Delta n^2 &= \langle n^2 \rangle - \langle n \rangle^2, \\
&= \langle n \rangle + \langle n \rangle^2.
\end{aligned}
\tag{19}
$$

Therefore, $\Delta n = \sqrt{\langle n \rangle + \langle n \rangle^2}$. From these relations we can plot the density distribution. Figure 3 shows the number photon distribution for the three studied cases with their computed uncertainties. Interestingly enough, the most probable photon number is $n = 0$ zero photons.

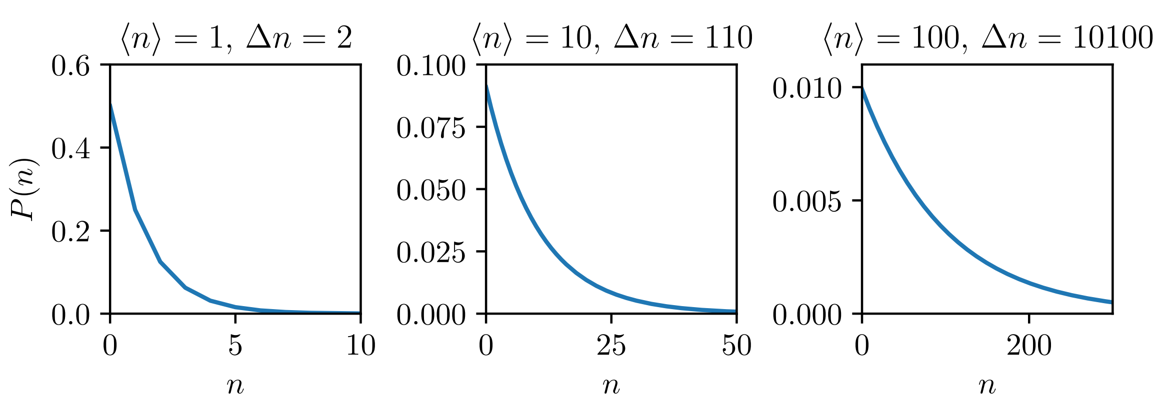

Figure 3: Photon number distribution for a thermal state.

Problem 2

Let the operators a_i and $a_i^\dagger$, for $i = 1, 2$ form two sets of creation and annihilation operators that satisfy the commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j] = 0. \tag{20}$$

Let us define two new sets of creation and annihilation operators b_i and $b_j^\dagger$ which also satisfy the same commutation relations such that

$$\begin{pmatrix} a_1 \\ a_2^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ v & u \end{pmatrix} \begin{pmatrix} b_1 \\ b_2^\dagger \end{pmatrix}. \tag{21}$$

What relationship must the real numbers u and v satisfy?

Using the given matrix relationship, we can easily derive the following two equations

$$\begin{aligned} a_1 &= ub_1 + vb_2^\dagger, \\ a_2^\dagger &= vb_1 + ub_2^\dagger. \end{aligned} \tag{22}$$

Together with their hermitian conjugates, we find four independent equations that can be summarized as follows

$$a_i = ub_i + vb_j^\dagger, \quad i \neq j \tag{23}$$

Let us consider then the commutator, and without loss of generality

$$\begin{aligned} [a_i, a_k^\dagger] &= \left[ub_i + vb_j^\dagger, ub_k^\dagger + vb_l\right], \\ &= u^2 b_i b_k^\dagger + uv(b_i b_l + b_j^\dagger b_k^\dagger) + v^2 b_j^\dagger b_l \\ &- u^2 b_k^\dagger b_i - uv(b_k^\dagger b_j^\dagger + b_l b_i) - v^2 b_l b_j^\dagger, \\ &= u^2 [b_i, b_k^\dagger] + v^2 [b_j^\dagger, b_l] + uv[b_i, b_l] + uv[b_j^\dagger, b_k^\dagger], \\ &= u^2 \delta_{ik} - v^2 \delta_{lj} \end{aligned} \tag{24}$$

Therefore we find that

$$\delta_{ik} = u^2 \delta_{ik} - v^2 \delta_{lj}, \quad i \neq j, k \neq l. \tag{25}$$

Taking the only non-zero equation we find that the parameters form a hyperbola in uv-space.

$$1 = u^2 - v^2. \tag{26}$$

Problem 3

An electric field mode has an equal probability of 1/3 to be found in each of the states $|0\rangle$ $|2\rangle$ and the superposition $4|0\rangle + 3|1\rangle$(before normalization). Find the corresponding density matrix $\hat{\rho}$.

A density matrix is defined as

$$\hat{\rho} = \sum_k p_k |\psi_k\rangle\langle\psi_k| , \tag{27}$$

So we should have

$$\begin{aligned}
\hat{\rho} &= \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}|2\rangle\langle 2| + \frac{1}{3}|\phi\rangle\langle\phi| , \\
&= \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}|2\rangle\langle 2| \\
&, + \frac{1}{3}\left(\frac{4}{5}|0\rangle + \frac{3}{5}|1\rangle\right)\left(\frac{4}{5}\langle 0| + \frac{3}{5}\langle 1|\right), \\
&= \frac{1}{3}|0\rangle\langle 0| + \frac{1}{3}|2\rangle\langle 2| \\
&+ \frac{1}{3}\left(\frac{16}{25}|0\rangle\langle 0| + \frac{12}{25}|0\rangle\langle 1| + \frac{12}{25}|1\rangle\langle 0| + \frac{9}{25}|1\rangle\langle 1|\right), \\
&= \frac{41}{75}|0\rangle\langle 0| + \frac{9}{75}|1\rangle\langle 1| + \frac{1}{3}|2\rangle\langle 2| + \frac{12}{75}|0\rangle\langle 1| + \frac{12}{75}|1\rangle\langle 0| .
\end{aligned} \tag{28}$$

This, in matrix form gives

$$\hat{\rho} = \begin{pmatrix} 41/75 & 12/25 & 0 \\ 12/25 & 9/75 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \tag{29}$$