

# Algebraic and Transcendental functions

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**Summary** Class sketches for the subject Algebraic and Transcendental Functions. Semester January-May 2025

## Recap

### Parabola

$$\sqrt{(x-a)^2 + y^2} = x + a \rightarrow (y-k)^2 = 4a(x-h)$$
$$\sqrt{x^2 + (y-a)^2} = y + a \rightarrow (x-h)^2 = 4a(y-k)$$

#### Note 1: Expanded equation

$$(x-h)^2 = 4a(y-k)$$
$$x^2 - 2hx + h^2 = 4ay - 4ak$$
$$\frac{1}{4a}[4ay] = \frac{1}{4a}[x^2 - 2hx + h^2 + 4ak]$$
$$y = \frac{x^2}{4a} - \frac{hx}{2a} + \frac{h^2}{4a} + k$$

In this case this function (a parabola with horizontal directrix) is bijective and has the following domain and range  $x \in (-\infty, \infty)$  and  $f(x, y) \in (-\infty, \infty)$ . However, we need to be careful, because the parabola with a vertical directrix is not a bijective function, because for each value of  $x$  it corresponds two values of  $y$ .

### Circle

$$a^2 + b^2 = c^2 \rightarrow (x-h)^2 + (y-k)^2 = r^2$$

#### Note 2: Expanded equation

We are going to expand the equation(??) to get familiarize ourselves with different expressions of the same mathematical object.

$$(x-h)^2 + (y-k)^2 = r^2$$
$$(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) = r^2$$
$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2$$

As we can see, this can be approach as a function of two independent variables  $x$  and  $y$  ( $f(x, y)$ ) equated to a constant value  $r^2$ ,  $f(x, y) = r^2$ . This function is not bijective, indicating that there is no inverse function and the domain and range are bounded.

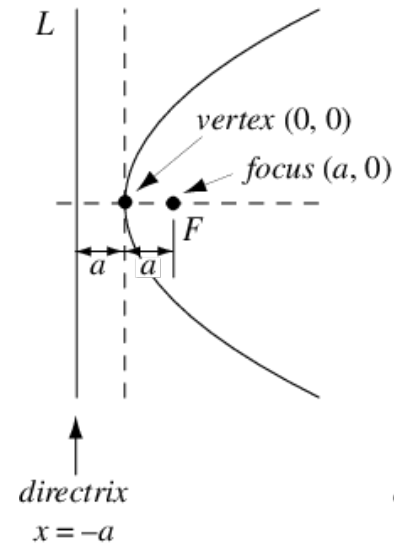


Figure 1: Parabola

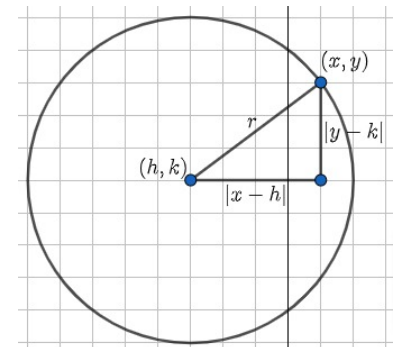


Figure 2: Circle

### Ellipse

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a \rightarrow \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

#### Note 3: Expanded equation

Now we are going to expand the expression,

$$\begin{aligned} \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} &= 1 \\ \frac{1}{a^2}(x^2 - 2xh + h^2) + \frac{1}{b^2}(y^2 - 2yk + k^2) &= 1 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xh}{a^2} - \frac{2yk}{b^2} + \frac{h^2}{a^2} + \frac{k^2}{b^2} &= 1 \end{aligned}$$

As same with the circle equation, we can approach this as a two variable function  $f(x,y)$  equated to 1,  $f(x,y) = 1$ . However, we need to take into account that this is not a bijective function, which leads to have a bounded domain and range with no inverse function.

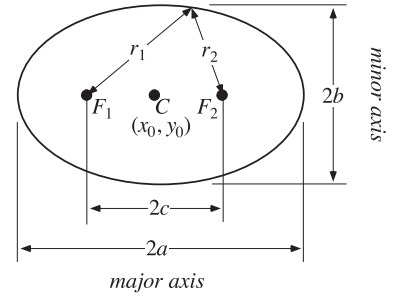


Figure 3: Ellipse

### Hyperbola

$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a \rightarrow \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

#### Note 4: Expanded equation

For completeness, we are going to expand the quadratic terms,

$$\begin{aligned} \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1 \\ \frac{1}{a^2}[x^2 - 2hx + h^2] - \frac{1}{b^2}[y^2 - 2ky + k^2] &= 1 \\ \frac{x^2}{a^2} - \frac{2hx}{a^2} + \frac{h^2}{a^2} - \frac{y^2}{b^2} + \frac{2ky}{b^2} - \frac{k^2}{b^2} &= 1 \end{aligned}$$

This equation ( $f(x,y) = 1$ ) is not a bijective function. On the other hand, the range can go from plus infinity to minus infinity  $f(x,y) \in (-\infty, \infty)$ , however, the domain is not defined between the distance of the two vertex  $x \in (-\infty, h-a] \cup [h+a, \infty)$ .

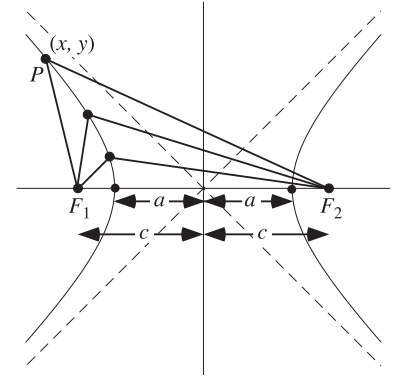


Figure 4: Hyperbola

#### Note 5: Asymptotes of the hyperbola

As mentioned in the note before, the domain is not defined in the interval  $x \in (h-a, h+a)$ , indicating the existence of asymptotes. Here we are going to see the derivation of the asymptotes. To do

that, we are going to equate the function to zero  $f(x,y) = 0$ ,

$$\begin{aligned}\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1 \rightarrow \frac{(x-h)^2}{a^2} = \frac{(y-k)^2}{b^2} \\ \left[(x-h)^2\right]^{1/2} &= \left[\frac{a^2}{b^2}(y-k)^2\right]^{1/2} \\ x-h &= \pm \frac{a}{b}(y-k)\end{aligned}$$

## Exercises

### Conic sections

1. With help of the pythagorean theorem find the function of a circle with origin at  $O = (-3/2, -1)$  and the point  $(3/2, 3/2)$ .
2. Write the equation of the circle with center at  $(2, 4)$  and radius 3.
3. Find the origin, radius of the following circle equation  $x^2 - 4x + y^2 - 5y + 2 = 0$ , then write the factored equation.
4. Find the vertex, focus and directrix of the parabola  $-x^2 + 3x - y + 5 = 0$ .
5. find the equation of the parabola with vertex  $(4, 3)$  and directrix  $x = 5$
6. Find the intersection points (2 points) of between these two parabolas,  $x^2 = 4(2/3)y$  and  $y^2 = 4(2/3)x$ .
7. Construct the hyperbola equation from it's asymptotes,  $y = \pm 2(x + 2) - 6$ .
8. Find the center, vertices, foci and asymptotes of the following hyperbola  $(x+1)^2/16 - (y+3)^2/2 = 1$ .
9. Find the center, vertices and foci of the following ellipse

$$\frac{(x+1)^2}{9} + \frac{(y-5)^2}{25} = 1$$

10. Write the compacted equation of the ellipse  $8x^2 - 24x + 3y^2 + 4y + 7 = 0$ .

## References

James Stewart, L. Redlin, Saleem Watson, and Phyllis Panman. *Precalculus: mathematics for calculus*. Cengage Learning, Boston, MA, seventh edition edition, 2016. ISBN 9781305071759.

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