Multivariable Feedback Control

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1 Control of a Spring-Mass-Damper System

Let's consider the three systems (**System 1**) shown in *Fig*.1.1. This system is described by the following set of linear differential equations:

$$\begin{cases} m_1\ddot{\delta}_1 = -k_1\big(\delta_1 - \delta_1(0)\big) - \big(\delta_2 - \delta_2(0)\big) - \mu_1\big(\dot{\delta}_1 - \dot{\delta}_2\big) - d_1\dot{\delta}_1 + f_1 \\ m_2\ddot{\delta}_2 = k_1\big(\delta_1 - \delta_1(0)\big) - \big(\delta_2 - \delta_2(0)\big) - k_2\big(\delta_2 - \delta_2(0)\big) - \big(\delta_3 - \delta_3(0)\big) + \\ \qquad\qquad +\mu_1\big(\dot{\delta}_1 - \dot{\delta}_2\big) - \mu_2\big(\dot{\delta}_2 - \dot{\delta}_3\big) - d_2\dot{\delta}_2 \\ m_3\ddot{\delta}_3 = k_2\big(\delta_2 - \delta_2(0)\big) - \big(\delta_3 - \delta_3(0)\big) + \mu_2\big(\dot{\delta}_2 - \dot{\delta}_3\big) - d_3\dot{\delta}_3 + f_3 \\ y_1 = \big(\delta_1 - \delta_1(0)\big) \\ y_2 = \big(\delta_2 - \delta_2(0)\big) \end{cases}$$

where m_1, m_2, m_3 are the masses, k_1, k_2 are the elastic coefficients, μ_1, μ_2 are the damped coefficients between masses, d_1, d_2, d_3 are the frictions with the terrain coefficient, f_1 is a control force on the mass m_1 and f_3 is a second control force on the mass m_3.
We rewrite the above set of linear differential equations in terms of the relative displacements:

$$\begin{cases} m_1\ddot{z}_1 = -k_1(z_1 - z_2) - \mu_1(\dot{z}_1 - \dot{z}_2) - d_1\dot{z}_1 + f_1 \\ m_2\ddot{z}_2 = k_1(z_1 - z_2) - k_2(z_2 - z_3) + \mu_1(\dot{z}_1 - \dot{z}_2) - \mu_2(\dot{z}_2 - \dot{z}_3) - d_2\dot{z}_2 \\ m_3\ddot{z}_3 = k_2(z_2 - z_3) + \mu_2(\dot{z}_2 - \dot{z}_3) - d_3\dot{z}_3 + f_3 \\ y_1 = z_1 \\ y_2 = z_2 \end{cases}$$

A variant (**System 2**) of this system consists of adding a spring from to a wall on its left with elastic coefficient. This leads to a change in only the first equation:

$$m_1\ddot{z}_1 = -k_0 z_1 - k_1(z_1 - z_2) - \mu_1(\dot{z}_1 - \dot{z}_2) - d_1\dot{z}_1 + f_1$$

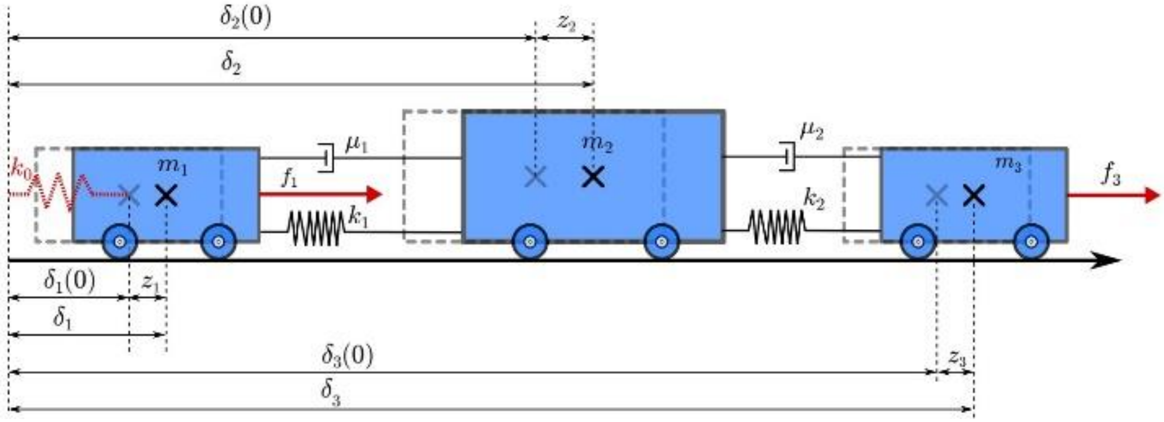

Figure 1.1: *Considered*

1.1 State-Space representation

Let's find the **state-space representation** of the ***System 1***. Since we want to write the equations in the first-order differential form, we introduce and define three new variables as:

$$\begin{aligned}\dot{z}_1 &= \xi_1\\ \dot{z}_2 &= \xi_2\\ \dot{z}_3 &= \xi_3\end{aligned}$$

Thus, the system becomes:

$$\begin{cases} m_1\dot{\xi}_1 = -(\mu_1 + d_1)\xi_1 + \mu_1\xi_1 - k_1z_1 + k_1z_2 + f_1 \\ m_2\dot{\xi}_2 = \mu_1\xi_1 - (-\mu_1 + \mu_2 + d_2)\xi_2 + \mu_2\xi_3 + l_1z_1 - (k_1 + k_2)z_2 + k_2z_3 \\ m_3\dot{\xi}_3 = \mu_2\xi_2 - (\mu_2 + d_3)\xi_3 + k_2z_2 - k_2z_3 + f_2 \\ \dot{z}_1 = \xi_1 \\ \dot{z}_2 = \xi_2 \\ \dot{z}_3 = \xi_3 \\ y_1 = z_1 \\ y_2 = z_2 \end{cases}$$

Given the fact that the system is LTI (Linear-Time-Invariant), the differential equations may be written in matrix form:

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -\frac{(\mu_1+d_1)}{m_1} & \frac{\mu_1}{m_1} & 0 & -\frac{k_1}{m_1} & \frac{k_1}{m_1} & 0 \\ \frac{\mu_1}{m_2} & -\frac{(\mu_1+\mu_2+d_2)}{m_2} & \frac{\mu_2}{m_2} & \frac{k_1}{m_2} & -\frac{(k_1+k_2)}{m_2} & \frac{k_2}{m_2} \\ 0 & 0 & -\frac{(\mu_2+d_3)}{m_3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_3} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

where the matrix A, which is the **dynamic matrix**, the matrix B, which is the **input matrix**, the matrices C, which is the **state-output matrix** and the matrix D, which is the **input-output matrix**, are:

$$A_1 = \begin{bmatrix} -\frac{(\mu_1+d_1)}{m_1} & \frac{\mu_1}{m_1} & 0 & -\frac{k_1}{m_1} & \frac{k_1}{m_1} & 0 \\ \frac{\mu_1}{m_2} & -\frac{(\mu_1+\mu_2+d_2)}{m_2} & \frac{\mu_2}{m_2} & \frac{k_1}{m_2} & -\frac{(k_1+k_2)}{m_2} & \frac{k_2}{m_2} \\ 0 & 0 & -\frac{(\mu_2+d_3)}{m_3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_3} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \qquad D_1 = 0$$

The **state-space** representation, in matrix form, of the ***System 2*** is:

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -\frac{(\mu_1+d_1)}{m_1} & \frac{\mu_1}{m_1} & 0 & -\frac{(k_0+k_1)}{m_1} & \frac{k_1}{m_1} & 0 \\ \frac{\mu_1}{m_2} & -\frac{(\mu_1+\mu_2+d_2)}{m_2} & \frac{\mu_2}{m_2} & \frac{k_1}{m_2} & -\frac{(k_1+k_2)}{m_2} & \frac{k_2}{m_2} \\ 0 & 0 & -\frac{(\mu_2+d_3)}{m_3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_3} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

where the matrices B, C, D are the same, while the matrix A is:

$$A_2 = \begin{bmatrix} -\frac{(\mu_1 + d_1)}{m_1} & \frac{\mu_1}{m_1} & 0 & -\frac{(k_0 + k_1)}{m_1} & \frac{k_1}{m_1} & 0 \\ \frac{\mu_1}{m_2} & -\frac{(\mu_1 + \mu_2 + d_2)}{m_2} & \frac{\mu_2}{m_2} & \frac{k_1}{m_2} & -\frac{(k_1 + k_2)}{m_2} & \frac{k_2}{m_2} \\ 0 & 0 & -\frac{(\mu_2 + d_3)}{m_3} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Finally, let's consider another system which is the system 2 but, in this case, we consider only one output, in particular $y_1 = z_2$. The only matrix that changes is C and it is equal to:

$$C2_{TISO} = [0\ 0\ 0\ 0\ 1\ 0]$$

1.2 Parameter's values

The values that we have chosen are the following:

Parameters	Values
$\boldsymbol{m_1, m_2, m_3}$	$1\ Kg$, $1.5\ Kg$, $0.5\ Kg$
$\boldsymbol{k_0, k_1, k_2}$	$5\ \frac{N}{m}, 10\ \frac{N}{m}, 100\ \frac{N}{m}$
$\boldsymbol{\mu_1, \mu_2}$	$5 \frac{Ns}{m}, 15\ \frac{Ns}{m}$
$\boldsymbol{d_1, d_2, d_3}$	$0.1\ \frac{Ns}{m}, 0.2\ \frac{Ns}{m}, 0.1\ \frac{Ns}{m}$

Table 1: *Parameter's Values*

1.3 Transfer Function Matrix

Let's compute the **transfer function matrix**, for the three systems, through the following relationship:

$$F(s) = C(sI - A)^{-1}B + D$$

1.3.1 System 1

The transfer function matrix for **system 1** is:

$$F_1(s) = \begin{bmatrix} F_1(1,1)(s) & F_1(1,2)(s) \\ F_1(2,1)(s) & F_1(2,2)(s) \end{bmatrix}$$

where:

$$F_1(1,1)(s) = \frac{(s + 32.92)(s + 8.109)(s^2 + \ 2.633s\ 4.994)}{s(s + 33.13)(s + 7.66)(s + 0.1333)(s^2 + 7.848s + 15.76)}$$

$$F_1(1,2)(s) = \frac{100(s + 6.667)(s + 2)}{s(s + 33.13)(s + 7.66)(s + 0.1333)(s^2 + \ 7.848s + 15.76)}$$

$$F_1(2,1)(s) = \frac{3.3333(s + 20.39)\ (s + 9.808)\ (s + 2)}{s(s + 33.13)(s + 7.66)(s + 0.1333)(s^2 + 7.848s + 15.76)}$$

$$F_1(2,2)(s) = \frac{20(s + 6.667)(s\text{^}2\ +\ 5.1s + 10)}{s(s + 33.13)(s + 7.66)(s + 0.1333)(s^2 + 7.848s + 15.76)}$$

1.3.2 System 2

The transfer function matrix for **system 2** is:

$$F_2(s) = \begin{bmatrix} F_2(1,1)(s) & F_2(1,2)(s) \\ F_2(2,1)(s) & F_2(2,2)(s) \end{bmatrix}$$

where:

$$F_2(1,1)(s) = \frac{(s + 32.92)(s + 8.109)(s\text{^}2 + 2.633s\ 4.994)}{(s + 33.12)(s + 7.75)(s^2 + 0.2631s + 1.416)(s\text{^}2 + 7.629s + 18.34)}$$

$$F_2(1,2)(s) = \frac{100(s + 6.667)(s + 2)}{(s + 33.12)(s + 7.75)(s\text{^}2 + 0.2631s + 1.416)(s\text{^}2 + 7.629s + 18.34)}$$

$$F_2(2,1)(s) = \frac{3.3333(s + 20.39)(s + 9.808)(s + 2)}{(s + 33.12)(s + 7.75)(s\text{^}2 + 0.2631s + 1.416)(s\text{^}2 + 7.629s + 18.34)}$$

$$F_2(2,2)(s) = \frac{20(s + 6.667)(s\text{^}2 + 5.1s + 15)}{(s + 33.12)(s + 7.75)(s^2 + 0.2631s\ 1.416)(s\text{^}2 + 7.629s + 18.34)}$$

1.3.3 System 2 TISO

The transfer function matrix for **system 2 TISO** is:

$$F_{2_{TISO}}(s) = \left[F_{2_{TISO}}(1,1)(s) \quad F_{2_{TISO}}(1,2)(s)\right]$$

where:

$$F_{2_{TISO}}(1,1)(s) = \frac{3.3333(s + 20.39)(s + 9.808)(s + 2)}{(s + 33.12)(s + 7.75)(s^2 + 0.2631s + 1.416)(s^2 + 7.629s + 18.34)}$$

$$F_{2_{TISO}}(1,2)(s) = \frac{20(s + 6.667)(s^2 + 5.1s + 15)}{(s + 33.12)(s + 7.75)(s^2 + 0.2631s + 1.416)(s^2 + 7.629s + 18.34)}$$

At this point we wonder: what changes by introducing, in the system 2, a spring on the mass m_1? The answer is found in the:

1.4 Eigenvalue's Analysis

Let's focus on the eigenvalues of the first two system. Through the MATLAB.*Command* `eig`, we obtain:

	Eigenvalues of system 1
1	$-33.1258 + 0.0000i$
2	$-7.6599 + 0.0000i$
3	$0.0000 + 0.0000i$
4	$-0.1333 + 0.0000i$
5	$-3.9238 + 0.6058i$
6	$-3.9238 - 0.6058i$

Table 2: *Eigenvalues of system1*

	Eigenvalues of system 2
1	$-33.1247 + 0.0000i$
2	$-7.7496 + 0.0000i$
3	$-0.1315 + 1.1827i$
4	$-0.1315 - 1.1827i$
5	$-3.8146 + 1.9462i$
6	$-3.8146 - 1.9462i$

Table 3: *Eigenvalues of system2*

We note that, the important difference between the two systems, in terms of eigenvalues, is that the system 1 has one pole in zero. This means that this system is of type 1 and this yields a steeper sensitivity function, therefore, an higher distrubance attenuation with respect to a 0 type system. Moreover, a type 1 system guarantees 0 steady-state error for a step input.

2 Static gain matrix and singular value decomposition (SVD)

In this section we will compute the **static gain matrix**, the **SVD** and the **condition number** the for each system.

Definition 2.1: The ***static gain matrix*** is defined as the transfer function matrix, computed in zero: $G(0)$.

Definition 2.2: Any complex $p \times q$ matrix G may be decomposed into its ***singular value decomposition***, and we write:

$$G = U\,\Sigma\,V^H$$

where:

- U is an $p \times p$ unitary matrix of **output singular vectors**;
- V is an $q \times q$ unitary matrix of **input singular vectors**;
- Σ is an $p \times q$ matrix with **non-negative singular values**, σ_i , arranged in descending order along its main diagonal; the other entries are zero.

Remark 1: The column vectors of U, denoted u_i, represent the output direction of the plant, while the column vectors od V, denoted v_i, represent the input direction. These input and output directions are related through the singular values and we can see this, according to the following relationship:

$$Gv_i = \sigma_i u_i$$

If we consider an input in the direction v_i, then the output is in the direction u_i, thus, the σ_i gives directly the gain of the matrix G in this direction. In other word:

$$\sigma_i(G) = \frac{||Gv_i||_2}{||v_i||_2}$$

Remark 2: The largest gain for any input, d, direction, is equal to the maximum singular value:

$$\overline{\sigma}(G) \equiv \sigma_1(G) = \max_{d \neq 0} \frac{||Gd||_2}{||d||_2} = \frac{||Gv_1||_2}{||v_1||_2}$$

While the smallest gain for any input direction, is equal to the minimum singular value:

$$\underline{\sigma}(G) \equiv \sigma_k(G) = \min_{d \neq 0} \frac{||Gd||_2}{||d||_2} = \frac{||Gv_k||_2}{||v_k||_2}$$

Thus, for any vector d, we have that:

$$\underline{\sigma}(G) \leq \frac{||Gd||_2}{||d||_2} \leq \overline{\sigma}(G)$$

Defining $u_1 = \overline{u}, v_1 = \overline{v}, u_k = \underline{u}, v_k = \underline{v}$, then it follows that

$$G\overline{v} = \overline{\sigma}\,\overline{u}, \qquad G\underline{v} = \underline{\sigma}\,\underline{u}$$

The vector $\overline{v}$ corresponds to the input direction with **largest amplification**, and $\overline{u}$ is the corresponding output direction in which the inputs are most effective. The directions involving $\overline{v}$ and $\overline{u}$ are sometimes referred to as the **"strongest"**, **"high-gain"** or **"most important"** directions.

The vector $\underline{v}$ corresponds to the input direction with **smallest amplification**, and $\underline{u}$ is the corresponding output direction in which the inputs are less effective. The directions involving $\underline{v}$ and $\underline{u}$ are sometimes referred to as the **"weak"**, **"low-gain"** or **"least important"** directions.

Remark 3: In MIMO systems, the behaviour at steady-state depends on the **STATIC GAIN MATRIX**, $G(0)$, but also on the direction of d:

$$y_{ss} = G(0)d$$

Definition 2.3: The ***condition number*** is the ratio between the maximum and minimum singular values:

$$\gamma = \frac{\overline{\sigma}}{\underline{\sigma}}$$

This quantity quantifies the difficulty in controlling the system, in particular a large condition number means to have a system which is hard to control.

2.1 System 1

The **static gain matrix** is equal to:

$$\mathrm{F}_1(0) = \begin{bmatrix} -0.6912 \cdot 10^{14} & 4.9167 \cdot 10^{14} \\ -0.6912 \cdot 10^{14} & 4.9167 \cdot 10^{14} \end{bmatrix}$$

As we see, the dimension of this matrix is 2, while the rank is 1, therefore we expect to have 2 singular values where one, clearly the minimum singular value, must be equal to 0.

The **SVD** is:

$$\mathrm{F}_1 = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \begin{bmatrix} 7.0217 \cdot 10^{14} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.1392 & -0.9903 \\ -0.9903 & -0.1392 \end{bmatrix}$$

where the singular values are $\overline{\sigma}_1 = 7.0217 \cdot 10^{14}$ and $\underline{\sigma}_1 = 0$. According to **Remark 2**, the largest gain of $7.0217 \cdot 10^{14}$ is for an input in the direction $\overline{v} = \begin{bmatrix} 0.1392 \\ -0.9903 \end{bmatrix}$, while the smallest gain of 0 is for an input in the direction $\overline{v} = \begin{bmatrix} -0.9903 \\ -0.1392 \end{bmatrix}$.

Notice that, we have a huge different between this 2 values, so this means a large difference in amplification. When we have a large difference between smallest and largest singular value, the system is said to be **ILL-CONDITIONED**. Clearly, we will have a high condition number, indeed we have that the **condition number** is equal to:

$$\gamma_1 = \frac{\overline{\sigma}_1}{\underline{\sigma}_1} = 3.4667 \cdot 10^{15}$$

This means that it is very hard to control this system.

In *Fig.*2.1 is shown the principal gain for system 1.

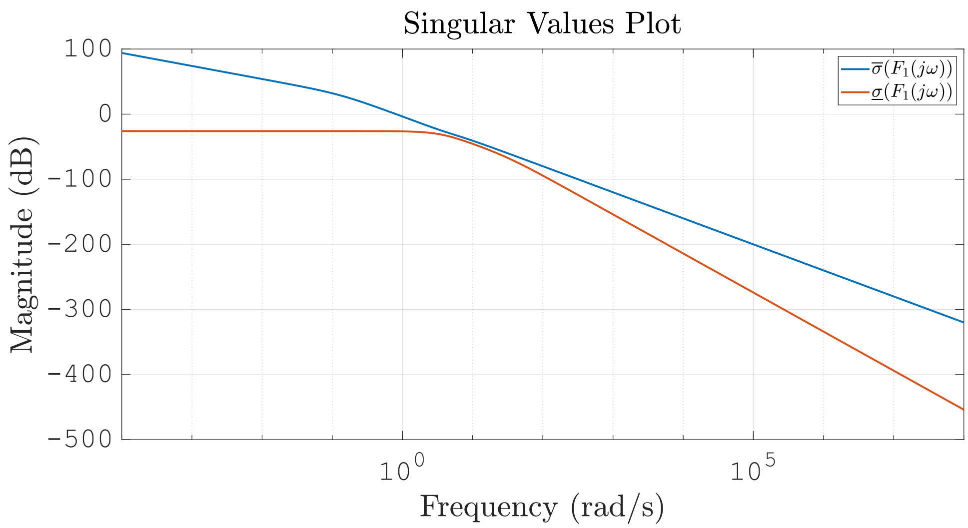

Figure 2.2: *Principal gain of system 1*

2.2 System 2

The **static gain matrix** is equal to:

$$F_2(0) = \begin{bmatrix} 0.2 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}$$

As we see, the dimension of this matrix is 2 and the rank is 2, therefore we will have 2 non-zero singular values. Indeed, the **SVD** is:

$$F_2 = \begin{bmatrix} -0.6154 & -0.7882 \\ -0.7882 & 0.6154 \end{bmatrix} \begin{bmatrix} 0.4562 & 0 \\ 0 & 0.0438 \end{bmatrix} \begin{bmatrix} -0.6154 & -0.7882 \\ -0.7882 & 0.6154 \end{bmatrix}$$

where the singular values are $\overline{\sigma}_2 = 0.4562$ and $\underline{\sigma}_2 = 0.0438$. In this case, the largest gain of 0.4562 is for an input in the direction $\overline{v} = \begin{bmatrix} -0.6154 \\ -0.7882 \end{bmatrix}$, while the smallest gain of 0.0438 is for an input in the direction $\overline{v} = \begin{bmatrix} -0.7882 \\ 0.6154 \end{bmatrix}$. In this case, the **condition number** is equal to:

$$\gamma_2 = \frac{\overline{\sigma}_2}{\underline{\sigma}_2} = 10.4039$$

In *Fig.2.2* is shown the principal gain for system 2.

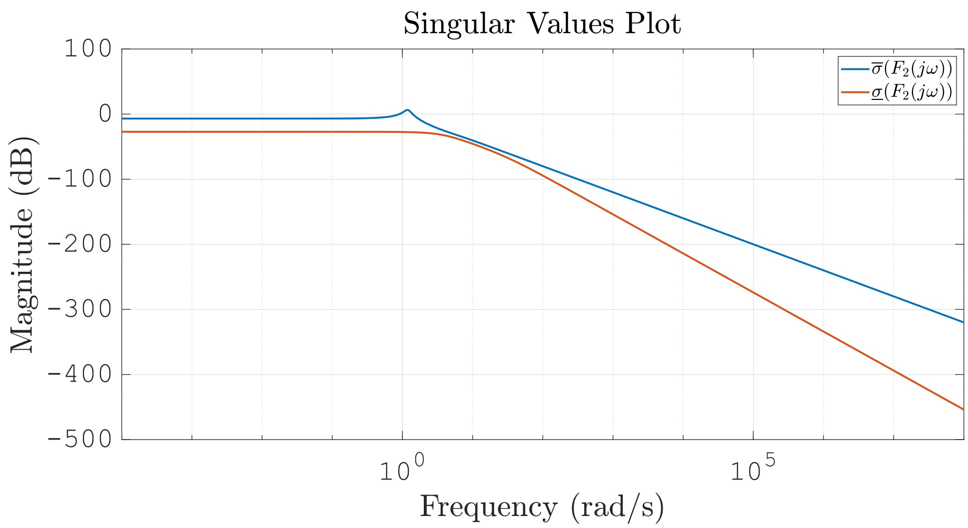

***Figure 2.2:** Principal gain of system 2*

2.3 System 2 TISO

The **static gain matrix** is equal to:

$$F_2(0) = [0.2 \quad 0.3]$$

In this case, the dimension of this matrix is 1 and the rank is 1, therefore we will have 1 singular value, thus the smallest singular value will coincide with the greater singular value. The **SVD** is:

$$F_{2_{TISO}} = 1\,[0.3606 \quad 0] \begin{bmatrix} 0.5547 & -0.8321 \\ 0.8321 & 0.5547 \end{bmatrix}$$

where the singular values are $\overline{\sigma}_{2_{TISO}} = \underline{\sigma}_{2_{TISO}} = 0.3606$. The largest gain coincides with the smallest gain.

Clearly here, the **condition number** is equal to:

$$\gamma_{2_{TISO}} \frac{\overline{\sigma}_{2_{TISO}}}{\underline{\sigma}_{2_{TISO}}} = 1$$

In *Fig.2.3* is shown the principal gain for system 2 TISO.

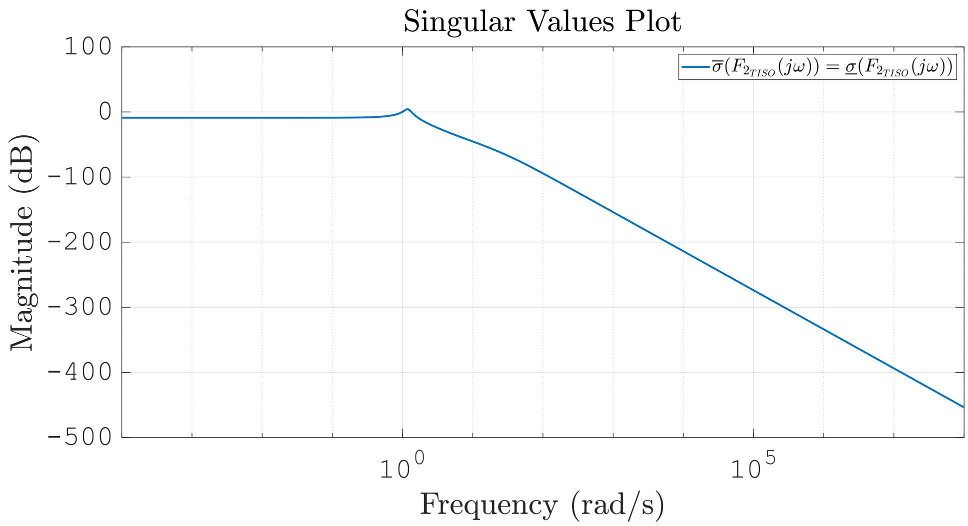

Figure 2.3: *Principal gain of system 2 TISO*

3 $\mathcal{H}_\infty$ Nominal Control Design Problem for system 2

The aim of this section is to find a MIMO controller which guarantees the nominal performances, for the nominal control problem of the system 2, in which we will consider a measurement noise and a disturbance on the mass m_1.

To design the controller of interest, let's use the $\mathcal{H}_\infty$ approach, therefore we have to correctly choose the weights performance and then we have to build the augmented plant. Choosing $B_{3S} = 6$, $A_S = 0.001$, $M_S = 2$, $B_{3L} = 400$, $k = 0.002$, we obtain that:

- $w_S(s) = \begin{bmatrix} \frac{s+12}{2s+0.012} & 0 \\ 0 & \frac{s+12}{2s+0.012} \end{bmatrix}$
- $w_U(s) = \begin{bmatrix} \frac{0.002s+0.8}{0.001+400} & 0 \\ 0 & \frac{0.002s+0.8}{0.001+400} \end{bmatrix}$

By these choices, we find a controller such that:

$$\left[\left\| \begin{matrix} w_S(s)S(s) \\ w_U(s)S_u(s) \end{matrix} \right\|_\infty \right] \leq 1.0664$$

where $S(s), S_u(s), w_S(s), w_U(s)$ are matrices. Once this is done, we can see the result in terms of output responses, disturbance responses and control actions, see *Fig.3.1*. Let's consider as input:

- ***step signal***, for the **first input reference**, defined as follow:

$$\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ 0.25, & t \geq 1 \end{cases}$$

- ***step signal***, for the **second input reference**, defined as follow:

$$\delta_{-1}(t) = \begin{cases} 0, & t < 0 \\ 0.30, & t \geq 1 \end{cases}$$

- ***sinusoidal signal***, for the **disturbance**, defined as follow:

$$d = 0.075 \sin(0.1\, t)$$

- ***sinusoidal signal***, for the **measurement noise**, defined as follow:

$$n = 0.03 \sin(1000\, t)$$

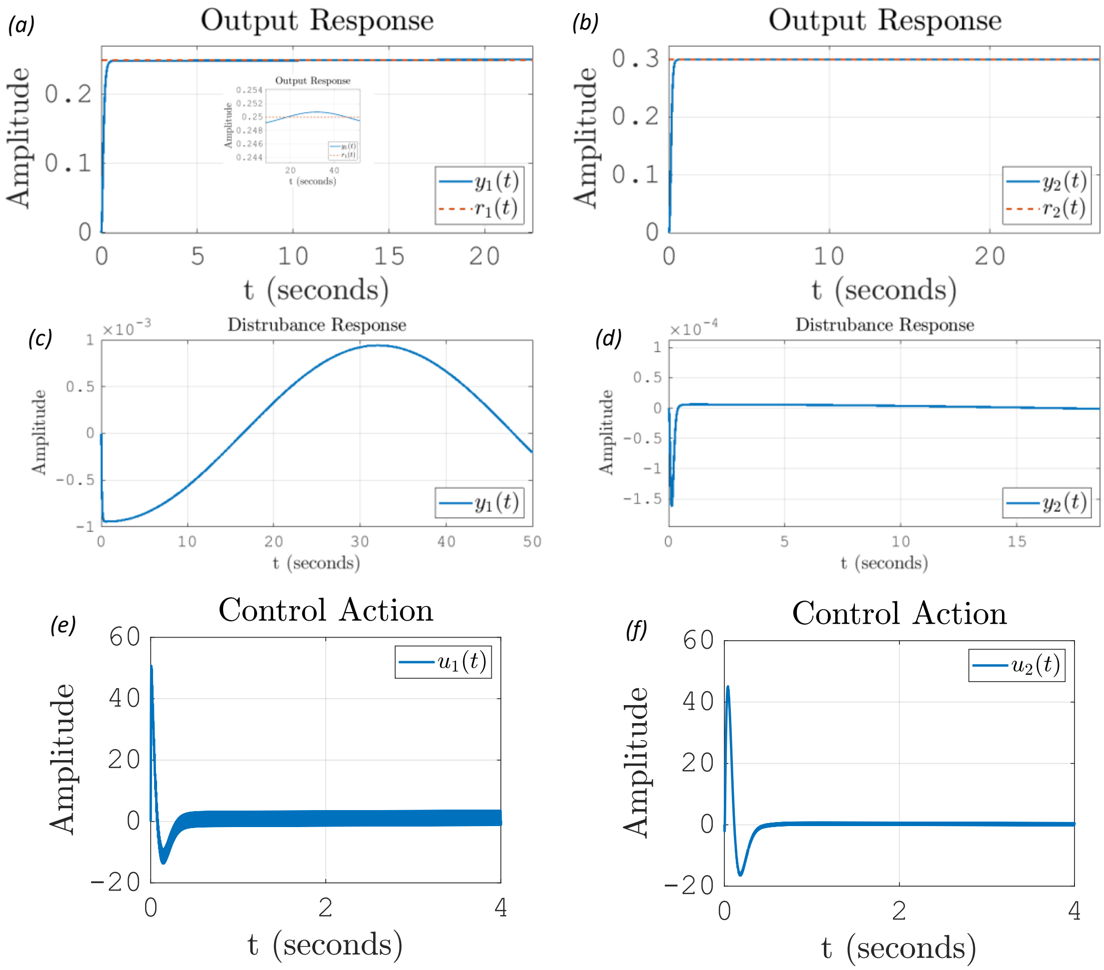

Figure 3.1: *(a) Output response y_1.(b) Output response y_2.(c) Disturbance response y_1. (d) Disturbance response y_2. (e) Control action u_1. (f) Control action u_2.*

We see that our MIMO controller is able to guarantees the nominal performance and, clearly, also the nominal stability. Moreover, it is able to reject the sinusoidal disturbance very well, as we see in *Fig.*3.1(c). However, the control effort is not very low, as we see in *Fig.*3.1(e)–(f).

3.1 Robust stability and performance of the previous $\mathcal{H}_\infty$ controller

In this section, we want to analyse if the previous $\mathcal{H}_\infty$ controller, **robustly stabilizes** the system or not. In the case in which the answer is 'yes', we have to verify the **robust performance** of this controller, otherwise no since the robust stability is the necessary condition for robust performance. For this assignment, we have to consider the **unstructured** and **structured uncertainty**, and then see the difference between them.

Let's suppose that our uncertain values varies as follow:

Parameter	Nominal Value	Uncertain Value
$\mathbf{k_0}$	$5\frac{N}{m}$	$k_0 \in [4.1, 6.65]$
$\mathbf{k_1}$	$10\frac{N}{m}$	$k_1 \in [7.87, 12.2]$
$\mathbf{k_2}$	$100\frac{N}{m}$	$k_2 \in [98.35, 102.14]$
$\boldsymbol{\mu_1}$	$5\frac{N\,s}{m}$	$\mu_1 \in [4.11, 6.11]$
$\boldsymbol{\mu_2}$	$15\frac{N\,s}{m}$	$\mu_2 \in [13.11, 15.99]$

Table 1: *Uncertain Parameters*

So, the starting point for our robustness analysis is a system representation in which the uncertain perturbations are "pulled out". If we also pull out the controller K, we get the plant P, as shown in *Fig.3.2*. Alternatively, if we want to analyse the robust performance of the uncertain system, we use the $N\Delta$ - structure in *Fig.3.3*, where N is related to P and K by a lower LFT:

$$N = F_l(P, K)$$

Similarly, the uncertain closed-loop transfer function from, is related to N and Δ by an upper LFT:

$$F = F^u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$$

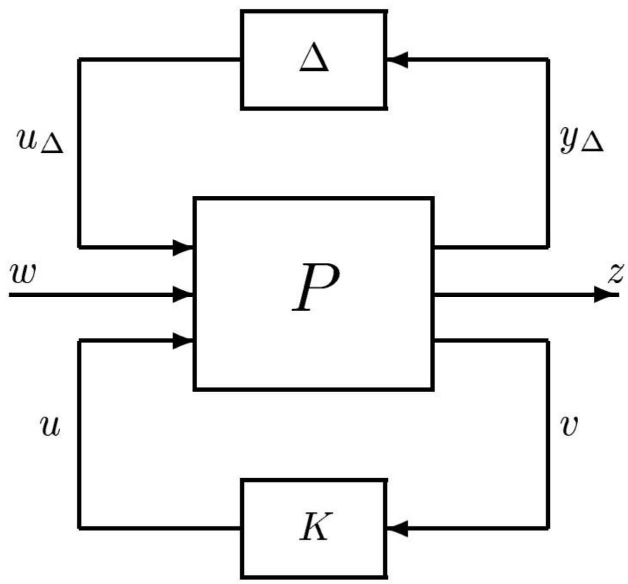

Figure 3.2: *General control configuration (for controller synthesis)*

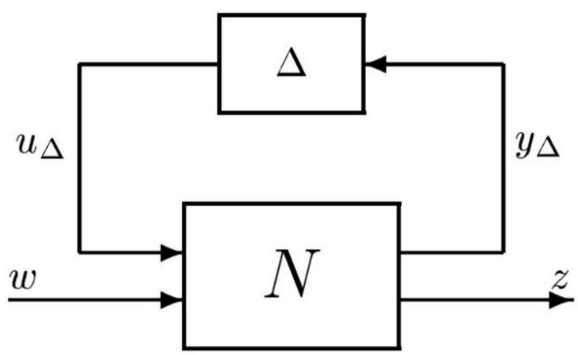

Figure 3.3: *NΔ – structure for RP analysis*

Instead, to analyse robust stability of F, we can rearrange the system into the M[1]Δ – structure, shown in *Fig*.3.4, since, supposing that $N_{22}, N_{21}, N_{12}, N_{11}, \Delta$ are

[1] $M = N_{11}$

asymptotically stable, then W is asymptotically stable if and only if $[I + N_{11}\Delta]^{-1}$ is asymptotically stable.

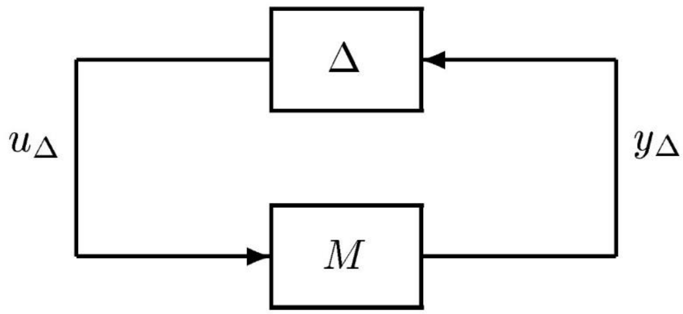

Figure 3.4: *$M\Delta$ – structure for RS analysis*

3.1.1 Unstructured uncertainty

When we lump into one single perturbation block Δ the dynamic perturbations that may occur in different parts of a system, as for instance, some unmodeled, high-frequency dynamics, what we are doing, is to consider an **unstructured uncertainty** representation. In this case, the block Δ may be represented by an **unknown** transfer function matrix. The unstructured dynamics uncertainty, in a control system, can be described in different ways, such as:

- Additive uncertainty;
- Multiplicative uncertainty.

In our problem, we will consider, as unstructured uncertainty, the **multiplicative uncertainty**, thus our perturbed system, can be described by the following relationship:

$$F_p(s) = F_n(s)[I + w_m \Delta_m]$$

where $F_n(s)$ is the nominal system, w_m is the multiplicative weight matrix and Δ_m is the full complex perturbation matrix in which $||\Delta_m||_\infty \leq 1$. In order to write our uncertain system as above, we have to compute only the multiplicative weight matrix.

Using the command `ucover`, we find the following multiplicative weight matrix:

$$w_m = \begin{bmatrix} \dfrac{0.003386s + 1.709}{s + 1.571} & \dfrac{0.1943s + 1.836}{s + 1.532} \\ \dfrac{0.283s + 1.795}{s + 1.498} & \dfrac{0.1327s + 1.829}{s + 1.541} \end{bmatrix}$$

In *Fig.3.5* are shown the magnitude Bode diagrams, for each component of the transfer function matrix, and the magnitude Bode diagrams of their multiplicative weight.

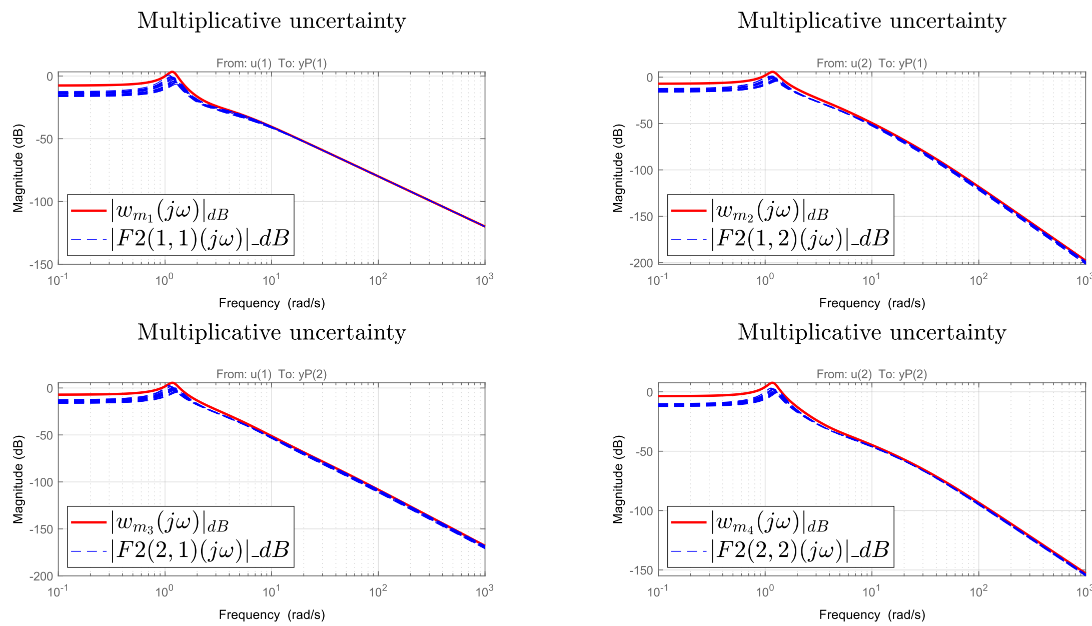

Figure 3.5: *Bode diagram*

Now, we are ready to verify if our nominal controller robustly stabilizes the perturbed system described by this type of uncertainty representation.

To check the **robust stability** for MIMO systems, we directly use the **Robust Stability for unstructured perturbation theorem**, that says: Assume that the nominal system $M(s)$ is stable (NS) and that the perturbations $\Delta(s)$ are stable. Then the $M\Delta -$ structure is stable for all perturbations Δ satisfying $||\Delta||_\infty \leq 1$ (i.e. we have RS) if and only if:

$$\overline{\sigma}\big(M(j\omega)\big) < 1, \forall\omega \;\Leftrightarrow\; ||M||_\infty < 1$$

In $Fig.3.6$ is shown the plot of $\overline{\sigma}\big(M(j\omega)\big)$ for all ω. We see that, the above condition is not verified and so this controller is not able to robust stabilizes the considered perturbed system.

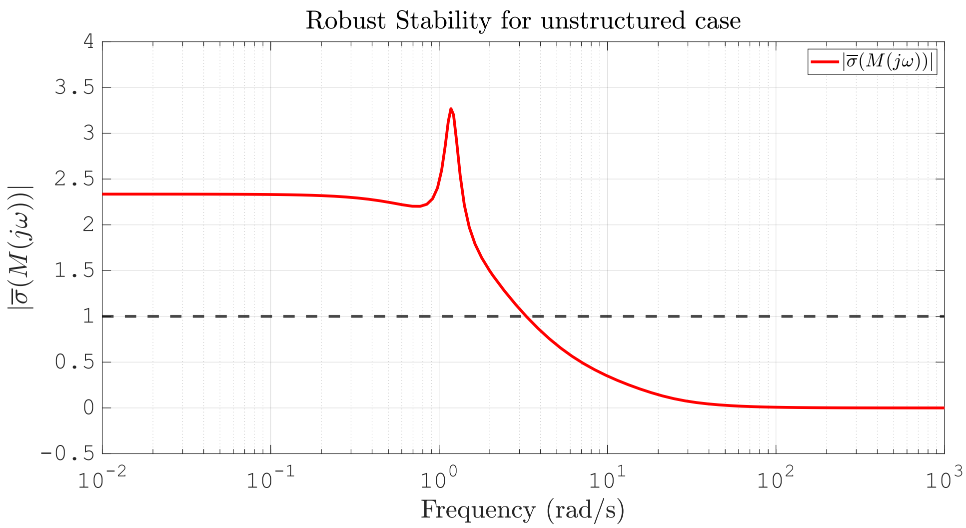

***Figure 3.6:** $\overline{\sigma}(M(j\omega))$ plot*

Since the necessary condition for robust performance is not satisfied, then it cannot be verified.

3.1.2 Structured uncertainty

In the **structured uncertainty** representation, we represent the uncertainty as a complex diagonal matrix:

$$\Delta = \begin{bmatrix} \Delta_1 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & 0 & \Delta_i & 0 \\ 0 & 0 & 0 & \dots \end{bmatrix}$$

where each Δ_i represents a specific source of uncertainty. In particular, let's define the family delta which characterize the uncertainties with a specific diagonal structure:

$$\Delta_F = \{\Delta = \mathrm{diag}\{\Delta_1, \dots, \Delta_s\}, \text{where } \Delta_i \text{ is asymptotically stable and } ||\Delta_i||_\infty \leq \delta\}$$

In this case, to check the **robust stability** for MIMO systems, we use the **Robust Stability for block-diagonal perturbation theorem**, that says: Assume that the nominal system $M(s)$ and that the perturbations $\Delta(s)$ are stable. Then the

$M\Delta$ – structure is stable for all allowed perturbations Δ with $\overline{\sigma}(\Delta) \leq 1, \forall\omega$, if and only if:

$$\mu\big(M(j\omega)\big) < 1, \forall\omega$$

where μ is the **structured singular value** (**SSV**) which is equal to:

$$\mu(M) = \frac{1}{\delta_{min}(\Delta)}$$

and δ_{min} is the **smallest structured destabilizer**.

In *Fig*.3.7 is shown the plot of $\mu\big(M(j\omega)\big)$ for all ω. We see that, the above condition is verified and so this controller is able to robust stabilizes the perturbed system described by the structured uncertainty.

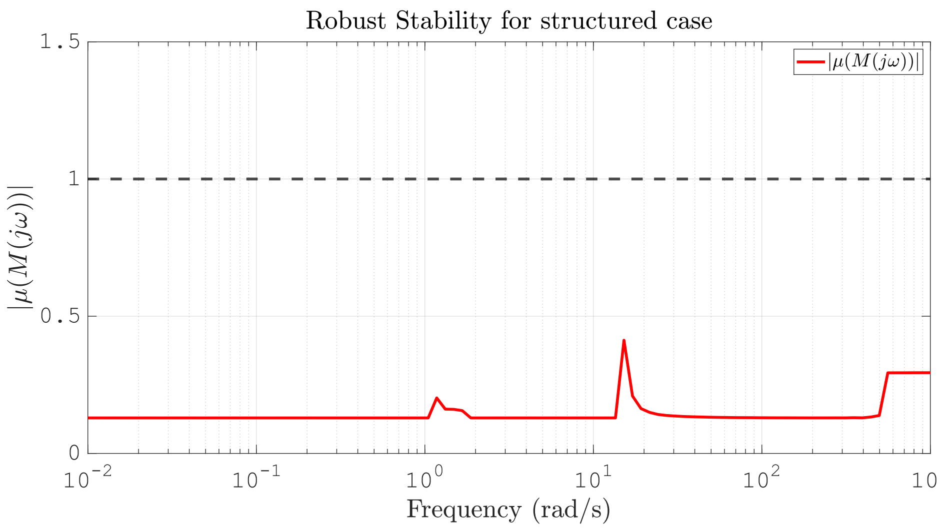

Figure 3.7: *$\mu(M(j\omega))$ plot*

To check the **robust performance** for MIMO systems, we use the **Robust Performance theorem**. Rearrange the uncertain system into the $N\Delta$ - structure of *Fig*.3.8. Assume nominal stability such that N is (internally) stable, then we have **robust performance** if and only if:

$$\mu_{\hat{\Delta}}\big(N(j\omega)\big) < 1, \qquad \forall\omega$$

where μ is computed with respect to the following structure:

$$\widehat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_P \end{bmatrix}$$

and Δ_P is a full complex perturbation with the same dimension as F^T.

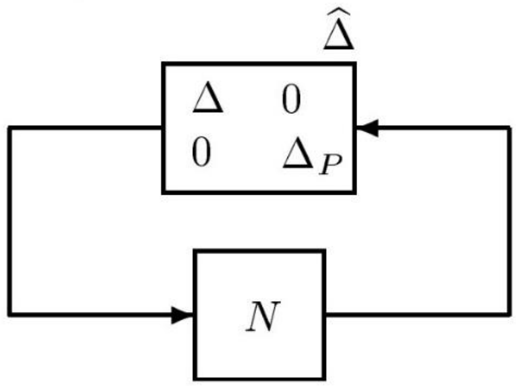

Figure 3.8: *$N\hat{\Delta}$ – structure for RP analysis*

In $Fig.3.9$ is shown the plot of $\mu_{\widehat{\Delta}}(N(j\omega))$ for all ω. We see that, the above condition is not verified and so this controller does not guarantee the robust performance.

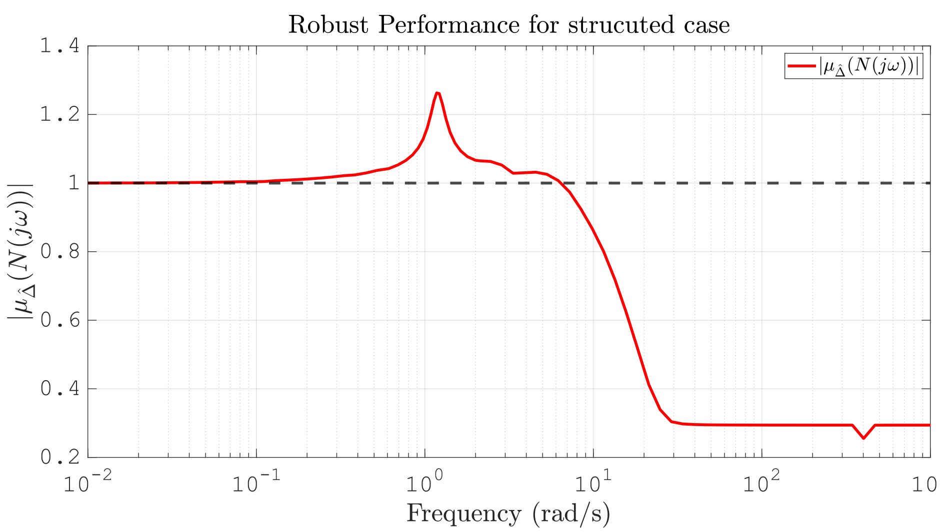

Figure 3.9: *$\mu_{\widehat{\Delta}}(N(j\omega))$ plot*

In particular we see that this function, at a critical frequency of 1.20Hz, has a peak equal to 1.17.

We can also check the robust performance, directly using the information that the MATLAB.*Command* `robustperf` gives us, as for example looking the performance margin. This controller gets the following robust performance margin:

```
           LowerBound: 0.7908
           UpperBound: 0.8414
    CriticalFrequency: 1.1754
```

Thus:

```
The trade-off of model uncertainty and system gain is balanced at a
level of 79.1% of the modeled uncertainty.

A model uncertainty of 84.1% can lead to input/output gain of 1.19
at 1.18 rad/seconds.
```

3.1.3 Worst-case gain

For uncertain systems, it is of interest to determine the largest value of the maximum of the frequency response of the largest singular value of the sensitivity or complementary sensitivity transfer function matrix, for the allowed uncertainty. This value, representing the largest possible gain in the frequency domain, is defined as the "**worst-case**" gain.

In *Fig.*3.10, are shown the nominal and worst case of the output sensitivity function and the output complementary sensitivity function.

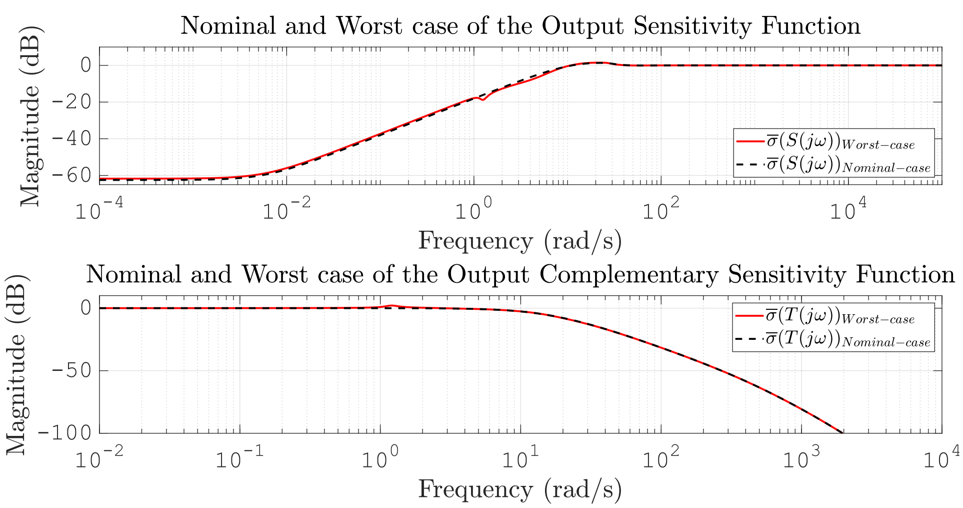

Figure 3.10: *Nominal and worst case of the output sensitivity function and of the output complementary sensitivity function*

In *Fig.3.11*, are shown the output response of the nominal system and the output response of the worst case.

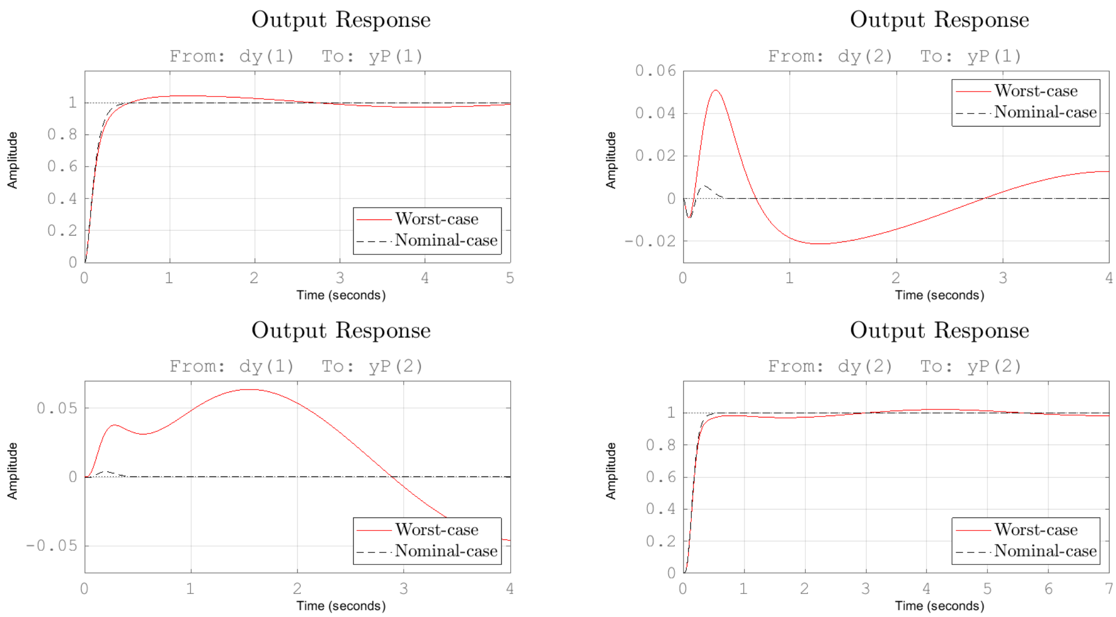

***Figure 3.11:** Output response of the nominal system and output response of the worst case*

To illustrate the influence of uncertainty on the closed-loop system, in *Fig.3.12* we show the singular value plots of sensitivity function singular value and the singular value plots of output complementary sensitivity function, for random value and worst-case uncertain elements.

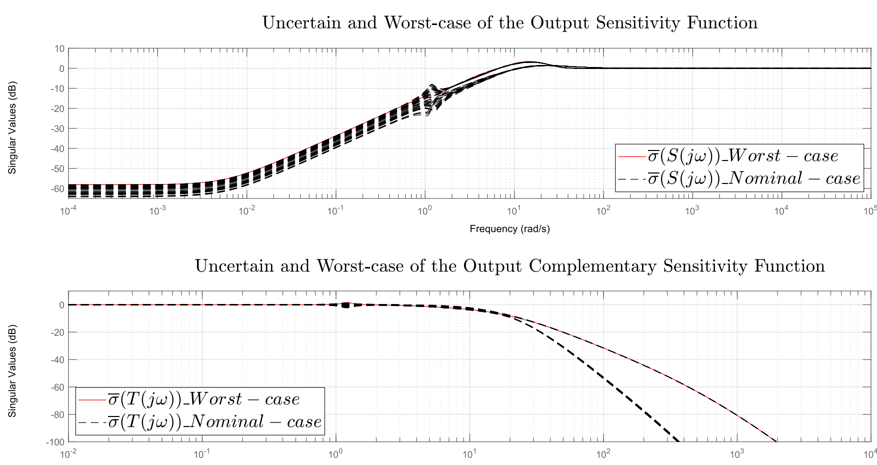

***Figure 3.12:** Random and worst case of the output sensitivity function and of the output complementary sensitivity function*

In *Fig.3.13*, are shown the output response of the random system and the output response of the worst case.

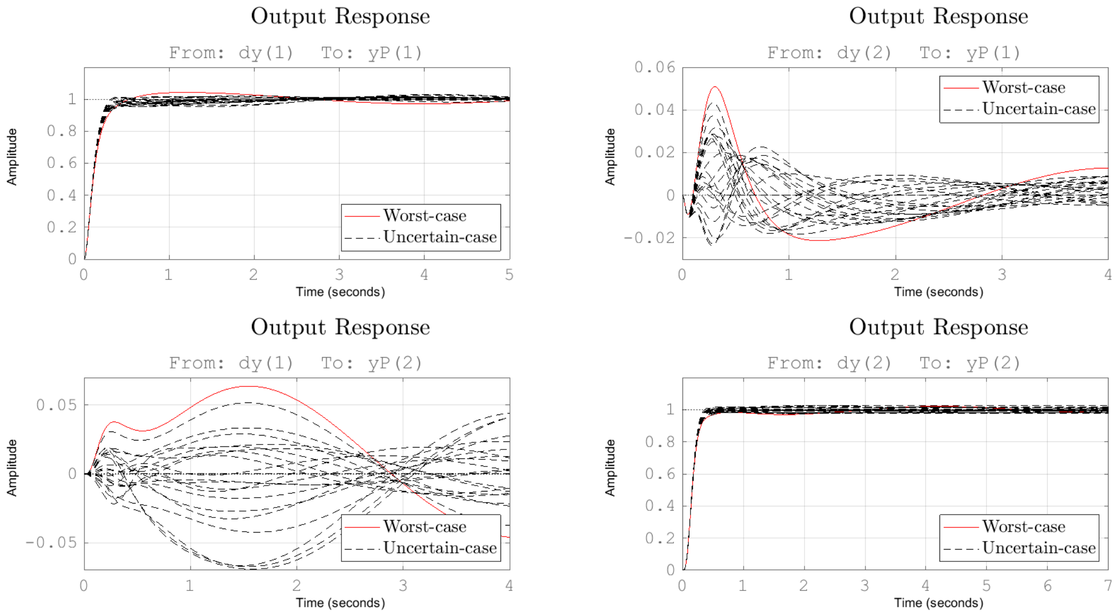

Figure 3.13: *Output response of the random system and output response of the worst case*

3.2 $\mathcal{H}_\infty$ Robust Control Design Problem for system 2

In this section we want to find a MIMO robust controller which guarantees the robust performances of the uncertain system 2. To design this controller, let's use the μ-synthesis approach which designs a robust controller for an uncertain plant using D-K iteration.

By the same choices of weights, we find:

	Robust performance			Fit order	
Iter	K Step	Peak MU	DG Fit	D	G
1	4.112	3.479	3.536	28	30
2	2.247	2.008	2.025	30	30
3	1.762	1.553	1.579	32	38
4	1.505	1.406	1.426	34	40
5	1.387	1.344	1.358	36	40
6	1.287	1.265	1.278	42	48
7	1.24	1.225	1.237	44	48
8	1.209	1.199	1.211	48	54
9	1.186	1.177	1.201	44	42
10	1.172	1.163	1.179	38	46

3.3 Robust stability and performance of this new controller

Now, let's analyse if this robust controller guarantees the **robust stability** and the **robust performance**, always considering the **unstructured** and **structured uncertainty**.

3.3.1 Unstructured uncertainty

In *Fig*.3.14 is shown the plot of $\overline{\sigma}\big(M(j\omega)\big)$ for all ω. We see that, the above condition is not verified and so this controller is not able to robust stabilizes the perturbed system described by unstructured uncertainty.

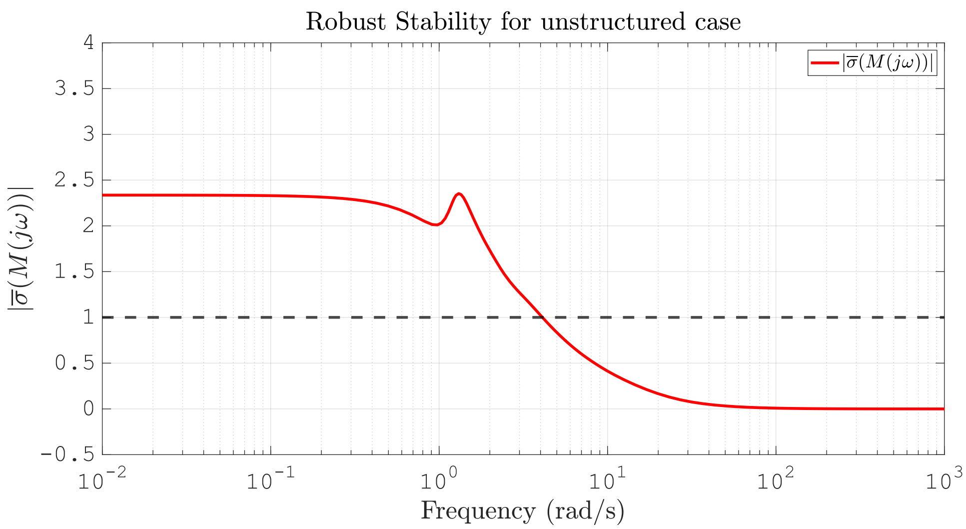

***Figure 3.14:** $\overline{\sigma}(M(j\omega))$ plot*

Since the necessary condition for robust performance is not satisfied, then it cannot be verified.

3.3.2 Structured uncertainty

In $Fig.3.15$ is shown the plot of $\mu\big(M(j\omega)\big)$ for all ω. We see that, the above condition is verified and so this controller is able to robust stabilizes the perturbed system described by the structured uncertainty.

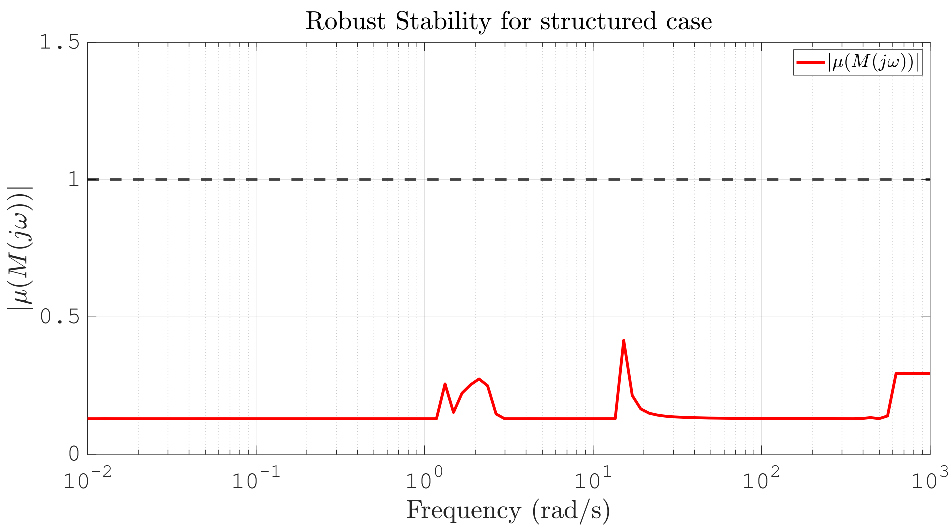

Figure 3.15: *$\mu(M(j\omega))$ plot*

In $Fig.3.16$ is shown the plot of $\mu_{\hat{\Delta}}\big(N(j\omega)\big)$ for all ω. We see that, the above condition is not verified and so this also controller does not guarantee the robust performance.

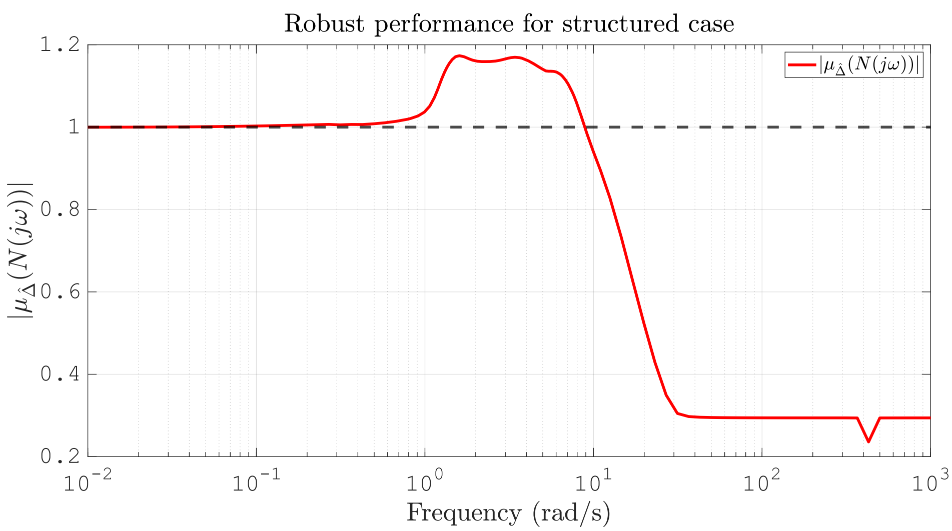

Figure 3.16: *$\mu_{\hat{\Delta}}(N(j\omega))$ plot*

In particular we see that this function, at a critical frequency of 3.07Hz, has a peak equal to 1.17.

What can we say about the performance margin? Through this controller, we get the following robust performance margin:

```
           LowerBound: 0.8516
           UpperBound: 0.8585
    CriticalFrequency: 3.0726
```

Thus:

```
The trade-off of model uncertainty and system gain is balanced at a
level of 85.2% of the modeled uncertainty.

A model uncertainty of 85.9% can lead to input/output gain of 1.16
at 3.07 rad/seconds.
```

3.3.3 Worst-case gain

Also here, let's show the comparison between the worst case and the nominal, or random, case.

In *Fig.3.17*, are shown the nominal and worst case of the output sensitivity function and the output complementary sensitivity function.

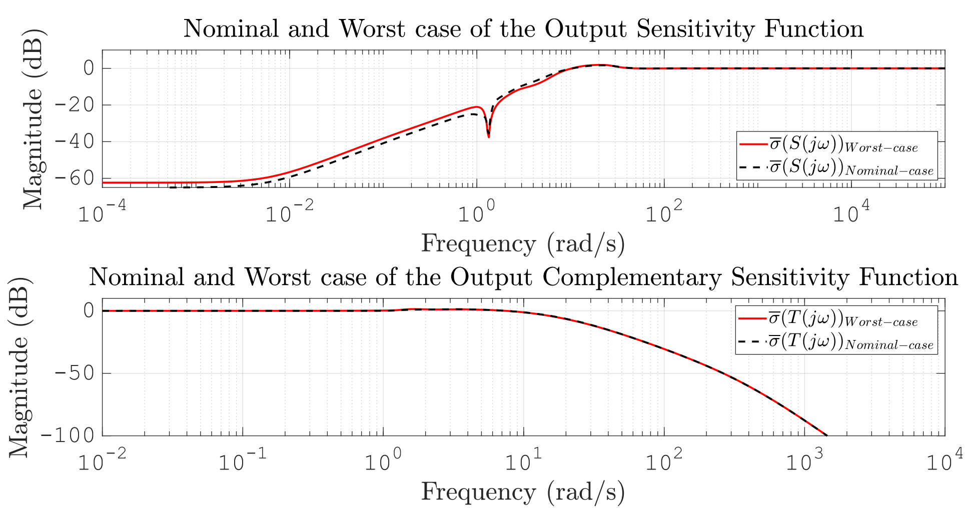

Figure 3.17: *Nominal and worst case of the output sensitivity function and of the output complementary sensitivity function*

In *Fig.*3.18, are shown the output response of the nominal system and the output response of the worst case.

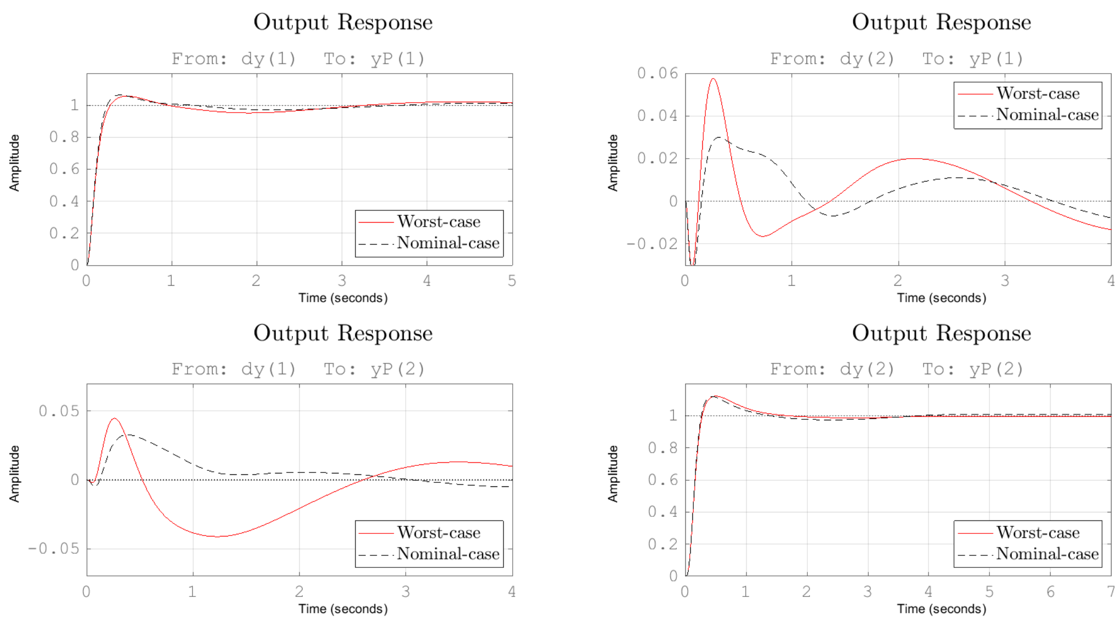

Figure 3.18: *Output response of the nominal system and output response of the worst case*

In *Fig.*3.19 we show the singular value plots of sensitivity function singular value and the singular value plots of output complementary sensitivity function, for random value and worst-case uncertain elements.

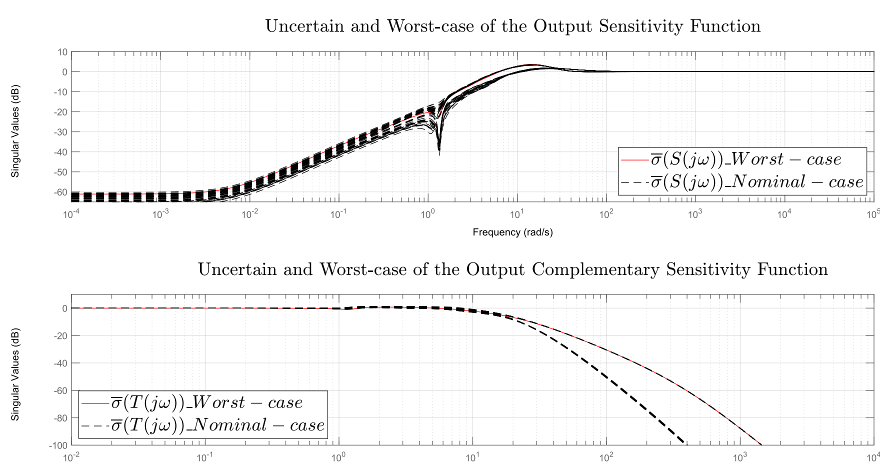

Figure 3.19: *Random and worst case of the output sensitivity function and of the output complementary sensitivity function*

In *Fig.3.20*, are shown the output response of the random system and the output response of the worst case.

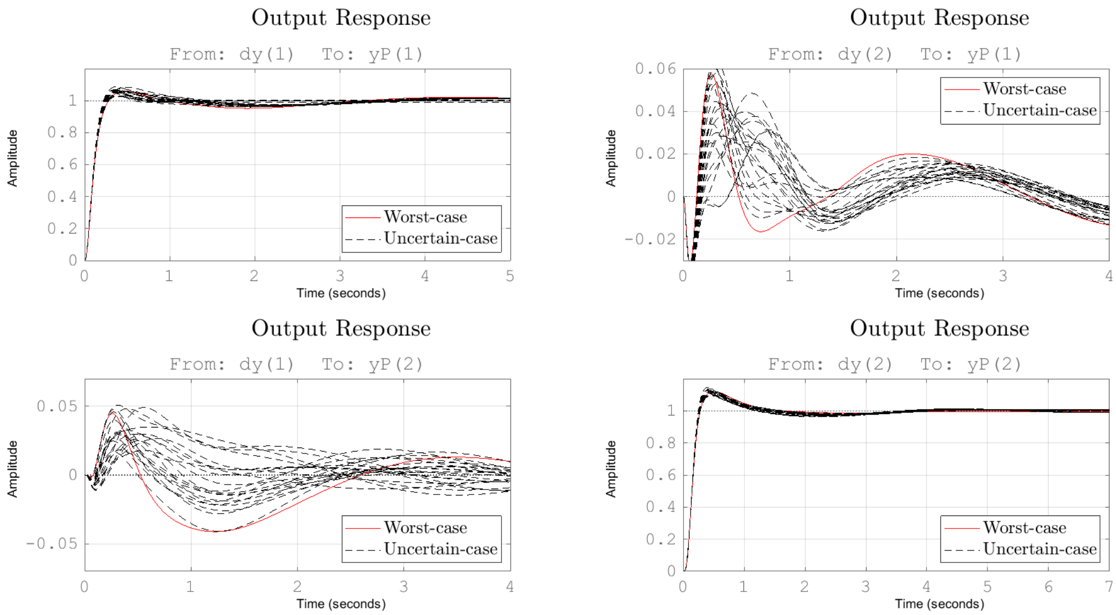

Figure 3.20: *Output response of the nominal system and output response of the worst case*

4 Comparison between TISO and SISO system

As final assignment, let's focus, using the system 2, on the comparison between the SISO case (input f_1, output z_2) and the TISO case. Is there any advantage of using two inputs instead of one?

The advantage in using two inputs instead of one, translates in terms of **control effort** indeed, as shown in *Fig.*4.1, the SISO system must make more effort to reach a specific target compared to the TISO system.
Moreover, we know that the greater the control action, the more energy the system needs, and so, the TISO system requires **less energy** than the SISO system.

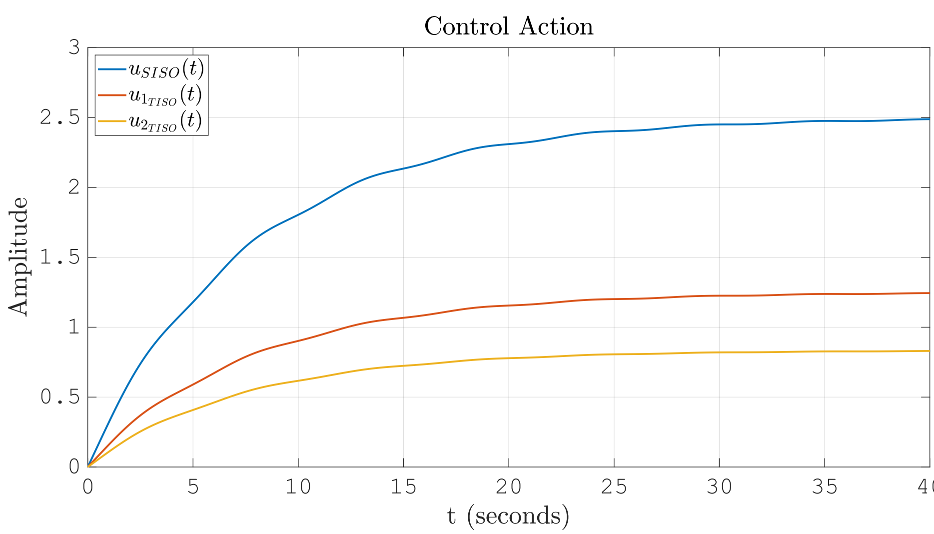

Figure 4.1: *Comparison of the control actions*