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> #France Paquet-Nadeau
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  #code for the recurrences S_1(k,d) in Algorithmica, earlier
  version of S_2(k,d).

  #This is essentially the same code as for the recurrences of S_2
  (k,d).
> restart:
  libname:= "/Users/France Paquet-Nadeau/Desktop/Maple/gfun",
  "/Users/France Paquet-Nadeau/Desktop/Maple/algolib", libname:

  with(gfun):
  gfun:-version();
```

$$3.76 \tag{1}$$

```
> d:=5:
> S:=proc(T,d,s)
  option remember;

  if d=0 then 1
  elif type(T, nonnegint) then
     if (d=1 and T=1) then 2*s
     elif (d>1 and T=1) then (2*s-1)*s^(d-1)
        else
           procname(T-1,d,s) + procname(T-1,d-1,s) + (2*s-1)*add(
  (s^(j-1))*procname(T-1,d-j,s), j=1..d)
     fi;
  else 'S'(args);
  fi:
  end proc:

> GenRec:=proc(T,d,s)
  if d=0 then 1
  elif (d=1 and T=1) then return(2*s)
    elif (d>1 and T=1) then return( (2*s-1)*s^(d-1) )
      else
      R(T-1,d,s) + R(T-1,d-1,s) + (2*s-1)*add( s^(j-1)*R(T-1,d-j,
  s), j=1..d)
  fi;
  end proc:
> # This is how we obtain the differential equation to solve F2[1]
> eval(subs(R=S, GenRec(k, 1, s)));
```

$$S(k-1, 1, s) + 2\,s \tag{2}$$

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> #-----------------------------------------------------
```

Transforming the recurrences of S into generating function. We will denote by F2[d] the functions obtain from this method.
To obtain F2[1] we had to compute S_1(k,1) and input the resulting expression manually into rectodiffeq to solve the differential expression
For the following values of d, we construct 'expression' with the recurrences of 'S'.
We have to take every term in the recurrences, of the form S(k,d), take the summation for k from d to infinity times z^k. Then we perform some manipulations to obtain the desired form.
We remove the constant terms because we know the summation of z^k will translate to z^k/(z-1), hence we multiply by the constant to obtain the generating function equivalent.

We leave s as a constant, but we can easily substitute a value to obtain a specific serie.

```
>
F2:='F2':
F2[1]:=solve(rectodiffeq({a(n)= a(n-1)+2*s, a(0)=0}, a(n), f(z)),
f(z));    #In this case a(0)=0 because when t=0 there are no
strings.
print(series(F2[1],z,9));

for dPrime from 2 to d do:
              #we use dPrime as an index to build the table of
functions

expression:= S(k, dPrime, s)=collect(eval(subs(R=S, GenRec(k,
dPrime, s))), S): #this gives the RHS of the equation of S_1(k,d)
with

              # similar terms together

const:= eval(subs(S=(()-> 0),expression)):  # collect the
constant terms
expression:= expression - const:            # remove the constant
terms from the equation to keep only terms of the form 'S(k,d)'

# this part transforms 'expression' into generating function;

for dd from 1 to dPrime do                       # dd corresponds
to the distance in the terms 'S(k,d)'
  for j from 0 to dPrime do                      # j is the index
for terms substracted to 'k'


  expression:= subs(S(k-j, dd, s)=z^j*(F2[dd]-(add (eval(S(l, dd,
s))*z^l, l=1..(dPrime-j-1)))), expression):


  od:
od:

expression:= expression +  const * z^dPrime/(1-z): #add back the
constant terms


F2[dPrime]:=solve(expression, F2[dPrime]);
print(F2[dPrime]);

print(series(F2[dPrime],z,9));
od:
```

$$F2_1 := \frac{2\,z\,s}{z^2-2\,z+1}$$

$$2\,s\,z+4\,s\,z^2+6\,s\,z^3+8\,s\,z^4+10\,s\,z^5+12\,s\,z^6+14\,s\,z^7+16\,s\,z^8+\mathrm{O}(z^9)$$

$$-\frac{(2\,z\,s+2\,s+z-1)\,z\,s}{(z-1)^3}$$

$$(2\,s-1)\,s\,z+(3\,(2\,s-1)\,s+(2\,s+1)\,s)\,z^2+(6\,(2\,s-1)\,s+3\,(2\,s+1)\,s)\,z^3+(10\,(2\,s$$

$$- 1) s + 6 (2 s + 1) s) z^4 + (15 (2 s - 1) s + 10 (2 s + 1) s) z^5 + (21 (2 s - 1) s + 15 (2 s + 1) s) z^6 + (28 (2 s - 1) s + 21 (2 s + 1) s) z^7 + (36 (2 s - 1) s + 28 (2 s + 1) s) z^8 + O(z^9)$$

$$\frac{(2 z^2 s + 4 z s + 3 z^2 + 2 s - 2 z - 1) s^2 z}{(z - 1)^4}$$

$$(2 s - 1) s^2 z + \left(4 (2 s - 1) s^2 + (4 s - 2) s^2\right) z^2 + \left(10 (2 s - 1) s^2 + 4 (4 s - 2) s^2 + (2 s + 3) s^2\right) z^3 + \left(20 (2 s - 1) s^2 + 10 (4 s - 2) s^2 + 4 (2 s + 3) s^2\right) z^4 + \left(35 (2 s - 1) s^2 + 20 (4 s - 2) s^2 + 10 (2 s + 3) s^2\right) z^5 + \left(56 (2 s - 1) s^2 + 35 (4 s - 2) s^2 + 20 (2 s + 3) s^2\right) z^6 + \left(84 (2 s - 1) s^2 + 56 (4 s - 2) s^2 + 35 (2 s + 3) s^2\right) z^7 + \left(120 (2 s - 1) s^2 + 84 (4 s - 2) s^2 + 56 (2 s + 3) s^2\right) z^8 + O(z^9)$$

$$-\frac{(2 s^2 z^3 + 6 s^2 z^2 + 5 s z^3 + 6 s^2 z + z^2 s + z^3 + 2 s^2 - 5 z s - 2 z^2 - s + z) s^2 z}{(z - 1)^5}$$

$$(2 s^2 - s) s^2 z + \left(5 (2 s^2 - s) s^2 + (6 s^2 - 5 s + 1) s^2\right) z^2 + \left(15 (2 s^2 - s) s^2 + 5 (6 s^2 - 5 s + 1) s^2 + (6 s^2 + s - 2) s^2\right) z^3 + \left(35 (2 s^2 - s) s^2 + 15 (6 s^2 - 5 s + 1) s^2 + 5 (6 s^2 + s - 2) s^2 + (2 s^2 + 5 s + 1) s^2\right) z^4 + \left(70 (2 s^2 - s) s^2 + 35 (6 s^2 - 5 s + 1) s^2 + 15 (6 s^2 + s - 2) s^2 + 5 (2 s^2 + 5 s + 1) s^2\right) z^5 + \left(126 (2 s^2 - s) s^2 + 70 (6 s^2 - 5 s + 1) s^2 + 35 (6 s^2 + s - 2) s^2 + 15 (2 s^2 + 5 s + 1) s^2\right) z^6 + \left(210 (2 s^2 - s) s^2 + 126 (6 s^2 - 5 s + 1) s^2 + 70 (6 s^2 + s - 2) s^2 + 35 (2 s^2 + 5 s + 1) s^2\right) z^7 + \left(330 (2 s^2 - s) s^2 + 210 (6 s^2 - 5 s + 1) s^2 + 126 (6 s^2 + s - 2) s^2 + 70 (2 s^2 + 5 s + 1) s^2\right) z^8 + O(z^9)$$

$$\frac{1}{(z - 1)^6} \left((2 s^2 z^4 + 8 s^2 z^3 + 7 s z^4 + 12 s^2 z^2 + 8 s z^3 + 4 z^4 + 8 s^2 z - 6 z^2 s - 6 z^3 + 2 s^2 - 8 z s - s + 2 z) s^3 z\right)$$

$$(2 s^2 - s) s^3 z + \left(6 (2 s^2 - s) s^3 + (8 s^2 - 8 s + 2) s^3\right) z^2 + \left(21 (2 s^2 - s) s^3 + 6 (8 s^2 - 8 s + 2) s^3 + (12 s^2 - 6 s) s^3\right) z^3 + \left(56 (2 s^2 - s) s^3 + 21 (8 s^2 - 8 s + 2) s^3 + 6 (12 s^2 - 6 s) s^3 + (8 s^2 + 8 s - 6) s^3\right) z^4 + \left(126 (2 s^2 - s) s^3 + 56 (8 s^2 - 8 s + 2) s^3 + 21 (12 s^2 - 6 s) s^3 + 6 (8 s^2 + 8 s - 6) s^3 + (2 s^2 + 7 s + 4) s^3\right) z^5 + \left(252 (2 s^2 - s) s^3 + 126 (8 s^2 - 8 s + 2) s^3 + 56 (12 s^2 - 6 s) s^3 + 21 (8 s^2 + 8 s - 6) s^3 + 6 (2 s^2 + 7 s + 4) s^3\right) z^6 + \left(462 (2 s^2 - s) s^3 + 252 (8 s^2 - 8 s + 2) s^3 + 126 (12 s^2 - 6 s) s^3 + 56 (8 s^2 + 8 s - 6) s^3 + 21 (2 s^2 + 7 s + 4) s^3\right) z^7 + \left(792 (2 s^2 - s) s^3 + 462 (8 s^2 - 8 s + 2) s^3 + 252 (12 s^2 - 6 s) s^3 + 126 (8 s^2 + 8 s - 6) s^3 + 56 (2 s^2 + 7 s + 4) s^3\right) z^8 + O(z^9) \qquad \textbf{(3)}$$

We evaluate each generating function for different alphabet sizes.
We evaluate to find th asymptotic by converting the coefficients of F2n[dd,s] into polynomials, multiplying by d! to get rid of the fraction terms.

```
> s:=2:
  for dd from 1 to d do
    F2n[dd,s]:=equivalent(F2[dd], z, k);
  print(factor(numer(F2[dd])), factor(denom(F2[dd])));
```

```
od:
```

$$4\,z,\ (z-1)^2$$
$$-2\,(5z+3)\,z,\ (z-1)^3$$
$$4\,(7z^2+6z+3)\,z,\ (z-1)^4$$
$$-4\,(19z^3+24z^2+15z+6)\,z,\ (z-1)^5$$
$$16\,(13z^4+21z^3+18z^2+9z+3)\,z,\ (z-1)^6 \tag{4}$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
  (convert(F2n[dd,s], polynom)*dd!, k, dd))*k^dd), dd=1..d);
```

$$(2)^2\,k$$
$$(2)^4\,k^2$$
$$(2)^6\,k^3$$
$$(2)^8\,k^4$$
$$(2)^{10}\,k^5 \tag{5}$$

```
> s:=3:
  for dd from 1 to d do
    F2n[dd,s]:=equivalent(F2[dd], z, k);
  print(factor(numer(F2[dd])), factor(denom(F2[dd])));
  od:
```

$$6\,z,\ (z-1)^2$$
$$-3\,(7z+5)\,z,\ (z-1)^3$$
$$9\,(9z^2+10z+5)\,z,\ (z-1)^4$$
$$-9\,(34z^3+55z^2+40z+15)\,z,\ (z-1)^5$$
$$27\,(43z^4+90z^3+90z^2+50z+15)\,z,\ (z-1)^6 \tag{6}$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
  (convert(F2n[dd,s], polynom), k, dd)*dd!)*k^dd), dd=1..d);
```

$$(2)\,(3)\,k$$
$$(2)^2\,(3)^2\,k^2$$
$$(2)^3\,(3)^3\,k^3$$
$$(2)^4\,(3)^4\,k^4$$
$$(2)^5\,(3)^5\,k^5 \tag{7}$$

```
> s:=4:
  for dd from 1 to d do
    F2n[dd,s]:=equivalent(F2[dd], z, k);
  print(factor(numer(F2[dd])), factor(denom(F2[dd])));
  od:
```

$$8\,z,\ (z-1)^2$$
$$-4\,(9z+7)\,z,\ (z-1)^3$$
$$16\,(11z^2+14z+7)\,z,\ (z-1)^4$$
$$-16\,(53z^3+98z^2+77z+28)\,z,\ (z-1)^5$$
$$128\,(32z^4+77z^3+84z^2+49z+14)\,z,\ (z-1)^6 \tag{8}$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
```

```
(convert(F2n[dd,s], polynom)*dd!, k, dd))*k^dd), dd=1..d);
```

$$(2)^3 k$$
$$(2)^6 k^2$$
$$(2)^9 k^3$$
$$(2)^{12} k^4$$
$$(2)^{15} k^5 \quad \textbf{(9)}$$

```
> s:=5:
  for dd from 1 to d do
    F2n[dd,s]:=equivalent(F2[dd], z, k);
  print(factor(numer(F2[dd])), factor(denom(F2[dd])));
  od:
```

$$10\, z,\ (z-1)^2$$
$$-5\ (11\, z+9)\, z,\ (z-1)^3$$
$$25\ \left(13\, z^2+18\, z+9\right) z,\ (z-1)^4$$
$$-25\ \left(76\, z^3+153\, z^2+126\, z+45\right) z,\ (z-1)^5$$
$$125\ \left(89\, z^4+234\, z^3+270\, z^2+162\, z+45\right) z,\ (z-1)^6 \quad \textbf{(10)}$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
  (convert(F2n[dd,s], polynom)*dd!, k, dd))*k^dd), dd=1..d);
```

$$(2)\ (5)\ k$$
$$(2)^2\ (5)^2\ k^2$$
$$(2)^3\ (5)^3\ k^3$$
$$(2)^4\ (5)^4\ k^4$$
$$(2)^5\ (5)^5\ k^5 \quad \textbf{(11)}$$

```
> s:=6:
  for dd from 1 to d do
    F2n[dd,s]:=equivalent(F2[dd], z, k);
  print(factor(numer(F2[dd])), factor(denom(F2[dd])));
  od:
```

$$12\, z,\ (z-1)^2$$
$$-6\ (13\, z+11)\, z,\ (z-1)^3$$
$$36\ \left(15\, z^2+22\, z+11\right) z,\ (z-1)^4$$
$$-36\ \left(103\, z^3+220\, z^2+187\, z+66\right) z,\ (z-1)^5$$
$$432\ \left(59\, z^4+165\, z^3+198\, z^2+121\, z+33\right) z,\ (z-1)^6 \quad \textbf{(12)}$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
  (convert(F2n[dd,s], polynom)*dd!, k, dd))*k^dd), dd=1..d);
```

$$(2)^2\ (3)\ k$$
$$(2)^4\ (3)^2\ k^2$$
$$(2)^6\ (3)^3\ k^3$$
$$(2)^8\ (3)^4\ k^4$$
$$(2)^{10}\ (3)^5\ k^5 \quad \textbf{(13)}$$

```
> s:=20:
```

```
for dd from 1 to d do
  F2n[dd,s]:=equivalent(F2[dd], z, k);
print(factor(numer(F2[dd])), factor(denom(F2[dd])));
od:
```

$$40\,z,\ (z-1)^2$$
$$-20\,(41\,z+39)\,z,\ (z-1)^3$$
$$400\,\left(43\,z^2+78\,z+39\right)\,z,\ (z-1)^4$$
$$-400\,\left(901\,z^3+2418\,z^2+2301\,z+780\right)\,z,\ (z-1)^5$$
$$16000\,\left(472\,z^4+1677\,z^3+2340\,z^2+1521\,z+390\right)\,z,\ (z-1)^6 \qquad \textbf{(14)}$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
  (convert(F2n[dd,s], polynom)*dd!, k, dd))*k^dd), dd=1..d);
```

$$(2)^3\,(5)\,k$$
$$(2)^6\,(5)^2\,k^2$$
$$(2)^9\,(5)^3\,k^3$$
$$(2)^{12}\,(5)^4\,k^4$$
$$(2)^{15}\,(5)^5\,k^5 \qquad \textbf{(15)}$$