```
> #France Paquet-Nadeau
  #July 25, 2017
  #Final version of the code to compute the asymptotic bound on the
  recurrences S_2(k,d) of Gene Myers as presented in "What's behind
  #BLAST".
  # We obtain the asymptotic bound on S_2(k,d) and N_D(k).
```

```
> restart:
  libname:= "/Users/France Paquet-Nadeau/Desktop/Maple/gfun",
  "/Users/France Paquet-Nadeau/Desktop/Maple/algolib", libname:

  with(gfun):
  gfun:-version();
```

$$3.76 \tag{1}$$

Variables:
k = no. symbols in the query (the 'real' variable)
s = size of alphabet (we start with 2)
d = neighbourhood size, number of errors allowed

```
> d:=5:
```

We use this function to evaluate S. The function S will be kept symbolic while GenRec will evaluate each term.

```
> GenRec:= proc(k,d::integer, s)
  if d=0 then 1
  elif type(k, posint) and k<= d then return 1
  else
        T(k-1,d,s)
          +(s-1)*T(k-1,d-1,s)
          +(s-1)*add(s^j*T(k-2,d-1-j,s), j=0..d-1)
          +(s-1)^2*add(s^j*T(k-2,d-2-j,s), j=0..d-2)
          +add(T(k-2-j,d-1-j,s), j=0..d-1);
  fi;
  end proc:
```

```
> S:= proc(k,d::integer,s)
  option remember;
      if d=0 then 1
      elif type(k, nonnegint) then
        if k<= d then return 1
        else
          procname (k-1,d, s)
          +(s-1)*procname(k-1,d-1,s)
          +(s-1)*add(s^j*procname(k-2,d-1-j, s), j=0..d-1)
          +(s-1)^2*add(s^j*procname(k-2,d-2-j,s), j=0..d-2)
          +add(procname(k-2-j,d-1-j, s), j=0..d-1);
        fi;
      else 'S'(args);
      fi;
  end proc:
```

Transforming the recurrences of S_2 into generating function. We will denote by F[d] the functions

obtain from this method.
To obtain F[1] we had to compute S(k,1) and input the resulting expression manually into rectodiffeq to solve the differential expression
For the following values of d, we construct 'expression' with the recurrences of 'S'.

```
> F:='F':  #initializing
  F[1]:=solve(rectodiffeq({a(n)= a(n-1)+2*s-1, a(0)=1},a(n),f(z)),f
  (z)); #This comes from 'eval(subs(T=S, GenRec(k, 1, s)))=S(k-1,1,
  s)+2s-1'
  print(convert(series(F[1],z,9),polynom));

  for dPrime from 2 to d do: #we use dPrime as an index to build
  the table of functions

  expression:= S(k, dPrime, s)=collect(eval(subs(T=S, GenRec(k,
  dPrime, s))), S): #this gives the RHS of the equation of S(k,d)
  with

                # similar terms together

  const:= eval(subs(S=(()-> 0),expression)):  # collect the
  constant terms
  expression:= expression - const;            # remove the constant
  terms from the equation to keep only terms of the form 'S(k,d)'

  # this part transforms 'expression' into generating function;

  for dd from 1 to dPrime do                       # dd corresponds
  to the distance in the terms 'S(k,d)'
    for j from 0 to dPrime do                      # j is the index
  for terms substracted to 'k'
    expression:= subs(S(k-j, dd, s)=z^j*(F[dd]-(add (eval(S(l+1,
  dd, s))*z^l, l=0..(dPrime-j-1)))), expression);  # substitute the
  term

                                              #S(k-j,dd) with the
  equation
                                                     #for F[dd]
  adjusted to thie

  #index

    od;
  od;
  expression:= expression +  const * z^dPrime/(1-z); #add back the
  constant terms transformed into generating function

  F[dPrime]:=solve(expression, F[dPrime]);
  print(F[dPrime]);
  print(convert(series(F[dPrime],z,9),polynom));
  od:
```

$$F_1 := \frac{2\,z\,s - 2\,z + 1}{z^2 - 2\,z + 1}$$

$$1 + 2\,z\,s + (-1 + 4\,s)\,z^2 + (6\,s - 2)\,z^3 + (-3 + 8\,s)\,z^4 + (10\,s - 4)\,z^5 + (-5 + 12\,s)\,z^6$$

$+ (14 s - 6) z^7 + (-7 + 16 s) z^8$

$$-\frac{4 s^2 z^2 - 4 s z^2 - z^3 + 3 z^2 - 2 z + 1}{(z-1)^3}$$

$1 + z + (4 s^2 - 4 s + 3) z^2 + (12 s^2 - 12 s + 6) z^3 + (24 s^2 - 24 s + 10) z^4 + (40 s^2 - 40 s + 15) z^5 + (60 s^2 - 60 s + 21) z^6 + (84 s^2 - 84 s + 28) z^7 + (112 s^2 - 112 s + 36) z^8$

$\frac{1}{(z-1)^4} (2 s^3 z^5 - 4 s^3 z^4 - s^2 z^5 + 10 s^3 z^3 + 6 s^2 z^4 - 3 s z^5 - 17 s^2 z^3 - 2 s z^4 + 3 z^5 + 11 s z^3 - 3 z^4 - 2 z^3 + 3 z^2 - 3 z + 1)$

$1 + z + z^2 + (10 s^3 - 17 s^2 + 11 s) z^3 + (36 s^3 - 62 s^2 + 42 s - 6) z^4 + (86 s^3 - 147 s^2 + 99 s - 18) z^5 + (168 s^3 - 284 s^2 + 188 s - 37) z^6 + (290 s^3 - 485 s^2 + 315 s - 64) z^7 + (460 s^3 - 762 s^2 + 486 s - 100) z^8$

$\frac{1}{(z-1)^5} (4 s^4 z^7 - 20 s^4 z^6 - 8 s^3 z^7 + 28 s^4 z^5 + 32 s^3 z^6 + 8 s^2 z^7 - 28 s^4 z^4 - 56 s^3 z^5 - 14 s^2 z^6 - 6 s z^7 + 64 s^3 z^4 + 40 s^2 z^5 + 4 s z^6 + 3 z^7 - 58 s^2 z^4 - 14 s z^5 - 6 z^6 + 24 s z^4 + 8 z^5 - 7 z^4 + 4 z^3 - 6 z^2 + 4 z - 1)$

$1 + z + z^2 + z^3 + (28 s^4 - 64 s^3 + 58 s^2 - 24 s + 7) z^4 + (112 s^4 - 264 s^3 + 250 s^2 - 106 s + 23) z^5 + (300 s^4 - 712 s^3 + 684 s^2 - 294 s + 57) z^6 + (656 s^4 - 1552 s^3 + 1492 s^2 - 644 s + 118) z^7 + (1260 s^4 - 2960 s^3 + 2830 s^2 - 1220 s + 216) z^8$

$\frac{1}{(z-1)^6} (14 s^5 z^9 - 72 s^5 z^8 - 29 s^4 z^9 + 156 s^5 z^7 + 164 s^4 z^8 + 20 s^3 z^9 - 152 s^5 z^6 - 354 s^4 z^7 - 140 s^3 z^8 - 5 s^2 z^9 + 86 s^5 z^5 + 380 s^4 z^6 + 290 s^3 z^7 + 52 s^2 z^8 - 241 s^4 z^5 - 368 s^3 z^6 - 93 s^2 z^7 + z^9 + 278 s^3 z^5 + 170 s^2 z^6 - 9 s z^7 - 9 z^8 - 164 s^2 z^5 - 30 s z^6 + 20 z^7 + 49 s z^5 - 10 z^6 - 4 z^5 + 5 z^4 - 10 z^3 + 10 z^2 - 5 z + 1)$

$1 + z + z^2 + z^3 + z^4 + (86 s^5 - 241 s^4 + 278 s^3 - 164 s^2 + 49 s - 2) z^5 + (364 s^5 - 1066 s^4 + 1300 s^3 - 814 s^2 + 264 s - 27) z^6 + (1050 s^5 - 3135 s^4 + 3920 s^3 - 2517 s^2 + 840 s - 102) z^7 + (2488 s^5 - 7476 s^4 + 9440 s^3 - 6120 s^2 + 2060 s - 266) z^8$ **(2)**

Defining the recurrences for N_D(k).

```
> for dd from 1 to d do:
     Z[dd]:=F[dd]+add(s^j*z*(F[dd-j]-1),j=1..dd-1)+s^dd/(1-z);
     print(convert(series(Z[dd],z,9),polynom));
  od;
```

$$Z_1 := \frac{2 z s - 2 z + 1}{z^2 - 2 z + 1} + \frac{s}{-z + 1}$$

$1 + s + 3 z s + (-1 + 5 s) z^2 + (7 s - 2) z^3 + (-3 + 9 s) z^4 + (11 s - 4) z^5 + (-5 + 13 s) z^6 + (15 s - 6) z^7 + (-7 + 17 s) z^8$

$$Z_2 := -\frac{4 s^2 z^2 - 4 s z^2 - z^3 + 3 z^2 - 2 z + 1}{(z-1)^3} + s z \left(\frac{2 z s - 2 z + 1}{z^2 - 2 z + 1} - 1 \right) + \frac{s^2}{-z + 1}$$

$$s^2+1+(s^2+1)z+(7s^2-4s+3)z^2+(13s^2-12s+6+s(-1+4s))z^3+(25s^2-24s+10+s(6s-2))z^4+(41s^2-40s+15+s(-3+8s))z^5+(61s^2-60s+21+s(10s-4))z^6+(85s^2-84s+28+s(-5+12s))z^7+(113s^2-112s+36+s(14s-6))z^8$$

$$Z_3 := \frac{1}{(z-1)^4}(2s^3z^5-4s^3z^4-s^2z^5+10s^3z^3+6s^2z^4-3sz^5-17s^2z^3-2sz^4+3z^5+11sz^3-3z^4-2z^3+3z^2-3z+1)+sz\left(-\frac{4s^2z^2-4sz^2-z^3+3z^2-2z+1}{(z-1)^3}-1\right)+s^2z\left(\frac{2zs-2z+1}{z^2-2z+1}-1\right)+\frac{s^3}{-z+1}$$

$$s^3+1+(s^3+1)z+(3s^3+s+1)z^2+(11s^3-17s^2+11s+s(4s^2-4s+3)+s^2(-1+4s))z^3+(37s^3-62s^2+42s-6+s(12s^2-12s+6)+s^2(6s-2))z^4+(87s^3-147s^2+99s-18+s(24s^2-24s+10)+s^2(-3+8s))z^5+(169s^3-284s^2+188s-37+s(40s^2-40s+15)+s^2(10s-4))z^6+(291s^3-485s^2+315s-64+s(60s^2-60s+21)+s^2(-5+12s))z^7+(461s^3-762s^2+486s-100+s(84s^2-84s+28)+s^2(14s-6))z^8$$

$$Z_4 := \frac{1}{(z-1)^5}(4s^4z^7-20s^4z^6-8s^3z^7+28s^4z^5+32s^3z^6+8s^2z^7-28s^4z^4-56s^3z^5-14s^2z^6-6sz^7+64s^3z^4+40s^2z^5+4sz^6+3z^7-58s^2z^4-14sz^5-6z^6+24sz^4+8z^5-7z^4+4z^3-6z^2+4z-1)+sz\left(\frac{1}{(z-1)^4}(2s^3z^5-4s^3z^4-s^2z^5+10s^3z^3+6s^2z^4-3sz^5-17s^2z^3-2sz^4+3z^5+11sz^3-3z^4-2z^3+3z^2-3z+1)-1\right)+s^2z\left(-\frac{4s^2z^2-4sz^2-z^3+3z^2-2z+1}{(z-1)^3}-1\right)+s^3z\left(\frac{2zs-2z+1}{z^2-2z+1}-1\right)+\frac{s^4}{-z+1}$$

$$s^4+1+(s^4+1)z+(3s^4+s^2+s+1)z^2+(1+s+s^2(4s^2-4s+3)+s^3(-1+4s)+s^4)z^3+(29s^4-64s^3+58s^2-24s+7+s(10s^3-17s^2+11s)+s^2(12s^2-12s+6)+s^3(6s-2))z^4+(113s^4-264s^3+250s^2-106s+23+s(36s^3-62s^2+42s-6)+s^2(24s^2-24s+10)+s^3(-3+8s))z^5+(301s^4-712s^3+684s^2-294s+57+s(86s^3-147s^2+99s-18)+s^2(40s^2-40s+15)+s^3(10s-4))z^6+(657s^4-1552s^3+1492s^2-644s+118+s(168s^3-284s^2+188s-37)+s^2(60s^2-60s+21)+s^3(-5+12s))z^7+(1261s^4-2960s^3+2830s^2-1220s+216+s(290s^3-485s^2+315s-64)+s^2(84s^2-84s+28)+s^3(14s-6))z^8$$

$$Z_5 := \frac{1}{(z-1)^6}(14s^5z^9-72s^5z^8-29s^4z^9+156s^5z^7+164s^4z^8+20s^3z^9-152s^5z^6-354s^4z^7-140s^3z^8-5s^2z^9+86s^5z^5+380s^4z^6+290s^3z^7+52s^2z^8-241s^4z^5$$

$$
\begin{aligned}
&-368\, s^3 z^6 - 93\, s^2 z^7 + z^9 + 278\, s^3 z^5 + 170\, s^2 z^6 - 9\, s\, z^7 - 9\, z^8 - 164\, s^2 z^5 - 30\, s\, z^6 + 20\, z^7 \\
&+ 49\, s\, z^5 - 10\, z^6 - 4\, z^5 + 5\, z^4 - 10\, z^3 + 10\, z^2 - 5\, z + 1\big) + s\, z \bigg(\frac{1}{(z-1)^5} \big(4\, s^4 z^7 \\
&- 20\, s^4 z^6 - 8\, s^3 z^7 + 28\, s^4 z^5 + 32\, s^3 z^6 + 8\, s^2 z^7 - 28\, s^4 z^4 - 56\, s^3 z^5 - 14\, s^2 z^6 - 6\, s\, z^7 \\
&+ 64\, s^3 z^4 + 40\, s^2 z^5 + 4\, s\, z^6 + 3\, z^7 - 58\, s^2 z^4 - 14\, s\, z^5 - 6\, z^6 + 24\, s\, z^4 + 8\, z^5 - 7\, z^4 + 4\, z^3 \\
&- 6\, z^2 + 4\, z - 1\big) - 1\bigg) + s^2 z \bigg(\frac{1}{(z-1)^4} \big(2\, s^3 z^5 - 4\, s^3 z^4 - s^2 z^5 + 10\, s^3 z^3 + 6\, s^2 z^4 \\
&- 3\, s\, z^5 - 17\, s^2 z^3 - 2\, s\, z^4 + 3\, z^5 + 11\, s\, z^3 - 3\, z^4 - 2\, z^3 + 3\, z^2 - 3\, z + 1\big) - 1\bigg) + s^3 z \bigg(\\
&- \frac{4\, s^2 z^2 - 4\, s\, z^2 - z^3 + 3\, z^2 - 2\, z + 1}{(z-1)^3} - 1\bigg) + s^4 z \left(\frac{2\, z\, s - 2\, z + 1}{z^2 - 2\, z + 1} - 1\right) + \frac{s^5}{-z+1}
\end{aligned}
$$

$$
\begin{aligned}
&s^5 + 1 + (s^5 + 1)\, z + (3\, s^5 + s^3 + s^2 + s + 1)\, z^2 + \big(1 + s + s^2 + s^3 (4\, s^2 - 4\, s + 3) + s^4 (-1 \\
&+ 4\, s) + s^5\big)\, z^3 + \big(1 + s + s^2 (10\, s^3 - 17\, s^2 + 11\, s) + s^3 (12\, s^2 - 12\, s + 6) + s^4 (6\, s \\
&- 2) + s^5\big)\, z^4 + \big(87\, s^5 - 241\, s^4 + 278\, s^3 - 164\, s^2 + 49\, s - 2 + s\, (28\, s^4 - 64\, s^3 + 58\, s^2 \\
&- 24\, s + 7) + s^2 (36\, s^3 - 62\, s^2 + 42\, s - 6) + s^3 (24\, s^2 - 24\, s + 10) + s^4 (-3 + 8\, s)\big)\, z^5 \\
&+ \big(365\, s^5 - 1066\, s^4 + 1300\, s^3 - 814\, s^2 + 264\, s - 27 + s\, (112\, s^4 - 264\, s^3 + 250\, s^2 \\
&- 106\, s + 23) + s^2 (86\, s^3 - 147\, s^2 + 99\, s - 18) + s^3 (40\, s^2 - 40\, s + 15) + s^4 (10\, s \\
&- 4)\big)\, z^6 + \big(1051\, s^5 - 3135\, s^4 + 3920\, s^3 - 2517\, s^2 + 840\, s - 102 + s\, (300\, s^4 - 712\, s^3 \\
&+ 684\, s^2 - 294\, s + 57) + s^2 (168\, s^3 - 284\, s^2 + 188\, s - 37) + s^3 (60\, s^2 - 60\, s + 21) \\
&+ s^4 (-5 + 12\, s)\big)\, z^7 + \big(2489\, s^5 - 7476\, s^4 + 9440\, s^3 - 6120\, s^2 + 2060\, s - 266 \\
&+ s\, (656\, s^4 - 1552\, s^3 + 1492\, s^2 - 644\, s + 118) + s^2 (290\, s^3 - 485\, s^2 + 315\, s - 64) \\
&+ s^3 (84\, s^2 - 84\, s + 28) + s^4 (14\, s - 6)\big)\, z^8
\end{aligned}
\tag{3}
$$

Now evaluating for different alphabet sizes for both S_2(k,d) and N_D(k) to obtain their respective generating functions.
We also compute the asymptotic for each value of s.

```
> s:=2:
```

First the generating functions for S_2(k,d) for each value of d.

```
> for dd from 1 to d do
    Fn[dd,s]:=equivalent(F[dd], z, k);
  print(factor(numer(F[dd])), factor(denom(F[dd])));
  od:
```

$$
\begin{gathered}
2\, z + 1,\ (z-1)^2 \\
z^3 - 11\, z^2 + 2\, z - 1,\ (z-1)^3 \\
9\, z^5 - 15\, z^4 + 32\, z^3 + 3\, z^2 - 3\, z + 1,\ (z-1)^4 \\
23\, z^7 - 118\, z^6 + 140\, z^5 - 127\, z^4 + 4\, z^3 - 6\, z^2 + 4\, z - 1,\ (z-1)^5 \\
125\, z^9 - 601\, z^8 + 1278\, z^7 - 1118\, z^6 + 558\, z^5 + 5\, z^4 - 10\, z^3 + 10\, z^2 - 5\, z + 1,\ (z-1)^6
\end{gathered}
\tag{1.1}
$$

Now the asymptotic of S_2(k,d).

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
  (convert(Fn[dd,s]*dd!, polynom), k, dd))*k^dd), dd=1..d);
```

$$(3)\ k$$
$$(3)^2\ k^2$$
$$(3)^3\ k^3$$
$$(3)^4\ k^4$$
$$(3)^5\ k^5 \tag{1.2}$$

For the same value of s, the generating function and asymptotic of N_D(k).

```
> for dd from 1 to d do
    Zn[dd,s]:=equivalent(Z[dd], z, k);
  print(factor(numer(Z[dd])), factor(denom(Z[dd])));
  od:
```

$$3\,z-3,\ (z-1)^3$$
$$-\left(2\,z^4-11\,z^3+23\,z^2-10\,z+5\right)\ (z-1)^2,\ (z-1)^5$$
$$\left(5\,z^5-7\,z^4+2\,z^3+45\,z^2-27\,z+9\right)\ (z-1)^2,\ (z-1)^6$$
$$\left(41\,z^7-176\,z^6+268\,z^5-281\,z^4+156\,z^3-140\,z^2+68\,z-17\right)\ (z-1)^2,\ (z-1)^7$$
$$\left(171\,z^9-847\,z^8+1640\,z^7-1268\,z^6+226\,z^5+637\,z^4-592\,z^3+408\,z^2-165\,z+33\right)\ (z-1)^2,\ (z-1)^8 \tag{1.3}$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
  (convert(Zn[dd,s]*dd!, polynom), k, dd))*k^dd), dd=1..d);
```

$$(3)\ k$$
$$(3)^2\ k^2$$
$$(3)^3\ k^3$$
$$(3)^4\ k^4$$
$$(3)^5\ k^5 \tag{1.4}$$

```
> s:=3:
> for dd from 1 to d do
    Fn[dd,s]:=equivalent(F[dd], z, k);
  print(factor(numer(F[dd])), factor(denom(F[dd])));
  od:
```

$$4\,z+1,\ (z-1)^2$$
$$z^3-27\,z^2+2\,z-1,\ (z-1)^3$$
$$39\,z^5-63\,z^4+148\,z^3+3\,z^2-3\,z+1,\ (z-1)^4$$
$$165\,z^7-876\,z^6+1082\,z^5-997\,z^4+4\,z^3-6\,z^2+4\,z-1,\ (z-1)^5$$
$$1549\,z^9-7533\,z^8+16220\,z^7-14662\,z^6+7550\,z^5+5\,z^4-10\,z^3+10\,z^2-5\,z+1,\ (z-1)^6 \tag{2.1}$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
```

```
(convert(Fn[dd,s]*dd!, polynom), k, dd))*k^dd), dd=1..d);
```

$$(5)\,k$$
$$(5)^2\,k^2$$
$$(5)^3\,k^3$$
$$(5)^4\,k^4$$
$$(5)^5\,k^5 \qquad (2.2)$$

```
> for dd from 1 to d do
    Zn[dd,s]:=equivalent(Z[dd], z, k);
  print(factor(numer(Z[dd])), factor(denom(Z[dd])));
  od:
```

$$(z+4)\,(z-1),\ (z-1)^3$$
$$-\left(3\,z^4-22\,z^3+54\,z^2-20\,z+10\right)\,(z-1)^2,\ (z-1)^5$$
$$\left(30\,z^5-63\,z^4+73\,z^3+141\,z^2-84\,z+28\right)\,(z-1)^2,\ (z-1)^6$$
$$\left(282\,z^7-1212\,z^6+1757\,z^5-1687\,z^4+655\,z^3-666\,z^2+328\,z-82\right)\,(z-1)^2,\ (z-1)^7$$
$$\left(2044\,z^9-10305\,z^8+20732\,z^7-17848\,z^6+6413\,z^5+4058\,z^4-3958\,z^3+2965\,z^2-1220\,z+244\right)\,(z-1)^2,\ (z-1)^8 \qquad (2.3)$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
  (convert(Zn[dd,s]*dd!, polynom), k, dd))*k^dd), dd=1..d);
```

$$(5)\,k$$
$$(5)^2\,k^2$$
$$(5)^3\,k^3$$
$$(5)^4\,k^4$$
$$(5)^5\,k^5 \qquad (2.4)$$

```
> s:=4:
> for dd from 1 to d do
    Fn[dd,s]:=equivalent(F[dd], z, k);
  print(factor(numer(F[dd])), factor(denom(F[dd])));
  od:
```

$$6\,z+1,\ (z-1)^2$$
$$z^3-51\,z^2+2\,z-1,\ (z-1)^3$$
$$103\,z^5-171\,z^4+410\,z^3+3\,z^2-3\,z+1,\ (z-1)^4$$
$$619\,z^7-3286\,z^6+4176\,z^5-3911\,z^4+4\,z^3-6\,z^2+4\,z-1,\ (z-1)^5$$
$$8113\,z^9-39881\,z^8+86176\,z^7-79330\,z^6+41728\,z^5+5\,z^4-10\,z^3+10\,z^2-5\,z+1,\ (z-1)^6 \qquad (3.1)$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
  (convert(Fn[dd,s]*dd!, polynom), k, dd))*k^dd), dd=1..d);
```

$$(7)\,k$$
$$(7)^2\,k^2$$
$$(7)^3\,k^3$$

$$(7)^4 k^4$$

$$(7)^5 k^5 \tag{3.2}$$

```
> for dd from 1 to d do
    Zn[dd,s]:=equivalent(Z[dd], z, k);
  print(factor(numer(Z[dd])), factor(denom(Z[dd])));
  od:
```

$$(2z+5)(z-1),\ (z-1)^3$$

$$-(4z^4-37z^3+99z^2-34z+17)(z-1)^2,\ (z-1)^5$$

$$(87z^5-203z^4+262z^3+327z^2-195z+65)(z-1)^2,\ (z-1)^6$$

$$(1031z^7-4450z^6+6456z^5-6043z^4+1908z^3-2074z^2+1028z-257)(z-1)^2,\ (z-1)^7$$

$$(10589z^9-53853z^8+109716z^7-97878z^6+39680z^5+16189z^4-15918z^3+12382z^2-5125z+1025)(z-1)^2,\ (z-1)^8 \tag{3.3}$$

```
> seq(print(map(z->ifactor(numer(z))/ifactor(denom(z)), coeff
  (convert(Zn[dd,s]*dd!, polynom), k, dd))*k^dd), dd=1..d);
```

$$(7)\,k$$

$$(7)^2 k^2$$

$$(7)^3 k^3$$

$$(7)^4 k^4$$

$$(7)^5 k^5 \tag{3.4}$$