Detailed sieve list study concerning if $X_0^*(N)$ is not bielliptic or need further work

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1 Introduction

Fix M a non-square free integer such that $X_0^*(M)$ has genus ≥ 2, we call such genus by g_M^*. We have a natural map $X_0(M) \to X_0(N)$, with N the biggest square-free integer such that $N|M$ which the property that all the prime $p|M$ its p-valuation in M is odd. By Lemma 3.1 [1] $X_0^*(M)$ is bielliptic then either $X_0^*(N)$ is bielliptic, is hyperelliptic (by Hasegawa this is equivalent that $X_0^*(N)$ is of genus 2), $X_0^*(N)$ is of genus 1 or has genus 0.

In order to study all possible levels M that may be bielliptic, we first consider that N is the biggest square-free integer such that $N|M$, (we have no natural map for $X_0^*(M) \to X_0^*(N)$ except only if all the primes $p|N$ satisfy that its p-valuation in M is odd)[1] with $X_0^*(N)$ bielliptic, hypereliptic or of genus 0 or 1. This is done in the first subsections. In this part we also assume that N is not a prime, because in such case $X_0^*(M) = X_0^+(M)$ is already studied by Jeon in the paper "Bielliptic modular curves $X_0^+(M)$" in Journal of Number Theory 185(2018) 319–338.

After, to recover all possible M with $X_0^*(M)$ bielliptic, (because by Lemma 3.1 [1] we only have a natural map for $X_0^*(Np^2) \to X_0^*(N)$ and not from $X_0^*(Np^2) \to X_0^*(Np)$), in the last subsection we study $\psi(M)$ for all M with gonality ≤ 4 (covering all the possible candidates to bielliptic curves $X_0^*(M)$), and by use of Prop.3.4 [1] and from the work of the previous subsections we list the remaining M where were not considerer previously and may need a further work to know if they are bielliptic or not.

As usual g_N, g_M denotes the genus of $X_0(N)$ and $X_0(M)$ respectively, and g_N^* the genus of $X_0^*(N)$.

If $X_0^*(M)$ may be bielliptic should map to an elliptic curve, and we mark in red the possibles over $\mathbb{Q}$ that can happen (which are the only possibilities if $g_M^* \geq 6$). Moreover, if is bielliptic we mark in blue the possibles that may happen over the rationals (which is the general case if the genus $g_M^* \geq 6$). If we know that is bielliptic we introduce the reference in our paper "Bielliptic of modular curves $X_0^*(M)$" which follow mainly from Hasegawa paper [2]..

The notation $n(M, E, p^k)$ (or sometimes by abuse of notation $n(E, p^k)$ denotes the difference between the number of $\mathbb{F}_{p^k}$-points of $X_0^*(M)$ minus two times the number of $\mathbb{F}_{p^k}$-points of the elliptic curve E, here we assume all defined over the rational field and $p \nmid N$. Of course if such $n(M, E, p^k) > 0$ we can not have a degree two map of $X_0^*(M)$ to E over the rational field.

If $g_M^* \geq 6$, $X_0^*(M)$ is only bielliptic to an elliptic curve E over $\mathbb{Q}$ which appears in the Jacobian decomposition of $X_0^*(M)$, by use of q-expansion given in Cremona tables and the Magma programme in html://mat.uab.cat/∼francesc/Magmaprogrammesxostar.pdf. We have,

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[1] We thank Andreas Schweizer for nice discussion on natural maps between modular curves $X_0^*(N)$ with different level, which helped to clarify such point

in particular, that the only possible elliptic curves E over $\mathbb{Q}$ that appear in $J_0^*(M)$ are the only possible situations if is bielliptic over $\mathbb{Q}$, therefore we need to study $n(M,E,p^k)$ (in some places we forget the level). Moreover if E has conductor exactly M, is a Weil parametrization, and the degree of such parametrization we denote by D_E which always should divide 2^{n+1} where n is the number of different prime divisors of M in order to be bielliptic curve (and we obtain D_E from Cremona tables).

In the last column, we introduce when the curve is not-bielliptic the justification why is not (making reference to our paper "Bielliptic modular curves $X_0^*(N)$" [1], where we only list the elliptic curves over the rationals that appears in the Jacobian of $X_0^*(M)$. Here we do not make explicit the Jacobian decomposition over $\mathbb{Q}$ for $X_0^*(M)$, but one easily can obtain it by use of Magma code available in html://mat.uab.cat/∼francesc/Magmaprogarmmesxostar.pdf.

The last subsection, because for general level M we have a natural map to $X_0^*(N)$ but N only have the primes p with odd valuation in M, we deal with the general levels $M = N\prod_i p_i^{2n_i}$; with p_i primes, such that $X_0^*(N)$ is bielliptic, hyperelliptic, or has genus 0 to recover all the conductors that could need a further study to be bielliptic or not. Because if $X_0^*(M)$ is bielliptic have a degree 4 to a projective line, then $M < 121337$. This is made following an argument of Ogg in [4], that in [3, Lemma 1,Proposition 2] is used to find an upper bound of the level for trigonal curves $X^*0(N)$. We compute for levels $M < 121337$ which satisfies the criterion Prop.3.4(i) [1],$\psi(N)$, and we present the list of the M which are not presented before which need further work.

1.1 $X_0^*(N)$ bielliptic, and not hyperelliptic:$g_N^* > 2$

Odd case

N	g_N	g_N^*	M	g_M	g_M^*	Is it non-bielliptic?
$183 = 3\cdot 61$	19	3	$9\cdot 61$	59	12	$Yes, Prop.3.4(iii)[1]$
						$n(594, 61a, 4) = 21-8, D_{549a,549b} \nmid 8$
			$3\cdot 61^2$	1199	290	$Yes, Prop.3.4(i)[1], \psi(N)$
$249 = 3\cdot 83$	27	3	$9\cdot 83$	81	15	$Yes, Lemma2.9[1], g_M > 72$
			$3\cdot 83^2$	2241	547	$Yes, Prop.3.4(i)[1], \psi(N)$
$303 = 3\cdot 101$	33	3	$9\cdot 101$	99	18	$Yes, Lemma2.9.[1], g_M > 72$
			$3\cdot 101^2$	3333	817	$Yes, Prop.3.4(i)[1]\psi(N)$
$455 = 5\cdot 7\cdot 13$	53	3	$25\cdot 91$	269	30	$Yes, Prop.3.4(i)[1]\psi(N)$
			$49\cdot 65$	377	43	$Yes, Prop.3.4(i)[1]\psi(N)$
			$169\cdot 35$	701	78	$Yes, Prop.3.4(i)[1]\psi(N)$

Even N,

N	g_N	g_N^*	M	g_M	g_M^*	Is it not-bielliptic?
$178 = 2 \cdot 89$	21	3	$4 \cdot 89$	43	8	$Yes, Prop.3.4(iii)[1]$
						$n(356, E89a\ o\ E178a, 9) = 32 - 30$
			$2 \cdot 89^2$	1913	462	$Yes, Prop.3.4(i)[1], \psi(N)$
$246 = 2 \cdot 3 \cdot 41$	39	3	$4 \cdot 123$	79	7	$Unknown, 123b n(492, 82a, 13) = 29 - 20$
						$n(492, 246g, 5) = 13 - 10, n(492, 246d, 7) = 20 - 12$
			$9 \cdot 82$	119	12	$Yes, Prop.3.4(iii)[1] D_{738a} \nmid 16$
						$n(738, 82a\ , 123b\ , 246c\ , 246d, 25)) = 92 - 54;$
						$n(738, 246f, 5) = 12 - 6$
			$6 \cdot 41^2$	1639	190	$Yes, Prop.3.4(i)[1], \psi(N)$
$290 = 2 \cdot 5 \cdot 29$	41	3	$4 \cdot 145$	85	8	$Yes, Prop.3.4(iii)[1]$
						$n(580, 145a, 290a, 9) = 33 - 32,$
						$n(580, 58b, 9) = 33 - 32, n(580, 58a, 9) = 33 - 14$
			$25 \cdot 58$	213	24	$Yes, Prop.3.4(i)[1], \psi(N)$
			$29^2 \cdot 10$	1245	147	$Yes, Prop.3.4(i)[1], \psi(N)$
$318 = 2 \cdot 3 \cdot 53$	51	3	$4 \cdot 159$	103	7	$Unknown : 53a; n(636, 106a, 5) = 8 - 6$
						$n(636, 106b, 25) = 70 - 40, n(636, 318a, 7) = 17 - 14$
						$n(636, 318c, 7) = 17 - 16, n(636, 106c, 13) = 20 - 18$
			$9 \cdot 106$	155	17	$Yes, Prop.3.4(i)[1], \psi(N)$
			$53^2 \cdot 6$	2755	325	$Yes, Prop.3.4(i)[1], \psi(N)$
$370 = 2 \cdot 5 \cdot 37$	53	4	$4 \cdot 185$	109	12	$Yes, Prop.3.4(iii)[1]$
						$n(740, 37a, 9) = 31 - 4, n(740, 185a, 3) = 13 - 6$
						$n(740, 185c, 3) = 13 - 12, n(740, 370a, 3) = 13 - 8$
			$25 \cdot 74$	273	30	$Yes, Prop.3.4(i)[1], \psi(N)$
			$37^2 \cdot 10$	2033	243	$Yes, Prop.3.4(i)[1], \psi(N)$
$430 = 2 \cdot 5 \cdot 43$	63	3	$4 \cdot 215$	127	9	$Yes, Prop.3.4(i)[1], \psi(N)$
			$25 \cdot 86$	319	34	$Yes, Prop.3.4(i)[1], \psi(N)$
			$43^2 \cdot 10$	2751	331	$Yes, Prop.3.4(i)[1], \psi(N)$
$510 = 2 \cdot 3 \cdot 5 \cdot 17$	101	3	$4 \cdot 255$	205	8	$Unknown 102a, n(1020, 34a, 49 = 110 - 96$
						$n(1020, 170c, 510c, 7) = 18 - 12$
			$9 \cdot 170$	309	17	$Yes, Prop.3.4(iii)[1]$
						$n(1530, 30a, 51a, 49) = 140 - 96$
						$n(1530, 102a, 102c, 153a, 7) = 24 - 20$
			$25 \cdot 102$	517	28	$Yes, Prop.3.4(i)[1], \psi(N)$
			$17^2 \cdot 30$	1765	100	$Yes, Prop.3.4(i)[1], \psi(N)$

1.2 $X_0^*(N)$ is hyperelliptic: $g_N^* = 2$

N	g_N	g_N^*	M	g_M	g_M^*	Is it not-bielliptic?
$85 = 5 \cdot 17$	7	2	$25 \cdot 17 = 425$	39	7	$Yes, Prop.3.4(iii)[1], n(425c, 425d, 4) = 18 - 16, n(425a, 2) = 6 - 4$
			$5 \cdot 17^2 = 1445$	135	28	$Yes, Prop.3.4(i)[1], \psi(N)$
$93 = 3 \cdot 31$	9	2	$9 \cdot 31 = 279$	29	5	$Yes, over, \mathbb{Q}, closure?, Hyperelliptic$
			$3 \cdot 31^2 = 2883$	299	70	$Yes, Prop.3.4(i)[1], \psi(N)$
			$27 \cdot 31 = 837$	91	21	$Yes, Prop.3.4(i)[1], \psi(N)$
$106 = 2 \cdot 53$	12	2	$4 \cdot 53 = 212$	25	5	$Yes, over\mathbb{Q}, closure? : n(212, 106a, 5) = 7 - 6, n(212, 106b, 3) = 12 - 10$
						$n(212, 106c, 3) = 12 - 4, n(212, 53a, 3) = 12 - 8$
			$8 \cdot 53 = 424$	51	12	$Yes, Prop.3.4(iii)[1], n(106c; 3) = 16 - 6, n(106b; 3) = 16 - 10$
						$n(106a; 3) = 16 - 12, n(212b, 3) = 16 - 4, n(53a; 9) = 32 - 14, n(53a; 3) = 16 - 14$
			$16 \cdot 53 = 848$	103	23	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 53^2 = 5618$	662	156	$Yes, Prop.3.4(i)[1], \psi(N)$
$115 = 5 \cdot 23$	11	2	$25 * 23 = 575$	55	10	$Yes, Prop.3.4(iii)[1], D_{575a,b} \nmid 8$
			$5 * 23^2 = 2645$	253	55	$Yes, Prop.3.4(i)[1], \psi(N)$
$122 = 2 \cdot 61$	14	2	$4 \cdot 61 = 244$	29	6	$Unknown, 61a, n(244, 122a, 7) = 7 - 6$
			$8 \cdot 61 = 488$	59	13	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 61^2 = 7442$	884	210	$Yes, Prop.3.4(i)[1], \psi(N)$
$3 \cdot 43 = 129$	13	2	$9 \cdot 43 = 387$	41	9	$Yes, Prop.3.4(iii)[1], D_{387b,c} \nmid 8, n(43a, 4) = 16 - 10$
						$n(387, 129b, 2) = 8 - 4, n(129a, 2) = 8 - 6$
			$27 \cdot 43 = 1161$	127	27	$Yes, Prop.3.4(i)[1], \psi(N)$
			$3 \cdot 43^2 = 5547$	587	140	$Yes, Prop.3.4(i)[1], \psi(N)$
$133 = 7 \cdot 19$	11	2	$7^2 \cdot 19 = 931$	85	18	$Yes, Prop.3.4(i)[1], \psi(N)$
			$7 \cdot 19^2$	233	54	$Yes, Prop.3.4(i)[1], \psi(N)$
$134 = 2 \cdot 67$	16	2	$4 \cdot 67 = 268$	32	7	$Yes, Prop.3.4.(iii)$
			$8 \cdot 67 = 536$	65	13	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 67^2$	1072	256	$Yes, Prop.3.4(i)[1], \psi(N)$
$146 = 2 \cdot 73$	17	2	$4 \cdot 73 = 292$	35	8	$Yes, Prop.3.4.(iii)$
			$8 \cdot 73 = 584$	71	14	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 73^2$	1277	306	$Yes, Prop.3.4(i)[1], \psi(N)$
$154 = 2 \cdot 7 \cdot 11$	21	2	$4 \cdot 7 \cdot 11 = 308$	43	4	$Yes, over\mathbb{Q}, closure?, n(154a, b, 3 =) = 10 - 8, n(77a, 9) = 16 - 14$
			$8 \cdot 7 \cdot 11 = 616$	89	10	$Yes, Prop.3.4(iii)[1], n(77a, 88a; 9) = 30 - 14$
						$n(154a, 154b, 616a; 3) = 12 - 8, n(44a; 3) = 12 - 6$
			$2 \cdot 7^2 \cdot 11 = 1078$	153	15	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 7 \cdot 11^2$	241	25	$Yes, Prop.3.4(i)[1], \psi(N)$
$2 \cdot 79 = 158$	19	2	$4 \cdot 79 = 316$	38	5	$Yes, over\mathbb{Q}, closure?n(79a, 158b, c; 9) = 32 - 30, n(158e; 3) = 6 - 4$
			$8 \cdot 79 = 632$	77	13	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 79^2$	1501	361	$Yes, Prop.3.4(i)[1], \psi(N)$
$161 = 7 \cdot 23$	15	2	$7^2 \cdot 23 = 1127$	105	21	$Yes, Prop.3.4(i)[1], \psi(N)$
			$7 \cdot 23^2$	345	81	$Yes, Prop.3.4(i)[1], \psi(N)$
$165 = 3 \cdot 5 \cdot 11$	21	2	$3^2 \cdot 5 \cdot 11 = 495$	65	5	$Unknown 99a$
			$3 * 5^2 * 11 = 825$	109	10	$Yes, Prop.3.4(iii)[1], D_{825a} \nmid 16, n(15a, 275a; 4) = 23 - 16, n(55a; 2) = 7 - 4$
			$3 * 5 * 11^2$	241	25	$Yes, Prop.3.4(i)[1], \psi(N)$
$166 = 2 \cdot 83$	20	2	$4 \cdot 83 = 332$	40	6	$Unknown 83a, 166a$
			$8 \cdot 83 = 664$	81	18	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 83^2$	1660	400	$Yes, Prop.3.4(i)[1], \psi(N)$
$170 = 2 \cdot 5 \cdot 17$	23	2	$4 \cdot 5 \cdot 17 = 340$	49	6	$Yes, Prop.3.4(iii)[1], n(34a, 2) = 13 - 12, n(170c; 2) = 13 - 6$
			$8 \cdot 5 \cdot 17 = 680$	101	11	$Yes, Prop.3.4(iii)[1], n(170c; 3) = 13 - 6$
						$n(20a, 34a; 9) = 31 - 24$
			$2 \cdot 25 \cdot 17 = 850$	23	12	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 5 \cdot 17^2$	423	48	$Yes, Prop.3.4(i)[1], \psi(N)$
$177 = 3 \cdot 59$	19	2	$9 \cdot 59 = 531$	57	9	$Yes, Prop.3.4.(iii)[1]$
			$3 \cdot 59^2$	1121	271	$Yes, Prop.3.4(i)[1], \psi(N)$
$186 = 2 \cdot 3 \cdot 31$	29	2	$4 \cdot 3 \cdot 31 = 372$	59	7	$Yes, n(372, 62a, 11) = 25 - 24$
			$8 \cdot 3 \cdot 31 = 744$	121	12	$Yes, Prop.3.4(iii)[1], n(62a; 25) = 77 - 64$
						$n(124b, 744a, 248a; 25) = 77 - 70$
			$2 \cdot 9 \cdot 31 = 558$	89	7	$Unknown 558a, n(186c, 5) = 10 - 6$
			$2 \cdot 3 \cdot 31^1$	929	105	$Yes, Prop.3.4(i)[1], \psi(N)$

N	g_N	g_N^*	M	g_M	g_M^*	Is it not-bielliptic
$205 = 5 \cdot 41$	19	2	$25 \cdot 41 = 1025$	99	19	$Yes, Prop.3.4(i)[1], \psi(N)$
			$5 \cdot 41^2$	819	190	$Yes, Prop.3.4(i)[1], \psi(N)$
$206 = 2 \cdot 103$	25	2	$4 \cdot 103 = 412$	50	8	$Yes, Prop.3.4(iii)[1]$
			$8 \cdot 103 = 824$	101	16	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 103^2$	2575	625	$Yes, Prop.3.4(i)[1], \psi(N)$
$209 = 11 \cdot 19$	19	2	$11^2 \cdot 19$	209	47	$Yes, Prop.3.4(i)[1], \psi(N)$
			$11 \cdot 19^2$	361	81	$Yes, Prop.3.4(i)[1], \psi(N)$
$213 = 3 \cdot 71$	23	2	$3^3 \cdot 71 = 639$	69	11	$Yes, Prop.3.4(i)[1], \psi(N)$
			$3 \cdot 71^2$	1633	397	$Yes, Prop.3.4(i)[1], \psi(N)$
$215 = 5 \cdot 43$	21	2	$25 \cdot 43 = 1075$	105	24	$Yes, Prop.3.4(i)[1], \psi(N)$
			$5 \cdot 43^2$	903	210	$Yes, Prop.3.4(i)[1], \psi(N)$
$221 = 13 \cdot 17$	19	2	$13^2 \cdot 17$	259	57	$Yes, Prop.3.4(i)[1], \psi(N)$
			$13 \cdot 17^2$	339	79	$Yes, Prop.3.4(i)[1], \psi(N)$
$230 = 2 \cdot 5 \cdot 23$	33	2	$4 \cdot 5 \cdot 23 = 460$	67	7	$Yes, Prop.3.4(iii)[1]$
			$5 \cdot 8 \cdot 23 = 920$	137	14	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 25 \cdot 23$	169	16	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 5 \cdot 23^2$	781	91	$Yes, Prop.3.4(i)[1], \psi(N)$
$255 = 3 \cdot 5 \cdot 17$	33	2	$9 \cdot 5 \cdot 17 = 765$	101	12	$Yes, Prop.3.4(iii)[1], n(51a; 2) = 7 - 6, n(153a; 4) = 17 - 10$
			$3 \cdot 25 \cdot 17$	169	16	$Yes, Prop.3.4(i)[1], \psi(N)$
			$3 \cdot 5 \cdot 17^2$	577	64	$Yes, Prop.3.4(i)[1], \psi(N)$
$266 = 2 \cdot 7 \cdot 19$	37	2	$4 \cdot 7 \cdot 19 = 532$	75	8	$Yes, Prop.3.4(iii)[1], n(38b; 3) = 12 - 10$
			$2 \cdot 7^2 \cdot 19$	265	27	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 7 \cdot 19^2$	721	81	$Yes, Prop.3.4(i)[1], \psi(N)$
$285 = 3 \cdot 5 \cdot 19$	37	2	$9 \cdot 5 \cdot 19 = 855$	113	9	$Yes, Prop.3.4(iii)[1], n(57b, 285b; 4) = 18 - 16, n(57a, 57c; 4) = 18 - 10$
			$3 \cdot 5^2 \cdot 19$	189	22	$Yes, Prop.3.4(i)[1], \psi(N)$
			$3 \cdot 5 \cdot 19^2$	721	81	$Yes, Prop.3.4(i)[1], \psi(N)$
$286 = 2 \cdot 11 \cdot 13$	39	2	$4 \cdot 11 \cdot 13 = 572$	79	5	$Yes, over, \mathbb{Q} closure?, n(143a, 286c; 25) = 71 - 70 closure$
						$n(286e; 3) = 8 - 4, n(26b; 9) = 24 - 14$
			$2 \cdot 11^1 \cdot 13$	439	49	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 11 \cdot 13^2$	519	57	$Yes, Prop.3.4(i)[1], \psi(N)$
$287 = 7 \cdot 41$	27	2	$49 \cdot 41$	189	39	$Yes, Prop.3.4(i)[1], \psi(N)$
			$7 \cdot 41^2$	1107	267	$Yes, Prop.3.4(i)[1], \psi(N)$
$299 = 13 \cdot 23$	27	2	$13^2 \cdot 23$	351	78	$Yes, Prop.3.4(i)[1], \psi(N)$
			$13 \cdot 23^2$	621	147	$Yes, Prop.3.4(i)[1], \psi(N)$
$330 = 2 \cdot 3 \cdot 5 \cdot 11$	65	2	$4 \cdot 3 \cdot 5 \cdot 11 = 660$	133	7	$Unknown 110b n(66b; 17) = 33 - 32, n(330d; 7) = 17 - 8$
			$8 \cdot 3 \cdot 5 \cdot 11 = 1320$	273	14	$Yes, Prop.3.4(iii)[1], n(110b; 49) = 137 - 126, n(66b, 330d; 49) = 137 - 96$
						$n(20a, 44a, 88a, 132b, 440a, 660a, 1320a; 49) = 137 - 120$
			$2 \cdot 9 \cdot 5 \cdot 11 = 990$	201	8	$Unknown 66a, 99a, n(30a; 49) = 106 - 96$
						$D_{990a} = 64 \nmid 2^5$
			$2 \cdot 3 \cdot 5^5 \cdot 11$	685	37	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 3 \cdot 5 \cdot 11^2$	1513	85	$Yes, Prop.3.4(i)[1], \psi(N)$
$357 = 3 \cdot 7 \cdot 17$	45	2	$9 \cdot 7 \cdot 17 = 1071$	137	11	$Yes, Prop.3.4(i)[1], \psi(N)$
			$3 \cdot 7^2 \cdot 17$	321	33	$Yes, Prop.3.4(i)[1], \psi(N)$
			$3 \cdot 7 \cdot 17^2$	781	91	$Yes, Prop.3.4(i)[1], \psi(N)$
$390 = 2 \cdot 3 \cdot 5 \cdot 13$	77	2	$4 \cdot 3 \cdot 5 \cdot 13 = 780$	157	6	$Unknown 130c, 65a, n(26b; 7) = 17 - 14$
						$n(390a; 7) = 17 - 16, n(390e; 7) = 17 - 12$
			$8 \cdot 3 \cdot 5 \cdot 13 = 1560$	321	16	$Yes, Prop.3.4(iii)[1], n(20a, 52a, 260a, 390e; 49) = 139 - 120, n(26b; 7) = 19 - 14$
						$n(65a, 130c, 312b; 49) = 139 - 96, n(390a, 520a; 7) = 19 - 16$
			$2 \cdot 9 \cdot 5 \cdot 13 = 1170$	237	12	$Yes, Prop.3.4(iii)[1], n(30a, 65a; 49) = 108 - 96, n(195c; 7) = 22 - 18$
						$n(234c; 7) = 22 - 20, n(390a; 7) = 22 - 16$
						$n(390g; 7) = 22 - 8, n(585a; 7) = 22 - 12$
			$2 \cdot 3 \cdot 25 \cdot 13$	397	19	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 3 \cdot 5 \cdot 13^2$	1037	57	$Yes, Prop.3.4(i)[1], \psi(N)$

1.3 $X_0^*(N)$ is an elliptic curve: $g_N^* = 1$

N	g_N	g_N^*	M	g_M	g_M^*	Is it a not-bielliptic curve?
$57 = 3 \cdot 19$	5	1	$9 \cdot 19 = 171$	17	3	$Unknown 57a, 57c, n(57b; 2) = 6 - 4$ $Hyperelliptic$
			$513 = 3^3 \cdot 19$	55	12	$Yes, Prop.3.4(iii)[1], n(57a, 57c, 171c, 171d; 4) = 20 - 10$
						$n(57b, 171a, 513a, 513b; 4) = 20 - 16$
			$3 \cdot 19^2$	107	14	$Yes, Prop.3.4(i)[1], \psi(N)$
$58 = 2 \cdot 29$	6	1	$4 \cdot 29 = 116$	13	2	$No, Cor.3.9[1] 58a, 58b$
			$8 \cdot 29 = 232$	27	7	$Yes, Prop.3.4(iii)[1], n(58a, 116a; 9) = 18 - 14, n(116c; 3) = 12 - 4$
						$n(58b, 232a; 3) = 12 - 10, n(116b; 3) = 12 - 6$
			$464 = 16 \cdot 29$	55	11	$Yes : X_0(232) NO - bielliptic, Lemma 3.1[1]$
			$2 \cdot 29^2$	188	42	$Yes, Prop.3.4(i)[1], \psi(N)$
$65 = 5 \cdot 13$	5	1	$25 \cdot 13 = 325$	29	6	$Yes, Prop.3.4(iii)[1], n(65a; 9) = 29 - 24, D = 12 \nmid 2^3$
			$5 \cdot 13^2$	77	15	$Yes, g_M, Lemma 2.9(i)[1]$
$74 = 2 \cdot 37$	8	1	$4 \cdot 37 = 148$	17	4	$No, Cor.3.9[1] over, \mathbb{Q} : 37a$
			$8 \cdot 37 = 296$	35	7	$Yes, Prop.3.4(iii)[1], n(37a; 9) = 21 - 14, n(296a; 3) = 13 - 10$
			$2 \cdot 37^2$	314	72	$Yes, Prop.3.4(i)[1], \psi(N)$
$77 = 7 \cdot 11$	7	1	$49 \cdot 11 = 539$	49	9	$Yes, Prop.3.4(ii)[1], Q_2(27) = 147$
			$7 \cdot 11^2$	77	17	$Yes, Prop.3.4(i)[1], \psi(N)$
$82 = 2 \cdot 41$	9	1	$4 \cdot 41 = 164$	19	3	$No, Cor.3.9[1], over, \mathbb{Q} : 82a$
			$8 \cdot 41 = 328$	39	9	$Yes, Prop.3.4(iii)[1], n(82a; 9) = 28 - 24, 16 = D \nmid 2^3$
			$2 \cdot 41^2$	389	90	$Yes, Prop.3.4(i)[1], \psi(N)$
$86 = 2 \cdot 43$	10	1	$4 \cdot 43 = 172$	20	4	$No, Cor.3.9[1], over, \mathbb{Q} : 43a$
			$8 \cdot 43 = 344$	41	8	$Yes, Prop.3.4(iii)[1], n(43a; 9) = 27 - 24$
			$2 \cdot 43^2$	430	100	$Yes, Prop.3.4(i)[1], \psi(N)$
$91 = 7 \cdot 13$	7	1	$7^2 \cdot 13 = 637$	57	12	$Yes, Prop.3.4(iii)[1], n(91a; 4) = 21 - 10, n(637; 2) = 7 - 4$
			$7 \cdot 13^2$	107	24	$Yes, Prop.3.4(i)[1], \psi(N)$
$102 = 2 \cdot 3 \cdot 17$	15	1	$4 \cdot 3 \cdot 17 = 204$	31	2	$No, Cor.3.9 34a, 102a$
			$8 \cdot 3 \cdot 17 = 408$	65	8	$Unknown 102a, n(204a; 5) 16 - 14, n(34a; 5) 16 - 12$
			$2 \cdot 9 \cdot 17 = 306$	47	4	$No, Cor.3.11[1] over, \mathbb{Q} : 102a, 102c, n(51a; 5) = 8 - 6$
			$4 \cdot 9 \cdot 17 = 612$	97	10	$Yes, Prop.3.4(iii)[1], n(34a, 102c; 25) = 78 - 72, n(51a; 25) = 78 - 54$
						$n(153a; 25) = 78 - 70, n(102a; 25) = 78 - 40$
			$2 \cdot \cdot 17^2$	271	28	$Yes, Prop.3.4(i)[1], \psi(N)$
$111 = 3 \cdot 37$	11	1	$9 \cdot 37 = 333$	35	8	$Yes, Prop.3.4(iii)[1], n(37a; 4) = 15 - 10$
						$n(333b; 2) = 7 - 4, n(333c; 5) = 12 - 8$
			$3 \cdot 37^2$	431	102	$Yes, Prop.3.4(i)[1], \psi(N)$
$2 \cdot 3 \cdot 19 = 114$	17	1	$4 \cdot 3 \cdot 19 = 228$	34	4	$No, Cor.3.9[1], over, \mathbb{Q} 57a n(114c; 5) = 14 - 8, n(38b; 7) = 15 - 10$
			$8 \cdot 3 \cdot 19 = 456$	73	7	$Unknown, 57a, 76a, 152a n(114c, 228a; 5) = 14 - 8$
						$n(38b; 25) = 52 - 40$
			$2 \cdot 9 \cdot 19 = 342$	53	4	$No, Cor.3.11[1], 57a, b, 342e, n(57c; 5) = 12 - 10$
			$4 \cdot 9 \cdot 19 = 684$	109	11	$Yes, Prop.3.4(iii)[1], n(114c, 342d; 5) 20 - 8, n(38b; 25) = 64 - 40$
						$n(57c; 5) = 20 - 10, n(57b, 342e; 5) = 20 - 16, n(57a; 25) = 64 - 54$
			$2 \cdot 3 \cdot 19^2$	341	36	$Yes, Prop.3.4(i)[1], \psi(N)$
$118 = 2 \cdot 59$	14	1	$4 \cdot 59 = 236$	28	3	$No, Cor.3.9[1], 118a, b, c$
			$8 \cdot 59 = 472$	57	13	$Yes, Prop.3.4(iii)[1], n(118a, 118b, 236b; 9) = 42 - 30$
						$n(118c; 3) = 8 - 4, n(472a; 9) = 42 - 14$
			$2 \cdot 59^2$	826	196	$Yes, Prop.3.4(i)[1], \psi(N)$
$123 = 3 \cdot 41$	13	1	$9 \cdot 41 = 369$	39	6	$Yes, Prop.3.4(iii)[1], n(123b; 49) = 98 - 96$
			$3 \cdot 41^2$	533	127	$Yes, Prop.3.4(i)[1], \psi(N)$
$130 = 2 \cdot 5 \cdot 13$	17	1	$4 \cdot 5 \cdot 13 = 260$	37	4	$No, Cor.3.9[1], 65a n(26b; 7) = 19 - 14$
						$n(130c; 3) = 9 - 4$
			$8 \cdot 5 \cdot 13 = 520$	77	8	$Yes, Prop.3.4(iii)[1], n(20a, 65a, 130c, 260a; 9) = 29 - 24$
						$n(52a, 520a; 3) = 9 - 8, n(26b; 9) = 29 - 14$
			$2 \cdot 25 \cdot 13 = 650$	93	9	$Yes, Prop.3.4(iii)[1], n(65a, 650b; 9) = 34 - 24, n(325b; 9) 34 - 30$
						$n(650a; 3) = 10 - 8, n(650c; 9) = 34 - 14$
			$2 \cdot 5 \cdot 13^2$	245	27	$Yes, Prop.3.4(i)[1], \psi(N)$
$138 = 2 \cdot 3 \cdot 23$	21	1	$4 \cdot 3 \cdot 23 = 276$	43	2	$No, Cor.3.9[1], 138a, c$
			$8 \cdot 3 \cdot 23 = 552$	89	8	$Yes, n(92a; 11) = 28 - 24$
						$n(138a, 138c, 184b, 552a; 25) = 66 - 64$
			$2 \cdot 9 \cdot 23 = 414$	65	5	$Unknown, over \mathbb{Q} : 69a, 138a$
						$n(138b; 7) = 14 - 12$
			$2 \cdot 3 \cdot 23^2$	1561	187	$Yes, Prop.3.4(i)[1], \psi(N)$
$141 = 3 \cdot 47$	15	1	$9 \cdot 47 = 423$	45	7	$Yes, Prop.3.4(iii)[1], n(141e, 423g; 4) = 14 - 10$
						$n(141c; 5) = 10 - 8, n(141d; 25) = 72 - 70$
			$3 \cdot 47^2$	705	169	$Yes, Prop.3.4(i)[1], \psi(N)$
$142 = 2 \cdot 71$	17	1	$4 \cdot 71 = 284$	34	2	$No, Cor.3.9[1], 142b, d$
			$8 \cdot 71 = 568$	69	10	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 71^2$	1207	289	$Yes, Prop.3.4(i)[1], \psi(N)$

N	g_N	g_N^*	M	g_M	g_M^*	Is it not-bielliptic?
$143 = 11 \cdot 13$	13	1	$11^2 \cdot 13$	143	32	$Yes, Prop.3.4(i)[1], \psi(N)$
			$11 \cdot 13^2$	169	36	$Yes, Prop.3.4(i)[1], \psi(N)$
$145 = 5 \cdot 29$	13	1	$25 \cdot 29 = 725$	69	13	$Yes, Prop.3.4(i)[1], \psi(N)$
			$5 \cdot 29^2$	405	91	$Yes, Prop.3.4(i)[1], \psi(N)$
$155 = 5 \cdot 31$	15	1	$25 \cdot 31$	75	15	$Yes, Prop.3.4(i)[1], \psi(N)$
			$5 \cdot 31^2$	465	105	$Yes, Prop.3.4(i)[1], \psi(N)$
$159 = 3 \cdot 53$	17	1	$9 \cdot 53 = 477$	51	10	$Yes, Prop.3.4(iii)[1], n(53a; 4) = 21 - 16$
			$3 \cdot 53^2$	901	217	$Yes, Prop.3.4(i)[1], \psi(N)$
$174 = 2 \cdot 3 \cdot 29$	27	1	$4 \cdot 3 \cdot 29 = 348$	55	3	$Cor.3.9[1], 58a, b; 174c$
			$8 \cdot 3 \cdot 29 = 696$	113	10	$Yes, Prop.3.4(iii)[1], D_{696b} = 48 \nmid 16, n(58b, 174c; 5) = 11 - 10$
						$n(58a, 116a, 116b, 232a; 25) = 71 - 54, n(116c, 348c; 25) = 71 - 64$
			$2 \cdot 9 \cdot 29 = 522$	83	9	$Yes, Prop.3.4(iii)[1], n(174a; 5) = 12 - 8$
						$n(58a, 174a, 522a; 25) = 56 - 54$
			$2 \cdot 3 \cdot 29^2$	811	91	$Yes, Prop.3.4(i)[1], \psi(N)$
$182 = 2 \cdot 7 \cdot 13$	25	1	$4 \cdot 7 \cdot 13 = 364$	51	5	$No, Cor.3.9[1], 91a, n(182a; 5) = 15 - 8$
						$n(26b, 182d; 9) = 22 - 14$
			$8 \cdot 7 \cdot 13 = 728$	105	10	$Yes, n(26b, 182d; 9) = 32 - 14, n(91a; 25) = 63 - 54$
						$n(52a, 182a; 5) = 15 - 8$
			$2 \cdot 7^2 \cdot 13$	181	19	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 7 \cdot 13^2$	337	36	$Yes, Prop.3.4(i)[1], \psi(N)$
$190 = 2 \cdot 5 \cdot 19$	27	1	$4 \cdot 5 \cdot 19 = 380$	55	2	$No, Cor.3.9[1], 38b, 190b$
			$8 \cdot 5 \cdot 19 = 760$	113	10	$Yes, Prop.3.4(iii)[1], n(20a, 76, 152a, 380b; 9) = 38 - 24$
						$n(38b, 190b; 9) = 38 - 30$
			$2 \cdot 25 \cdot 19 = 950$	139	13	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 5 \cdot 19^2$	531	61	$Yes, Prop.3.4(i)[1], \psi(N)$
$195 = 3 \cdot 5 \cdot 13$	25	1	$9 \cdot 5 \cdot 13 = 585$	77	8	$Yes, n(39a; 2) = 7 - 4, n(195c; 2) = 7 - 2$
						$n(585a; 7) = 20 - 12$
			$3 \cdot 25 \cdot 13 = 975$	129	13	$Yes, Prop.3.4(i)[1], \psi(N)$
			$3 \cdot 5 \cdot 13^2$	129	13	$Yes, Prop.3.4(i)[1], \psi(N)$
$210 = 2 \cdot 3 \cdot 5 \cdot 7$	41	1	$4 \cdot 3 \cdot 5 \cdot 7 = 420$	85	3	$No, Cor.3.9[1], 42a, 70a, 210d$
			$8 \cdot 3 \cdot 5 \cdot 7 = 840$	177	10	$Unknown, 20a, 140b, 210dn(42a, ; 13) = 22 - 16$
						$n(280a; 17) = 34 - 30, n(84b, 420a; 11) = 22 - 20, n(70a, 840a; 11) = 22 - 16$
			$1680 = 2^4 \cdot 3 \cdot 5 \cdot 7$	361	19	$Yes, Lemma2.9(i)[1], g_M$
			$2 \cdot 9 \cdot 5 \cdot 7 = 630$	129	5	$Unknown21a, 210dn(30a; 17) = 28 - 24$
			$1890 = 2 \cdot 3^3 \cdot 5 \cdot 7$	409	22	$Yes, Lemma2.9.(i)[1], g_M$
			$1260 = 2^2 \cdot 3^2 \cdot 5 \cdot 7$	265	13	$Unknown, 21a; 70a; 90b; 210dn(42a, 630g; 13) = 30 - 16, n(30a; 13) = 30 - 24$
			$2 \cdot 3 \cdot 5^2 \cdot 7 = 1050$	217	10	$Unknown175bn(15a, 210d; 121) = 265 - 256$
						$n(350c; 11) = 25 - 18, n(525a, 1050a; 11) = 25 - 24$
			$2 \cdot 3 \cdot 5 \cdot 7^2 = 1470$	305	15	$Yes, Lemma2.9.(i)[1], g_M$
$222 = 2 \cdot 3 \cdot 37$	35	1	$4 \cdot 3 \cdot 37 = 444$	71	5	$No, Cor.3.9[1], \mathbb{Q}37a, rep; 222b$
			$2 \cdot 9 \cdot 37 = 666$	107	11	$Yes, Prop.3.4(iii)[1], n(222d; 5) = 12 - 4$
						$n(37a, 333b, 333c; 25) = 76 - 64$
			$2 \cdot 3 \cdot 37^2$	1331	153	$Yes, Prop.3.4(i)[1], \psi(N)$
$231 = 3 \cdot 7 \cdot 11$	29	1	$9 \cdot 7 \cdot 11 = 693$	89	9	$Yes, Prop.3.4(iii)[1], n(21a, 99a; 4) = 20 - 16, n(77a; 4) = 20 - 18$
			$3 \cdot 7^2 \cdot 11$	209	21	$Yes, Prop.3.4(i)[1], \psi(N)$
			$3 \cdot 7 \cdot 11^2$	329	37	$Yes, Prop.3.4(i)[1], \psi(N)$
$238 = 2 \cdot 7 \cdot 17$	33	1	$4 \cdot 7 \cdot 17 = 476$	67	3	$No, Cor.3.9[1], a, \mathbb{Q} : 34a, 238b, 238d$
			$2 \cdot 49 \cdot 17$	237	25	$Yes, Prop.3.4(i)[1], \psi(N)$
			$2 \cdot 7 \cdot 17^2$	577	64	$Yes, Prop.3.4(i)[1], \psi(N)$

1.4 $X_0^*(N)$ is projective line: $g_N^* = 0$

N	g_N	g_N^*	M	g_M	g_M^*	Is it not-bielliptic?
6	0	0	$144 = 2^4 \cdot 3^2$	13	3	$No, Cor.3.11[1], a, \mathbb{Q}: 24a, 36a, 48a.$
			$162 = 2 \cdot 3^4$	16	3	$Unknown, a, \mathbb{Q}: 27a, 54a, D_{162a} = 12 \nmid 8$
			$192 = 2^6 \cdot 3$	21	5	$Unknown, 24a, 32a, 192an(96b; 5) = 12 - 8$
			$216 = 2^3 \cdot 3^3$	25	5	$Unknown54b, *36an(72a; 5) = 11 - 8, n(216a; 25) = 47 - 40, n(108b, 7) = 11 - 6$
			$288 = 2^5 \cdot 3^2$	33	7	$Unknown36an(24a, 48a, 96a; 25) = 68 - 64$
						$n(288a; 25) = 68 - 40, n(144a; 7) = 16 - 8$
			$384 = 2^7 \cdot 3$	49	9	$Yes, Prop.3.4(iii)[1], D_{384d} = 16 \nmid 8$
						$n(24a, 32a, 48a, 64a, 96b, 128a, 192a, 192d; 25) = 84 - 64$
			$324 = 2^2 \cdot 3^4$	37	7	$Unknown, 27an(162b; 7) = 18 - 12$
						$n(54a, 54b, 162a, 162d; 25) = 60 - 54$
			$432 = 2^4 \cdot 3^3$	55	13	$Yes, Prop.3.4(i)[1], \psi(N)$
			$486 = 2 \cdot 3^5$	64	12	$Yes, Prop.3.4(i)[1], \psi(N)$
			$576 = 2^6 \cdot 3^2$	73	17	$Yes, Prop.3.4(i)[1], \psi(N)$
			$648 = 2^3 \cdot 3^4$	85	20	$Yes, Prop.3.4(i)[1], \psi(N)$
10	0	0	$160 = 2^5 \cdot 5$	17	4	$Unknown20a, 160an(80b, 3) = 8 - 4$
			$200 = 2^3 \cdot 5^2$	19	4	$Unknown, 40a, 50bn(100a, 20a; 9) = 25 - 24$
			$250 = 2 \cdot 5^3$	28	5	$Yes, over, \mathbb{Q}, n(50a; 3) = 9 - 6clousure?$
			$320 = 2^6 \cdot 5$	37	7	$Unknown, 32a, n(320b; 49) = 112 - 96$
						$n(20a, 80b, 160a, 160b; 9) = 32 - 24$
			$400 = 2^4 \cdot 5^2$	43	10	$Yes, Prop.3.4(iii)[1], n(40a, 80a, 200c, 400a; 9) = 42 - 32$
						$n(50b; 9) = 42 - 30, n(20a, 100a; 9) = 42 - 24, n(200e, 400h; 9) = 42 - 14$
			$500 = 2^2 \cdot 5^3$	61	12	$Yes, Prop.3.4(i)[1], \psi(N)$
			$640 = 2^7 \cdot 5$	81	19	$Yes, Prop.3.4(i)[1], \psi(N)$
14	1	0	$112 = 2^4 \cdot 7$	11	2	$Unknown, 56a, 112a$
			$196 = 2^2 \cdot 7^2$	17	3	$No, Cor.3.9[1], \mathbb{Q}: 14a$
			$224 = 2^5 \cdot 7$	25	4	$Unknown, over, \mathbb{Q}: 56a, 224an(112c; 2) = 6 - 4, n(112a; 25) = 46 - 40$
			$392 = 2^3 \cdot 7^2$	41	9	$Yes, Prop.3.4(iii)[1], n(14a(rep), 56b; 9) = 30 - 24, n(196b; 5) = 8 - 6, D_{392c} = 24 \nmid 8$
			$448 = 2^6 \cdot 7$	53	12	$Yes, Prop.3.4(i)[1], \psi(N)$
			$686 = 2 \cdot 7^3$	85	15	$Yes, Prop.3.4(i)[1], \psi(N)$
15	1	0	$135 = 3^3 \cdot 5$	13	2	$Unknown, 45aD_{135a} = 12 \nmid 8$
			$225 = 3^2 \cdot 5^2$	19	4	$Unknown, over, \mathbb{Q}: 15a, 225a$
			$375 = 3 \cdot 5^3$	41	6	$Yes, Prop.3.4(iii)[1], n(15a; 4) = 18 - 16, n(75a; 4) = 18 - 10$
			$405 = 3^4 \cdot 5$	43	8	$Yes, Prop.3.4(iii)[1], n(27a, 405b; 4) = 20 - 18$
						$n(45a, 405c; 4) = 20 - 16, n(135a, 405f; 4) = 20 - 10$
			$675 = 3^3 \cdot 5^2$	73	16	$Yes, Prop.3.4(i)[1], \psi(N)$
$21 = 3 \cdot 7$	1	0	$147 = 3 \cdot 7^2$	11	2	$Yes, over, \mathbb{Q}: closure?$
			$189 = 3^3 \cdot 7$	19	3	$Unknown, 21a, n(189a; 4) = 11 - 10, n(63a; 2) = 5 - 4$
			$441 = 3^2 \cdot 7^2$	41	9	$Yes, Prop.3.4(iii)[1], D_{441b} = 24 \nmid 2^3, n(21a; 4) = 18 - 16, n(147b; 2) = 6 - 2$
			$567 = 3^4 \cdot 7$	61	13	$Yes, Prop.3.4(i)[1], \psi(N)$
$22 = 2 \cdot 11$	2	0	$88 = 2^3 \cdot 11$	9	2	$Unknown, 44a, 88a$
			$176 = 2^4 \cdot 11$	19	4	$Unknown, a\mathbb{Q}: 44a, Is - Hyperelliptic, n(88a; 9) = 15 - 14$
			$242 = 2 \cdot 11^2$	22	4	$Unknown, over, \mathbb{Q}: 11a, 121b$
			$352 = 2^5 \cdot 11$	41	9	$Yes, Prop.3.4(iii)[1], n(88a; 9) = 32 - 14$
						$n(44a, 176b, 352c, 352d; 9) = 32 - 30$
			$484 = 2^2 \cdot 11^2$	29	10	$Yes, Prop.3.4(i)[1], \psi(N)$
			$704 = 2^6 \cdot 11$	85	19	$Yes, Prop.3.4(i)[1], \psi(N)$
$26 = 2 \cdot 13$	2	0	$104 = 2^3 \cdot 13$	11	2	$Unknown, 26b, 52a$
			$208 = 2^4 \cdot 13$	23	5	$Yes, over, \mathbb{Q}, Prop.3.4(iii)[1]: n(52a; 3) = 10 - 8, n(104a; 3) = 10 - 6$
						$n(26b; 9) = 20 - 14, D_{208b} = 16 \nmid 8clausura$
			$338 = 2 \cdot 13^2$	32	6	$Yes, Prop.3.4(iii)[1], n(36a; 3) = 8 - 6, n(338f; 9) = 30 - 14, D_{338a} = 12 \nmid 8$
			$416 = 2^5 \cdot 13$	49	10	$Yes, Prop.3.4(iii)[1], n(104a, 208b; 9) = 32 - 30, n(26b(rep), 208d; 9) = 32 - 14$
						$n(52a(rep), 208c; 25) = 96 - 64$
			$676 = 2^2 \cdot 13^2$	71	15	$Yes, Prop.3.4(i)[1], \psi(N)$
$30 = 2 \cdot 3 \cdot 5$	3	0	$180 = 2^2 \cdot 3^2 \cdot 5$	25	2	$No, Cor.3.9[1]30a, 90b$
			$240 = 2^4 \cdot 3 \cdot 5$	37	3	$No, Cor.3.11.[1]20a, 24a, 240c$
			$270 = 2 \cdot 3^3 \cdot 5$	43	3	$Unknown, 30a, 45a, 135a$
			$300 = 2^2 \cdot 3 \cdot 5^2$	43	4	$No, Cor.3.9[1], 15a, rep; 50b, 150c$
			$360 = 2^3 \cdot 3^2 \cdot 5$	57	7	$Unknown, 20a, 30an(36a; 19) = 28 - 24, n(120b; 7) = 10 - 8, n(90b; 121) = 240 - 216$
			$480 = 2^5 \cdot 3 \cdot 5$	81	8	$Unknown, 20a, 24a, 80b, 160aD_{480a} = 32 \nmid 2^4, n(240b, 240c; 49) = 114 - 96$
			$450 = 2 \cdot 3^2 \cdot 5^2$	67	7	$Unknown, 15a, 75b, n(30a; 17) = 26 - 24, D_{450f} = 192 \nmid 2^4$
						$n(90a; 7) = 14 - 12, n(75c; 7) = 14 - 10, n(225a; 49) = 90 - 78$
			$540 = 2^2 \cdot 3^3 \cdot 5$	91	8	$Unknown, 45a, 54bn(90b, 270b; 7) = 15 - 12$
						$n(135a; 49) = 119 - 110, n(30a; 49) = 119 - 96$
			$600 = 2^3 \cdot 3 \cdot 5^2$	97	10	$Yes, Prop.3.4(iii)[1], n(15a, 600a; 49) = 136 - 128, n(300a; 49) = 136 - 126, n(600b; 49) = 136 - 110$
						$n(20a, 50b, 100a; 49) = 136 - 120, n(40a, 150c; 49) = 136 - 96$
			$720 = 2^4 \cdot 3^2 \cdot 5$	121	13	$Yes, Prop.3.4(iii)[1], n(20a, 90b, 360b, 720a; 49) = 128 - 120$
						$n(30a, 36a, 120b, 240c; 49) = 128 - 96, n(24a, 48a; 121) = 296 - 256$
			$750 = 2 \cdot 3 \cdot 5^3$	131	12	$Yes, Prop.3.4(iii)[1], n(50a, 150b; 7) = 16 - 12, n(15a; 49) = 154 - 128$
						$n(75a; 49) = 154 - 110$
			$810 = 2 \cdot 3^4 \cdot 5$	139	15	$Yes, Prop.3.4(iii)[1], n(30a, 162a; 49) = 139 - 96, n(135a, 405c; 49) = 139 - 110$
						$n(270a, 405b; 7) = 21 - 12, n(27a, 54a; 7) = 21 - 18, n(45a, 405f; 7) = 21 - 16$
$33 = 3 \cdot 11$	3	0	$297 = 3^3 \cdot 11$	31	6	$Yes, Prop.3.4(iii)[1], n(99d; 2) = 6 - 2, n(99a; 25) = 53 - 40, n(297b, 99b; 5) = 9 - 8$
			$363 = 3 \cdot 11^2$	33	7	$Yes, n(121b; 2) = 7 - 6, n(33a; 2) = 7 - 4, n(11a; 4) = 15 - 10$
$34 = 2 \cdot 17$	3	0	$136 = 2^3 \cdot 17$	15	3	$Unknown, over, \mathbb{Q}: \mathbb{Q}: 34aHyperelliptic$
			$272 = 2^4 \cdot 17$	31	6	$Unknown, 34an(136b; 3) = 6 - 4, D_{272a} = 16 \nmid 2^3$
			$578 = 2 \cdot 17^2$	59	12	$Yes, Prop.3.4(i)[1], \psi(N)$
$35 = 5 \cdot 7$	3	0	$175 = 5^2 \cdot 7$	15	3	$a\mathbb{Q}: Yes, Prop.3.4(iii)[1], D_{175b} = 16 \nmid 2^3clausura$
			$245 = 5 \cdot 7^2$	21	3	$Unknown, over\mathbb{Q}: 35a$
$38 = 2 \cdot 19$	4	0	$152 = 2^3 \cdot 19$	17	3	$No, Cor.3.11[1], a\mathbb{Q}: 38b, 152a, n(76a; 3) = 5 - 4$
			$304 = 2^4 \cdot 19$	35	8	$Yes, Prop.3.4(iii9[1], n(152a, 76a; 9) = 26 - 24, n(38b; 25) = 51 - 40, D_{304c} = 16 \nmid 2^3$
			$2 \cdot 19^2 = 722$	76	16	$Yes, Prop.3.4(i)[1], \psi(N)$
$39 = 3 \cdot 13$	3	0	$117 = 3^2 \cdot 13$	11	2	$Yes, over\mathbb{Q}, Prop.3.4(iii)[1], clausure?$
			$351 = 3^3 \cdot 13$	37	6	$Yes, Prop.3.4(iii)[1]$
			$507 = 3 \cdot 13^2$	47	10	$Yes, Prop.3.4(iii)[1], n(39a, 507abc; \mathbb{F}_4) = 18 - 16$

N	g_N	g_N^*	M	g_M	g_M^*	Is it not-bielliptic?
$42 = 2 \cdot 3 \cdot 7$	5	0	$168 = 2^3 \cdot 3 \cdot 7$	25	2	$Unknown, 42a, 84b$
			$252 = 2^2 \cdot 3^2 \cdot 7$	37	3	$No, Cor.3.9[1] 21a, rep; 42a Hyperelliptic$
			$294 = 2 \cdot 3 \cdot 7^2$	41	3	$Unknown, over, \mathbb{Q} : 14a$
			$336 = 2^4 \cdot 3 \cdot 7$	53	6	$Unknown, 42a, 112a n(56a; 5) = 10 - 8, n(84b; 5) = 10 - 4, n(24a; 11) = 18 - 16$
			$378 = 2 \cdot 3^3 \cdot 7$	61	5	$Unknown, 63a, 189a n(21a, 126b; 11) = 17 - 16, D_{378d} = 24 \nmid 2^4$
			$504 = 2^3 \cdot 3^2 \cdot 7$	81	7	$Unknown, 21a, 36a, 42a 168d, n(84b; 5) = 8 - 4, D_{504a} = 32 \nmid 16$
			$588 = 2^2 \cdot 3 \cdot 7^2$	89	9	$Yes, Prop.3.4(iii)[1], n(42a; 25) = 71 - 64, n(14a; 5) = 15 - 12, n(294a; 5) = 15 - 10$
			$672 = 2^5 \cdot 3 \cdot 7$	113	11	$Unknown, 112c, 224a D_{672a} = 32 \nmid 2^4, n(84b(rep), 112a; 25) = 72 - 40$
						$n(24a, 42a, 56a; 25) = 72 - 64, n(336a; 11) = 16 - 12$
46	5	0	$184 = 2^3 \cdot 23$	21	2	$Unknown, 92a, 184b$
			$368 = 2^4 \cdot 23$	43	7	$Yes, Prop.3.4(iii)[1], n(92a, 184b; 9) = 31 - 30, D_{368a} = 24, D_{368d} = 16, D_{368g} = 48 \nmid 8$
$51 = 3 \cdot 17$	5	0	$153 = 3^2 \cdot 17$	15	2	$Unknown, 51a, 153a$
			$459 = 3^3 \cdot 17$	49	9	$Yes, Prop.3.4(iii)[1], n(153a, 459b; 4) = 19 - 10, n(153c, 459a; 4) = 19 - 16, n(51a; 4) = 19 - 18$
$55 = 5 \cdot 11$	5	0	$275 = 5^2 \cdot 11$	25	4	$Unknown, over, \mathbb{Q} : 55a n(275a; 8) = 13 - 8$
			$605 = 5 \cdot 11^2$	55	10	$Yes, Prop.3.4(ii)[1], Q_3(29) = 139$
$62 = 2 \cdot 31$	7	0	$248 = 2^3 \cdot 31$	29	3	$Unknown, 62a, 124b, D_{248a} = 84 \nmid 8,$
$66 = 2 \cdot 3 \cdot 11$	9	0	$198 = 2 \cdot 3^2 \cdot 11$	29	2	$Unknown, 66a, 99a$
			$264 = 2^3 \cdot 3 \cdot 11$	41	4	$No, Cor.3.11[1], a, \mathbb{Q} : 44a, 66b, 88a, 132b$
			$396 = 2^2 \cdot 3^2 \cdot 11$	61	5	$Unknown, over, \mathbb{Q} : 66a, 99a, rep, 198c, n(66b; 5) = 12 - 8$
			$528 = 2^4 \cdot 3 \cdot 11$	85	9	$Yes, Prop.3.4(iii)[1], n(24a, 66b, 132b; 25) = 69 - 64, D_{528a} = 32 \nmid 2^4 = 16$
						$n(44a, 88a; 25) = 69 - 54$
			$594 = 2 \cdot 3^3 \cdot 11$	97	9	$Yes, Prop.3.4(iii)[1], n(99a, 198e; 25) = 63 - 40, n(66a; 7) = 16 - 12$
						$n(594a; 7) = 16 - 14, n(99b; 7) = 16 - 8; n(297b; 49) = 104 - 78, n(99b; 13) = 22 - 20$
			$726 = 2 \cdot 3 \cdot 11^2$	109	10	$Yes, Prop.3.4(iii)[1], n(726a; 5) = 15 - 12, n(11a; 5) = 15 - 10$
						$n(33a; 25) = 85 - 64, n(121b; 25) = 85 - 54$
$69 = 3 \cdot 23$	7	0	$207 = 3^2 \cdot 23$	21	3	$No, Cor.3.12[1], over, \mathbb{Q} : 69a Hyperelliptic$
$70 = 2 \cdot 5 \cdot 7$	9	0	$280 = 2^3 \cdot 5 \cdot 7$	41	4	$No, Cor.3.11[1], a, \mathbb{Q} : 20a, 70a, 280a, n(140b; 9) = 20 - 14$
			$350 = 2 \cdot 5^2 \cdot 7$	49	4	$Unknown, over, \mathbb{Q} : 175b, 350c$
			$490 = 2 \cdot 5 \cdot 7^2$	69	7	$Yes, Prop.3.4(iii)[1], n(35a, 490a : 3) = 10 - 6, n(14a; 9) = 28 - 24$
			$560 = 2^4 \cdot 5 \cdot 7$	85	9	$Unknown, 56a, 70a, 280a n(20a, 112a; 9) = 29 - 24$
						$n(140b; 3) = 7 - 1;$
			$700 = 2^2 \cdot 5^2 \cdot 7$	103	10	$Yes, Prop.3.4(iii)[1], n(50b, 175b, 350c; 9) = 44 - 30$
						$n(350d; 3) = 8 - 4, n(70a; 9) = 44 - 32$
$78 = 2 \cdot 3 \cdot 13$	11	0	$234 = 2 \cdot 3^2 \cdot 13$	35	3	$No, Cor.3.11[1], over, \mathbb{Q} : 234c$
			$312 = 2^3 \cdot 3 \cdot 13$	49	3	$No, Cor.3.11[1], a, \mathbb{Q} : 26b, 52a, 312b$
			$468 = 2^2 \cdot 3^2 \cdot 13$	73	8	$Unknown, 26b, 234c n(234b; 11) = 24 - 16$
			$624 = 2^4 \cdot 3 \cdot 13$	101	10	$Yes, Prop.3.4(iii)[1], n(24a, 52a; 25) = 82 - 64$
						$n(624a, 312b; 25) = 82 - 71, n(26b, 104a, 208b; 25) = 82 - 70, n(312e, 624b; 25) = 82 - 40$
			$702 = 2 \cdot 3^3 \cdot 13$	115	10	$Yes, Prop.3.4(iii)[1], n(234a, 702a; 5) = 11 - 10, n(234c, 702b; 25) = 77 - 64$
$87 = 3 \cdot 29$	9	0	$261 = 3^2 \cdot 29$	27	4	$Yes, over, \mathbb{Q}, Prop.3.4(iii)[1], clausura$
$94 = 2 \cdot 47$	11	0	$376 = 2^3 \cdot 47$	45	5	$Unknown, over, \mathbb{Q}, 94a$
$95 = 5 \cdot 19$	9	0	$475 = 5^2 \cdot 19$	45	9	$Yes, Prop.3.4(ii)[1], Q_2(9) = 25$
$105 = 3 \cdot 5 \cdot 7$	13	0	$315 = 3^2 \cdot 5 \cdot 7$	41	3	$No, Cor.3.12[1], over\mathbb{Q} : 21a Hyperelliptic$
			$525 = 3 \cdot 5^2 \cdot 7$	69	7	$Yes, n(15a; 8) = 10 - 8, n(175b; 2) = 7 - 6, 96 = D_{525a} \nmid 2^4$
			$735 = 3 \cdot 5 \cdot 7^2$	97	9	$Yes, n(35a; 16) = 37 - 18.$
$110 = 2 \cdot 5 \cdot 11$	15	0	$440 = 2^3 \cdot 5 \cdot 11$	65	5	$Yes, over, \mathbb{Q}, Prop.3.4(iii)[1], closure?$
						$n(44a, 110b; 3) = 7 - 6; n(20a, 9) = 25 - 24; n(88a; 9) = 25 - 14; D_{440a} = 48$
			$550 = 2 \cdot 5^2 \cdot 11$	79	7	$Unknown, 55a, 275a, 550a$
$119 = 7 \cdot 17$	11	0				$Yes, all\, \mathrm{N}, Prop.3.4(i)[1], \psi(N)$

1.5 General study $X_0^*(N \prod p_i^{2k})$

First we list all the levels where $Prop.3.4(i)[1], \psi(M)$ does not apply and the levels M for $X_0^*(M)$ needs further study, we observe that we runt a Magma programme until level $M \leq 121337$ (concerning gonality 4).

$$4, 8, 9, 12, 16, 18, 20, 24, 25, 27, 28, 32, 36, 38, 39, 40, 42, 44, 45, 48, 49, 50, 52, 54, 56, 60, 63, 64, 67, 68,$$

$$72, 73, 75, 76, 80, 81, 84, 85, 88, 90, 92, 93, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108,$$

$$109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126,$$

$$127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139,$$

$$140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152,$$

$$153, 154, 155, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165,$$

$$166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178,$$

$$179, 180, 181, 182, 183, 184, 185, 186, 187, 188, 189, 190, 191,$$

$$192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 203, 204,$$

$205, 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217,$
$218, 219, 220, 221, 222, 223, 224, 225, 226, 227, 228, 229, 230,$
$231, 232, 233, 234, 235, 236, 237, 238, 239, 240, 241, 242, 243,$
$244, 245, 246, 247, 248, 249, 250, 251, 252, 253, 254, 255, 257,$
$258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270,$
$271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283,$
$284, 285, 286, 287, 288, 289, 290, 291, 292, 293, 294, 295, 296,$
$297, 298, 299, 300, 301, 302, 303, 304, 305, 306, 307, 308, 309,$
$310, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322,$
$323, 324, 325, 326, 327, 328, 329, 330, 331, 332, 333, 334, 335,$
$336, 337, 338, 339, 340, 341, 342, 343, 344, 345, 346, 347, 348,$
$349, 350, 351, 352, 353, 354, 355, 356, 357, 358, 359, 360, 361,$
$362, 363, 364, 365, 366, 367, 368, 369, 370, 371, 372, 373, 374,$
$375, 376, 377, 378, 379, 380, 381, 382, 383, 384, 385, 386, 387,$
$388, 389, 390, 391, 392, 393, 394, 395, 396, 397, 398, 399, 400,$
$401, 402, 403, 404, 405, 406, 407, 408, 410, 411, 412, 413, 414,$
$415, 416, 417, 418, 420, 422, 423, 424, 425, 426, 427, 428, 429,$
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$450, 451, 452, 453, 454, 455, 456, 458, 459, 460, 462, 464, 465,$
$466, 468, 469, 470, 471, 472, 473, 474, 475, 476, 477, 478, 480,$
$481, 482, 483, 485, 489, 490, 492, 493, 494, 495, 497, 498, 501,$
$504, 505, 506, 507, 510, 511, 513, 515, 516, 517, 518, 519, 520,$
$522, 525, 527, 528, 530, 531, 532, 533, 534, 535, 537, 539, 540,$
$543, 545, 546, 549, 550, 551, 552, 553, 555, 558, 559, 560, 561,$
$564, 565, 570, 572, 573, 574, 575, 579, 580, 581, 582, 583, 585,$
$588, 589, 590, 591, 594, 595, 597, 598, 600, 602, 605, 606, 609,$
$610, 611, 612, 615, 616, 618, 620, 623, 624, 627, 629, 630, 635,$
$636, 637, 638, 642, 644, 645, 646, 649, 650, 651, 654, 655, 658,$
$660, 663, 665, 666, 667, 670, 671, 672, 678, 679, 680, 682, 684,$
$689, 690, 693, 696, 697, 700, 702, 703, 705, 708, 710, 713, 714,$
$715, 720, 726, 728, 730, 731, 732, 735, 738, 740, 741, 742, 744,$
$748, 750, 754, 759, 760, 762, 765, 770, 774, 777, 780, 782, 786,$
$790, 795, 798, 804, 805, 806, 810, 812, 814, 819, 822, 825, 826,$
$834, 836, 840, 855, 858, 861, 870, 874, 885, 897, 903, 910, 915,$
$924, 930, 935, 957, 966, 969, 987, 990, 1001, 1015, 1020, 1023,$

$$1045, 1050, 1085, 1092, 1105, 1110, 1122, 1140, 1155, 1170, 1190,$$

$$1218, 1230, 1254, 1260, 1290, 1302, 1320, 1326, 1330, 1365, 1380,$$

$$1410, 1470, 1482, 1530, 1560, 1590, 1680, 1890, 2310, 2730, 3570$$

We need to arise from the above list the genus $0, 1$ curves, the ones that is a power of a prime, the square-free cases, and the non-square-free levels already studied in the previous subsections and we obtain the following table (for the remaining levels) to study bielllipticity or not:

M	g_M	g_M^*	Is it not-bielliptic?
$388 = 2^2 \cdot 97$	47	11	$Yes, Prop.3.4(iii)[1], n(\lvert a_3\rvert \geq 0, \mathbb{F}_9) \geq 34 - 32$
$404 = 2^2 \cdot 101$	49	9	$Yes, Prop.3.4(iii)[1], n(a_3 \geq -1, \mathbb{F}_3) \geq 12 - 10, n(\lvert a_3\rvert > 1, \mathbb{F}_9) > 0$
$428 = 2^2 \cdot 107$	52	9	$Yes, Prop.3.4(iii)[1], n(\lvert a_3\rvert \geq 0, \mathbb{F}_9) \geq 36 - 32$
$436 = 2^2 \cdot 109$	53	12	$Yes, Prop.3.4(iii)[1], n(\lvert a_3\rvert \geq 0, \mathbb{F}_9) \geq 35 - 32$
$452 = 2^2 \cdot 113$	55	12	$Yes, Prop.3.4(iii)[1], n(\lvert a_3\rvert \geq 0, \mathbb{F}_9) \geq 36 - 32$
$516 = 2^2 \cdot 3 \cdot 43$	83	9	$Yes, Prop.3.4(iii)[1], n(a_5 \geq -3, \mathbb{F}_5) \geq 19 - 18, n(\lvert a_5\rvert \geq 3, \mathbb{F}_9) \geq 55 - 54$
$564 = 2^2 \cdot 3 \cdot 47$	91	6	$Remains, 94a, Prop.3.4(iii)[1], n(\lvert a_5\rvert \geq 1; \mathbb{F}_{25}) > 0$
$620 = 2^2 \cdot 5 \cdot 31$	91	6	$Remains, 62a, Prop.3.4(iii)[1], n(\lvert a_3\rvert \geq 1; \mathbb{F}_9) > 0$
$644 = 2^2 \cdot 7 \cdot 23$	91	9	$Yes, No, dim = 1, Q - factor, Jacobian$
$708 = 2^2 \cdot 3 \cdot 59$	115	10	$Yes, Prop.3.4(iii)[1], n(\lvert a_5\rvert \geq 0; \mathbb{F}_{25}) \geq 76 - 72$
$732 = 2^2 \cdot 3 \cdot 61$	119	11	$Yes, Prop.3.4(iii)[1], n(\lvert a_5\rvert \geq 0; \mathbb{F}_{25}) \geq 122 - 72$
$748 = 2^2 \cdot 11 \cdot 17$	103	11	$Yes, Prop.3.4(iii)[1], n(\lvert a_3\rvert \geq 0; 9) \geq 37 - 32$
$774 = 2 \cdot 3^2 \cdot 43$	125	12	$Yes, Prop.3.4(iii)[1], n(a_5 = 0, 5) = 20 - 12, n(\lvert a_5\rvert \geq 1, 25) > 0$
$804 = 2^2 \cdot 3 \cdot 67$	131	15	$Yes, Prop.3.4(iii)[1], n(\lvert a_5\rvert \geq 0; 25) \geq 75 - 72$
$812 = 2^2 \cdot 7 \cdot 29$	115	11	$Yes, Prop.3.4(iii)[1], n(\lvert a_3\rvert \geq 0; 9) \geq 36 - 32$
$819 = 3^2 \cdot 7 \cdot 13$	105	11	$Yes, Prop.3.4(iii)[1], n(a_2; 2) = 4, n(\lvert a_2\rvert \geq 1; 4) > 0$
$836 = 2^2 \cdot 11 \cdot 19$	115	11	$Yes, Prop.3.4(iii)[1], n(\lvert a_3\rvert \geq 0; 9) \geq 40 - 32$
$924 = 2^2 \cdot 3 \cdot 7 \cdot 11$	181	7	$Remains : 77a, 462a, n(\lvert a_5\rvert \geq 2; 25) \geq 66 - 64$
$1092 = 2^2 \cdot 3 \cdot 7 \cdot 13$	213	13	$Yes, Prop.3.4(iii)[1], n(a_5 \geq -2; 5) \geq 20 - 16, n(\lvert a_5\rvert \geq 2; 25) \geq 70 - 64$
$1140 = 2^2 \cdot 3 \cdot 5 \cdot 19$	229	9	$Remains : 190b, 285b; Prop.3.4(iii)[1], n(a_7 \geq 0; 7) \geq 18 - 16, n(\lvert a_7\rvert \geq 4; 49) \geq 104 - 96$
$1380 = 2^2 \cdot 3 \cdot 5 \cdot 23$	277	15	$Yes, Prop.3.4(iii)[1], n(\lvert a_7\rvert \geq 0; 49) \geq 141 - 128$

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