Images as continuous functions

It is convenient to think of an image as a continuous function, which maps a region of the 2D space into the color space.

$$f : 2D \rightarrow Colors$$

If the colors are represented as three channels (R, G, B) we can think f either as a vector function or as a set of three scalar functions.

In my solution, I am not working internally with colored images, but I convert the image to a grey scale image, so we can think of f as one single scalar function.

Second derivatives

Second derivatives of a function can be used to detect (enhance, emphasize) sharp changes in the function, or more precisely in its gradient.

When an image is shredded, what we are introducing is a sharp variation in the color along the boundaries of contiguous shreds. So second derivatives can help in:

- determining when two shreds were contiguous in the un-shredded image (because there is no abrupt discontinuity)
- where the shreds boundaries lay (because there is abrupt discontinuity)

Derivatives of a function require the function to be continuous, and that is why I was commenting it is convenient to think of an image as a continuous function.

Let's take one single row of the image, and consider the function restricted to that row.

Image 1

Remember I said I was working internally with a grey scale version of the shredded image.

The function across the image row, is no longer on a 2D region but a scalar function on a 1D segment. $f: R \rightarrow R$

Image 2

The formula of the first derivative of a function f (along x axis) in a point x is:

$$f'(x) = \lim_{\partial x \to 0} \frac{f(x+\partial x)-f(x)}{\partial x} \qquad \text{(eq. 1)}$$

And the second derivative is the derivative of the first derivative:

$$f''(x) = \lim_{\partial x \to 0} \frac{f\prime(x+\partial x) - f\prime(x)}{\partial x} \qquad \text{(eq. 2)}$$

The first derivative measures the slope (gradient) of the function at point *x*.
The second derivative measures the curvature of the function at point *x*.

Calculation of the second derivative on discretized functions

Even if the underlying (theoretical) function were continuous, in practice in order to capture and represent the image on a computer it had to be discretized. There are two dimensions of this discretization of any image:

- **Sampling.** Instead of storing the colors of the infinite points of the region in 2D space, the image is measured only in a finite set of locations called pixels.
- **Quantization.** Instead of having a theoretically infinite set of colors, color is encoded with a finite number of bytes (in any representation, for instance RGB). And by having a finite number of bytes, we can represent only a finite number of colors.

So we will have to calculate the second derivative using the available information, which is a discretization of the underlying continuous image. And at the end the formulas get very simple.

Despite the little bit of mathematics introduced so far, at the end the calculation of the second derivative will be just a linear combination of weighted pixel values (addition and subtraction of pixel values, each one multiplied by some factor).

Consider, for convenience, that pixels are separated a distance of 1.

Retaking the formula of the first derivative (eq. 1), despite it says limit when δx tends to zero, the smallest δx available is $\delta x = 1$ (distance between two consecutive pixels).
And then the first derivative is, on a discretized function:

$$f'(x) = \frac{f(x+1) - f(x)}{1} = f(x+1) - f(x) \qquad \text{(eq. 3)}$$

And the second derivative, (derivative of the first derivative), in *x* along the *x* direction, becomes:

$$f''(x) = f'(x+1) - f'(x) = f(x+2) - f(x+1) - f(x+1) + f(x)$$

$$f''(x) = f(x+2) - 2\,f(x+1) + f(x) \qquad \text{(eq. 4)}$$

This represents the curvature of *f* at point *x.*

What I do then is to calculate the difference between:

- The second derivative in x along the x axis, coming from the left (positive direction)

- The second derivative in x+1 along the x axis (negative direction)

When this amount is small in absolute value, it means that between x and x + 1 the function is not only continuous, but it is also smooth (no big differences in curvature).
Let's see the formula.

Second derivative in x along the x axis, coming from the left (positive direction) is as in equation 4:

$$f''_{-}(x) = f'(x+1) - f'(x) = f(x+2) - 2\,f(x+1) + f(x) \qquad \text{(eq. 5)}$$

Second derivative in $x + 1$ along the x axis, coming from the right (negative direction) is analogous:

$$f''_{+}(x+1) = f'(x) - f'(x+1) = f(x-1) - 2f(x) + f(x+1) \qquad \text{(eq. 6)}$$

And the absolute value of the difference between both expressions is

$$|f''_{-}(x) - f''_{+}(x+1)| = |f(x+2) - 3\,f(x+1) + 3\,f(x) - f(x-1)| \qquad \text{(eq. 7)}$$

"Distance" function between two given image shreds

Putting together (side by side) any two shreds of the image, if they are contiguous in the un-shredded image, the underlying image function will be continuous and smooth (same curvature) along the shred boundaries.
But if the shreds are not contiguous, the function will not be smooth at all; there will be a vertical edge along the shred boundaries.

The thing to do then is to take equation 7, calculated where at the shred boundaries, and not any longer for a single row but added for all the rows in the image. We took absolute value precisely to avoid positive and negative values in different rows to cancel each other.

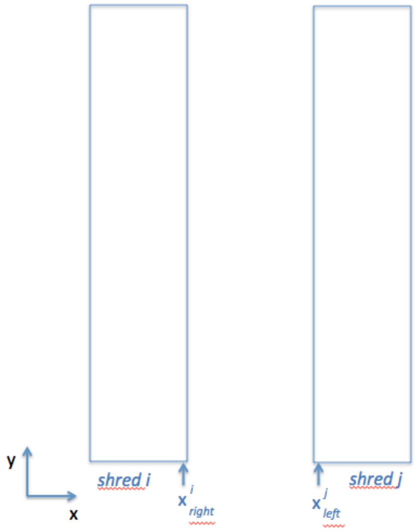

Image 3

Given two any shreds, *i* and *j,*

$$distance(i, j) = \sum_{y=1}^{N} |f\left(x_l^j + 1, y\right) - 3\, f\left(x_l^j + 1, y\right) + 3\, f\left(x_r^i, y\right) - f\left(x_r^i - 1\right)| \qquad \text{(eq. 8)}$$

where

N is the height of the image

x_l^j is the leftmost coordinate of shred *j*

x_r^i is the rightmost coordinate of shred *i*

$f(x, y)$ is the value of pixel at coordinates x, y

If shreds *i* and *j* are contiguous in the un-shredded image, the function *distance(i , j)* gives an small value. When they are not contiguous, the function gives a bigger value.

<u>For any given shred *i*, the shred *j* that will go contiguous to it in the un-shredded image is the one that minimizes *distance(i , j).*</u>

Note that only two pixels columns of each shred are necessary and enough to measure the distance between shreds.

Formulas as the one in equation 8 are used in image processing for edge detection. At the end of the day, this problem is looking for the presence or absence of vertical edges from the top to the bottom of the image.