$$\partial_t u = \partial_x^2 u - u + u^2 v$$
$$\partial_t v = d\,\partial_x^2 v + b - u^2 v \quad ,$$

Periodic BCs, with $d, b > 0$.

Homogeneous Steady State: Q1

$$0 = u(-1 + uv) \qquad\qquad u = 1/v,$$
$$\Rightarrow$$
$$0 = b - u^2 v \qquad\qquad b - u^2\, 1/u = 0$$

($u=0$, $b=0$ impossible as $b>0$)

$$\Rightarrow (u_*, v_*) = (b, 1/b)$$

Linearised operator: Q2

$$\mathcal{L}(u_*, v_*) = \begin{pmatrix} \partial_x^2 - 1 + 2u_* v_* & u_*^2 \\ -2u_* v_* & d\partial_x^2 - u_*^2 \end{pmatrix}$$

At $(u_*, v_*) = (b, 1/b)$

$$\mathcal{L}(u_*, v_*) = \begin{pmatrix} \partial_x^2 + 1 & b^2 \\ -2 & d\partial_x^2 - b^2 \end{pmatrix}, \qquad \partial_t \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \mathcal{L}(u_*, v_*) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}$$

QUESTION 3:

Eigenvalue problem: solve $\mathcal{L}(u_*, v_*)\varphi = \lambda\varphi$ on $\mathbb{R}$ (no BCs). One wa

Ansatz: $\varphi(x,t) = e^{\lambda t + ikx}\psi$, $\psi \in \mathbb{C}^2$. Leading to

$$\underbrace{\begin{pmatrix} -k^2+1 & b^2 \\ -2 & -dk^2-b^2 \end{pmatrix}}_{M}\psi = \lambda\psi \qquad \lambda^2 - \mathrm{Tr}(M)\lambda + \det(M) = 0.$$

$$\lambda^2 - [-k^2+1 \ -dk^2-b^2]\lambda + [dk^4 + b^2k^2 - dk^2 + b^2] = 0$$

$$\lambda^2 - [-(1+d)k^2 + (1-b^2)]\lambda + [dk^4 + (b^2-d)k^2 + b^2] = 0$$

$$\lambda^2 - \tau(b,d,k)\lambda + \Delta(b,d,k), \qquad \tau(b,d,k) = -(1+d)k^2 + (1-b^2) \ (= \text{trace of } M)$$

$$\Delta(b,d,k) = dk^4 + (b^2-d)k^2 + b^2 \ (= \det \text{ of } M)$$

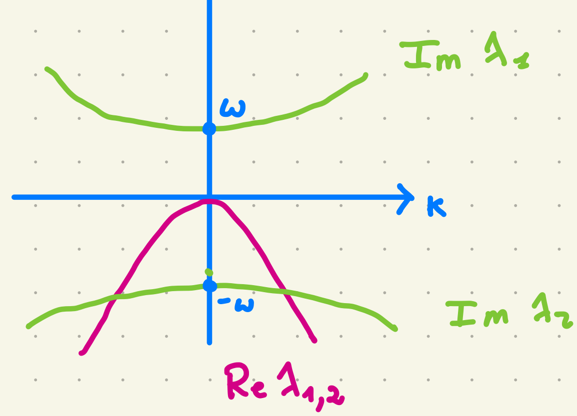
$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4}}{2}$$

QUESTION 4: when $b=2$, $d=1$, then

$$\tau = -2k^2 - 3, \quad \Delta = k^4 + 3k^2 + 4, \quad \tau^2 - 4\Delta = 4k^4 + 12k^2 + 9 - 4k^4 - 12k^2 - 16 = -7$$

$$\lambda_{1,2}(k) = -\frac{(2k^2+3)}{2} \pm i\frac{\sqrt{7}}{2}, \quad \text{hence} \quad \mathrm{Re}\,\lambda_{1,2}(k) < 0 \ \text{ for all } k \in \mathbb{R}.$$

Conditions for Hopf of the homogeneous steady state:



At some $(b_c, d_c) \in \mathbb{R}^2$ it holds

$$\mathrm{Re}\ \lambda_i(b_c, d_c, 0) = 0, \qquad \mathrm{Im}\ \lambda_i(b_c, d_c, 0) \neq 0.$$

At $k=0$ we therefore have

$$\lambda_{1,2}(b_c, d_c, 0) = \pm i\ \omega.$$

The steady state (u_*, v_*) becomes unstable to perturbations $\psi e^{i\omega t} + c.c.$ We expect the period of emerging solutions to be $T = 2\pi/\omega$. If the bifurcation is supercritical, we should be able to observe homogeneous oscillations with period T.

Conditions for Turing bifurcations

At some (b_c, d_c, k_c) we find

$$\mathrm{Re}\ \lambda_i(b_c, d_c, 0) < 0$$

(u_*, v_*) is stable in the absence of diffusion

$$\mathrm{Re}\ \lambda_i(b_c, d_c, k_c) = 0, \qquad \mathrm{Im}\ \lambda_i(b_c, d_c, k_c) = 0$$

Then (u_*, v_*) becomes unstable to perturbations $\psi e^{ik_c x}$,

where ψ is an eigenvector of M

$$\begin{pmatrix} -k_c^2+1 & b_c^2 \\ -2 & -d_c k^2 - b_c^2 \end{pmatrix} \psi = \lambda \psi$$

On the domain $[-L, L]$ with periodic BCs, if there exists $n_c \in \mathbb{N}$ such that

$k_c = n_c \frac{\pi}{L}$, then we expect the emanating pattern to have n_c peaks in $[-L, L]$.