Question 12

Allen-Cahn PDE

$$\partial_t u = \nu \partial_x^2 u + \lambda u + u^3 - u^5 \qquad x \in \Omega = (-L, L), \quad t \in \mathbb{R}_{>0}, \quad \nu, \gamma \in \mathbb{R}_{>0}, \quad \lambda \in \mathbb{R}$$

$$\partial_x u(-L, t) = 0, \quad \partial_x u(L, t) = 0 \qquad t \in \mathbb{R}_{\geq 0}$$

$$u(x, 0) = u_0(x) \qquad x \in [-1, 1].$$

Homogeneous steady states

$u(x,t) = U$, $U \in \mathbb{R}$, for all $(x,t) \in [-1,1] \times \mathbb{R}_{\geq 0}$.

$$0 = \lambda U + U^3 - U^5$$

$$0 = U(\lambda + U^2 - U^4).$$

$U_1 = 0$ is a solution

Set $V = U^2$ and solve $-\lambda - V + V^2 = 0 \Rightarrow V = \frac{1 \pm \sqrt{1 + 4\lambda}}{2}$

$$\Rightarrow \quad \begin{aligned} U_1 &= 0 \\ U_2 &= \sqrt{\frac{1 + \sqrt{1 + 4\lambda}}{2}} \\ U_3 &= -\sqrt{\frac{1 - \sqrt{1 + 4\lambda}}{2}} \\ U_4 &= \sqrt{\frac{1 - \sqrt{1 + 4\lambda}}{2}} \\ U_5 &= -\sqrt{\frac{1 + \sqrt{1 + 4\lambda}}{2}} \end{aligned}$$

Depending on the value of λ, we can have 1, 3, or 5 Homogeneous steady state.