QUESTION 12

Nonlinear equation:

$$\partial_t u = \nu \partial_x^2 u + N(u, \lambda, \alpha, \beta, \gamma) \quad \text{on} \quad (-5,5) \times \mathbb{R}_{>0}$$

$$\partial_x u = 0 \quad \text{on} \quad \{-5,5\} \times \mathbb{R}_{>0}$$

$$u = \varphi \quad \text{on} \quad [-5,5] \times \{0\}.$$

Where $N(u, \lambda, \alpha, \beta, \gamma) = \lambda u + \alpha u^2 + \beta u^3 - \gamma u^5$

Formal derivation of linearised equation around equilibrium u_*.

u_* satisfies $\quad \partial_x^2 u_* + N(u_*, \lambda, \alpha, \beta, \gamma) \quad$ on $(-5,5)$

$\quad \partial_x u_* = 0 \quad$ on $\{-5,5\}$.

Set $u = u_* + v$.

$$\partial_t v = \nu \partial_x u_* + \nu \partial_x v + N(u_* + v, \cdot) = \cancel{\nu \partial_x u_*} + \nu \partial_x v + \cancel{N(u_*, \cdot)} + (\lambda + 2\alpha u_* + 3\beta u_*^2 - 5\gamma u_*^4) v + \mathcal{O}(\|v\|^2)$$

Hence, the linearised eqn. is

$$\partial_t v = (\lambda + 2\alpha u_* + 3\beta u_*^2 - 5\gamma u_*^4) v + \partial_x^2 v \quad \text{on} \quad (-5,5) \times \mathbb{R}_{>0}$$

$$\partial_x v = 0 \quad \text{on} \quad \{-5,5\} \times \mathbb{R}_{>0}$$

$$v = \varphi - u_* \quad \text{on} \quad [-5,5] \times \{0\}$$

Evolution equation linearised around $u_*(x) \equiv 0$.

$$\partial_t v(x,t) = \partial_x^2 v(x,t) + \lambda v(x,t) =: \mathcal{L}\, v(x,t), \qquad \mathcal{L} = \partial_x^2 + \lambda$$

$$\partial_x^2 v(\pm 5, t) = 0$$

$$v(x,t) = \Psi(x)$$

Eigenvalues of $\mathcal{L}$: find (λ, Ψ) with $\Psi \neq 0$ s.t. $(\partial_x^2 + \lambda)\Psi = \mu \Psi$, on $(-5,5)$

$\partial_x \Psi = 0$ on $\{-5,5\}$

μ eigenvalues of $\mathcal{L}$

Method 1:

If (μ, ψ) is an eigenpair for $\mathcal{A} = \partial_x^2$, then $(\mu+\lambda, \psi)$ is an eigenpair for $\mathcal{L}$, because

$$\mathcal{A}\psi = \mu\psi \iff \mathcal{A}\psi + \lambda\psi = (\mu+\lambda)\psi \iff \mathcal{L}\psi = (\mu+\lambda)\psi.$$

So we can compute eigenpairs for $\mathcal{A}$ by solving

$$(1) \quad \begin{cases} \partial_x^2 \psi(x) = \mu \Psi(x) & x \in (-5,5) \\ \partial_x \Psi(\pm 5) = 0. \end{cases}$$

Case 1: $\mu > 0$ $\quad \Psi(x) = c_1 e^{s_1 x} + c_2 e^{s_2 x}$, $s_{1,2}$ satisfy $s^2 = \mu$.

$\Psi(x) = c_1 e^{\sqrt{\mu} x} + c_2 e^{-\sqrt{\mu} x}$ can <u>not</u> satisfy $\partial_x \Psi(\pm 5) = 0$, hence there is no solution to (1) when $\mu > 0$

Case 2: $\mu = 0$. $\quad \Psi(x) = c_1 x + c_2$. Which satisfy BCs only if $c_1 = 0$. Hence one eigenpair is

$$\mu = 0, \quad \Psi(x) \equiv 1 \qquad (\text{w.l.o.g. we can take } c_1 = 1)$$

Case 3: $\mu < 0$. Set $\mu = -\omega^2$. It holds $\Psi(x) = A\cos(\omega x) + B\sin(\omega x)$

To see why: $\Psi(x) = C e^{i\omega x} + \bar{C} e^{-i\omega x}$, $\quad C \in \mathbb{C}$. Hence

$$\Psi(x) = C\cos(\omega x) + iC\sin(\omega x) + \bar{C}\cos(\omega x) - i\bar{C}\sin(\omega x)$$

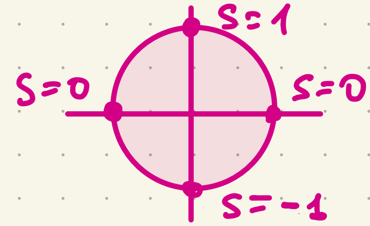
$$= \underbrace{(C + \bar{C})}_{= A \in \mathbb{R}} \cos(\omega x) + \underbrace{i(\bar{C} - \bar{C})}_{= B \in \mathbb{R}} \sin(\omega x)$$

Apply BCs: $\quad \partial_x \Psi(x) = \omega\left[-A\sin(\omega x) + B\cos(\omega 5)\right]$, hence

$$(2) \quad \begin{cases} -AS + BC = 0, \\ AS + BC = 0 \end{cases} \qquad \text{where } S = \sin(5\omega), \ C = \cos(5\omega)$$

System (2) is equivalent to

$$\begin{cases} BC = 0 \\ AS = 0 \end{cases}$$



Sol 1: if $B=0$, then $A \neq 0$ (or else $\Psi(x) \equiv 0$), hence $S=0$, $C=\pm 1$

$$\Psi_k(x) = A\cos(\omega_k x) \qquad 5\omega_k = k\pi, \quad k \in \mathbb{Z} \longrightarrow$$ this comes from $S=0$.

Sol 2: if $C=0$ then $S=\pm 1$ (because $S^2+C^2=1$), which implies $A=0$ and $B\neq 0$.

$$\Psi_k(x) = B\sin(\omega_k x) \qquad 5\omega_k = (2k+1)\frac{\pi}{2}, \quad k \in \mathbb{Z}.$$

Sol 3: if $A=0$, then $B\neq 0$, $C=0$, $S=\pm 1$, that is, sol 2.

Sol 4: if $S=0$, then $C=\pm 1$, $B=0$, $A\neq 0$, that is, sol 1.

Putting all solutions together, using symmetries of sin, cos, we have

$$\Psi_j(x) = \begin{cases} B\cos\left(j\frac{\pi}{10}x\right) & j \text{ even} \\ B\sin\left(j\frac{\pi}{10}x\right) & j \text{ odd} \end{cases} \qquad \text{with eigenvalue} \qquad \mu_j = -\left(j\frac{\pi}{10}\right)^2$$

Therefore we have, for $\mathcal{L} = \mathcal{A} + \mu\, Id$

$$\Psi_j(x) = \begin{cases} B\cos\left(j\frac{\pi}{10}x\right) & j \text{ even} \\ B\sin\left(j\frac{\pi}{10}x\right) & j \text{ odd} \end{cases} \qquad \text{with eigenvalue} \qquad \mu_j = -\left(j\frac{\pi}{10}\right)^2 + \lambda$$

BP_1: $\lambda = \lambda_0 := 0$, critical eigenf $\Psi_0(x) \equiv C$, $C \in \mathbb{R}$, $\qquad \Psi_0(x) \equiv 1$

BP_2: $\lambda = \lambda_1 := \left(\frac{\pi}{10}\right)^2$, critical eigenf $\Psi_1(x) = C \sin\left(\frac{\pi}{10} x\right)$, $C \in \mathbb{R}$, $\qquad \Psi_1(x) = \sin\left(\frac{\pi}{10} x\right)$

BP_3: $\lambda = \lambda_2 := \left(\frac{\pi}{5}\right)^2$, critical eigenf. $\Psi_2(x) = C \cos\left(\frac{\pi}{5} x\right)$, $C \in \mathbb{R}$ $\qquad \Psi_2(x) = \cos\left(\frac{\pi}{5} x\right)$

Method 2:

$$\partial_t v(x,t) = \partial_{xx} v(x,t) + \lambda\, v(x,t)$$

$$\partial_x v(\pm 5) = 0$$

note $\mu = \mu(k)$, $\hat{v} = \hat{v}(k)$.

Supports solutions of the form $\quad v(x,t) = e^{ikx + \mu t}\, \hat{v} + c.c.$

From PDE:

$$\mu\, e^{ikx+\mu t}\hat{v} = -k^2\, e^{ikx+\mu t}\hat{v} + \lambda\, e^{ikx+\mu t} v \quad \Rightarrow \quad \mu = -k^2 + \lambda$$

From BCs:

$ik\, e^{ik5}\, \hat{v}$

$$\hat{v}\left[ik \cos(k5) - k\sin(k5)\right] = 0$$

$$\hat{v}\left[ik \cos(k5) + k\sin(k5)\right] = 0 \quad \rightsquigarrow \text{compare this with (2) in method 1.}$$