NONLINEAR DYNAMICAL SYSTEMS PART 3
DYNAMICS IN PATTERN-FORMING SYSTEMS
TUTORIAL 3

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Abstract. Themes of this tutorial:

1. Working with systems of PDEs
2. Determining dispersion relations
3. Determining Hopf and Turing bifurcations
4. Predicting wavelengths and/or temporal periods of patterns near Hopf and Turing bifurcations.

1. Introduction. Let us consider the following system of PDEs, known as the Schnakenberg model,

$$
\begin{aligned}
\partial_t u &= \partial_x^2 u - u + u^2 v, \qquad (x,t) \in \mathbb{R}/2L\mathbb{Z} \times \mathbb{R}_{>0}, \\
\partial_t v &= d\partial_x^2 v + b - u^2 v, \qquad (x,t) \in \mathbb{R}/2L\mathbb{Z} \times \mathbb{R}_{>0}.
\end{aligned} \tag{1.1}
$$

where $d, b \in \mathbb{R}_{>0}$. By writing $x \in \mathbb{R}/2L\mathbb{Z}$ we compactly state that we are interested in $2L$-periodic functions: we can consider the equation posed on $[-L, L)$, or $[0, 2L)$, or $[nL, (n+2)L)$ with $n \in \mathbb{Z}$, subject to *periodic boundary conditions* (PBCs). Initial conditions to the problem will be specified in due course.

Question 1 (Homogeneous steady state). Show that for any $d, b > 0$ system (1.1) admits a unique homogeneous equilibrium (u_*, v_*), independent of d. Express (u_*, v_*) in terms of b.

Question 2 (Linearisation). Show that small perturbations $\varphi = (\tilde u, \tilde v)$ around (u_*, v_*) evolve according to a linear PDE

$$
\partial_t \varphi = \mathcal{L}(u_*, v_*)\varphi, \qquad \text{or, in extended form,} \qquad \partial_t \begin{pmatrix} \tilde u \\ \tilde v \end{pmatrix} = \mathcal{L}(u_*, v_*) \begin{pmatrix} \tilde u \\ \tilde v \end{pmatrix}, \tag{1.2}
$$

and, using the result in Question 1, express $\mathcal{L}$ in terms of b and d.

Question 3 (Dispersion relation). Let us study perturbations to the homogeneous equilibrium disregarding boundary conditions. Loosely speaking, one way to achieve that is to think of the problem (1.1) with $x \in \mathbb{R}$ as opposed to $x \in \mathbb{R}/2L\mathbb{Z}$. Show that the linear problem (1.2), posed on $\mathbb{R}$ admits solutions in the form

$$
\varphi(x,t) = \psi \exp(\lambda t + ikx) + \overline{\psi \exp(\lambda t + ikx)}, \qquad \psi \in \mathbb{C}^2,
$$

where the overbar denotes complex conjugation, provided λ, b, d, k satisfy the dispersion relation

$$
\lambda^2 - \tau(k,b,d)\lambda + \Delta(k,b,d) = 0, \tag{1.3}
$$

where τ and Δ are trace and determinant of the matrix

$$
M(k,b,d) = \begin{pmatrix} 1-k^2 & b^2 \\ -2 & -dk^2 - b^2 \end{pmatrix},
$$

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respectively, and (λ, ψ) is an eigenpair of $M(k, b, d)$.

Question 4 (Curves of eigenvalues). As we have seen in the lectures, we investigate linear stability of the homogeneous steady states as follows: let (u_*, v_*) be a homogeneous steady state of (1.1) posed on $x \in \mathbb{R}$ for $(b, d) = (b_*, d_*)$ and let $\gamma_{1,2}$ be the curves of eigenvalues

$$\gamma_{1,2} \colon \mathbb{R} \to \mathbb{C}, \qquad k \mapsto \lambda_{1,2}(k, b_*, d_*).$$

It is sometimes useful to omit the dependence on the parameters (b_*, d_*), and use $\lambda_{1,2}(k)$ and $\gamma_{1,2}(k)$ interchangeably. The steady state (u_*, v_*) is linearly stable if $\operatorname{Re} \gamma_{1,2}(k) < 0$, that is, if the eigenvalues $\lambda_{1,2}(k)$ obtained for fixed (b_*, d_*) have negative real parts for all wavelengths $k \in \mathbb{R}$.

Prove that that the homogeneous steady state $(u_*, v_*) \equiv (3, 1/3)$ to (1.1) posed on $\mathbb{R}$ is linearly stable for $(b_*, d_*) = (3, 1)$.

Question 5 (Plot eigenvalues). Write code that, for fixed value (b_*, d_*), plots graphs of the curves

$$\operatorname{Re} \gamma_1(k), \qquad \operatorname{Im} \gamma_1(k), \qquad \operatorname{Re} \gamma_2(k), \qquad \operatorname{Im} \gamma_2(k),$$

and test the code by verifying the analytical prediction of Question 4.

Question 6 (Periodic Boundary conditions). As in any system of PDEs, boundary conditions have an impact on the problem. Discuss with a colleague the following argument, and fill the gaps where necessary.

Let us consider problem (1.1), which has now the correct periodic boundary conditions, and consider the linearised problem

$$\partial_t \varphi(x, t) = \mathcal{L}(u_*, v_*) \varphi(x, t), \qquad (x, t) \in \mathbb{R}/2L\mathbb{Z} \times \mathbb{R}_{>0}, \tag{1.4}$$

Small perturbations $\varphi = (\tilde{u}, \tilde{v})$ to a homogeneous steady state (u_*, v_*) of (1.1) with $(b, d) = (b_*, d_*)$ can be written

$$\varphi(x, t) = \sum_{j \in \mathbb{Z}} \psi_j \exp(\lambda_j t + i k_j x) + \overline{\psi_j \exp(\lambda_j t + i k_j x)}$$

where

$$k_j = j \frac{\pi}{L}, \qquad \lambda_j^2 - \tau(k_j, b_*, d_*) \lambda_j + \Delta(k_j, b_*, d_*) = 0, \qquad j \in \mathbb{Z}.$$

You may assume, without proof, that the series above converges to φ.

Question 7 (Hopf bifurcation). Using the code in Question 5, show that the steady state $(u_*, v_*) = (0.99, 1/99)$ is linearly unstable when $(b_*, d_*) = (0.99, 1)$. With these data, argue that (u_*, v_*) is close to a Hopf bifurcation. What is the spatial wavelength and/or temporal period of the emerging pattern?

Question 8 (Hopf bifurcation). Produce numerical evidence in support of your predictions in Question 7. One way to do this is to perform a time simulation of (1.1) for (b_*, d_*) fixed as in Question 7, and with initial condition $(u(x, 0), v(x, 0)) = (u_*, v_*) + (\tilde{u}(x), \tilde{v}(x))$, for "small" $(\tilde{u}, \tilde{v})$. The analysis in Question 7 can not determine whether the bifurcation is sub- or super-critical. However, if it is supercritical, simulations should show a pattern with features predicted by the eigenvalues and eigenfunctions determined in Question 7.

Using a suitable differentiation matrix D_{xx} (see Tutorials 1 and 2), discretise (1.1) as a set of ODEs of the type

$$\begin{pmatrix} \dot U \\ \dot V \end{pmatrix} = \begin{pmatrix} D_{xx} & 0 \\ 0 & dD_{xx} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} f(U,V) \\ g(U,V,b) \end{pmatrix},$$

where $U, V \in \mathbb{R}^n$, $D_{xx} \in \mathbb{R}^{n\times n}$,

$$\begin{aligned} &f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, && (f(U,V))_i = -u_i + u_i^2 v_i, \quad i = 1,\dots,n, \\ &g\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n, && (g(U,V,b))_i = b - u_i^2 v_i, \quad i = 1,\dots,n. \end{aligned} \tag{1.5}$$

Time step the problem using 1501 evenly distributed grid points $\{x_j\} \subset [-L, L]$ with $L = 30$, and periodic boundary conditions and initial condition

$$\begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + 0.01 \begin{pmatrix} \sin(\pi x/5) \\ \sin(\pi x/10) \end{pmatrix}.$$

The time stepper `ode15s` or `ode23s` will perform better if you pass not only the right-hand side of (1.5), but also its Jacobian with respect to (U, V) (this is done by passing the option `'Jacobian'` to the time stepper, via Matlab's command `odeset`).

Produce numerical evidence that the model exhibits a pattern compatible with your analysis in Question 7, and in particular that the period of the emerging oscillations approximates the one predicted by your analysis. Note that the analysis predicts the period of the spatially continuous problem, whereas the numerical simulation timesteps an approximation to this system with $n = 1501$ grid points, hence you should expect some discrepancy between the predicted and the observed period.

Question 9 (Turing bifurcation). Using the code in Question 5, show that the steady state $(u_*, v_*) = (2, 1/2)$ is linearly unstable when $(b_*, d_*) = (2, 30)$. With these data, argue that (u_*, v_*) is close to a Turing bifurcation. What are the wavelengths of the unstable modes when $L = 30$? Which wavelength would you expect to observe in a time simulation with $L = 30$?

Question 10 (Turing bifurcation simulation). Produce numerical evidence that the model exhibits a pattern compatible with your analysis in Question 9, and in particular that the wavelength of the emerging pattern matches the one predicted by your analysis. Use an initial condition of the type

$$\begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} u_* \\ v_* \end{pmatrix} + \varepsilon \begin{pmatrix} \xi(x) \\ \eta(x) \end{pmatrix},$$

where $\xi(x)$ and $\eta(x)$ should also be selected using information from Question 9. Some fiddling with ε may be necessary to see the pattern.