Integral equation:

$$\nu(y) = \int_{\mathbb{X}} \varrho(x) K(x, y)\ dx \qquad \forall y \in \mathbb{Y}$$

with

$$\begin{aligned}
&\nu(y) = \mathcal{N}(y; \mu, \sigma_\nu^2 := \sigma_\rho^2 + \sigma_K^2)\\
&\varrho(x) = \mathcal{N}(x; \mu, \sigma_\rho^2)\\
&K(x, y) = \mathcal{N}(y; x, \sigma_K^2)
\end{aligned}$$

where $\mathbb{X} = \mathbb{Y} = \mathbb{R}$.

McKean-Vlasov SDE:

$$dX_t = -\int \nu(dy) \frac{\nabla K(X_t, y)}{\varrho_t K(y)}\ dt + \sqrt{2\lambda} dW_t, \qquad X_0 \sim \varrho_0$$

discretised using Euler-Maruyama method

$$X_{n+1} = X_n - \int \nu(dy) \frac{\nabla K(X_n, y)}{\varrho_n K(y)}\ \Delta t + \sqrt{2\lambda} \Delta W_n, \qquad X_0 \sim \varrho_0$$

where $\Delta W_n \sim \mathcal{N}(0, \Delta t^2)$.

Take $X_0 \sim \mathcal{N}(\mu, \sigma_0^2)$. Then

$$\begin{aligned}
\int \nu(dy) \frac{\nabla K(X_0, y)}{\varrho_0 K(y)} &= \int \mathcal{N}(y; \mu, \sigma_\nu^2) \frac{y - X_0}{\sigma_K^2} \frac{\mathcal{N}(y; X_0, \sigma_K^2)}{\mathcal{N}(y; \mu, \sigma_K^2 + \sigma_0^2)}\ dy\\
&= \frac{\sqrt{\sigma_K^2 + \sigma_0^2}}{\sqrt{2\pi} \sigma_K^3 \sigma_\nu} \int (y - X_0) \exp\left(-(y - \mu)^2 \left(\frac{1}{2\sigma_\nu^2} - \frac{1}{2(\sigma_K^2 + \sigma_0^2)} \right) \right) \exp\left(-\frac{(y - X_0)^2}{2\sigma_K^2} \right)\ dy\\
&= \exp\left(-\mu^2 \frac{\sigma_K^2 + \sigma_0^2 - \sigma_\nu^2}{2\sigma_\nu^2 (\sigma_K^2 + \sigma_0^2)} - \frac{X_0^2}{2\sigma_K^2} \right) \frac{\sqrt{\sigma_K^2 + \sigma_0^2}}{\sqrt{2\pi} \sigma_K^3 \sigma_\nu}\\
&\quad \int (y - X_0) \exp\left(\frac{-y^2(\beta + \gamma) + 2y(\mu\beta + X_0\gamma)}{2\sigma_K^2 \sigma_\nu^2 (\sigma_K^2 + \sigma_0^2)} \right)\ dy
\end{aligned}$$

where $\beta := \sigma_K^4 + \sigma_K^2 \sigma_0^2 - \sigma_\nu^2 \sigma_K^2$ and $\gamma := \sigma_\nu^2 (\sigma_K^2 + \sigma_0^2)$.

The exponential inside the integral becomes

$$\begin{aligned}
\exp\left(\frac{-y^2(\beta + \gamma) + 2y(\mu\beta + X_0\gamma)}{2\sigma_K^2 \sigma_\nu^2 (\sigma_K^2 + \sigma_0^2)} \right) &= \exp\left(\frac{-y^2 + 2y(\mu\beta + X_0\gamma)/(\beta + \gamma)}{2\sigma_K^2 \sigma_\nu^2 (\sigma_K^2 + \sigma_0^2)(\beta + \gamma)^{-1}} \right)\\
&= \exp\left(\frac{-\left(y - (\mu\beta + X_0\gamma)/(\beta + \gamma)\right)^2}{2\sigma_K^2 \sigma_\nu^2 (\sigma_K^2 + \sigma_0^2)(\beta + \gamma)^{-1}} \right) \exp\left(\frac{\left((\mu\beta + X_0\gamma)/(\beta + \gamma)\right)^2}{2\sigma_K^2 \sigma_\nu^2 (\sigma_K^2 + \sigma_0^2)(\beta + \gamma)^{-1}} \right)\\
&= \frac{\sqrt{2\pi} \sigma_K \sigma_\nu \sqrt{\sigma_K^2 + \sigma_0^2}}{\sqrt{\beta + \gamma}} \mathcal{N}\left(y; \frac{\mu\beta + X_0\gamma}{\beta + \gamma}, \sigma_K^2 \sigma_\nu^2 (\sigma_K^2 + \sigma_0^2)(\beta + \gamma)^{-1} \right)\\
&\quad \exp\left(\frac{\left((\mu\beta + X_0\gamma)/(\beta + \gamma)\right)^2}{2\sigma_K^2 \sigma_\nu^2 (\sigma_K^2 + \sigma_0^2)(\beta + \gamma)^{-1}} \right).
\end{aligned}$$

Thus

$$\int \nu(dy)\frac{\nabla K(X_0,y)}{\varrho_0 K(y)} = \exp\left(-\mu^2\frac{\sigma_K^2+\sigma_0^2-\sigma_\nu^2}{2\sigma_\nu^2(\sigma_K^2+\sigma_0^2)} - \frac{X_0^2}{2\sigma_K^2}\right)\frac{\sqrt{\sigma_K^2+\sigma_0^2}}{\sqrt{2\pi}\sigma_K^3\sigma_\nu}\exp\left(\frac{((\mu\beta+X_0\gamma)/(\beta+\gamma))^2}{2\sigma_K^2\sigma_\nu^2(\sigma_K^2+\sigma_0^2)(\beta+\gamma)^{-1}}\right)$$

$$\frac{\sqrt{2\pi}\sigma_K\sigma_\nu\sqrt{\sigma_K^2+\sigma_0^2}}{\sqrt{\beta+\gamma}}\int (y-X_0)\mathcal{N}\left(y;\frac{\mu\beta+X_0\gamma}{\beta+\gamma},\sigma_K^2\sigma_\nu^2(\sigma_K^2+\sigma_0^2)(\beta+\gamma)^{-1}\right)\,dy$$

$$= \frac{(\sigma_K^2+\sigma_0^2)}{\sigma_K^2\sqrt{\beta+\gamma}}\exp\left(-\frac{(\mu^2\beta+X_0^2\gamma)(\beta+\gamma)}{2\sigma_K^2\sigma_\nu^2(\sigma_K^2+\sigma_0^2)(\beta+\gamma)}\right)\exp\left(\frac{(\mu\beta+X_0\gamma)^2}{2\sigma_K^2\sigma_\nu^2(\sigma_K^2+\sigma_0^2)(\beta+\gamma)}\right)\left(\frac{\mu\beta+X_0\gamma}{\beta+\gamma}-X_0\right)$$

$$= \frac{(\sigma_K^2+\sigma_0^2)}{\sigma_K^2\sqrt{\beta+\gamma}}\exp\left(-\frac{\beta\gamma\,(\mu-X_0)^2}{2\sigma_K^2\sigma_\nu^2(\sigma_K^2+\sigma_0^2)(\beta+\gamma)}\right)\left(\frac{\mu\beta+X_0\gamma}{\beta+\gamma}-X_0\right)$$

$$= \frac{(\sigma_K^2+\sigma_0^2)}{\sigma_K^2\sqrt{\beta+\gamma}}\exp\left(-\frac{\beta\gamma\,(\mu-X_0)^2}{2\sigma_K^2\sigma_\nu^2(\sigma_K^2+\sigma_0^2)(\beta+\gamma)}\right)\frac{(\mu-X_0)\beta}{\beta+\gamma}$$

$$= \frac{(\sigma_K^2+\sigma_0^2)(\sigma_K^2+\sigma_0^2-\sigma_\nu^2)}{(\beta+\gamma)^{3/2}}\exp\left(-\frac{\beta\gamma\,(\mu-X_0)^2}{2\sigma_K^2\sigma_\nu^2(\sigma_K^2+\sigma_0^2)(\beta+\gamma)}\right)(\mu-X_0)$$

$$= \frac{(\sigma_K^2+\sigma_0^2)(\sigma_K^2+\sigma_0^2-\sigma_\nu^2)}{(\beta+\gamma)^{3/2}}\exp\left(-\frac{(\mu-X_0)^2}{2(\beta+\gamma)(\sigma_K^2+\sigma_0^2-\sigma_\nu^2)^{-1}}\right)(\mu-X_0)$$

Finally,

$$X_1 = X_0 - \frac{(\sigma_K^2+\sigma_0^2)(\sigma_K^2+\sigma_0^2-\sigma_\nu^2)}{(\beta+\gamma)^{3/2}}\exp\left(-\frac{(\mu-X_0)^2}{2(\beta+\gamma)(\sigma_K^2+\sigma_0^2-\sigma_\nu^2)^{-1}}\right)(\mu-X_0)\Delta t + \sqrt{2\lambda}\Delta W_1$$

But what is the distribution of X_1?

1. Non-explosion condition

$$\int \nu(dy)\nabla K(x,y) = \int \mathcal{N}(y;\mu,\sigma_\nu^2)\frac{y-x}{\sigma_K^2}\mathcal{N}(y;x,\sigma_K^2)\,dy$$

$$= \frac{(\mu-1)\sigma_K^2+(x-1)\sigma_\nu^2}{\sigma_K^2(\sigma_K^2+\sigma_\nu^2)^2}\mathcal{N}(x;\mu,\sigma_K^2+\sigma_\nu^2)$$

2. Ergodicity condition

π invariant measure (Gaussian)

$$\int \nu(dy)\frac{\Delta K(x,y)}{\pi K(y)} = \int \frac{\mathcal{N}(y;\mu,\sigma_\nu^2)}{\mathcal{N}(y;\mu,\sigma_K^2+\sigma_\pi^2)}\left(\frac{(y-x)^2}{\sigma_K^2}-1\right)\frac{\mathcal{N}(y;x,\sigma_K^2)}{\sigma_K^2}\,dy$$

$$= \frac{(\sigma_K^2+\sigma_\pi^2)}{\sigma_K^2\sqrt{(\beta+\gamma)^{3/2}}}\exp\left(-\frac{\beta\gamma\,(x-\mu)^2}{2\sigma_K^2\sigma_\nu^2(\sigma_K^2+\sigma_0^2)(\beta+\gamma)}\right)\left(\sigma_K^2\sigma_\nu^2(\sigma_K^2+\sigma_\pi^2)(\beta+\gamma)+\beta^2(x-\mu)^2\right)$$

3. KL

For this example we can also compute the value of

$$E(\rho) = \mathrm{KL}(\nu,\rho K) - \lambda\mathrm{Ent}(\rho)$$

if we assume that $\rho(x)$ is $\mathcal{N}(x;\mu,\sigma^2)$.

We have

$$E(\rho) = \frac{1}{2}\log\frac{\sigma^2+\sigma_K^2}{\sigma_\nu^2} + \frac{\sigma_\nu^2}{2(\sigma^2+\sigma_K^2)} - \frac{1}{2} - \lambda\left(\frac{1}{2}+\frac{1}{2}\log(2\pi\sigma^2)\right)$$

differentiating w.r.t. σ^2:

$$\begin{aligned}
&\frac{1}{2}\frac{1}{\sigma^2+\sigma_K^2}-\frac{\sigma_\nu^2}{2(\sigma^2+\sigma_K^2)^2}-\frac{\lambda}{2\sigma^2}=0\\
&\frac{1}{\sigma^2+\sigma_K^2}-\frac{\sigma_\nu^2}{(\sigma^2+\sigma_K^2)^2}-\frac{\lambda}{\sigma^2}=0\\
&\frac{\sigma^2(\sigma^2+\sigma_K^2)-\sigma^2\sigma_\nu^2-\lambda(\sigma^2+\sigma_K^2)^2}{\sigma^2(\sigma^2+\sigma_K^2)^2}=0\\
&\sigma^4(1-\lambda)+\sigma^2(\sigma_K^2-\sigma_\nu^2-2\lambda\sigma_K^2)-\lambda\sigma_K^4=0
\end{aligned}$$

setting $t=\sigma^2$ we get the second order equation

$$t^2(1-\lambda)+t(\sigma_K^2-\sigma_\nu^2-2\lambda\sigma_K^2)-\lambda\sigma_K^4=0.$$

For $\lambda=0$ we get $t^2+t(\sigma_K^2-\sigma_\nu^2)=0$ whose solution is the solution of the EM approach. For $\lambda=1$ we get $-t(\sigma_K^2+\sigma_\nu^2)-\sigma_K^4=0$ which has no solutions. For $\lambda\neq 1$

$$t_{1,2}=\frac{-(\sigma_K^2-\sigma_\nu^2-2\lambda\sigma_K^2)\pm\sqrt{\sigma_K^4+\sigma_\nu^4-2\sigma_K^2\sigma_\nu^2(1-2\lambda)}}{2(1-\lambda)}.$$

The solution is then given by

$$\sigma^2(\lambda)=\frac{-(\sigma_K^2-\sigma_\nu^2-2\lambda\sigma_K^2)+\sqrt{\sigma_K^4+\sigma_\nu^4-2\sigma_K^2\sigma_\nu^2(1-2\lambda)}}{2(1-\lambda)}$$

which is positive for $\lambda<1$.