Divide-and-Conquer sequential Monte Carlo with applications to high dimensional filtering

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Joint work with Adam M. Johansen

Outline

Spatio-temporal Data

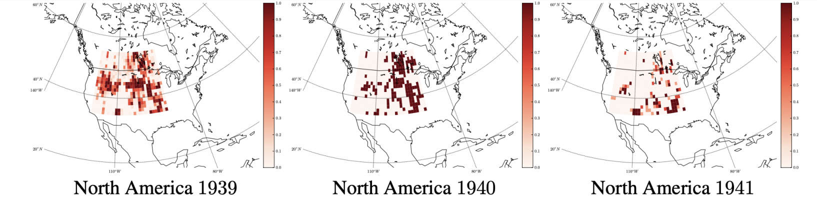
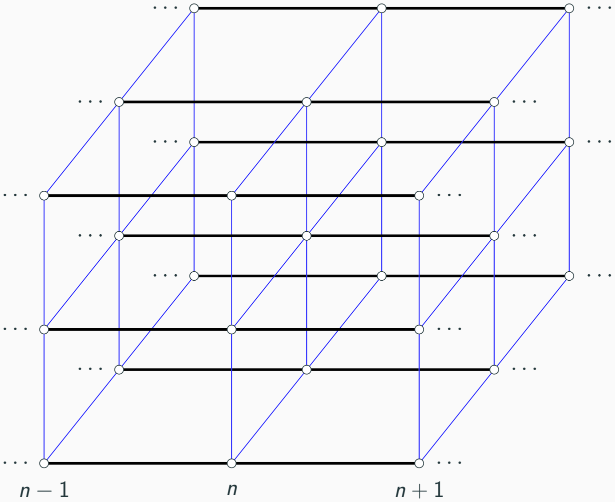


Figure 1: Drought Detection from Rain Precipitations (Næsseth et al., 2015).

- **Observe** precipitation (in millimeters) for each location and year
- **Recover** if there is a drought in that location

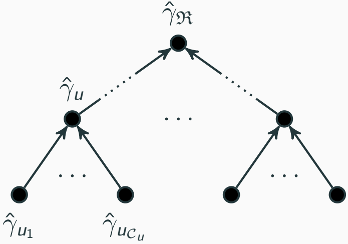
Spatial State Space Model



Outline

Divide-and-Conquer SMC (DaC-SMC)

An extension of standard SMC in which the sequence of target distributions $\{\hat{\gamma}_u\}_{u \in \mathbb{T}}$ evolves on a tree $\mathbb{T}$ rather than on a line (Lindsten et al., 2017).



Divide-and-Conquer SMC (DaC-SMC)

At each node u we have a set of N weighted particles $\{X_u^i, W_u^i\}_{i=1}^N$ approximating $\hat{\gamma}_u$.

To evolve use standard SMC ingredients

- transition/mutation: $X_u^i \sim M_u((X_{\ell(u)}^i, X_{r(u)}^i), \cdot)$
- reweighting: $W_u^i \propto G_u(X_u^i)$
- resampling

with the addition of a **merging** step when two branches of the tree merge.

The Merge Step

- **Aim:** Given two populations of weighted particles on the left and the right child, $\{X^i_{\ell(u)}, W^i_{\ell(u)}\}_{i=1}^N$ and $\{X^i_{r(u)}, W^i_{r(u)}\}_{i=1}^N$, approximating $\hat{\gamma}_{\ell(u)}$ and $\hat{\gamma}_{r(u)}$ build an approximation of $\hat{\gamma}_u$

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- **Ingredients:**
  - the (weighted) product form estimator (Kuntz et al., 2022)

$$\hat{\gamma}^N_{\ell(u)} \times \hat{\gamma}^N_{r(u)} = \frac{1}{N^2} \sum_{i_1=1}^N \sum_{i_2=1}^N W^{i_1}_{\ell(u)} W^{i_2}_{r(u)} \delta_{(X^{i_1}_{\ell(u)}, X^{i_2}_{r(u)})}$$

  - N^2 *mixture* (importance) weights

$$m_u^{(i_1,i_2)} := W^{i_1}_{\ell(u)} W^{i_2}_{r(u)} \frac{\hat{\gamma}_u(x^{i_1}_{\ell(u)}, x^{i_2}_{r(u)})}{\hat{\gamma}_{\ell(u)}(x^{i_1}_{\ell(u)})\hat{\gamma}_{r(u)}(x^{i_2}_{r(u)})}$$

  - a resampling scheme to obtain $\{\tilde{X}^i_u, 1/N\}_{i=1}^N$ from $\{(X^{i_1}_{\ell(u)}, X^{i_2}_{r(u)}), m_u^{(i_2,i_2)}\}_{i_1,i_2=1}^N$

Outline

Divide and Conquer SMC for Filtering

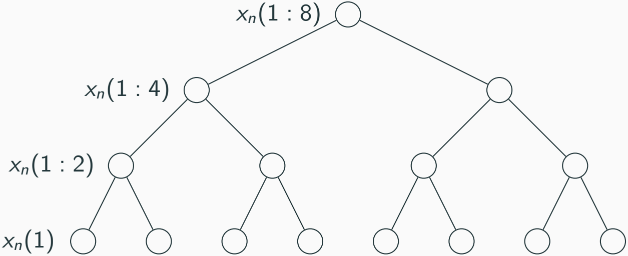

Figure 2: Space decomposition for $d = 8$.

Choice of Distributions

Fix n. We need $\{\hat{\gamma}_{n,u}\}_{u \in \mathbb{T}}$.

Constraints:

- $\hat{\gamma}_{n,\mathfrak{R}}(x_n) \propto p(x_n | y_{1:n})$
- for $u \neq \mathfrak{R}$, $\hat{\gamma}_{n,u}$ only depends on a subset of $x_n(1:d)$
- we can compute

$$\frac{\hat{\gamma}_{n,u}(x_{n,\ell(u)}, x_{n,r(u)})}{\hat{\gamma}_{n,\ell(u)}(x_{n,\ell(u)})\hat{\gamma}_{n,r(u)}(x_{n,r(u)})}$$

Choice of Distributions

Recall

$$p(x_n|y_{1:n}) \propto g_n(x_n, y_n) \int f_n(x_{n-1}, x_n) p(x_{n-1}|y_{1:n-1}) \mathrm{d}x_{n-1}.$$

Use auxiliary functions $f_{n,u}, g_{n,u}$ such that $f_{n,\mathfrak{R}} = f_n$, $g_{n,\mathfrak{R}} = g_n$ and for $u \in \mathbb{T} \setminus \mathfrak{R}$, $f_{n,u}$ and $g_{n,u}$ serve as proxies for marginals of f_n, g_n

$$\hat{\gamma}_{n,u}(x_{n,u}) = g_{n,u}(x_{n,u}, (y_n(i))_{i \in u}) \int f_{n,u}(x_{n-1}, x_{n,u}) \hat{\gamma}_{n-1,\mathfrak{R}}(x_{n-1}) \mathrm{d}x_{n-1}$$

Choice of Distributions

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The integral above is intractable, so we define approximate targets as in marginal particle filters (Klaas et al., 2005)

$$\hat{\gamma}_{n,u}(x_{n,u}) = g_{n,u}(x_{n,u}, (y_n(i))_{i \in u}) W_{n-1}^i \sum_{i=1}^{N} f_{n,u}(X_{n-1,\mathfrak{R}}^i, x_{n,u})$$

Divide and Conquer SMC for Filtering

At time n:

- start with $\{X^i_{n-1,\mathfrak{R}}, W^i_{n-1}\}_{i=1}^N$ approximating $p(x_{n-1}|y_{1:n-1})$ at the leaf level
- define targets

$$\hat{\gamma}_{n,u}(x_{n,u}) = g_{n,u}(x_{n,u}, (y_n(i))_{i\in u}) W^i_{n-1} \sum_{i=1}^{N} f_{n,u}(X^i_{n-1,\mathfrak{R}}, x_{n,u})$$

- use DaC to move up the space from the leaves to the root, i.e. from d particle populations approximating 1-dimensional to one d-dimensional population $\{X^i_{n,\mathfrak{R}}, W^i_n\}_{i=1}^N$ which approximates $p(x_n|y_{1:n})$

Linear Gaussian Model

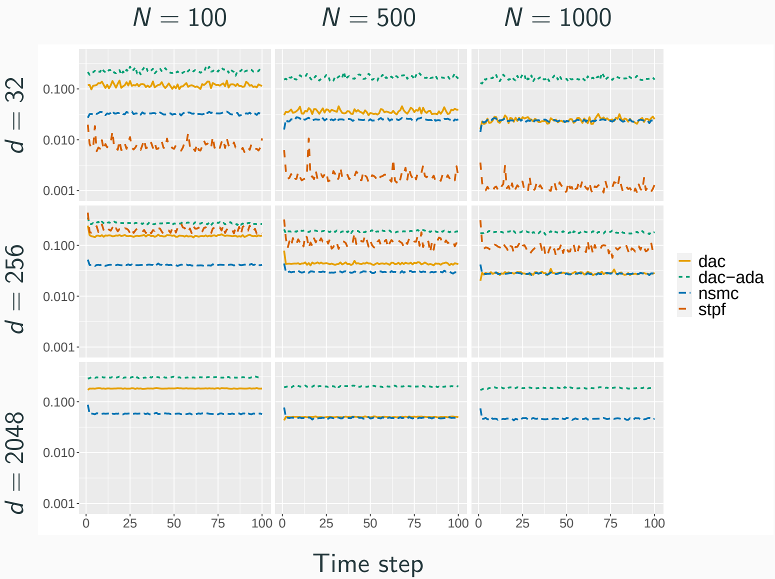
$$
\begin{aligned}
f_1(x_1) &= \mathcal{N}(x_1; m_1, \Sigma_1) \\
f_n(x_{n-1}, x_n) &= \mathcal{N}(x_n; Ax_{n-1}, \Sigma) \\
g_n(x_n, y_n) &= \mathcal{N}(y_n; x_n, \sigma_y^2 Id_d),
\end{aligned}
$$

with $\Sigma \in \mathbb{R}^{d\times d}$ a tridiagonal matrix.

We compare the RMSE

$$
\text{RMSE}(x_n(i)) := \frac{\mathbb{E}\left[(\bar{x}_n(i) - \mu_{n,i})^2\right]}{\sigma_{n,i}^2}
$$

Linear Gaussian Model - RMSE



Spatial Model

The components of X_t are indexed by the vertices $v \in V$ of a lattice, where $V = \{1, \dots, d\}^2$, and

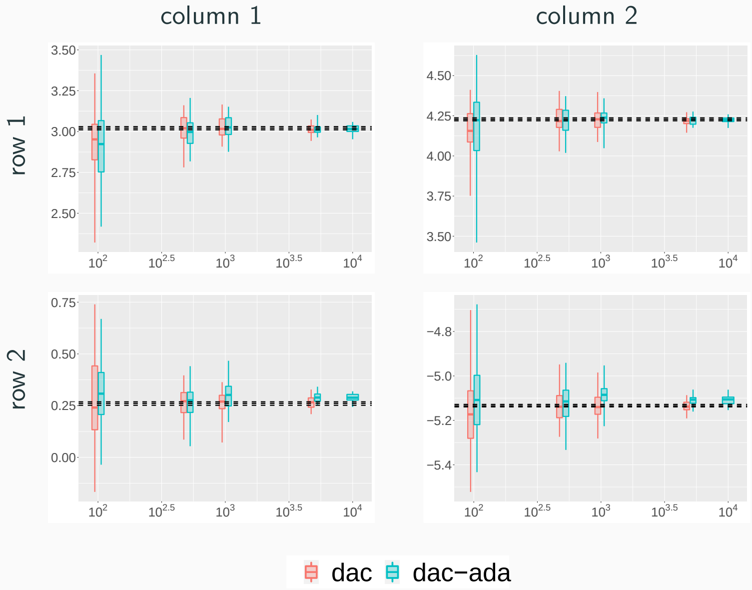
$$X_n(v) = X_{n-1}(v) + U_n(v), \qquad U_n(v) \sim \mathcal{N}(0, \sigma_x^2)$$

and

$$g_t(x_n, y_n) \propto$$

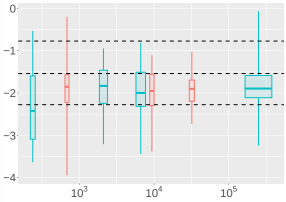
$$\left[1 + \nu^{-1} \sum_{v \in V} \left((y_n(v) - x_n(v)) \sum_{j : D(v,j) \leq r_y} \tau^{D(v,j)} (y_n(j) - x_n(j)) \right) \right]^{-(\nu + d^2)/2} .$$

2×2 grid



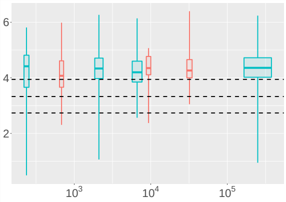
8×8 grid

$(1, 1)$



Runtime / s

$(8, 6)$



Runtime / s

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Pros and Cons

Pros

- No need for analytical form of $f_n(x_n(i) \mid x_n(1:i-1))$
- No need for factorised likelihoods
- Easy to parallelise and distribute

Cons

- Polynomial cost in N (can be mitigated via GPUs)
- Needs specification of $\hat{\gamma}_{n,u}$

Thank you!

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