Coupled Markov chains with applications to Approximate Bayesian Computation for model based clustering

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Abstract

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I. Introduction

Start explaining the initial problem

II. Methods

Parlare dei metodi di ABC e di Maximal coupling (con time average) in modo separato.

poi parlare qua di come siano stati messi insieme o nella sezione dopo?

i. Unbiased Markov chain Monte Carlo methods with couplings

Markov chain Monte Carlo (MCMC) methods provide consistent approximations of high dimensional integrals, namely as the number of iterations goes to infinity. However, these estimators can be potentially biased for any fixed number of iterations, hence the aim is to propose a general construction to produce unbiased estimators of integrals with respect to a target probability distribution.

Glynn and Rhee [?] illustrated a construction on Markov chains represented by iterated random functions; in their approach only two chains must be coupled for the proposed estimator to be unbiased, without further assumptions on the state space or target distribution.

The goal is to estimate

$$\mathbb{E}_\pi[h(X)] = \int h(x)\pi(dx).$$

The estimator is based on a coupled pair of Markov chains, $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq}$, which marginally start from π_0 and evolve accordingly to P.

It must be considered some assumptions:

1. as $t \to \infty$,

   $$\mathbb{E}[h(X_t)] \to \mathbb{E}_\pi[h(X)];$$

   and there exists $\eta > 0$ and $D < \infty$ such that $\mathbb{E}[|h(X_t)|^{2+\eta}] \leq D$ for all $t \geq 0$;

2. the chains are such that the meeting time

   $$\tau = \inf\{t \geq 1 : X_t = Y_{t-1}\}$$

   satisfies $\mathbb{P}(\tau > t) \leq C\delta^t$ for all $t \geq 0$, for some constants $C < \infty$ and $\delta \in (0, 1)$;

3. the chains stay together after meeting:
$$X_t = Y_{t-1} \forall t \geq \tau.$$

Thanks to the previous assumptions it can be proved that:

$$\mathbb{E}_\pi[h(X)] = \mathbb{E}[h(X_k) + \sum_{t=k+1}^{\tau-1} \{h(X_t) - h(Y_{t-1})\}];$$

and the Rhee–Glynn estimator can be defined as:

$$H_k(X,Y) = h(X_k) + \sum_{t=k+1}^{\tau-1} \{h(X_t) - h(Y_{t-1})\}$$

which is unbiased by construction.
time-averaged estimator:

$$H_{k:m}(X,Y) = MCMC_{k:m} + BC_{k:m}$$

where:

$$MCMC_{k:m} = \frac{1}{m-k+1} \sum_{l=k}^{m} h(X_l)$$

is the standard MCMC average;

$$BC_{k:m} = \sum_{l=k+1}^{\tau-1} \min(1, \frac{l-k}{m-k+1})\{h(X_l) - h(Y_{l-1})\}$$

is the bias correction.
The algorithm of the time-average estimator:

1. draw X_0 and Y_0 from an initial distribution π_0 and draw $X_1 \sim P(X_0, \cdot)$;
2. set $t = 1$: while $t < \max\{m, \tau\}$ and:
   a draw $(X_{t+1}, Y_t) \sim \bar{P}\{(X_t, Y_{t-1}), \cdot\}$;
   b set $t \leftarrow t + 1$;
3. compute the time-averaged estimator:
$$H_{k:m}(X,Y) = \frac{1}{m-k+1} \sum_{l=k}^{m} h(X_l)$$
$$+ \sum_{l=k+1}^{\tau-1} \min(1, \frac{l-k}{m-k+1})\{h(X_l) - h(Y_{l-1})\}.$$

Metropolis–Hasting algorithm allow us to calculate the coupled kernel $\bar{P}\{(X_t, Y_{t-1}), \cdot\}$:

1. sample $(X^\star, Y^\star)|(X_t, Y_{t-1})$ from a maximal coupling of $q(X_t, \cdot)$ and $q(Y_{t-1}, \cdot)$;

Table 1: *Example table*

Name		
First name	Last Name	Grade
John	Doe	7.5
Richard	Miles	2

2. sample $U \sim \mathcal{U}([0,1])$;
3. if
$$U \leq \min\left\{1, \frac{\pi(X^\star)q(X^\star, X_t)}{\pi(X_t)q(X_t, X^\star)}\right\}$$
then $X_{t+1} = X^\star$; otherwise $X_t = X_{t-1}$;
4. if
$$U \leq \min\left\{1, \frac{\pi(Y^\star)q(Y^\star, Y_t)}{\pi(Y_t)q(Y_t, Y^\star)}\right\}$$
then $Y_{t+1} = Y^\star$; otherwise $Y_t = Y_{t-1}$.

coupling of random walk mh chains
efficiency, parallel

ii. Approximate Bayesian Computation

Text requiring further explanation[1].

III. Implementation

IV. Results

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$$e = mc^2 \tag{1}$$

[1] Example footnote

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V. Discussion

i. Subsection One

A statement requiring citation [Figueredo and Wolf, 2009]. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Etiam lobortis facilisis sem. Nullam nec mi et neque pharetra sollicitudin. Praesent imperdiet mi nec ante. Donec ullamcorper, felis non sodales commodo, lectus velit ultrices augue, a dignissim nibh lectus placerat pede. Vivamus nunc nunc, molestie ut, ultricies vel, semper in, velit. Ut porttitor. Praesent in sapien. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Duis fringilla tristique neque. Sed interdum libero ut metus. Pellentesque placerat. Nam rutrum augue a leo. Morbi sed elit sit amet ante lobortis sollicitudin. Praesent blandit blandit mauris. Praesent lectus tellus, aliquet aliquam, luctus a, egestas a, turpis. Mauris lacinia lorem sit amet ipsum. Nunc quis urna dictum turpis accumsan semper.

ii. Subsection Two

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References

[Figueredo and Wolf, 2009] Figueredo, A. J. and Wolf, P. S. A. (2009). Assortative pairing and life history strategy - a cross-cultural study. *Human Nature*, 20:317–330.