Numerical Computation of the Ocean QG Baroclinic Modes and Rossby Radii

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The purpose of the following document, is to describe the theoretical background and the numerical resolution implemented in the software available here. The software has been developed to compute the Ocean QG baroclinic modes of motion and the baroclinic deformation radii, based on the Temperature and Salinity data provided by the user.

1 Theoretical Background: QG Potential-Vorticity Equation for Oceanic Synoptic Scales

The theoretical background has been extrapolated and adapted from Pedslosky [1] and Grilli, Pinardi [2].

1.1 Homogeneous Ocean

From the scaling of the primitive equations, we obtain the **non-dimensional** equation for the QG potential vorticity (for a homogeneous ocean):

$$\left[\frac{\partial}{\partial t} + \vec{u_0} \cdot \vec{\nabla}\right] \left[\xi_0' + \beta_0' y' - F\eta_0' + \eta_B'\right] = 0 \quad , \tag{1}$$

where

$$\vec{u_0} = (u_0, v_0, 0) \quad ,$$
$$F = \frac{{f_0}^2 L^2}{gD} \quad \text{inverse of the Burger number,}$$
$$\epsilon = \frac{U}{f_0 L} \qquad \text{(advective) Rossby number,}$$
$$\xi_0 = \frac{U}{L}\xi_0' \quad , \qquad h_B = \epsilon D \eta_B' \quad , \qquad \beta_0 = \frac{U}{L^2}\beta_0' \quad ,$$
$$y = Ly' \quad , \qquad \eta_0 = \frac{f_0 L U}{g}\eta_0' \quad .$$

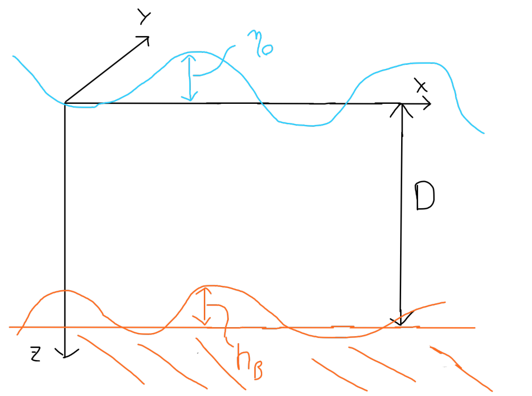

Figure 1: Ocean scheme, used for defining variables and scaling.

Here the superscript ' is used for "non-dimensional" variables; ξ_0 is the relative vorticity of the geostrophic field, η_0 is the free surface elevation respect to the rest level, h_B is the bottom variation respect to the flat bottom; β_0 is the term coming from the expansion of the coriolis parameter: $f = f_0 + \beta_0 y$. The subscript $_0$ refers to the geostrophic approximation. Lastly, D is the ocean depth in case of flat bottom and surface (see Fig.1); U, L are respectively the horizontal velocity and horizontal length scales.
If we now aim to obtain the dimensional equation, we need to substitute the non-dimensional parameters/variables. Therefore, we have

$$\xi_0' - F\eta_0' + \beta_0' y' + \eta_B' = const \qquad ,$$

$$\xi_0' - \frac{{f_0}^2 L^2}{gD}\eta_0' + \beta_0' y' + \frac{h_B}{D}\frac{f_0 L}{U} = const \qquad ,$$

$$(\xi_0' \frac{U}{L}) - \frac{f_0}{D}(\frac{f_0 LU}{g}\eta_0') + (\frac{U}{L^2}\beta_0')(Ly') + f_0\frac{h_B}{D} = const \qquad ,$$

$$\xi_0 - \frac{f_0}{D}\eta_0 + \beta_0 y + \frac{f_0}{D}h_B = const \qquad .$$

Eventually, the **dimensional** QG potential vorticity equation is

$$\boxed{\left[\frac{\partial}{\partial t} + \vec{u_0}\cdot\vec{\nabla}\right]\left[\xi_0 + \beta_0 y - f_0\frac{\eta_0 - h_B}{D}\right] = 0} \quad , \tag{2}$$

$$\Pi = \xi_0 + \beta_0 y - f_0\frac{\eta_0 - h_B}{D} \quad . \tag{3}$$

One may consider a different formulation which may be found in Pedslosky [1] (Eq. 3.17.8):

$$\Pi = \frac{\xi_0 + f_0 + (\beta_0 y + f_0 \frac{h_B}{D}) - f_0 \frac{\eta_0}{D}}{D} \quad ,$$

which is equivalent to Eq.3 considering that D, f_0 are constants.

1.2 Stratified Ocean

For a stratified ocean, in Eq.2 we should add the vortex-tube stretching term due to stratification effects:

$$\left[\frac{\partial}{\partial t} + \vec{u_0} \cdot \vec{\nabla}\right] \left[\xi_0 + \beta_0 y - f_0 \frac{\eta_0 - h_B}{D}\right] = \frac{f_0}{\rho_s} \frac{\partial}{\partial z}(\rho_s w_1) \quad , \tag{4}$$

where the right side term comes from the continuity equation

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{1}{\rho_s} \frac{\partial}{\partial z}(\rho_s w_1) = 0 \quad ; \tag{5}$$

u_1, v_1, w_1 are the first-order expansion term of the velocity components; ρ_s is the basic state upon which fluctuations due to the motion occur, so that the density profile may be described as

$$\begin{aligned} &\rho = \rho_s(z)[1 + \epsilon F \rho'] \quad (\rho' \text{ being non-dimensional}), \\ &\text{with } \rho' = \rho'_0 + \epsilon \rho'_1 + \ldots \qquad (\rho'_0 \text{ non-dimensional}). \end{aligned}$$

Besides, we want to take into account the thermodynamic equation for oceanic motions (considering null heat exchange):

$$\left[\frac{\partial}{\partial t} + \vec{u_0} \cdot \vec{\nabla}\right] \rho_0 + \epsilon w_1 \frac{\partial \rho_s}{\partial z} = 0 \quad , \tag{6}$$

$$\rho_0 = \rho_s \epsilon F \rho'_0$$

so that

$$\left[\frac{\partial}{\partial t} + \vec{u_0} \cdot \vec{\nabla}\right] \rho'_0 D - S w_1 = 0 \tag{7}$$

with

$$S(z) = \frac{{N_s}^2 D^2}{{f_0}^2 L^2} \quad , \qquad {N_s}^2(z) = -\frac{g}{\rho_s} \frac{\partial \rho_s}{\partial z} \quad .$$

If we now combine Eq.4 and 7, neglecting the sea surface and bathymetry contributions to QG potential vorticity (i.e. $\eta_0, h_B \ll D$), we obtain:

$$\left[\frac{\partial}{\partial t} + \vec{u_0} \cdot \vec{\nabla}\right] \left[\xi_0 + \beta_0 y - \frac{f_0 D}{\rho_s} \frac{\partial}{\partial z}\left(\rho_s \frac{\rho'_0}{S}\right)\right] = 0 \quad . \tag{8}$$

If we linearize Eq.8 and assume the hydrostatic balance of the geostrophic field (i.e. $\frac{\partial p_0'}{\partial z'} = -\rho_0'$), we obtain an equation in terms of pressure of the geostrophic field:

$$\frac{\partial}{\partial t}\Big[\frac{1}{\rho_0 f_0}\nabla_0{}^2 p_0 + \frac{f_0 D}{\rho_s}\frac{\partial}{\partial z}\Big(\frac{\rho_s}{S}\frac{\partial p_0'}{\partial z'}\Big)\Big] + \frac{1}{\rho_0 f_0}\beta_0\frac{\partial p_0}{\partial x} = 0 \quad ,$$

$$\frac{\partial}{\partial t}\Big[\nabla_0{}^2 p_0 + \frac{\rho_0 f_0{}^2 D}{\rho_s}\frac{\partial}{\partial z}\Big(\rho_s \frac{f_0{}^2 L^2}{N_s{}^2 D^2}\frac{\partial p_0'}{\partial z'}\Big)\Big] + \beta_0\frac{\partial p_0}{\partial x} = 0 \quad ,$$

$$\frac{\partial}{\partial t}\Big[\nabla_0{}^2 p_0 + \frac{1}{\rho_s}\frac{\partial}{\partial z}\Big(\rho_s \frac{f_0{}^2}{N_s{}^2}\frac{\partial(\rho_0 f_0{}^2 L^2 p_0')}{\partial (D z')}\Big)\Big] + \beta_0\frac{\partial p_0}{\partial x} = 0 \quad ,$$

$$\boxed{\frac{\partial}{\partial t}\Big[\nabla_0{}^2 p_0 + \frac{1}{\rho_s}\frac{\partial}{\partial z}\Big(\rho_s \frac{f_0{}^2}{N_s{}^2}\frac{\partial p_0}{\partial z}\Big)\Big] + \beta_0\frac{\partial p_0}{\partial x} = 0} \quad ; \tag{9}$$

with $\nabla_0{}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $z = Dz'$, $p_0 = \rho_0 f_0{}^2 L^2 p_0'$.
Decomposing the pressure field into one horizontal, time-dependent part and one vertical part

$$p_0(x, y, z, t) \propto \Re\{e^{i(kx+ly-\sigma t)}\}\Phi(z) \tag{10}$$

and substituting in Eq.9, we obtain the **vertical structure equation**:

$$\frac{1}{\rho_s}\frac{d}{dz}\Big(\rho_s \frac{f_0{}^2}{N_s{}^2}\frac{d\Phi}{dz}\Big) = -\lambda\Phi \quad \text{with} \quad \lambda = -\Big(\frac{\beta_0 k}{\sigma} + k^2 + l^2\Big) \quad . \tag{11}$$

In our software, we resolve Eq.11 assuming the **Boussinesq approximation**, i.e. considering $\rho_s(z) = \rho^* = 1025\frac{kg}{m^3}$ constant when not multiplied by g. Eventually, the equation solved is:

$$\frac{d}{dz}\Big(\frac{f_0{}^2}{N_s{}^2}\frac{d\Phi_n}{dz}\Big) = -\lambda_n\Phi_n \quad \text{where} \quad N_s{}^2 = \frac{g}{\rho^*}\frac{d\rho_s}{dz} \quad . \tag{12}$$

Eq. 12 is solved to find each eigenvalue/eigenvector $n = 0, 1, \dots N^*$ (where N^* is the number of vertical layers considered). While Φ_n is the vertical structure function for each mode, we use the eigenvalues to compute the **baroclinic deformation radius** R_n for each mode, defined as:

$$R_n = \frac{1}{\sqrt{\lambda_n}} \quad . \tag{13}$$

The Boundary Conditions (BCs) applied to Eq.s 11 and 12 are

$$\frac{\partial^2 p_0}{\partial t \partial z} = 0 \quad \text{i.e.} \quad \frac{d\Phi}{dz} = 0 \qquad \text{at } z = 0, -H \qquad . \tag{14}$$

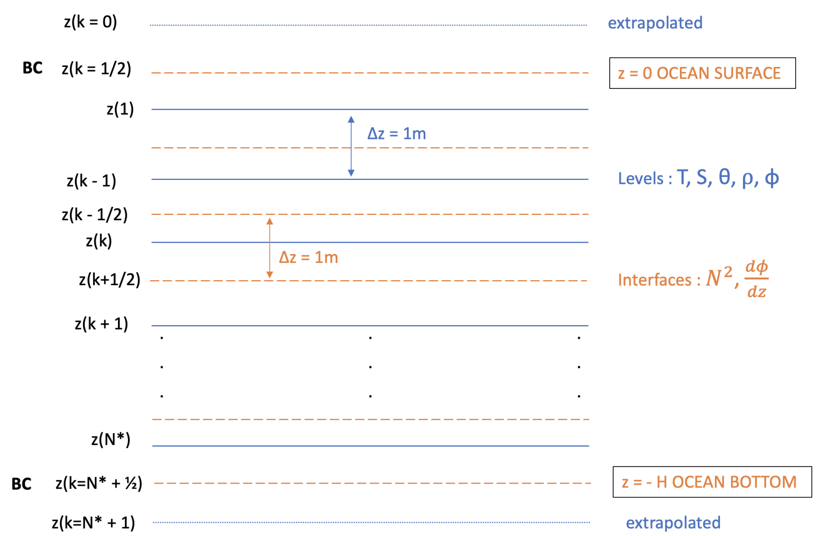

Figure 2: Scheme of the Charney-Phillips staggered grid used for numerical resolution.

2 Numerical Resolution

To solve numerically Eq.12, we use a *Charney-Phillips staggered grid* (Fig.2). On the one hand, a number of vertical levels is defined, where the integer values ($k = 1, 2, \dots N^*$) are the levels on which the ocean variables are defined (in-situ or potential temperature, salinity, potential density). The first level $k = 1$ will be the first value/data available below the sea surface, while the last level $k = N^*$ will be the last value/data available above the sea bottom. On the other hand, on the interfaces corresponding to fractional values ($k = \frac{1}{2}, \frac{3}{2}, \dots N^* + \frac{1}{2}$), we define the Brunt-Vaisala Frequency squared N^2 and the derivatives of the vertical structure function $\frac{d\Phi}{dz}$. The first interface $k = \frac{1}{2}$ corresponds to the sea surface, while the last interface $k = N^* + \frac{1}{2}$ to the sea bottom. This way, we can apply the BCs.
Both levels and interfaces are equally spaced with grid step $\Delta z = 1m$.

2.1 Software Workflow

The software workflow is described in Fig.3.

1. Potential temperature and salinity are read from user-provided NetCDF files. If in-situ temperature is provided, potential temperature can be computed using the software tools.

2. Bathymetry is read from GEBCO 2023 dataset.

3. Potential density is interpolated on the Charney-Phillips levels (1m grid step), from $k = 1$ up to $k = N^*$. In addition, it is extrapolated on two extra levels: $k = 0$ and $k = N^* + 1$. These two levels correspond to values/data respectively slightly above the sea surface and slightly below the sea bottom. These values will be used for computing N^2 at the sea surface and sea bottom interfaces.

4. The Brunt-Vaisala frequency squared is computed using the following centered differences equation:

$$N^2{}_{k+1/2} = -\frac{g}{\rho^*}\frac{\rho_k - \rho_{k+1}}{\Delta z} \qquad \Delta z = 1m \quad , \tag{15}$$

   so that N^2 is defined on the interfaces while computed from variables defined on the levels.

5. Negative values of N^2 (due to instabilities) are removed. N^2 is re-interpolated on the interfaces (to fill the gaps), and smoothed using a low-pass filter to avoid unrealistic spikes.

6. The problem matrix is computed, so that

$$\mathcal{A}\begin{bmatrix}\Phi_1\\ \Phi_2\\ \vdots\\ \Phi_{N^*}\end{bmatrix} = -\lambda\begin{bmatrix}\Phi_1\\ \Phi_2\\ \vdots\\ \Phi_{N^*}\end{bmatrix} \qquad ,$$

$$\mathcal{A} = \frac{{f_0}^2}{\Delta z^2}\begin{bmatrix} -\frac{1}{N^2(\frac{3}{2})} & \frac{1}{N^2(\frac{3}{2})} & \cdots & 0 \\ \frac{1}{N^2(\frac{3}{2})} & -\frac{1}{N^2(\frac{3}{2})} - \frac{1}{N^2(\frac{5}{2})} & \frac{1}{N^2(\frac{5}{2})} & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{N^2(N^*-\frac{1}{2})} & -\frac{1}{N^2(N^*-\frac{1}{2})} \end{bmatrix} \qquad .$$

   To notice, while the derivatives $\frac{d\Phi}{dz}$ are defined on the interfaces, Φ itself is defined on the **levels**. Therefore, the number of vertical modes is equal to the number of vertical levels N^*. Values at interfaces $k = \frac{1}{2}$ and $k = N^* + \frac{1}{2}$ are not used due to BCs.

7. Finally, the eigenvalues/eigenvectors are found using standard libraries (based on LAPACK/DSTEMR routines). The deformation radii are computed according to Eq.13. The vertical structure function is normalized dividing by its norm:

$$\|\Phi_n\| = \frac{1}{N^*\Delta z}\int_{z_{N^*}}^{z_1} \Phi_n^2 dz \qquad ,$$

so that

$$\int_{z_{N^*}}^{z_1} \frac{\Phi_n^2}{\|\Phi_n\|} dz = 1 \qquad .$$

8. A plotting tool is provided to visualize results.

References

[1] J. Pedlosky, *Geophysical Fluid Dynamics*. Springer New York, 1992. ISBN: 9780387963877. Available at: `https://books.google.com/books?id=FXs-uRSDBFYC`.

[2] F. Grilli and N. Pinardi, *Le Cause Dinamiche della Stratificazione Verticale nel Mediterraneo*. Technical Report 3, ISAO, CNR, 1999. (Cited on pp. 25–27).

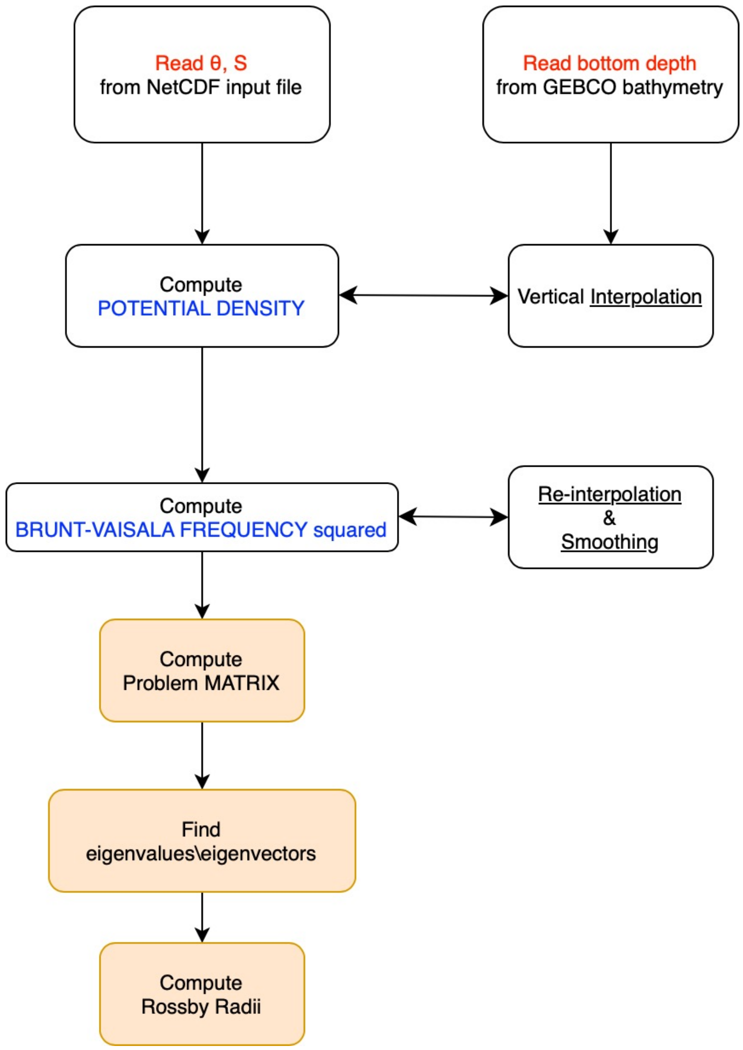

Figure 3: Scheme of the software workflow.