ROBOTICS VISION AND CONTROL

Assignments

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Contents

1 Polynomials

1.1 Cubic polynomials given ti and tf

In this case we use the equation when t $\in [t_i,t_f]$:

$$q(t) = a_3t^3 + a_2t^2 + a_1t + a_0 \tag{1}$$

Given the 4 unknowns a_3,a_2,a_1 and a_0, we need 4 constraints:

$$\begin{aligned} q_i(t) &= a_3t_i^3 + a_2t_i^2 + a_1t_i + a_0 \\ q_f(t) &= a_3t_f^3 + a_2t_f^2 + a_1t_f + a_0 \\ \dot{q_i}(t) &= 3a_3t_i^2 + 2a_2t_i + a_1 \\ \dot{q_f}(t) &= 3a_3t_f^2 + 2a_2t_f + a_1 \end{aligned} \tag{2}$$

1.2 Cubic polynomials given ΔT

In this case we use the equation when t $\in [0,\Delta T]$:

$$q(t) = a_3(t-t_i)^3 + a_2(t-t_i)^2 + a_1(t-t_i) + a_0 \tag{3}$$

Given the 4 unknowns a_3,a_2,a_1 and a_0, we need 4 constraints:

$$\begin{aligned} q_i &= a_0 \\ \dot{q}_i &= a_1 \\ q_f &= a_0 + a_1\Delta T + a_2\Delta T^2 + a_3\Delta T^3 \\ \dot{q}_f &= a_1 + 2a_2\Delta T + 3a_3\Delta T^2 \end{aligned} \tag{4}$$

1.3 5$^{\text{th}}$ order polynomials given ti and tf

In this case we use the equation when t $\in [t_i,t_f]$:

$$q(t) = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \tag{5}$$

Given the 6 unknowns a_5,a_4,a_3,a_2,a_1 and a_0, we need 6 constraints:

$$\begin{aligned} q_i(t) &= a_5t_i^5 + a_4t_i^4 + a_3t_i^3 + a_2t_i^2 + a_1t_i + a_0 \\ \dot{q_i}(t) &= 5a_5t_i^4 + 4a_4t_i^3 + 3a_3t_i^2 + 2a_2t_i \\ \ddot{q_i}(t) &= 20a_5t_i^3 + 12a_4t_i^2 + 6a_3t_i + 2a_2 \\ q_f(t) &= a_5t_f^5 + a_4t_f^4 + a_3t_f^3 + a_2t_f^2 + a_1t_f + a_0 \\ \dot{q_f}(t) &= 5a_5t_f^4 + 4a_4t_f^3 + 3a_3t_f^2 + 2a_2t_f \\ \ddot{q_f}(t) &= 20a_5t_f^3 + 12a_4t_f^2 + 6a_3t_f + 2a_2 \end{aligned} \tag{6}$$

1.4 5th order polynomials given ΔT

In this case we use the equation when t ∈ [0,ΔT]:

$$q(t) = a_5(t-ti)^5 + a_4(t-ti)^4 + a_3(t-ti)^3 + a_2(t-ti)^2 + a_1(t-ti) + a_0 \tag{7}$$

Given the 6 unknowns a_5,a_4,a_3,a_2,a_1 and a_0, we need 6 constraints:

$$\begin{aligned}
q_i &= a_0 \\
\dot{q}_i &= a_1 \\
\ddot{q}_i &= 2a_2 \\
q_f &= a_0 + a_1\Delta T + a_2\Delta T^2 + a_3\Delta T^3 + a_4\Delta T^4 + a_5\Delta T^5 \\
\dot{q}_f &= a_1 + 2a_2\Delta T + 3a_3\Delta T^2 + 4a_4\Delta T^3 + 5a_5\Delta T^4 \\
\ddot{q}_f &= 2a_2 + 6a_3\Delta T + 12a_4\Delta T^2 + 20a_5\Delta T^3
\end{aligned} \tag{8}$$

1.5 7th order polynomials given ti and tf

In this case we use the equation when t ∈ [t_i,t_f]:

$$q(t) = a_7t^7 + a_6t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \tag{9}$$

Given the 8 unknowns a_7,a_6,a_5,a_4,a_3,a_2,a_1 and a_0, we need 8 constraints:

$$\begin{aligned}
q_i(t) &= a_7t_i^7 + a_6t_i^6 + +a_5t_i^5 + a_4t_i^4 + a_3t_i^3 + a_2t_i^2 + a_1t_i + a_0 \\
\dot{q_i}(t) &= 7a_7t_i^6 + 6a_6t_i^5 + 5a_5t_i^4 + 4a_4t_i^3 + 3a_3t_i^2 + 2a_2t_i + a_1 \\
\ddot{q_i}(t) &= 42a_5t_i^5 + 30a_6t_i^4 + 20a_5t_i^3 + 12a_4t_i^2 + 6a_3t_i + 2a_2 \\
\dddot{q_i}(t) &= 210a_5t_i^4 + 120a_6t_i^3 + 60a_5t_i^2 + 24a_4t_i + 6a_3 \\
q_f(t) &= a_7t_f^7 + a_6t_f^6 + a_5t_f^5 + a_4t_f^4 + a_3t_f^3 + a_2t_f^2 + a_1t_f + a_0 \\
\dot{q_f}(t) &= 7a_7t_f^6 + 6a_6t_f^5 + 5a_5t_f^4 + 4a_4t_f^3 + 3a_3t_f^2 + 2a_2t_f + a_1 \\
\ddot{q_f}(t) &= 42t_f^5 + 30a_6t_f^4 + 20a_5t_f^3 + 12a_4t_f^2 + 6a_3t_f + 2a_2 \\
\dddot{q_f}(t) &= 210a_5t_f^4 + 120a_6t_f^3 + 60a_5t_f^2 + 24a_4t_f + 6a_3
\end{aligned} \tag{10}$$

1.6 7th order polynomials given ΔT

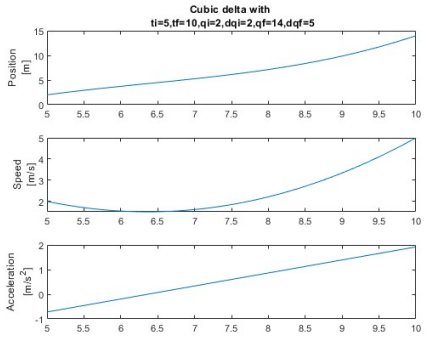
In this case we use the equation when t ∈ [0,ΔT]:

$$q(t) = a_7(t-ti)^7 + a_6(t-ti)^6 + a_5(t-ti)^5 + a_4(t-ti)^4 + a_3(t-ti)^3 + a_2(t-ti)^2 + a_1(t-ti) + a_0 \tag{11}$$

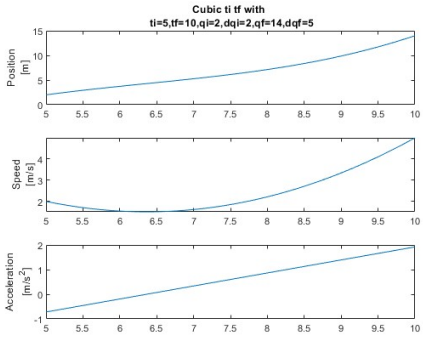
Given the 8 unknowns a_7,a_6,a_5,a_4,a_3,a_2,a_1 and a_0, we need 8 constraints:

$$
\begin{aligned}
q_i &= a_0 \\
\dot{q}_i &= a_1 \\
\ddot{q}_i &= 2a_2 \\
\dddot{q}_i &= 3a_3 \\
q_f &= a_0 + a_1\Delta T + a_2\Delta T^2 + a_3\Delta T^3 + a_4\Delta T^4 + a_5\Delta T^5 + a_6\Delta T^6 + a_7\Delta T^7 \\
\dot{q}_f &= a_1 + 2a_2\Delta T + 3a_3\Delta T^2 + 4a_4\Delta T^3 + 5a_5\Delta T^4 + 6a_6\Delta T^5 + 7a_7\Delta T^6 \\
\ddot{q}_f &= 2a_2 + 6a_3\Delta T + 12a_4\Delta T^2 + 20a_5\Delta T^3 + 30a_6\Delta T^4 + 42a_7\Delta T^5 \\
\dddot{q}_f &= 6a_3 + 24a_4\Delta T + 60a_5\Delta T^2 + 120a_6\Delta T^3 + 210\Delta T^4
\end{aligned}
\tag{12}
$$

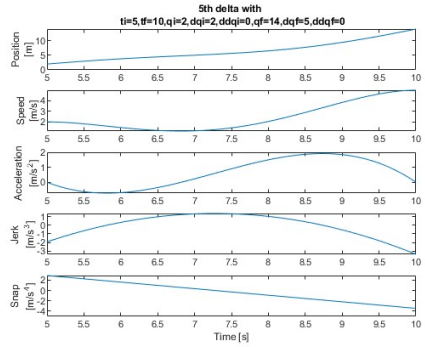
1.7 Results



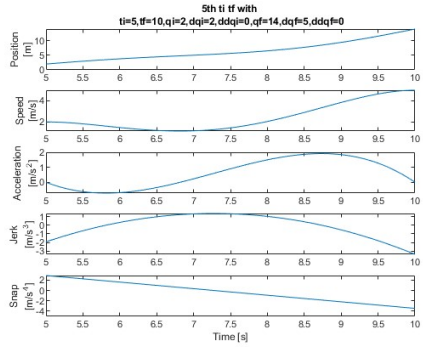
(a) 3^{th} order using ΔT



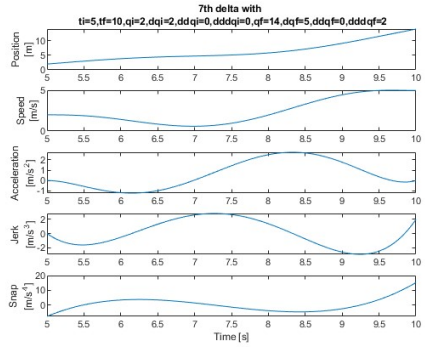
(b) 3^{th} order using t_i and t_f



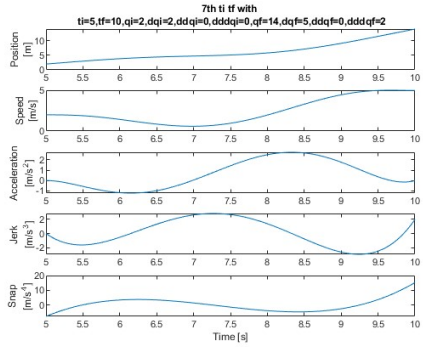
(c) 5^{th} order using ΔT



(d) 5^{th} order using t_i and t_f



(e) 7^{th} order using ΔT



(f) 7^{th} order using t_i and t_f

Figure 1: Results for polynomials trajectories

2 Trapeziodal

In this case we impose the following constraints:

- t_i,t_f
- q_i,q_f

using the following assumptions:

- $t_i = 0$
- $\dot{q_i} = 0$,$\dot{q_f} = 0$
- t_c = acceleration time = deceleration time

We have a first constant acceleration in the start phase (t $\in [0,t_C]$)

$$\begin{cases} q(t) = a_0 + a_1 t + a_2 t^2 \\ \dot{q(t)} = a_1 + 2a_2 t \\ \ddot{q(t)} = 2a_2 \end{cases} \tag{13}$$

Then a constant cruise velocity (t $\in [t_c,t_f - t_c]$)

$$\begin{cases} q(t) = b_0 + b_1 t \\ \dot{q(t)} = b_1 \\ \ddot{q(t)} = 0 \end{cases} \tag{14}$$

Finally a constant deceleration (t $\in [t_f - t_c,t_f]$)

$$\begin{cases} q(t) = c_0 + c_1 t + c_2 t^2 \\ \dot{q(t)} = c_1 + 2c_2 t \\ \ddot{q(t)} = 2c_2 \end{cases} \tag{15}$$

Using the following constraints:

- $q_m = \frac{q_i + q_f}{2}$
- $t_m = \frac{t_f - t_i}{2}$
- $q_c = q(t_i + t_c)$
- $\ddot{q_c} t + c = \frac{q_m - q_c}{t_m - t_c}$
- $q_c = q_i + \frac{1}{2}\ddot{q_c} t_c^2$

We end up with the following equation:

$$\ddot{q_c} t_c^2 - \ddot{q_c}(t_f - t_i) t_c + (q_f - q_i) = 0 \tag{16}$$

$$q(t) = \begin{cases} q_i + \frac{\dot{q_c}}{2t_C} t^2 & 0 \leq t \leq t_c \\ q_i + \dot{q_c}(t - \frac{t_c}{2}) & t_c < t \leq t_f - t_c \\ q_f - \frac{\dot{q_c}}{2t_c}(t_f - t)^2 & t_f - t_c < t \leq t_f \end{cases}$$

2.1 Trapezoidal given t_c with $t_i = 0$ assumption

If we know t_c we can obtain $\ddot{q_c}$ and $\dot{q_c}$ with:

$$\begin{aligned} \ddot{q_c} &= \frac{q_f - q_i}{t_c t_f - t_c^2} \\ \dot{q_c} &= \ddot{q_c} t_c \end{aligned} \tag{17}$$

If $\alpha = \frac{t_c}{t_f}$ with $0 < \alpha \leq \frac{1}{2}$:

$$\begin{aligned} \ddot{q_c} &= \frac{q_f - q_i}{\alpha(1-\alpha) t_f^2} \\ \dot{q_c} &= \frac{q_f - q_i}{(1-\alpha) t_f} \end{aligned} \tag{18}$$

2.2 Trapezoidal given $\ddot{q_c}$ with ti = 0 assumption

If we know $\ddot{q_c}$ and $t_i = 0$ we can obtain t_c as:

$$t_c = \frac{tf}{2} - \frac{1}{2}\sqrt{\frac{t_f^2 \ddot{q_c} - 4(q_f - q_i)}{\ddot{q_c}}} \tag{19}$$

2.3 Trapezoidal given $\dot{q_c}$ with ti = 0 assumption

If we know $\dot{q_c}$ and $t_i = 0$ we can obtain t_c and $\ddot{q_c}$ as:

$$\begin{aligned} t_c &= \frac{q_i - q_f + \dot{q_c} t_f}{\dot{q_c}} \\ \ddot{q_c} &= \frac{\dot{q_c}^2}{q_i - q_f + \dot{q_c} t_f} \end{aligned} \tag{20}$$

2.4 Trapezoidal with preassigned $\ddot{q_c}$ and $\dot{q_c}$

Given the condition $t_i = 0$

$$\begin{aligned} t_c &= \frac{\dot{q_c}}{\ddot{q_c}} \\ t_f &= \frac{\dot{q_c}^2 + \ddot{q_c}(q_f - q_i)}{\dot{q_c} \ddot{q_c}} \end{aligned} \tag{21}$$

We can derive the coefficients of the polynomials as a function of (t_c and $\ddot{q_c}$)

$$q(t) = \begin{cases} q_i + \frac{1}{2}\ddot{q_c} t^2 & 0 \leq t \leq t_c \\ q_i + \ddot{q_c} t_c (t - \frac{t_c}{2}) & t_c < t \leq t_f - t_c \\ q_f - \frac{1}{2}\ddot{q_c}(t_f - t) & t_c < t \leq (t_f - t_c) \end{cases}$$

If $q_f - q_i \geq \frac{\dot{q_c}^2}{\ddot{q_c}}$ we have a linear segment; if not the velocity profile has a triangular shape.

2.5 Trapezoidal with preassigned duration ΔT = t_f - t_i and maximum acceleration $\ddot{q_c}^{\max}$

First we should check the feasibility, i.e $\ddot{q(t)} \leq \ddot{q_c}^{\max}$

$$\ddot{q_c}^{\max} \Delta q > \frac{\left|\dot{q_i}^2 - \dot{q_f}^2\right|}{2} \tag{22}$$

After the check we can compute the constant velocity:

$$\dot{q_c} = \frac{1}{2}(\dot{q_i} + \dot{q_f} + \ddot{q_c}^{\max}\Delta T - \sqrt{(\ddot{q_c}^{\max})^2\Delta T^2 - 4\ddot{q_c}^{\max}\Delta q + 2\ddot{q_c}^{\max}(\dot{q_i} + \dot{q_f})\Delta T - (\dot{q_i} - \dot{q_f})^2}) \tag{23}$$

where $\ddot{q_c}^{\max}$ must satisfy

$$(\ddot{q_c}^{\max})^2\Delta T^2 - 4\ddot{q_c}^{\max}\Delta q + 2\ddot{q_c}^{\max}(\dot{q_i} + \dot{q_f})\Delta T - (\dot{q_i} - \dot{q_f})^2 > 0 \tag{24}$$

The max acceleration must be larger than a limit value:

$$\ddot{q_c}^{\max} \geq \ddot{q_c}^{\lim} \quad = \frac{2\Delta q - (\dot{q_i} + \dot{q_f})\Delta T + \sqrt{4\Delta q^2 - 4\Delta q(\dot{q_i}^2 + \dot{q_f}^2)\Delta T^2}}{\Delta T^2} \tag{25}$$

We can compute the acceleration and deceleration periods:

$$\begin{aligned} t_a &= \frac{\dot{q_c} - \dot{q_i}}{\ddot{q_c}^{\max}} \\ t_d &= \frac{\dot{q_c} - \dot{q_f}}{\ddot{q_c}^{\max}} \end{aligned} \tag{26}$$

2.6 Trapezoidal with preassigned maximum acceleration $\ddot{q_c}^{\max}$ and maximum velocity $\dot{q_c}^{\max}$

First we check for feasibility:

$$\ddot{q_c}^{\max}\Delta q > \frac{|\dot{q_i}^2 - \dot{q_f}^2|}{2} \tag{27}$$

If the trajectory exists,two cases are possible according to the fact tha the maximum velocity is reached ($>$) or not ($<$):

$$\ddot{q_c}^{\max}\Delta q \underset{<}{\geq} (\dot{q_c}^{\max})^2 - \frac{\dot{q_i}^2 + \dot{q_f}^2}{2} \tag{28}$$

If $\dot{q_c}^{\max}$ is reached:

$$\begin{aligned} \dot{q_c} &= \dot{q_c}^{\max} \\ t_a &= \frac{\dot{q_c}^{\max} - \dot{q_i}}{\ddot{q_c}^{\max}} \\ t_d &= \frac{\dot{q_c}^{\max} - \dot{q_f}}{\ddot{q_c}^{\max}} \\ \Delta T &= \frac{\Delta q}{\dot{q_c}} + \frac{\dot{q_c}^{\max}}{2\ddot{q_c}^{\max}}(1 - \frac{\dot{q_i}}{\dot{q_c}^{\max}})^2 + \frac{\dot{q_c}^{\max}}{2\ddot{q_c}^{\max}}(1 - \frac{\dot{q_f}}{\dot{q_c}^{\max}})^2 \end{aligned} \tag{29}$$

If $\dot{q_c}^{\max}$ is not reached:

$$\begin{aligned} \dot{q_c} = \dot{q_c}^{\lim} &= \sqrt{\ddot{q_c}^{\max}\Delta q + \frac{\dot{q_i}^2 + \dot{q_f}^2}{2}} < \dot{q_c}^{\max} \\ t_a &= \frac{\dot{q_c}^{\max} - \dot{q_i}}{\ddot{q_c}^{\max}} \\ t_d &= \frac{\dot{q_c}^{\max} - \dot{q_f}}{\ddot{q_c}^{\max}} \\ \Delta T &= t_a + t_d \end{aligned} \tag{30}$$

2.7 Trajectory with $\dot{q}_i \neq 0, \dot{q}_f \neq 0, \ddot{q}_i = 0, \ddot{q}_f = 0$

Acceleration phase:

$$\begin{aligned} q(t) &= q_i + \dot{q}_i(t - t_i) + \frac{\dot{q}_c - \dot{q}_i}{2t_a}(t - t_i)^2 \\ \dot{q}(t) &= \dot{q}_i + \frac{\dot{q}_c - \dot{q}_i}{t_a}(t - t_i) \\ \ddot{q}(t) &= \frac{\dot{q}_c - \dot{q}_i}{t_a} =: \ddot{q}_c \end{aligned} \tag{31}$$

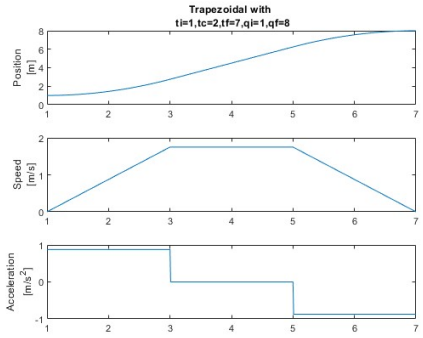
Constant velocity phase:

$$\begin{aligned} q(t) &= q_i + \dot{q}_i \frac{t_a}{2} + \dot{q}_c(t - t_i - \frac{t_a}{2}) \\ \dot{q}(t) &= \dot{q}_c \\ \ddot{q}(t) &= 0 \end{aligned} \tag{32}$$

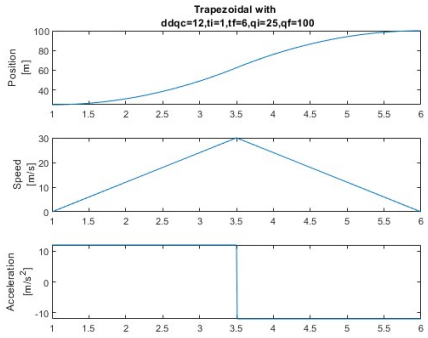
Deceleration phase:

$$\begin{aligned} q(t) &= q_f - \dot{q}_f(t_f - t) - \frac{\dot{q}_c - \dot{q}_f}{2t_d}(t_f - t)^2 \\ \dot{q}(t) &= \dot{q}_f + \frac{\dot{q}_c - \dot{q}_f}{t_d}(t_f - t) \\ \ddot{q}(t) &= -\frac{\dot{q}_c - \dot{q}_f}{t_d} =: -\ddot{q}_c \end{aligned} \tag{33}$$

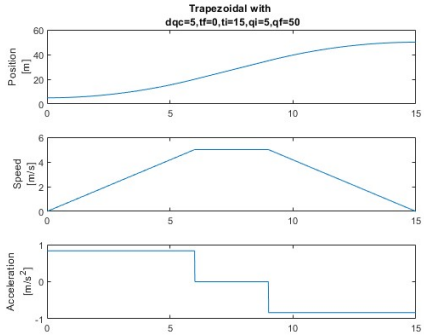
2.8 Results



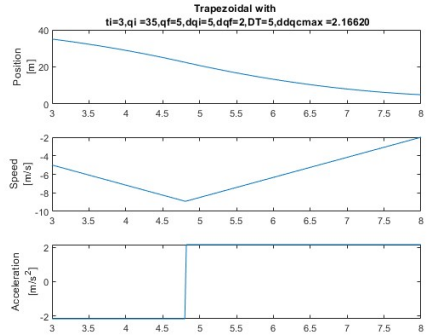
(a) t_i=0 given t_c



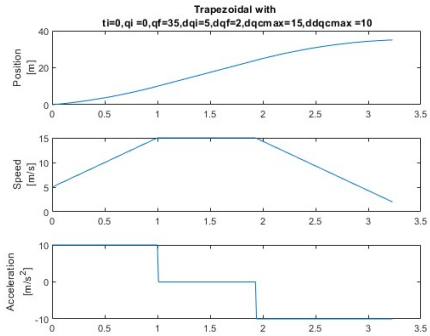
(b) t_i=0 given $\ddot{q_c}$



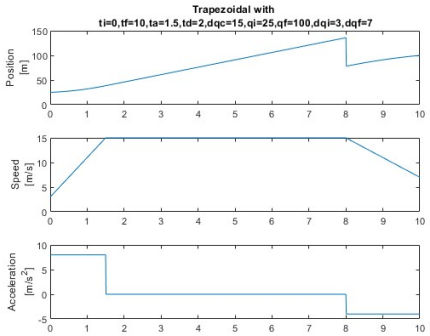
(c) t_i=0 given $\dot{q_c}$



(d) Given ΔT and $q_{c\_\ddot{max}}$



(e) Given $q_{c\_\ddot{max}}$ and $q_{c\_\dot{max}}$



(f) Generic case

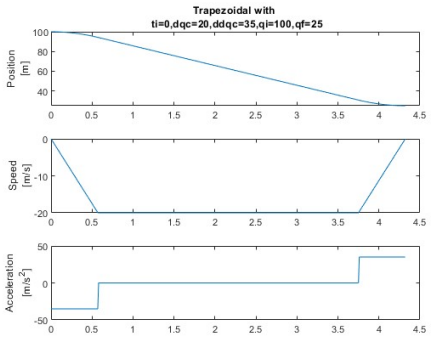


Figure 3: t_i=0 given $\ddot{q_c}$ and $\dot{q_c}$

3 Joint Space Trajectories - Sequence of Points

Given n+1 (q_k, t_k), we want to design a trajectory such that the end-effector passes by each point q_k at a specific instant of time t_k.

3.1 Interpolating polynomials with imposed velocities at path / initial / final points

For k = 1,...,n-1 we compute

$$vk := \frac{q_k - q_{k-1}}{t_k - t_{k-1}} \tag{34}$$

The velocity in each path point is computed as

$$\dot{q_k} = \begin{cases} 0 & if sgn(v_k) \neq sgn(v_{k+1}) \\ \frac{v_k + v_{k+1}}{2} & if sgn(v_k) = sgn(v_{k+1}) \end{cases}$$

We can the use the solution of the system given the conditions on positions and velocity:

$$\begin{aligned} a_0^k &= q_k \\ a_1^k &= \dot{q_k} \\ a_2^k &= \frac{1}{T_k}[\frac{3(q_{k+1} - q_k)}{T_k} - 2\dot{q_k} - \dot{q_{k+1}}] \\ a_3^k &= \frac{1}{T_k^2}[\frac{2(q_k - q_{k+1})}{T_k} + \dot{q_k} + \dot{q_{k+1}}] \end{aligned} \tag{35}$$

Finally we can compute the trajectories:

$$\begin{aligned} q_{k+1} &= a_3^k T_k^3 + a_2^k T_k^2 + a_1^k T_k + a_0^k \\ \dot{q_{k+1}} &= 3a_3^k T_k^2 + 2a_2^k T_k + a_1^k \\ \ddot{q_{k+1}} &= 6a_3^k T_k + 2a_2^k \end{aligned} \tag{36}$$

3.2 Interpolating polynomials with imposed velocities at path points and imposed velocity at initial and final points

We start by initializing the matrix c:

$$c(k) = 3\frac{T_{k+1}}{T_k}(q_{k+1} - q_k) + 3\frac{T_k}{T_{k+1}}(q_{k+2} - q_{k+1}) \tag{37}$$

We then set the matrix A:

$$\begin{aligned} A(0,0) &= 2(T_0 + T_1) \\ A(0,1) &= T(0) \\ A(k,k-1) &= T_{k+1} \\ A(k,k) &= 2(T_k + T_{k+1}) \\ A(k,k+1) &= T_k \\ A(end,end) &= 2(T_{n-2} + T_{n-1}) \\ A(end,end-1) &= T_{n-1} \end{aligned} \tag{38}$$

We apply the Forward elimination part of the Thomas algorithm: For every path point:

$$\begin{aligned} m &= \frac{A(k,k-1)}{A(k-1,k-1)} \\ A(k,k) &= A(k,k) - m * A(k-1,k) \\ c(k) &= c(k) - m * c(k-1) \end{aligned} \tag{39}$$

then we know $dqk(end) = \frac{c(dim)}{A(dim,dim)}$ whilst for the other we apply the backward substitution: Going back from last element:

$$dqk(k) = \frac{c(k) - A(k,k+1) * dqk(k+1)}{A(k,k)} \tag{40}$$

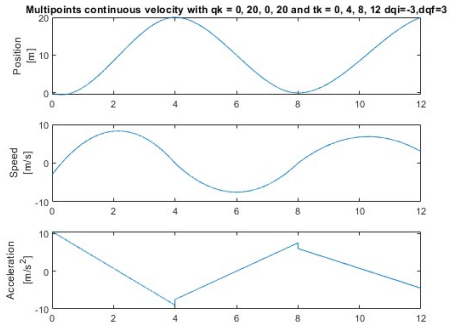
Finally We can the use the solution of the system given the conditions on positions and velocity:

$$\begin{aligned} a_0^k &= q_k \\ a_1^k &= \dot{q_k} \\ a_2^k &= \frac{1}{T_k}[\frac{3(q_{k+1} - q_k)}{T_k} - 2\dot{q_k} - \dot{q_{k+1}}] \\ a_3^k &= \frac{1}{T_k^2}[\frac{2(q_k - q_{k+1})}{T_k} + \dot{q_k} + \dot{q_{k+1}}] \end{aligned} \tag{41}$$

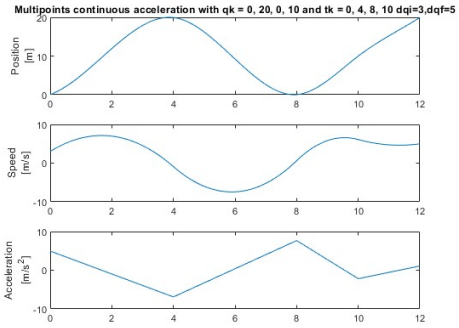
and calculate the trajectories:

$$\begin{aligned} q_{k+1} &= a_3^k T_k^3 + a_2^k T_k^2 + a_1^k T_k + a_0^k \\ \dot{q_{k+1}} &= 3a_3^k T_k^2 + 2a_2^k T_k + a_1^k \\ \ddot{q_{k+1}} &= 6a_3^k T_k + 2a_2^k \end{aligned} \tag{42}$$

3.3 Results



(a) Multipoints with continuous velocity



(b) Multipoints with continuous acceleration

4 Cubical Splines

4.1 Cubic splines based on the accelerations with assigned initial and final velocities

We start by defining the initial and final conditions of the matrices c and A:

$$\begin{aligned} c(1) &= 6(\frac{q_1 - q_0}{T_0} - \dot{q_0}) \\ c(end) &= 6(\dot{q_f} - \frac{q_n - q_{n-1}}{T_{n-1}}) \\ A(0,0) &= 2T_0 \\ A(0,1) &= T_0 \\ A(end, end-1) &= T_{end-1} \\ A(end, end) &= 2T_{end-1} \end{aligned} \tag{43}$$

and the other points:

$$\begin{aligned} A(k, k-1) &= T_{k-1} \\ A(k,k) &= 2T_{k-1} + 2T_i \\ A(k, k+1) &= T_k \\ c(n) &= 6(\frac{q_n - q_{n-1}}{T_{n-1}} - \frac{q_{n-1} - q_{n-2}}{T_{n-2}}) \end{aligned} \tag{44}$$

We then apply the Thomas algorithm to get $\ddot{q_k}$ and we use that in the solved system:

$$\begin{aligned} a_0^k &= q_k \\ a_1^k &= \frac{q_{k+1} - q_k}{T_k} - \frac{\ddot{q_{k+1}} + 2\ddot{q_k}}{6} T_k \\ a_2^k &= \frac{\ddot{q_k}}{2} \\ a_3^2 &= \frac{\ddot{q_{k+1}} - \ddot{q_k}}{6T_k} \end{aligned} \tag{45}$$

Finally we can solve for the trajectories:

$$\begin{aligned} q_k &= a_0^k \\ \ddot{q_k} &= 2a_2^k \\ q_{k+1} &= a_3^k T_k^3 + a_2^k T_k^2 + a_1^k T_k + a_0^k \\ \dot{q_{k+1}} &= 3a_3^k T_k^2 + 2a_2^k T_k + a_1^k \\ \ddot{q_{k+1}} &= 6a_3^k T_k + 2a_2^k \end{aligned} \tag{46}$$

4.2 Smoothing cubic splines

Smoothing cubic splines are designed to approximate and not interpolate a set of given data points, we set:

- w_k: parameters which can be arbitrarily chosen in order to modify the weight of the k-th quadratic error on the global optimization problem
- μ: weights the trade-off between the two conflicting goals (fitting the points and smoothing the trajectory)
- $\lambda = \frac{1-\mu}{6\mu}$

We set the matrices A and c:

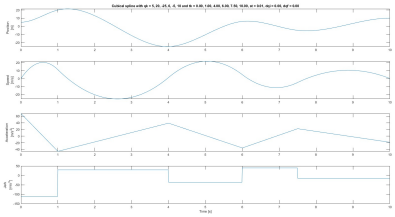
$$\begin{aligned}
A(0,0) &= 2T_0 \\
A(0,1) &= T_0 \\
C(0,0) &= -\frac{6}{T_0} \\
C(0,1) &= \frac{6}{T_0} \\
A(k,k-1) &= T_{k-1} \\
A(k,k) &= 2T_{k-1} + 2T_k \\
A(k,k+1) &= T_k \\
C(k,k-1) &= \frac{6}{T_k - 1} \\
C(k,k) &= -(\frac{6}{T_{k-1}} + \frac{6}{T_k}) \\
C(k,k+1) &= \frac{6}{T_k} \\
A(end,end-1) &= T_{end-1} \\
A(end,end) &= 2T_{end-1} \\
C(end,end-1) &= -\frac{6}{T_{k-1}}
\end{aligned} \tag{47}$$

We calculate:

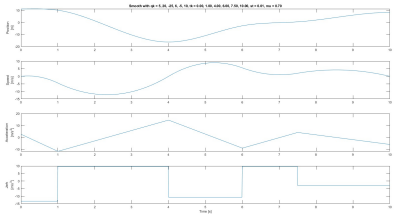
$$\begin{aligned}
\ddot{s} &= Cq * (A + \lambda C W^{-1} C^T)^{-1} \\
s &= q - \lambda W^{-1} C^T \ddot{s}
\end{aligned} \tag{48}$$

Then we compute the trajectories using s as the path points.

4.3 Results



(a) Cubical spline



(b) Smoothed version

5 3D Trajectory using linear and circular motion primitives

The trajectory composed by a set of linear segments is continuous but it is characterized by discontinuous derivatives at the intermediate points.

For the rectilinear path we define the arc length and the velocity as:

$$\begin{aligned} p(s) &= p_i + s\frac{p_f - p_i}{||p_f - p_i||} \\ \frac{dp}{ds} &= \frac{p_f - p_i}{||p_f - p_i||} \end{aligned} \tag{49}$$

For the circular path we first get the z axis of the circumnference cut by the point1-point2 plane by:

$$\begin{aligned} \mathbf{z'} &= \mathbf{p1\_c} \times \mathbf{p2\_c} \\ angle &= arccos(\mathbf{v}_1 \cdot \mathbf{v}_2) \end{aligned} \tag{50}$$

If the result is equal to 0 then we force one direction by modifying the **p2** vector as needed with angle between p1 and p2 equal to π.
We now get the unit vectors:

$$\begin{aligned} \mathbf{x'} &= \frac{\mathbf{p1\text{-}centre}}{||\mathbf{p1\text{-}centre}||} \\ \mathbf{y'} &= \mathbf{x} \times \mathbf{z} \\ \mathbf{z'} &= \frac{\mathbf{z'}}{||\mathbf{z'}||} \end{aligned} \tag{51}$$

We calculate now the parametric representation of the circle in Σ':

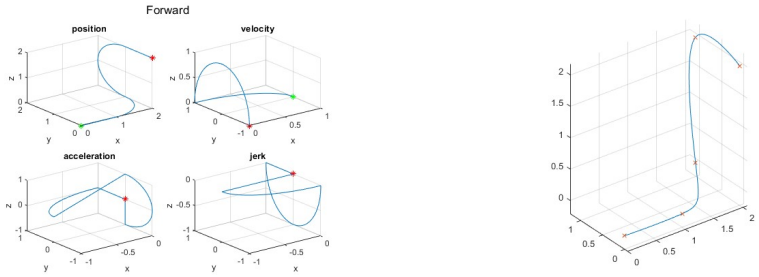
$$p'(u) = \begin{pmatrix} \rho\cos(u) \\ \rho\sin(u) \\ 0 \end{pmatrix} \tag{52}$$

and bring it in Σ:

$$\begin{aligned} p(u) &= \mathbf{c} + R\mathbf{p'}(u) \\ R &= [\mathbf{x'}\ \mathbf{y'}\ \mathbf{z'}] \end{aligned} \tag{53}$$

For the comparison with the spline method we apply the algorithm on x,y and z coordinates separately and then put them together to get the 3d path.

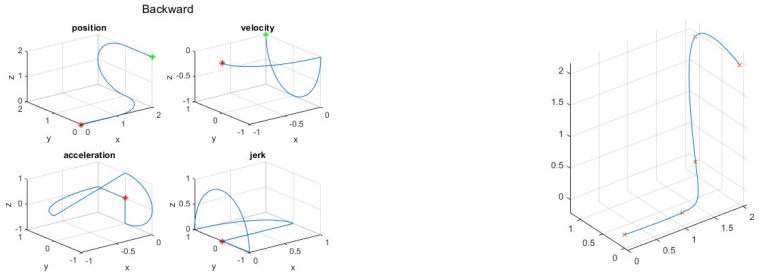
5.1 Results



(a) Linear and circular primitives

(b) Using splines

Figure 6: Forward trajectory



(a) Linear and circular primitives

(b) Using splines

Figure 7: Backward trajectory

6 Trajectory perpendicular to a sphere

The code here is equal to the previous assignment, except for the addition of the 3 Frenet frames T,N,B.

$$
\begin{aligned}
T &= R \begin{pmatrix} -r\ \sin(\text{angles}) \\ r\ \cos(\text{angles}) \\ 0 \end{pmatrix} \\
N &= R \begin{pmatrix} -r\ \cos(\text{angles}) \\ -r\ \sin(\text{angles}) \\ 0 \end{pmatrix} \\
B &= \mathbf{T} \times \mathbf{N}
\end{aligned}
\tag{54}
$$

where r is the radius, R is the matrix described in the previous assignment and **T,N,B** are then reduced to unit vectors.

6.1 Results

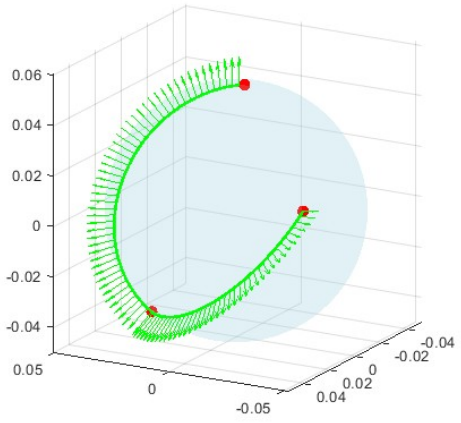


Figure 8: Trajectory perpendicular to a sphere

7 Pick and Place in Unity environment

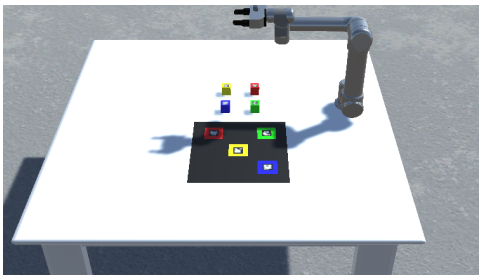
The task that needs to be accomplished is the pick and place of the cubes. Every trajectory in this assignment was calculated with the linear primitive for simplicity and to avoid collisions with the environment.

To complete this task we used a finite state machine approach, where the first phase involves getting the poses with the camera attached to the ur5 robot, to do this the robot does a rectangular like trajectory to scan the environment, saving the initial and final poses for each cube.

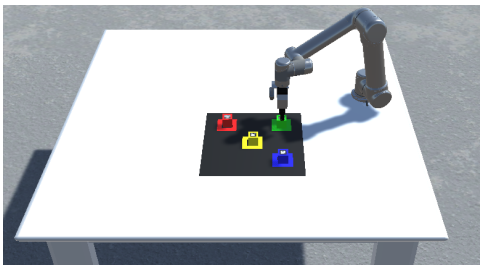
Then we follow a pre-decided order assigning a trajectory that goes from the starting pose to the final pose. To avoid collisions we added a first way point which brings the cube up from its starting pose, then goes horizontally over the destination and then down again; same for the trajectory that is calculated after placing a cube to reach the new cube to pick.

A video of the final result can be found in the GitHub folder.

7.1 Results



(a) Start of simulation



(b) End of the simulation