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# ROBOTICS VISION AND CONTROL

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## Assignments

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# Contents

<b>1</b>	<b>Polynomials</b>	<b>3</b>
1.1	Cubic polynomials given $t_i$ and $t_f$ . . . . .	3
1.2	Cubic polynomials given $\Delta T$ . . . . .	3
1.3	5 <sup>th</sup> order polynomials given $t_i$ and $t_f$ . . . . .	3
1.4	5 <sup>th</sup> order polynomials given $\Delta T$ . . . . .	4
1.5	7 <sup>th</sup> order polynomials given $t_i$ and $t_f$ . . . . .	4
1.6	7 <sup>th</sup> order polynomials given $\Delta T$ . . . . .	4
1.7	Results . . . . .	6
<b>2</b>	<b>Trapezoidal</b>	<b>7</b>
2.1	Trapezoidal given $t_c$ with $t_i = 0$ assumption . . . . .	8
2.2	Trapezoidal given $\ddot{q}_c$ with $t_i = 0$ assumption . . . . .	8
2.3	Trapezoidal given $\dot{q}_c$ with $t_i = 0$ assumption . . . . .	8
2.4	Trapezoidal with preassigned $\ddot{q}_c$ and $\dot{q}_c$ . . . . .	8
2.5	Trapezoidal with preassigned duration $\Delta T = t_f - t_i$ and maximum acceleration $\ddot{q}_c^{\max}$ . . . . .	9
2.6	Trapezoidal with preassigned maximum acceleration $\ddot{q}_c^{\max}$ and maximum velocity $\dot{q}_c^{\max}$ . . . . .	10
2.7	Trajectory with $\dot{q}_i \neq 0, \dot{q}_f \neq 0, \ddot{q}_i = 0, \ddot{q}_f = 0$ . . . . .	11
2.8	Results . . . . .	12
<b>3</b>	<b>Joint Space Trajectories - Sequence of Points</b>	<b>13</b>
3.1	Interpolating polynomials with continuous velocity . . . . .	13
3.2	Interpolating polynomials with continuous acceleration . . . . .	14
3.3	Results . . . . .	15
<b>4</b>	<b>Cubical Splines</b>	<b>16</b>
4.1	Cubic splines based on the accelerations with assigned initial and final velocities . . . . .	16
4.2	Smoothing cubic splines . . . . .	17
4.3	Results . . . . .	18
<b>5</b>	<b>3D Trajectory using linear and circular motion primitives</b>	<b>19</b>
5.1	Results . . . . .	20
<b>6</b>	<b>Trajectory perpendicular to a sphere</b>	<b>21</b>
6.1	Results . . . . .	21
<b>7</b>	<b>Pick and Place in Unity environment</b>	<b>22</b>
7.1	Results . . . . .	22

# 1 Polynomials

## 1.1 Cubic polynomials given ti and tf

In this case we use the equation when  $t \in [t_i, t_f]$ :

$$q(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0 \quad (1)$$

Given the 4 unknowns  $a_3, a_2, a_1$  and  $a_0$ , we need 4 constraints:

$$\begin{aligned} q_i(t) &= a_3 t_i^3 + a_2 t_i^2 + a_1 t_i + a_0 \\ q_f(t) &= a_3 t_f^3 + a_2 t_f^2 + a_1 t_f + a_0 \\ \dot{q}_i(t) &= 3a_3 t_i^2 + 2a_2 t_i + a_1 \\ \dot{q}_f(t) &= 3a_3 t_f^2 + 2a_2 t_f + a_1 \end{aligned} \quad (2)$$

## 1.2 Cubic polynomials given $\Delta T$

In this case we use the equation when  $t \in [0, \Delta T]$ :

$$q(t) = a_3 (t - t_i)^3 + a_2 (t - t_i)^2 + a_1 (t - t_i) + a_0 \quad (3)$$

Given the 4 unknowns  $a_3, a_2, a_1$  and  $a_0$ , we need 4 constraints:

$$\begin{aligned} q_i &= a_0 \\ \dot{q}_i &= a_1 \\ q_f &= a_0 + a_1 \Delta T + a_2 \Delta T^2 + a_3 \Delta T^3 \\ \dot{q}_f &= a_1 + 2a_2 \Delta T + 3a_3 \Delta T^2 \end{aligned} \quad (4)$$

## 1.3 5<sup>th</sup> order polynomials given ti and tf

In this case we use the equation when  $t \in [t_i, t_f]$ :

$$q(t) = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \quad (5)$$

Given the 6 unknowns  $a_5, a_4, a_3, a_2, a_1$  and  $a_0$ , we need 6 constraints:

$$\begin{aligned} q_i(t) &= a_5 t_i^5 + a_4 t_i^4 + a_3 t_i^3 + a_2 t_i^2 + a_1 t_i + a_0 \\ \dot{q}_i(t) &= 5a_5 t_i^4 + 4a_4 t_i^3 + 3a_3 t_i^2 + 2a_2 t_i \\ \ddot{q}_i(t) &= 20a_5 t_i^3 + 12a_4 t_i^2 + 6a_3 t_i + 2a_2 \\ q_f(t) &= a_5 t_f^5 + a_4 t_f^4 + a_3 t_f^3 + a_2 t_f^2 + a_1 t_f + a_0 \\ \dot{q}_f(t) &= 5a_5 t_f^4 + 4a_4 t_f^3 + 3a_3 t_f^2 + 2a_2 t_f \\ \ddot{q}_f(t) &= 20a_5 t_f^3 + 12a_4 t_f^2 + 6a_3 t_f + 2a_2 \end{aligned} \quad (6)$$

#### 1.4 5<sup>th</sup> order polynomials given $\Delta T$

In this case we use the equation when  $t \in [0, \Delta T]$ :

$$q(t) = a_5(t - ti)^5 + a_4(t - ti)^4 + a_3(t - ti)^3 + a_2(t - ti)^2 + a_1(t - ti) + a_0 \quad (7)$$

Given the 6 unknowns  $a_5, a_4, a_3, a_2, a_1$  and  $a_0$ , we need 6 constraints:

$$\begin{aligned} q_i &= a_0 \\ \dot{q}_i &= a_1 \\ \ddot{q}_i &= 2a_2 \\ q_f &= a_0 + a_1\Delta T + a_2\Delta T^2 + a_3\Delta T^3 + a_4\Delta T^4 + a_5\Delta T^5 \\ \dot{q}_f &= a_1 + 2a_2\Delta T + 3a_3\Delta T^2 + 4a_4\Delta T^3 + 5a_5\Delta T^4 \\ \ddot{q}_f &= 2a_2 + 6a_3\Delta T + 12a_4\Delta T^2 + 20a_5\Delta T^3 \end{aligned} \quad (8)$$

#### 1.5 7<sup>th</sup> order polynomials given $ti$ and $tf$

In this case we use the equation when  $t \in [t_i, t_f]$ :

$$q(t) = a_7t^7 + a_6t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \quad (9)$$

Given the 8 unknowns  $a_7, a_6, a_5, a_4, a_3, a_2, a_1$  and  $a_0$ , we need 8 constraints:

$$\begin{aligned} q_i(t) &= a_7t_i^7 + a_6t_i^6 + a_5t_i^5 + a_4t_i^4 + a_3t_i^3 + a_2t_i^2 + a_1t_i + a_0 \\ \dot{q}_i(t) &= 7a_7t_i^6 + 6a_6t_i^5 + 5a_5t_i^4 + 4a_4t_i^3 + 3a_3t_i^2 + 2a_2t_i + a_1 \\ \ddot{q}_i(t) &= 42a_7t_i^5 + 30a_6t_i^4 + 20a_5t_i^3 + 12a_4t_i^2 + 6a_3t_i + 2a_2 \\ \ddot{\ddot{q}}_i(t) &= 210a_7t_i^4 + 120a_6t_i^3 + 60a_5t_i^2 + 24a_4t_i + 6a_3 \\ q_f(t) &= a_7t_f^7 + a_6t_f^6 + a_5t_f^5 + a_4t_f^4 + a_3t_f^3 + a_2t_f^2 + a_1t_f + a_0 \\ \dot{q}_f(t) &= 7a_7t_f^6 + 6a_6t_f^5 + 5a_5t_f^4 + 4a_4t_f^3 + 3a_3t_f^2 + 2a_2t_f + a_1 \\ \ddot{q}_f(t) &= 42a_7t_f^5 + 30a_6t_f^4 + 20a_5t_f^3 + 12a_4t_f^2 + 6a_3t_f + 2a_2 \\ \ddot{\ddot{q}}_f(t) &= 210a_7t_f^4 + 120a_6t_f^3 + 60a_5t_f^2 + 24a_4t_f + 6a_3 \end{aligned} \quad (10)$$

#### 1.6 7<sup>th</sup> order polynomials given $\Delta T$

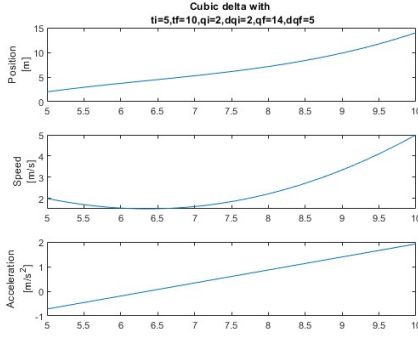
In this case we use the equation when  $t \in [0, \Delta T]$ :

$$q(t) = a_7(t - ti)^7 + a_6(t - ti)^6 + a_5(t - ti)^5 + a_4(t - ti)^4 + a_3(t - ti)^3 + a_2(t - ti)^2 + a_1(t - ti) + a_0 \quad (11)$$

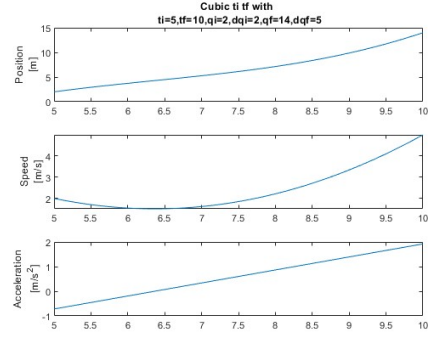
Given the 8 unknowns  $a_7, a_6, a_5, a_4, a_3, a_2, a_1$  and  $a_0$ , we need 8 constraints:

$$\begin{aligned}
q_i &= a_0 \\
\dot{q}_i &= a_1 \\
\ddot{q}_i &= 2a_2 \\
\dddot{q}_i &= 3a_3 \\
q_f &= a_0 + a_1\Delta T + a_2\Delta T^2 + a_3\Delta T^3 + a_4\Delta T^4 + a_5\Delta T^5 + a_6\Delta T^6 + a_7\Delta T^7 \\
\dot{q}_f &= a_1 + 2a_2\Delta T + 3a_3\Delta T^2 + 4a_4\Delta T^3 + 5a_5\Delta T^4 + 6a_6\Delta T^5 + 7a_7\Delta T^6 \\
\ddot{q}_f &= 2a_2 + 6a_3\Delta T + 12a_4\Delta T^2 + 20a_5\Delta T^3 + 30a_6\Delta T^4 + 42a_7\Delta T^5 \\
\dddot{q}_f &= 6a_3 + 24a_4\Delta T + 60a_5\Delta T^2 + 120a_6\Delta T^3 + 210a_7\Delta T^4
\end{aligned} \tag{12}$$

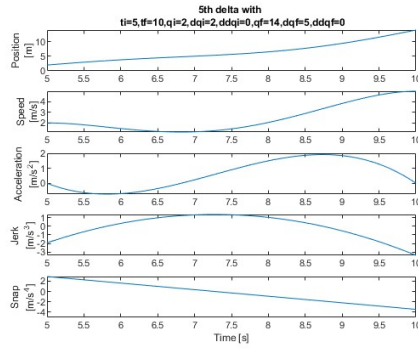
## 1.7 Results



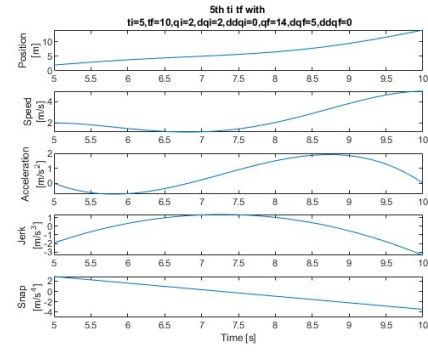
(a) 3<sup>th</sup> order using  $\Delta T$



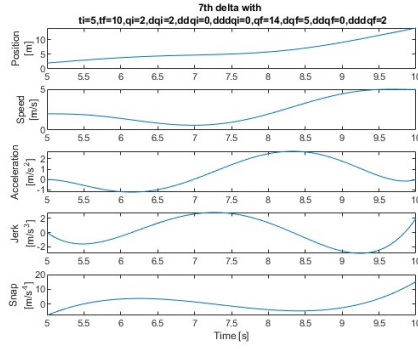
(b) 3<sup>th</sup> order using  $t_i$  and  $t_f$



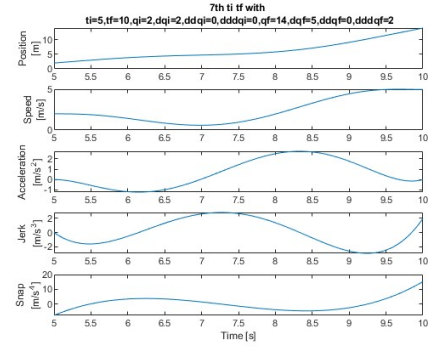
(c) 5<sup>th</sup> order using  $\Delta T$



(d) 5<sup>th</sup> order using  $t_i$  and  $t_f$



(e) 7<sup>th</sup> order using  $\Delta T$



(f) 7<sup>th</sup> order using  $t_i$  and  $t_f$

Figure 1: Results for polynomials trajectories

## 2 Trapeziodal

In this case we impose the following constraints:

- $t_i, t_f$
- $q_i, q_f$

using the following assumptions:

- $t_i = 0$
- $\dot{q}_i = 0, \dot{q}_f = 0$
- $t_c = \text{acceleration time} = \text{deceleration time}$

We have a first constant acceleration in the start phase ( $t \in [0, t_c]$ )

$$\begin{cases} q(t) = a_0 + a_1 t + a_2 t^2 \\ \dot{q}(t) = a_1 + 2a_2 t \\ \ddot{q}(t) = 2a_2 \end{cases} \quad (13)$$

Then a constant cruise velocity ( $t \in [t_c, t_f - t_c]$ )

$$\begin{cases} q(t) = b_0 + b_1 t \\ \dot{q}(t) = b_1 \\ \ddot{q}(t) = 0 \end{cases} \quad (14)$$

Finally a constant deceleration ( $t \in [t_f - t_c, t_f]$ )

$$\begin{cases} q(t) = c_0 + c_1 t + c_2 t^2 \\ \dot{q}(t) = c_1 + 2c_2 t \\ \ddot{q}(t) = 2c_2 \end{cases} \quad (15)$$

Using the following constraints:

- $q_m = \frac{q_i + q_f}{2}$
- $t_m = \frac{t_f - t_i}{2}$
- $q_c = q(t_i + t_c)$
- $\ddot{q}_c t + c = \frac{q_m - q_c}{t_m - t_c}$
- $q_c = q_i + \frac{1}{2} \ddot{q}_c t_c^2$

We end up with the following equation:

$$\ddot{q}_c t_c^2 - \ddot{q}_c (t_f - t_i) t_c + (q_f - q_i) = 0 \quad (16)$$

$$q(t) = \begin{cases} q_i + \frac{\dot{q}_c}{2t_c}t^2 & 0 \leq t \leq t_c \\ q_i + \dot{q}_c(t - \frac{t_c}{2}) & t_c < t \leq t_f - t_c \\ q_f - \frac{\dot{q}_c}{2t_c}(t_f - t)^2 & t_f - t_c < t \leq t_f \end{cases}$$

## 2.1 Trapezoidal given $t_c$ with $t_i = 0$ assumption

If we know  $t_c$  we can obtain  $\ddot{q}_c$  and  $\dot{q}_c$  with:

$$\begin{aligned} \ddot{q}_c &= \frac{q_f - q_i}{t_c t_f - t_c^2} \\ \dot{q}_c &= \ddot{q}_c t_c \end{aligned} \tag{17}$$

If  $\alpha = \frac{t_c}{t_f}$  with  $0 < \alpha \leq \frac{1}{2}$ :

$$\begin{aligned} \ddot{q}_c &= \frac{q_f - q_i}{\alpha(1 - \alpha)t_f^2} \\ \dot{q}_c &= \frac{q_f - q_i}{(1 - \alpha)t_f} \end{aligned} \tag{18}$$

## 2.2 Trapezoidal given $\ddot{q}_c$ with $t_i = 0$ assumption

If we know  $\ddot{q}_c$  and  $t_i = 0$  we can obtain  $t_c$  as:

$$t_c = \frac{t_f}{2} - \frac{1}{2} \sqrt{\frac{t_f^2 \ddot{q}_c - 4(q_f - q_i)}{\ddot{q}_c}} \tag{19}$$

## 2.3 Trapezoidal given $\dot{q}_c$ with $t_i = 0$ assumption

If we know  $\dot{q}_c$  and  $t_i = 0$  we can obtain  $t_c$  and  $\ddot{q}_c$  as:

$$\begin{aligned} t_c &= \frac{q_i - q_f + \dot{q}_c t_f}{\dot{q}_c} \\ \ddot{q}_c &= \frac{\dot{q}_c^2}{q_i - q_f + \dot{q}_c t_f} \end{aligned} \tag{20}$$

## 2.4 Trapezoidal with preassigned $\ddot{q}_c$ and $\dot{q}_c$

Given the condition  $t_i = 0$

$$\begin{aligned} t_c &= \frac{\dot{q}_c}{\ddot{q}_c} \\ t_f &= \frac{\dot{q}_c^2 + \ddot{q}_c(q_f - q_i)}{\dot{q}_c \ddot{q}_c} \end{aligned} \tag{21}$$

We can derive the coefficients of the polynomials as a function of  $(t_c$  and  $\ddot{q}_c)$

$$q(t) = \begin{cases} q_i + \frac{1}{2}\ddot{q}_c t^2 & 0 \leq t \leq t_c \\ q_i + \ddot{q}_c t_c(t - \frac{t_c}{2}) & t_c < t \leq t_f - t_c \\ q_f - \frac{1}{2}\ddot{q}_c(t_f - t)^2 & t_c < t \leq (t_f - t_c) \end{cases}$$



If  $q_f - q_i \geq \frac{\dot{q}_c^2}{\ddot{q}_c}$ , we have a linear segment; otherwise the velocity profile has a triangular shape.

## 2.5 Trapezoidal with preassigned duration $\Delta T = t_f - t_i$ and maximum acceleration $\ddot{q}_c^{\max}$

First we should check the feasibility, i.e.  $\ddot{q}(t) \leq \ddot{q}_c^{\max}$

$$\ddot{q}_c^{\max} \Delta q > \frac{|\dot{q}_i^2 - \dot{q}_f^2|}{2} \quad (22)$$

After the check we can compute the constant velocity:

$$\dot{q}_c = \frac{1}{2}(\dot{q}_i + \dot{q}_f + \ddot{q}_c^{\max} \Delta T - \sqrt{(\ddot{q}_c^{\max})^2 \Delta T^2 - 4\ddot{q}_c^{\max} \Delta q + 2\ddot{q}_c^{\max}(\dot{q}_i + \dot{q}_f) \Delta T - (\dot{q}_i - \dot{q}_f)^2}) \quad (23)$$

where  $\ddot{q}_c^{\max}$  must satisfy

$$(\ddot{q}_c^{\max})^2 \Delta T^2 - 4\ddot{q}_c^{\max} \Delta q + 2\ddot{q}_c^{\max}(\dot{q}_i + \dot{q}_f) \Delta T - (\dot{q}_i - \dot{q}_f)^2 > 0 \quad (24)$$

The max acceleration must be larger than a limit value:

$$\ddot{q}_c^{\max} \geq \ddot{q}_c^{\lim} = \frac{2\Delta q - (\dot{q}_i + \dot{q}_f) \Delta T + \sqrt{4\Delta q^2 - 4\Delta q(\dot{q}_i^2 + \dot{q}_f^2) \Delta T^2}}{\Delta T^2} \quad (25)$$

We can compute the acceleration and deceleration periods:

$$\begin{aligned} t_a &= \frac{\dot{q}_c - \dot{q}_i}{\ddot{q}_c^{\max}} \\ t_d &= \frac{\dot{q}_c - \dot{q}_f}{\ddot{q}_c^{\max}} \end{aligned} \quad (26)$$

## 2.6 Trapezoidal with preassigned maximum acceleration $\ddot{q}_c^{\max}$ and maximum velocity $\dot{q}_c^{\max}$

First we check for feasibility:

$$\ddot{q}_c^{\max} \Delta q > \frac{|\dot{q}_i^2 - \dot{q}_f^2|}{2} \quad (27)$$

If the trajectory exists, two cases are possible according to the fact that the maximum velocity is reached ( $>$ ) or not ( $<$ ):

$$\ddot{q}_c^{\max} \Delta q \begin{cases} \geq \\ \leq \end{cases} (\dot{q}_c^{\max})^2 - \frac{\dot{q}_i^2 + \dot{q}_f^2}{2} \quad (28)$$

If  $\dot{q}_c^{\max}$  is reached:

$$\begin{aligned} \dot{q}_c &= \dot{q}_c^{\max} \\ t_a &= \frac{\dot{q}_c^{\max} - \dot{q}_i}{\ddot{q}_c^{\max}} \\ t_d &= \frac{\dot{q}_c^{\max} - \dot{q}_f}{\ddot{q}_c^{\max}} \\ \Delta T &= \frac{\Delta q}{\dot{q}_c} + \frac{\dot{q}_c^{\max}}{2\ddot{q}_c^{\max}} \left(1 - \frac{\dot{q}_i}{\dot{q}_c^{\max}}\right)^2 + \frac{\dot{q}_c^{\max}}{2\ddot{q}_c^{\max}} \left(1 - \frac{\dot{q}_f}{\dot{q}_c^{\max}}\right)^2 \end{aligned} \quad (29)$$

If  $\dot{q}_c^{\max}$  is not reached:

$$\begin{aligned} \dot{q}_c &= \dot{q}_c^{\lim} = \sqrt{\ddot{q}_c^{\max} \Delta q + \frac{\dot{q}_i^2 + \dot{q}_f^2}{2}} < \dot{q}_c^{\max} \\ t_a &= \frac{\dot{q}_c^{\lim} - \dot{q}_i}{\ddot{q}_c^{\max}} \\ t_d &= \frac{\dot{q}_c^{\lim} - \dot{q}_f}{\ddot{q}_c^{\max}} \\ \Delta T &= t_a + t_d \end{aligned} \quad (30)$$

## 2.7 Trajectory with $\dot{q}_i \neq 0, \dot{q}_f \neq 0, \ddot{q}_i = 0, \ddot{q}_f = 0$

Acceleration phase:

$$\begin{aligned}
 q(t) &= q_i + \dot{q}_i(t - t_i) + \frac{\dot{q}_c - \dot{q}_i}{2t_a}(t - t_i)^2 \\
 \dot{q}(t) &= \dot{q}_i + \frac{\dot{q}_c - \dot{q}_i}{t_a}(t - t_i) \\
 \ddot{q}(t) &= \frac{\dot{q}_c - \dot{q}_i}{t_a} =: \ddot{q}_c
 \end{aligned} \tag{31}$$

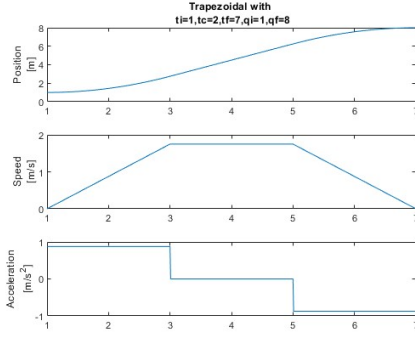
Constant velocity phase:

$$\begin{aligned}
 q(t) &= q_i + \dot{q}_i \frac{t_a}{2} + \dot{q}_c(t - t_i - \frac{t_a}{2}) \\
 \dot{q}(t) &= \dot{q}_c \\
 \ddot{q}(t) &= 0
 \end{aligned} \tag{32}$$

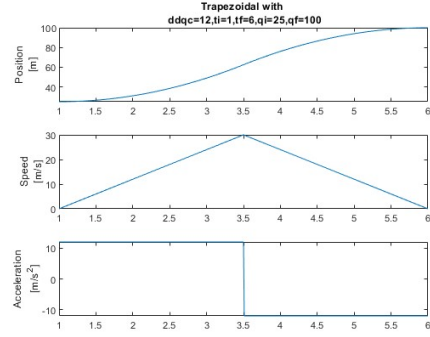
Deceleration phase:

$$\begin{aligned}
 q(t) &= q_f - \dot{q}_f(t_f - t) - \frac{\dot{q}_c - \dot{q}_f}{2t_d}(t_f - t)^2 \\
 \dot{q}(t) &= \dot{q}_f + \frac{\dot{q}_c - \dot{q}_f}{t_d}(t_f - t) \\
 \ddot{q}(t) &= -\frac{\dot{q}_c - \dot{q}_f}{t_d} =: -\ddot{q}_c
 \end{aligned} \tag{33}$$

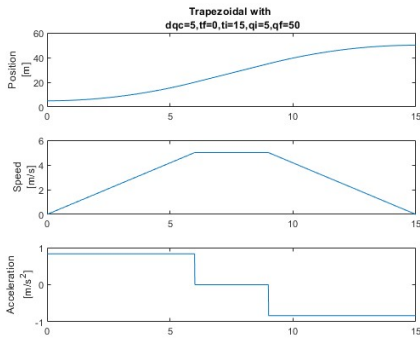
## 2.8 Results



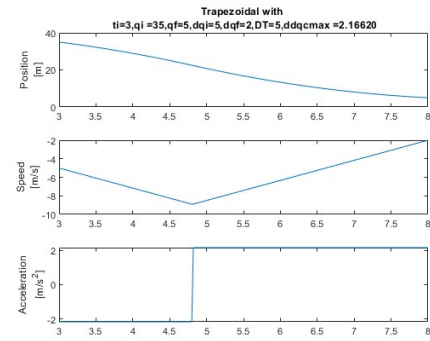
(a)  $t_i=0$  given  $t_c$



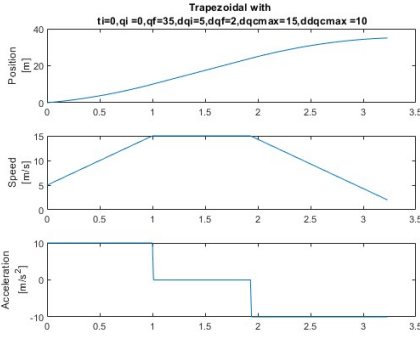
(b)  $t_i=0$  given  $\ddot{q}_c$



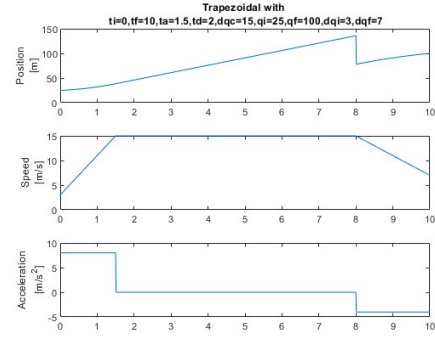
(c)  $t_i=0$  given  $\dot{q}_c$



(d) Given  $\Delta T$  and  $\ddot{q}_{c,max}$



(e) Given  $\ddot{q}_{c,max}$  and  $\dot{q}_{c,max}$



(f) Generic case

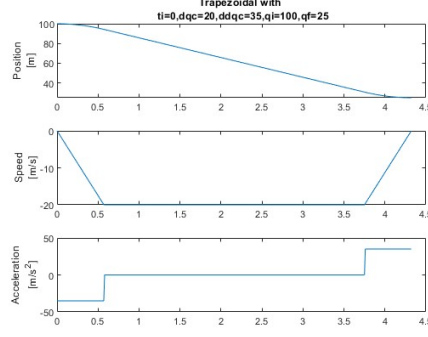


Figure 3:  $t_i=0$  given  $\ddot{q}_c$  and  $\dot{q}_c$

### 3 Joint Space Trajectories - Sequence of Points

Given  $n+1$   $(q_k, t_k)$ , we want to design a trajectory such that the end-effector passes by each point  $q_k$  at a specific instant of time  $t_k$ .

#### 3.1 Interpolating polynomials with continuous velocity

For  $k = 1, \dots, n-1$  we compute:

$$vk := \frac{q_k - q_{k-1}}{t_k - t_{k-1}} \quad (34)$$

The velocity in each path point is computed as:

$$\dot{q}_k = \begin{cases} 0 & \text{if } \text{sgn}(v_k) \neq \text{sgn}(v_{k+1}) \\ \frac{v_k + v_{k+1}}{2} & \text{if } \text{sgn}(v_k) = \text{sgn}(v_{k+1}) \end{cases}$$

We can use the solution of the system given the conditions on positions and velocity:

$$\begin{aligned} a_0^k &= q_k \\ a_1^k &= \dot{q}_k \\ a_2^k &= \frac{1}{T_k} \left[ \frac{3(q_{k+1} - q_k)}{T_k} - 2\dot{q}_k - \dot{q}_{k+1} \right] \\ a_3^k &= \frac{1}{T_k^2} \left[ \frac{2(q_k - q_{k+1})}{T_k} + \dot{q}_k + \dot{q}_{k+1} \right] \end{aligned} \quad (35)$$

Finally we can compute the trajectories:

$$\begin{aligned} q_{k+1} &= a_3^k T_k^3 + a_2^k T_k^2 + a_1^k T_k + a_0^k \\ \dot{q}_{k+1} &= 3a_3^k T_k^2 + 2a_2^k T_k + a_1^k \\ \ddot{q}_{k+1} &= 6a_3^k T_k + 2a_2^k \end{aligned} \quad (36)$$

### 3.2 Interpolating polynomials with continuous acceleration

We start by initializing the matrix  $c$ :

$$c(k) = 3 \frac{T_{k+1}}{T_k} (q_{k+1} - q_k) + 3 \frac{T_k}{T_{k+1}} (q_{k+2} - q_{k+1}) \quad (37)$$

We then set the matrix  $A$ :

$$\begin{aligned} A(0, 0) &= 2(T_0 + T_1) \\ A(0, 1) &= T(0) \\ A(k, k-1) &= T_{k+1} \\ A(k, k) &= 2(T_k + T_{k+1}) \\ A(k, k+1) &= T_k \\ A(end, end) &= 2(T_{n-2} + T_{n-1}) \\ A(end, end-1) &= T_{n-1} \end{aligned} \quad (38)$$

We apply the Forward elimination part of the Thomas algorithm:  
For every path point:

$$\begin{aligned} m &= \frac{A(k, k-1)}{A(k-1, k-1)} \\ A(k, k) &= A(k, k) - m * A(k-1, k) \\ c(k) &= c(k) - m * c(k-1) \end{aligned} \quad (39)$$

then we know  $dqk(end) = \frac{c(end)}{A(end, end)}$  whilst for the other we apply the backward substitution:

Going back from last element:

$$dqk(k) = \frac{c(k) - A(k, k+1) * \dot{q}_k(k+1)}{A(k, k)} \quad (40)$$

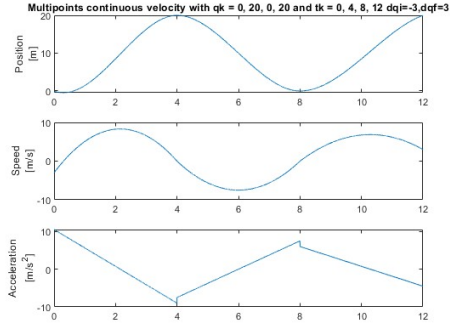
Finally we can use the solution of the system given the conditions on positions and velocity:

$$\begin{aligned} a_0^k &= q_k \\ a_1^k &= \dot{q}_k \\ a_2^k &= \frac{1}{T_k} \left[ \frac{3(q_{k+1} - q_k)}{T_k} - 2\dot{q}_k - \dot{q}_{k+1} \right] \\ a_3^k &= \frac{1}{T_k^2} \left[ \frac{2(q_k - q_{k+1})}{T_k} + \dot{q}_k + \dot{q}_{k+1} \right] \end{aligned} \quad (41)$$

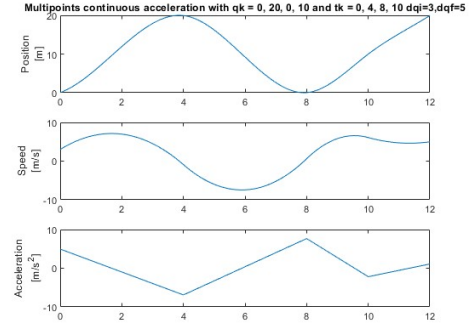
and calculate the trajectories:

$$\begin{aligned} q_{k+1} &= a_3^k T_k^3 + a_2^k T_k^2 + a_1^k T_k + a_0^k \\ \dot{q}_{k+1} &= 3a_3^k T_k^2 + 2a_2^k T_k + a_1^k \\ \ddot{q}_{k+1} &= 6a_3^k T_k + 2a_2^k \end{aligned} \quad (42)$$

### 3.3 Results



(a) Multipoints with continuous velocity



(b) Multipoints with continuous acceleration

## 4 Cubical Splines

### 4.1 Cubic splines based on the accelerations with assigned initial and final velocities

We start by defining the initial and final conditions of the matrices  $c$  and  $A$ :

$$\begin{aligned}
c(0) &= 6\left(\frac{q_1 - q_0}{T_0} - \dot{q}_0\right) \\
c(end) &= 6\left(\dot{q}_f - \frac{q_n - q_{n-1}}{T_{n-1}}\right) \\
A(0, 0) &= 2T_0 \\
A(0, 1) &= T_0 \\
A(end, end - 1) &= T_{end-1} \\
A(end, end) &= 2T_{end-1}
\end{aligned} \tag{43}$$

and the other points:

$$\begin{aligned}
A(k, k - 1) &= T_{k-1} \\
A(k, k) &= 2T_{k-1} + 2T_i \\
A(k, k + 1) &= T_k \\
c(n) &= 6\left(\frac{q_n - q_{n-1}}{T_{n-1}} - \frac{q_{n-1} - q_{n-2}}{T_{n-2}}\right)
\end{aligned} \tag{44}$$

We then apply the Thomas algorithm to get  $\ddot{q}_k$  and we use that in the solved system:

$$\begin{aligned}
a_0^k &= q_k \\
a_1^k &= \frac{q_{k+1} - q_k}{T_k} - \frac{q_{k+1}^{\ddot{}} + 2\ddot{q}_k}{6} T_k \\
a_2^k &= \frac{\ddot{q}_k}{2} \\
a_3^k &= \frac{q_{k+1}^{\ddot{}} - \ddot{q}_k}{6T_k}
\end{aligned} \tag{45}$$

Finally we can solve for the trajectories:

$$\begin{aligned}
q_k &= a_0^k \\
\ddot{q}_k &= 2a_2^k \\
q_{k+1} &= a_3^k T_k^3 + a_2^k T_k^2 + a_1^k T_k + a_0^k \\
\dot{q}_{k+1} &= 3a_3^k T_k^2 + 2a_2^k T_k + a_1^k \\
\ddot{q}_{k+1} &= 6a_3^k T_k + 2a_2^k
\end{aligned} \tag{46}$$



## 4.2 Smoothing cubic splines

Smoothing cubic splines are designed to approximate and not interpolate a set of given data points, we set:

- $w_k$ : parameters which can be arbitrarily chosen in order to modify the weight of the  $k$ -th quadratic error on the global optimization problem
- $\mu$ : weights the trade-off between the two conflicting goals (fitting the points and smoothing the trajectory)
- $\lambda = \frac{1-\mu}{6\mu}$

We set the matrices A and c:

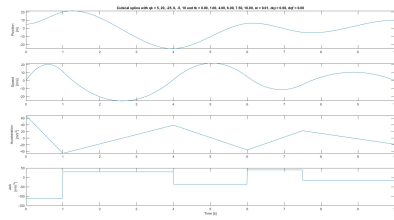
$$\begin{aligned}
A(0, 0) &= 2T_0 \\
A(0, 1) &= T_0 \\
C(0, 0) &= -\frac{6}{T_0} \\
C(0, 1) &= \frac{6}{T_0} \\
A(k, k-1) &= T_{k-1} \\
A(k, k) &= 2T_{k-1} + 2T_k \\
A(k, k+1) &= T_k \\
C(k, k-1) &= \frac{6}{T_k - 1} \\
C(k, k) &= -(\frac{6}{T_{k-1}} + \frac{6}{T_k}) \\
C(k, k+1) &= \frac{6}{T_k} \\
A(end, end-1) &= T_{end-1} \\
A(end, end) &= 2T_{end-1} \\
C(end, end-1) &= -\frac{6}{T_{k-1}}
\end{aligned} \tag{47}$$

We calculate:

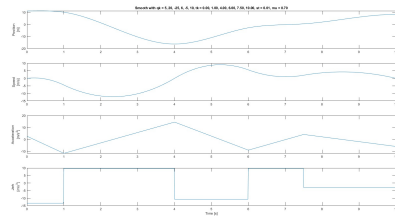
$$\begin{aligned}
\ddot{s} &= Cq * (A + \lambda CW^{-1}C^T)^{-1} \\
s &= q - \lambda W^{-1}C^T \ddot{s}
\end{aligned} \tag{48}$$

Then we compute the trajectories as before using s as the new path points.

### 4.3 Results



(a) Cubical spline



(b) Smoothed version

## 5 3D Trajectory using linear and circular motion primitives

The trajectory composed by a set of linear segments is continuous but it is characterized by discontinuous derivatives at the intermediate points.

For the rectilinear path we define the arc length and the velocity as:

$$\begin{aligned} p(s) &= p_i + s \frac{p_f - p_i}{\|p_f - p_i\|} \\ \frac{dp}{ds} &= \frac{p_f - p_i}{\|p_f - p_i\|} \end{aligned} \quad (49)$$

For the circular path we first get the z axis of the circumference cut by the point1-point2 plane by:

$$\begin{aligned} \mathbf{z}' &= \mathbf{p1\_c} \times \mathbf{p2\_c} \\ angle &= \arccos(\mathbf{v}_1 \cdot \mathbf{v}_2) \end{aligned} \quad (50)$$

where  $v_1$  and  $v_2$  are the versors of  $\mathbf{p1\_c}$  and  $\mathbf{p2\_c}$ ; if the result is equal to 0, we force the direction.

We now get the unit vectors:

$$\begin{aligned} \mathbf{x}' &= \frac{\mathbf{p1\_centre}}{\|\mathbf{p1\_centre}\|} \\ \mathbf{y}' &= \mathbf{x}' \times \mathbf{z}' \\ \mathbf{z}' &= \frac{\mathbf{z}'}{\|\mathbf{z}'\|} \end{aligned} \quad (51)$$

To move along a circular arc we can rely on the parametric representation:

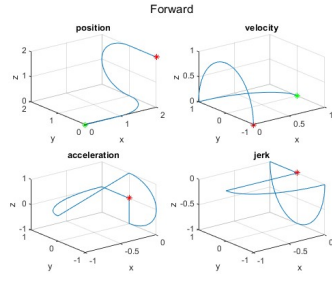
$$p'(u) = \begin{pmatrix} \rho \cos(u) \\ \rho \sin(u) \\ 0 \end{pmatrix} \quad (52)$$

and bring it in  $\Sigma$ :

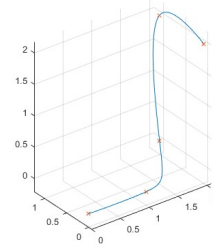
$$\begin{aligned} p(u) &= \mathbf{c} + R\mathbf{p}'(u) \\ R &= [\mathbf{x}' \ \mathbf{y}' \ \mathbf{z}'] \end{aligned} \quad (53)$$

For the comparison with the spline method we apply the algorithm on x,y and z coordinates separately and then put them together to get the 3d path.

## 5.1 Results

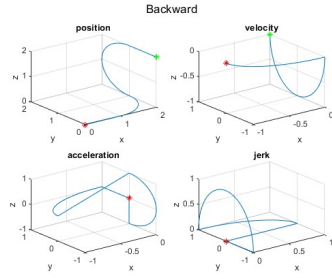


(a) Linear and circular primitives

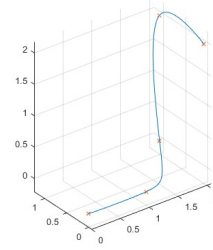


(b) Using splines

Figure 6: Forward trajectory



(a) Linear and circular primitives



(b) Using splines

Figure 7: Backward trajectory

## 6 Trajectory perpendicular to a sphere

The code here is equal to the previous assignment, except for the addition of the 3 Frenet frames  $\mathbf{T}, \mathbf{N}, \mathbf{B}$ :

$$\begin{aligned} T &= \frac{d\mathbf{P}}{ds} = R \begin{pmatrix} -\rho \sin(s) \\ \rho \cos(s) \\ 0 \end{pmatrix} \\ N &= \frac{d^2\mathbf{P}}{d^2s} = R \begin{pmatrix} -\rho \cos(s) \\ -\rho \sin(s) \\ 0 \end{pmatrix} \\ B &= \mathbf{T} \times \mathbf{N} \end{aligned} \tag{54}$$

where  $\rho$  is the radius,  $R$  is the matrix described in the previous assignment and  $\mathbf{T}, \mathbf{N}, \mathbf{B}$  are then reduced to unit vectors.

### 6.1 Results

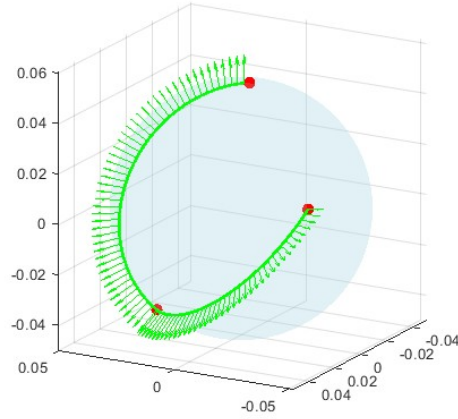


Figure 8: Trajectory perpendicular to a sphere

## 7 Pick and Place in Unity environment

In this assignment we are requested to perform a pick and place task.

Every trajectory in this assignment was calculated with the linear primitive for simplicity and to avoid collisions with the environment.

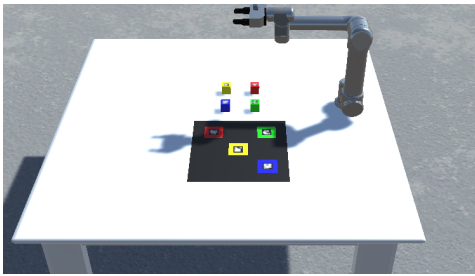
To complete this task we used a finite state machine approach, where the first phase involves getting the poses with the camera attached to the ur5 robot.

To do this the robot does a rectangular-like trajectory to scan the environment, saving the initial and final poses for each cube.

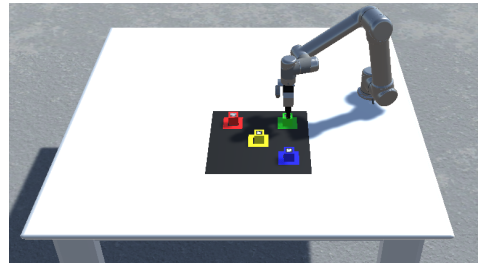
Then we follow a predecided order assigning a trajectory that goes from the starting pose to the final pose. To avoid collisions we added a first way point which brings the cube up from its starting pose, then goes horizontally over the destination and then down again; same for the trajectory that is calculated after placing a cube to reach the new cube to pick.

A video of the final result can be found in the GitHub folder.

### 7.1 Results



(a) Start of simulation



(b) End of simulation