Profit & Loss

Francesco Pascullo

March 2025

1 Introduction

In these exercises, we investigate the pricing and risk analysis of a European put option under two perspectives: *(i)* the actual market dynamics, modeled via the Heston stochastic volatility framework; and *(ii)* the Black–Scholes model, which we (incorrectly) assume to be the "true" market model for risk assessment.

Concretely, we consider a put with strike $\kappa = 1.03$ and maturity $T \approx 1.13$ years, written on a stock that currently trades at $S(0) = 1.0$. Under the market's (hidden) Heston dynamics, the *today* value of this option is first computed by integrating the appropriate payoff under the risk-neutral measure. Next, believing instead in a simpler Black–Scholes world, we back out the implied volatility σ_I that reproduces the Heston-based put price. This implied volatility lets us gauge how "expensive" or "cheap" the option would appear under a naive Black–Scholes lens.

Finally, to examine the portfolio's risk, we evolve many future scenarios up to an intermediate time $t_M = 0.5$ years. In each scenario, we price the residual option from t_M to maturity using *Black–Scholes with the implied vol* σ_I. Comparing each path's hypothetical "marked-to-model" value at t_M against its initial cost (discounted back to the present) leads to a distribution of Profit & Loss (P&L) outcomes. We then compute Value at Risk (VaR) at various confidence levels (10%, 5%, 1%) from the resulting P&L histogram.

Structure of This Document

- **Exercise 1.0.1** Compute the current (time 0) fair value of the put option under the true (Heston) model.
- **Exercise 1.0.2** Back out the implied volatility σ_I consistent with that Heston-based price.
- **Exercise 1.0.3 Part a)** Simulate the stock's future paths up to an intermediate time t_M under the Heston model, then using our BS+σ_I approach (conditioning on each Heston path) to reprice the option at t_M. Compute VaR at several levels with the distribution of P&L.

- **Exercise 1.0.3 Part b)** Repeat the analysis using various shifts of σ away from σ_I (e.g. $\pm 5\%, \pm 10\%, \pm 50\%$) to show how the perceived risk changes with different volatility assumptions.

Through these steps, we gain both the correct Heston-based price of the put and a deeper look at the potential mispricing and risk assessment that arise when modeling all risk solely within a simpler Black–Scholes framework.

2 EXERCISE 1.01

2.0.1 Model Setup

We assume the market follows a *Heston* stochastic volatility process, in which the underlying $S(t)$ evolves according to:

$$\{ \, d\,S(t) \; = \; \mu\,S(t)\,dt + \sqrt{\nu(t)}\,S(t)\; dW_S(t), d\nu(t) \; = \; \lambda\left(\overline{\nu} - \nu(t)\right)dt + \eta\,\sqrt{\nu(t)}\; dW_\nu(t),$$

where

- $\nu(t)$ is the instantaneous variance process,
- $\lambda > 0$ is the mean-reversion speed,
- $\overline{\nu} > 0$ is the long-run variance level,
- $\eta > 0$ is the volatility of volatility, and
- W_S and W_ν are correlated Brownian motions with correlation ρ.

Under the risk-neutral measure (adjusting μ to be $r - q$, where r is the risk-free rate and q the dividend yield), the fair price of a European put with strike κ and maturity T is:

$$\Pi(0) \; = \; e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\big[\big(\kappa - S(T)\big)^+\big].$$

2.0.2 Monte Carlo (MC) Pricing

To compute $\Pi(0)$ via Monte Carlo, we sampled N paths from the Heston model. In particular:

1. We generated the variance paths $\nu(t)$ using two distinct schemes:
   - *Andersen's Quadratic-Exponential (QE) scheme,* which is a popular discretization for the CIR-type process;
   - *Fast exact CIR,* based on the method in, which draws exact realizations of the CIR process increments.
2. Given $\nu(t)$, we evolved $S(t)$ with Andersen's QE step for the asset, ensuring at each step that the correlation structure ρ between dW_S and dW_ν was properly handled.

3. On each simulated path, we recorded the terminal stock price $S_j(T)$ and computed the payoff $(\kappa - S_j(T))^+$.
4. Averaging and discounting gave the MC estimator:

$$\widehat{\Pi}_{\mathrm{MC}}(0) \; = \; e^{-rT} \, \frac{1}{N} \sum_{j=1}^{N} \big(\kappa - S_j(T)\big)^+.$$

The current fair value of the put option under the Heston model was computed in file "heston_opt.py" using nv = 10, ns = 10, nt = 12, dt = 1day, Heston parameters as provided in the file, seed = 29283

$$\Pi(0) = P(0,T)\,\mathbb{E}\left[(\kappa - S(T))^+\right] = 0.1033$$

2.1 Martingale Check.

As a consistency test, we verified the *martingale property* of $S(t)$ under the risk-neutral measure by checking numerically that the sample mean of $S(t_n)$ remained close to $S(0)\,e^{(r-q)t_n}$ for each time t_n across many simulations.

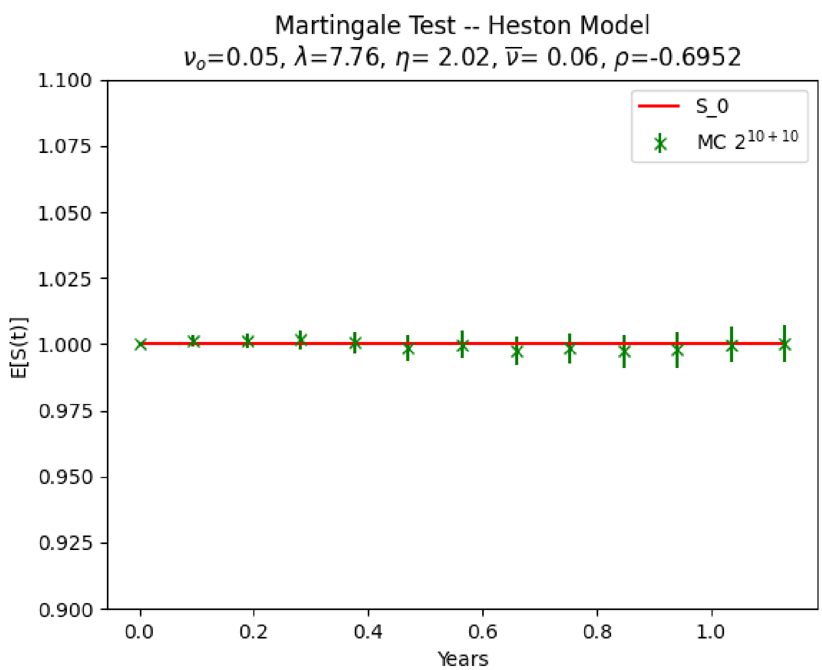

Figure 1: Martingality check for the Andersen's QE vol evo

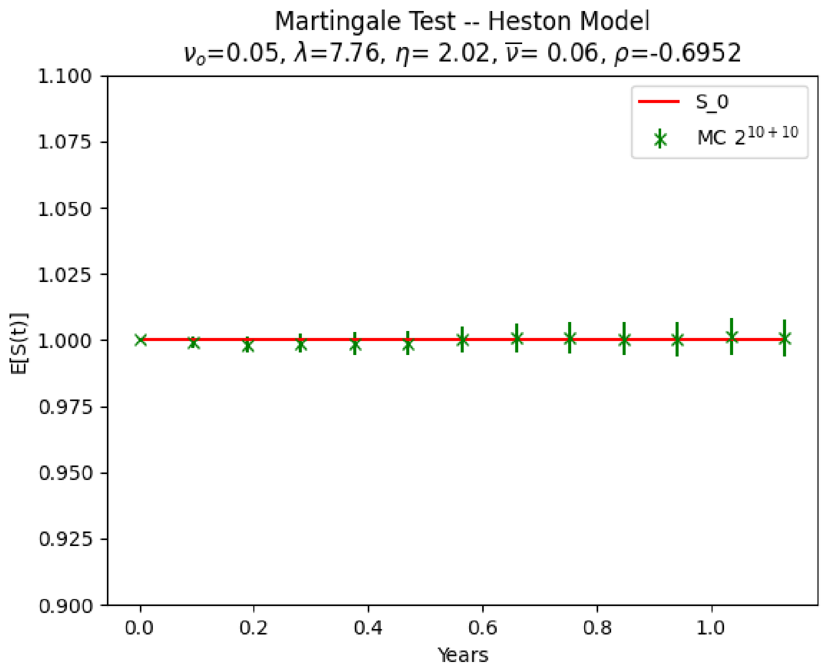

Figure 2: Martingality check for Fast Exact Cir vol evolution

Both discretizations for $\nu(t)$ (QE and Fast exact CIR) preserved the martingale property successfully, giving confidence in our code correctness.

2.2 Comparison with Fourier-Transform (FT) Approaches

In addition to MC, the Heston model admits semi-analytic prices via characteristic functions. We implemented:

- A *SINC-based Fourier method,* which uses Shannon sampling expansions to invert the characteristic function and compute digital/vanilla payoffs;
- A *COS-based Fourier method.* It uses a truncated Fourier-cosine expansion of the payoff and exploits the model's characteristic function to efficiently compute vanilla option prices. It converges very quickly, making it especially accurate for Heston-type models.

Both yielded the same put option price to within 10^{-6}, reassuring us that our numerical integrations were accurate.
FT price (SINC method) = 0.09753913, time = 0.0022
FT price (COS method) = 0.09753913, time = 0.0015

Consistency Checks. We additionally cross-checked:

- **Monte Carlo vs. FT**: Both the Andersen-QE and fast CIR variants produced MC estimates that matched the SINC/COS benchmark prices to within small Monte Carlo sampling error.
- **SINC vs. COS**: The two transforms agreed to at least six decimal places for the final put value.

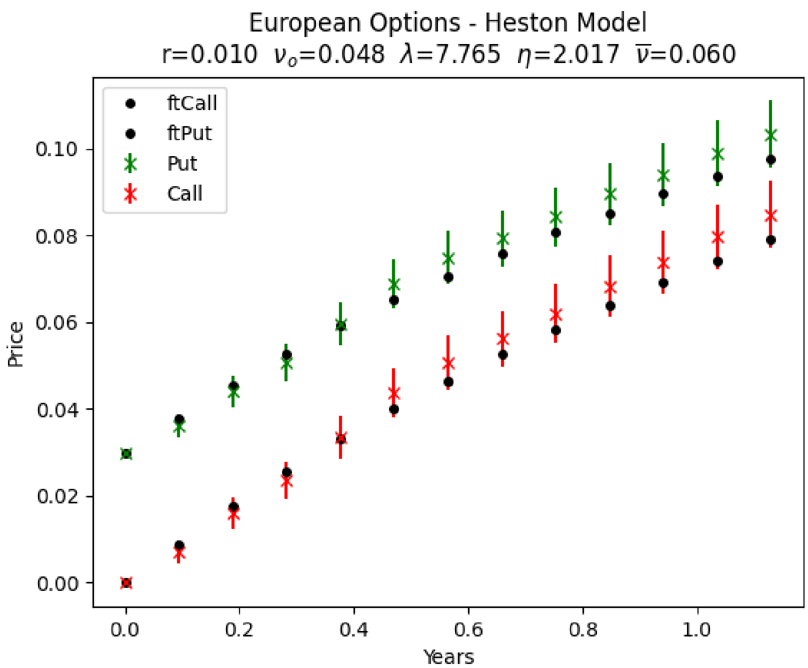

Figure 3: FT prices vs MC prices QE scheme

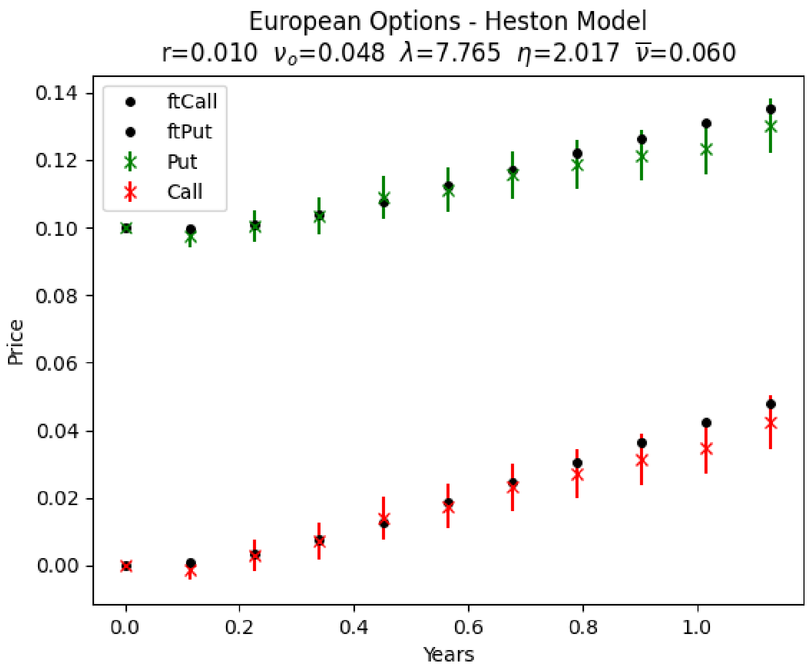

Figure 4: FT prices vs MC prices Fast Exact CIR

Finally, in Exercise 1.0.2 we compare with the professor's reference results. Our code reproduced the same put price values and confirmed that the approach is correctly implemented.

3 Exercise 1.02

3.1 Implied Volatility

Once we have obtained the Heston-based put price $\Pi_{\text{Heston}}(0)$ from Exercise 1.0.1, the implied volatility σ_I is found by solving

$$\text{BSPut}\big(S(0),\, \kappa,\, \sigma_I,\, r,\, q,\, T\big) \;=\; \Pi_{\text{Heston}}(0).$$

Practically, we use the bisection as a root-finding procedure to find σ_I that forces the Black–Scholes formula to match the Heston price. Symbolically,

$$\sigma_I \;=\; \arg\min_{\sigma} \Big|\, \text{BSPut}\big(S(0),\, \kappa,\, \sigma,\, r,\, q,\, T\big) \;-\; \Pi_{\text{Heston}}(0) \Big|.$$

In our implementation, we observed rapid convergence to the unique solution (consistent with no-arbitrage constraints), obtaining

$$\sigma_I \approx 0.2196.$$

This calibrated implied volatility is then adopted in subsequent risk assessments (see Exercise 1.0.3) to reprice the option under a naïve Black–Scholes assumption.

We compared our FT prices and impVol with the prices and impVol in the "heston_benchmark.txt" file for the same set of maturities and strikes.
The average difference for the Prices is = 0.003743508
The average difference for the impVol is = -0.001770381

4 Exercise 1.03:

4.1 Profit & Loss Distribution and VaR

Having determined the "true" price under Heston and the corresponding Black–Scholes implied volatility σ_I, we now assess the portfolio's Profit & Loss (P&L) over a six-month horizon $t_M = 0.5\,\text{yr}$.

Step 1: Generate Heston Paths. We simulate N independent paths $\{S_j(t)\}$ up to t_M under the Heston model parameters. For each path, we record $S_j(t_M)$.

Step 2: Reprice the Option at t_M Using BS + σ_I. On each path, we treat $S_j(t_M)$ as the current underlying price in a Black–Scholes setting with volatility σ_I to compute the time-t_M put value, which we denote $\Pi_j(t_M)$. We then discount from $t = 0$ to t_M via $P(0, t_M)$ and measure

$$V_j \;=\; \Pi_{\text{Heston}}(0) \;-\; P(0, t_M)\,\Pi_j(t_M).$$

This V_j is the realized P&L of holding the option from $t = 0$ to t_M under actual Heston dynamics but marking to BS prices at t_M.

Step 3: Build the P&L Histogram and Compute VaR. We plot the empirical distribution of V_j, as shown below, and estimate the Value at Risk (VaR) at 10%, 5%, and 1%. Specifically, the α% VaR is the loss threshold L_α such that $\mathbb{P}(V < -L_\alpha) = \alpha\%$.
Mean of the P&L: -0.00223
Std Dev of P&L : 0.09761
Var 10%: -0.11811
VaR 5%: -0.20544
Var 1%: -0.39418

Δ	-50%		σ	0.1098	Δ	5%		σ	0.2306
Mean P&L:	0.02476				Mean P&L:	-0.00522			
Std Dev		0.10623			Std Dev		0.09678		
VaR	10%	-0.11089			VaR	10%	-0.11955		
VaR	5%	-0.2041			VaR	5%	-0.2059		
VaR	1%	-0.39417			VaR	1%	-0.39418		
Δ	-10%		σ	0.1976	Δ	10%		σ	0.2416
Mean P&L:	0.00364				Mean P&L:	-0.00825			
Std Dev		0.09929			Std Dev		0.09597		
VaR	10%	-0.11558			VaR	10%	-0.1211		
VaR	5%	-0.20475			VaR	5%	-0.20646		
VaR	1%	-0.39417			VaR	1%	-0.39419		
Δ	-5%		σ	0.2086	Δ	50%		σ	0.3294
Mean P&L:	0.00073				Mean P&L:	-0.03337			
Std Dev		0.09845			Std Dev		0.08988		
VaR	10%	-0.11678			VaR	10%	-0.13663		
VaR	5%	-0.20506			VaR	5%	-0.21398		
VaR	1%	-0.39418			VaR	1%	-0.3946		

Step 4: Sensitivity to Different σ. Figure 5 summarizes the P&L statistics when we replace the true implied volatility $\sigma_I = 0.2196$ by $\sigma = \sigma_I(1+\Delta)$, with Δ from -50% up to $+50\%$. Three observations stand out:

1. **Mean P&L Shifts.** For $\Delta < 0$ (using a lower volatility), the mean P&L becomes slightly positive (e.g. $+0.02476$ at $\Delta = -50\%$), indicating we are "underpaying" initially relative to the Heston-based fair price. Conversely, for $\Delta > 0$ (too high a volatility), the mean P&L turns negative (down to about -0.03337 at $\Delta = +50\%$), reflecting an overpayment at inception.

2. **Standard Deviation and VaR.** As Δ increases, the standard deviation of P&L tends to decrease mildly, while the 10% VaR moves from about -0.11089 (understated vol) to -0.13663 (overstated vol). Interestingly, the 1% VaR remains near -0.394, suggesting a similar extreme downside tail for all tested vol shifts. Still, the slightly more negative 10% VaR at high Δ implies greater typical losses from holding an overpriced option.

3. **Volatility Mis-Specification Risk.** These results highlight how a different assumed volatility σ can systematically bias the average P&L (*positive* if σ is below the true market level, *negative* if σ is above it), while also altering risk measures like VaR in the less extreme quantiles. In practice, this underscores the sensitivity of hedging outcomes and profit/loss distributions to one's volatility estimate.

Overall, if we are uncertain about the "true" volatility, the table shows that even moderate differences (e.g. $\pm 5\%$ or $\pm 10\%$) can shift P&L expectations and typical downside risk significantly. Extreme mismatches ($\pm 50\%$) clearly amplify these effects.

5 Codes used:

- CFLib_25.py: Heston.py for the heston model; CIR.py for the CIR model with different volatility path (QE, FEC); euro_opt.py for the implVol; heston_evol.py for the heston asset evolution; Ft_opt.py for the FT (sinc) price
- heston_opt.py : compute Option price in exercise 1.01, implVol in exercise 1.02 and graph MC prices vs FT prices
- heston_martingale.py: check the martingality and to plot it.
- heston_cf.py: compute FT SINC price
- cos_price2.py: compute FT COS price. The original code was written by Prof. Lech A. Grzelak. I modified it to complete our task.
- exercise_1.03: solve exercise 1.03