ARIMM group 5

Francesco Pascullo & Sara Farnedi

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1 Exercise 1: Cap-Floor Parity demonstration, Caplet-Floorlet Parity demonstration

a) Prove analytically the parity relations for Caplets/Floorlets and Caps/Floors prices.
b) Is it possible to verify the parity relations above using available market quotes?

1.1 Caplet-Floorlet parity

Assuming that AOA is satisfied, considering a portfolio composed of a long Caplet position and a short Floorlet position we have:

$$
\begin{aligned}
&\mathrm{Caplet}(T_i) - \mathrm{Floorlet}(T_i)\\
&= N\tau_x(T_{i-1},T_i)\max\{L_x(T_{i-1},T_i)-K,0\} - N\tau_x(T_{i-1},T_i)\max\{K-L_x(T_{i-1},T_i),0\}\\
&= N\tau_x(T_{i-1},T_i)\Big[\max\{L_x(T_{i-1},T_i)-K,0\} - \max\{K-L_x(T_{i-1},T_i),0\}\Big]\\
&= N\tau_x(T_{i-1},T_i)\Big[\big(L_x(T_{i-1},T_i)-K\big)^+ - \big(K-L_x(T_{i-1},T_i)\big)^+\Big]\\
&= N\tau_x(T_{i-1},T_i)\Big[(L_x(T_{i-1},T_i)-K)\mathbf{1}_{L_x\geq K} - (K-L_x(T_{i-1},T_i))\mathbf{1}_{L_x<K}\Big]\\
&= N\tau_x(T_{i-1},T_i)\Big[L_x(T_{i-1},T_i)-K\Big]\\
&= \mathrm{FRA}_{\mathrm{Std}}(T_i,T,K,\omega=+1).
\end{aligned}
$$

Where $\mathrm{FRA}_{\mathrm{Std}}(T_i,T,K)$ is intended from the payer side and the floating and fixed leg have the same time schedule.

1.2 Cap-Floor Parity

The cap floor parity states that holding a long position in a Cap and a short position in a floor, with the same strike and maturity is equivalent to paying the fixed leg in an interest rate swap (IRS) where the fixed rate is equal to the strike rate. The proof follows:

$$
\begin{aligned}
&\mathrm{Cap}(t) - \mathrm{Floor}(t)\\
&= \sum_{i=1}^{n} \mathrm{Caplet}(t) - N \sum_{i=1}^{n} \mathrm{Floorlet}(t)\\
&= N \sum_{i=1}^{n} P_d(t,T_i)\tau_x(T_{i-1},T_i)\mathbb{E}^{\mathbb{Q}}_{T_i}\Big[\max\big(L_x(T_{i-1},T_i) - K, 0\big)\Big]\\
&\quad - N \sum_{i=1}^{n} P_d(t,T_i)\tau_x(T_{i-1},T_i)\mathbb{E}^{\mathbb{Q}}_{T_i}\Big[\max\big(K - L_x(T_{i-1},T_i), 0\big)\Big]\\
&= N \sum_{i=1}^{n} P_d(t,T_i)\tau_x(T_{i-1},T_i)\mathbb{E}^{\mathbb{Q}}_{T_i}\Big[(L_x(T_{i-1},T_i) - K)\mathbf{1}_{L_x \geq K} - (K - L_x(T_{i-1},T_i))\mathbf{1}_{L_x < K}\Big]\\
&= N \sum_{i=1}^{n} P_d(t,T_i)\tau_x(T_{i-1},T_i)\mathbb{E}^{\mathbb{Q}}_{T_i}\Big[(L_x(T_{i-1},T_i) - K)\mathbf{1}_{L_x \geq K} + (L_x(T_{i-1},T_i) - K)\mathbf{1}_{L_x < K}\Big]\\
&= N \sum_{i=1}^{n} P_d(t,T_i)\tau_x(T_{i-1},T_i)\mathbb{E}^{\mathbb{Q}}_{T_i}\Big[L_x(T_{i-1},T_i) - K\Big]\\
&= N \sum_{i=1}^{n} P_d(t,T_i)\tau_x(T_{i-1},T_i)\Big[F_{x,i}(t) - K\Big]\\
&= \mathrm{IRS}(t; T = S; K; \omega = +1).
\end{aligned}
$$

This follows from the fact that $F_{x,i}(t) = F_x(t; T_{i-1}, T_i) = \mathbb{E}^{\mathbb{Q}}_{T_i}\big[L_x(T_{i-1},T_i)\big]$ and from the definition of IRS from the perspective of the payer of the fixed leg:

$$
\begin{aligned}
\mathrm{IRS}(t,T,S,K) &= \mathrm{IRS}_{\mathrm{float}}(t,S,K) - \mathrm{IRS}_{\mathrm{fix}}(t,S,K)\\
&= N\Bigg[\sum_{i=1}^{n} P_d(t,T_i)F_{x,i}(t)\tau_\lambda(T_{i-1},T_i) - \sum_{j=1}^{m} P_d(t,S_j)\tau_k(S_{j-1},S_j)K\Bigg].
\end{aligned}
$$

1.3 Market Verification of Cap-Floor Parity

The cap-floor parity relationship states that if the market is arbitrage-free, the difference between the cap and floor price (with the same strike and maturity) must be equal to the present value of the payer IRS.

$$
\mathrm{Cap}(K) - \mathrm{Floor}(K) = \text{Payer IRS}(K)
$$

where the right-hand side represents the present value of a fixed-for-floating interest rate swap (IRS) with fixed rate K. The value of this IRS is given by:

$$\sum_{i=1}^{n} P_d(t, T_i)\tau_x(T_{i-1}, T_i)\Big[F_{x,i}(t) - K\Big]$$

where $P_d(t, T_i)$ are the discount factors, K is the fixed strike, and F is the forward swap rate: $F = \frac{1}{\tau_x(T_{i-1},T_i)}\left[\frac{P_x(t,T_{i-1})}{P_x(t,T_i)} - 1\right]$

In practice, the only available market quotations in the excel file *AIRMM-MarketData31Oct2019* for cap and floor prices with the same combination of strike and maturity are the values at-the-money (ATM). Given that quoted ATM cap and floor prices are equal for any combination of maturity and strike, the left-hand side of the parity equation is always zero. This means we must verify whether the right-hand side, representing the payer swap value, is also zero. Applying the formula for IRS above for every combination of strike and maturity we get the following results:

Maturity	K ATM	IRS(K)
1Y	-0,4	-0.00030228887
18M	-0,4	-0.00058925834
2Y	-0,4	-0.00074879212
3Y	-0,3	-0.0006168433
4Y	-0,2	-0.00305756372
5Y	-0,2	-0.00119506786
6Y	-0,2	0.001833222533
7Y	-0,1	-0.00062285737
8Y	0	-0.00371562222
9Y	0	0.00136155101
10Y	0,04	0.003436646174
12Y	0,16	0.0028507807
15Y	0,29	0.003320427435
20Y	0,41	0.002924012286
25Y	0,45	0.001107838012
30Y	0,44	0.002929897778

Figure 1: Cap Floor Parity

1.4 Caplet-Floorlet Parity and Market Data Limitations

For caplet-floorlet parity, direct market verification is challenging due to the absence of quotations for individual caplets and floorlets. To reconstruct these prices, we would need a more detailed cap and floor quotation matrix, in this case one approach would be to derive caplet and floorlet prices from the difference in cap and floor prices with varying maturities and the interpolate these values to find all the prices we need. However, since the difference between maturities in market quotes is typically at least one year, this method does not provide sufficient granularity to extract caplet and floorlet prices accurately.

An alternative approach could be to recover the prices of caplets and floorlets using forward volatilities but market quotes refer to term volatilities: such volatilities are solely used for quotation purposes and they cannot be directly used for pricing. The procedure to recover forward volatilities from Cap/Floor market prices is based on a recursive approach a complete caplet/floorlet matrix would be needed, but in this case we do not have. If the market quotations does not form a complete caplet/floorlet matrix, one must resort to volatility interpolation during the bootstrapping procedure, to obtain the unknown volatilities but again the cap and floor prices do not provide sufficient granularity to extract these data.

1.5 Conclusion

While cap-floor parity can be checked using available ATM market quotes, caplet-floorlet parity requires additional information or model calibration techniques to obtain the necessary granular data for verification.

2 Exercise 2: Implied term volatility surfaces

> Using the relevant yield curves and Caps/Floors market prices provided in the market data sheets, build the corresponding normal (Bachelier) and shifted-lognormal (Black) implied term volatility surfaces. Compare them for different values of the lognormal shift and discuss possible cases where the numerical inversion of the Bachelier or Black formulas is problematic

To compute the normal implied term volatility and the shifted lognormal implied term volatility, we numerically invert the Bachelier and shifted Black formulas. The equations are:

$$
\begin{aligned}
CF(t, T, K, \omega) &= N \sum_{i=1}^{n} cf(t, T_{i-1}, T_i, K, \omega) \\
&= N \sum_{i=1}^{n} P_d(t, T_i)\tau_x(T_{i-1}, T_i)\text{Black}\left(\overline{F}_{x,i}(t), \overline{K}, \lambda_x, v_x(t, Ti-1), \omega\right) \\
&= N \sum_{i=1}^{n} P_d(t, T_i)\tau_x(T_{i-1}, T_i)\text{Black}\left(\overline{F}_{x,i}(t), \overline{K}, \lambda_x, v_{x,i,n}(t, T_{i-1}, T_n), \omega\right)
\end{aligned}
$$

$$
\begin{aligned}
CF(t,T,K,\omega) &= N\sum_{i=1}^{n} cf(t,T_{i-1},T_i,K,\omega) \\
&= N\sum_{i=1}^{n} P_d(t,T_i)\tau_x(T_{i-1},T_i)\mathrm{Bach}\left(\overline{F}_{x,i}(t),\overline{K},\lambda_x,v_x(t,Ti-1),\omega\right) \\
&= N\sum_{i=1}^{n} P_d(t,T_i)\tau_x(T_{i-1},T_i)\mathrm{Bach}\left(\overline{F}_{x,i}(t),\overline{K},\lambda_x,v_{x,i,n}(t,T_{i-1},T_n),\omega\right)
\end{aligned}
$$

where:

- cf represents the caplet/floorlet price depending on ω

- $v_x(t,T_{i-1})$ is the shifted lognormal implied forward variance $v_x(t,T_{i-1}) = \sigma_x(t,T_{i-1})^2\tau_x(t,T_{i-1})$. In this case each caplet/floorlet has a distinct forward volatility

- $v_{x,i,n}(t,T_{i-1},T_n) = \sigma_x(t,T_n)^2\tau_x(t,T_{i-1})$ is the shifted lognormal implied term variance, providing a single term volatility figure for the entire cap/floor

2.1 Market Data

The following inputs are extracted from the excel file *AIRMM-MarketData31Oct2019*, in particular Cap/Floor quoted prices at time t are extracted from the sheet *Cap&Floor*:

16:10 31OCT19 ICAP UK69580 VCAP4

EUR Caps - Premium Mids (Eonia disc)

Please call +44 (0)20 7532 3080 for further details

	STK	ATM	-1.50	-1.25	-1.0	-0.5	-0.25	0.0	0.25	0.50	1.0	2.0	3.0	5.0	10.0
1Y	-0.4	3					1								
18M	-0.4	7					1								
2Y	-0.4	12					4	1							
3Y	-0.3	26					18	7	3	1					
4Y	-0.2	51					45	23	13	8	3	1			
5Y	-0.2	85						51	32	21	10	4	2	1	
6Y	-0.2	130						94	64	45	24	9	4	1	
7Y	-0.1	184						152	110	81	46	18	8	3	1
8Y	-0.0	247						225	170	130	78	32	16	5	1
9Y	-0.0	317						312	243	191	121	54	27	9	2
10Y	0.04	396							328	263	174	82	44	16	3
12Y	0.16	569							529	438	304	156	87	34	7
15Y	0.29	851								741	536	293	172	73	17
20Y	0.41	1342								1276	959	559	346	158	42
25Y	0.45	1847								1796	1381	840	538	258	73
30Y	0.44	2362								2295	1791	1118	732	362	106

1y,18m and 2y vs 3m, 3y and above vs 6m

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ICAP Global Index <ICAP> Forthcoming changes <ICAPCHANGE>

Figure 2: Cap mid premia

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EUR Floors - Premium Mids (Eonia disc)

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	STK	ATM	-1.5	-1.25	-1.0	-0.5	-0.25	0.0	0.25	0.50	1.0	2.0	3.0	5.0	10.0
1Y	-0.4	3				2									
18M	-0.4	7				4									
2Y	-0.4	12				7									
3Y	-0.3	26			1	10									
4Y	-0.2	51	1	2	4	21									
5Y	-0.2	85	3	5	9	36	85								
6Y	-0.2	130	6	10	17	54	112								
7Y	-0.1	184	10	16	26	74	141								
8Y	-0.0	247	15	23	37	96	171								
9Y	-0.0	317	20	31	48	118	201								
10Y	0.04	396	27	41	62	144	235	369							
12Y	0.16	569	47	67	96	200	306	459							
15Y	0.29	851	88	118	160	297	424	602	814						
20Y	0.41	1342	188	238	304	503	668	890	1153						
25Y	0.45	1847	317	394	490	765	976	1249	1568						
30Y	0.44	2362	478	586	717	1076	1336	1663	2041						

1y,18m and 2y vs 3m, 3y and above vs 6m

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Figure 3: Floor mid premia

In the first column we have all the maturities, the second column lists ATM strikes (equal to IBOR spot rate from zero to maturity), the third column contains the mid premia ATM, and subsequent columns show mid premia for various strikes. Mid premia does not include the first caplet/floorlet since it is fixed.

- Premia are expressed in basis points and are divided by 10,000.
- The notional N is assumed to be 1 for simplicity.
- The summation includes all caplet/florlet prices contributing to the overall cap/floor price.
- 1Y, 18M and 2Y tenors are based on EURIBOR3M (quarterly frequency), while all other maturities reference EURIBOR6M (semiannual frequency).

The discount factors P_d are obtained from the OIS curve and shown in the following figure. We compute the discount factors using the continuously compounded formula
$P_d(t,T) = e^{-r_{OIS}(T)T}$ where r_{OIS} is the annualized OIS zero rate, and T is measured in years according to the Act/365 convention. The discount curve is interpolated using a cubic spline.
The year fraction τ_x depends on the floating rate index:

- EURIBOR3M: $\tau_x = 0.25$ (quarterly)
- EURIBOR6M: $\tau_x = 0.5$ (semiannual)

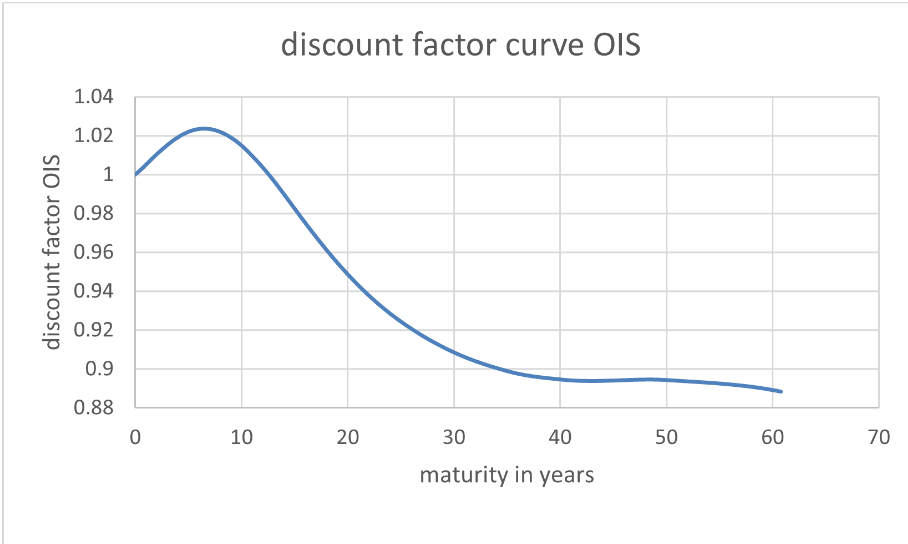

Figure 4: OIS Discount factor Curve

Since IBOR forward rates can be implied from market quotations of interest rate instruments, $F_{x,i}$ are computed as: $F_{x,i}(t) = \frac{1}{\tau_x(T_{i-1},T_i)}\left[\frac{P_x(t,T_{i-1})}{P_x(t,T_i)} - 1\right]$ where P_x represents the discount factors for EURIBOR3M or EURIBOR6M.

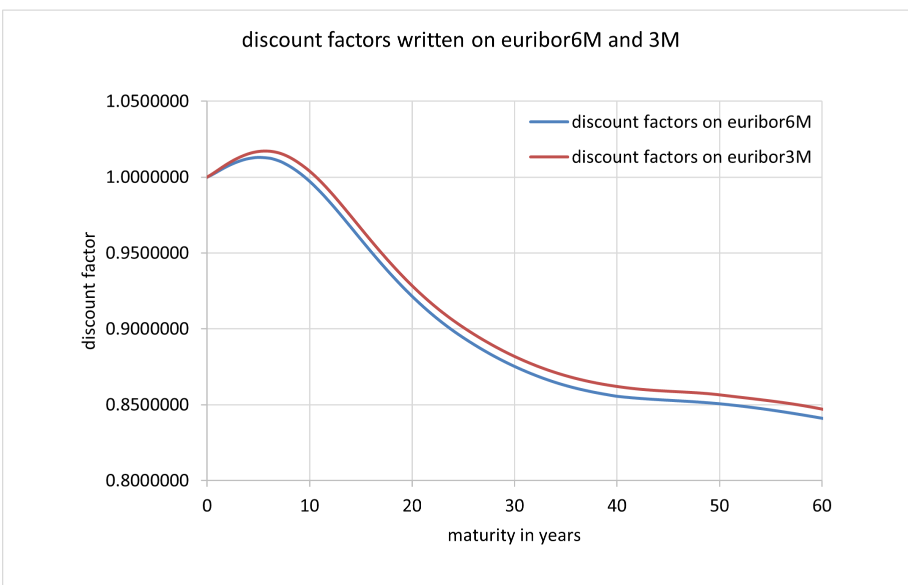

Figure 5: Discount factors on Euribor6M curve and Euribor3M

The strike price K is fixed for each caplet/floorlet, equal to the strike of the corresponding cap/floor. These values are taken from the mid premia matrix and divided by 100.The variable ω distinguishes between Caps (+1) and Floors (-1). The implied term variance $v_{x,i,n}(t)$ is used to compute a single term volatility figure for the entire cap/floor.

2.2 Implied term normal volatility surface

To obtain the implied term normal volatility for each Cap/Floor price, we numerically invert the Bachelier pricing formula. The goal is to solve the following equation for the implied volatility

$$MarketPrice - BachelierPrice(\sigma) = 0$$

This requires a root-finding algorithm. We initially used Brent's method, which is a hybrid approach combining bisection and secant method. The bisection method ensures robustness and convergence even when the function behaves unpredictably, while the secant method speeds up convergence by using an approximation of the derivative of the price with respect to volatility. Brent's method computes the option price at the two extremes of our volatility search bracket $[10^9, 5]$, then iteratively refines the solution. We set the maximum number of iterations to 500. The obtained volatility surface is shown in the matrix below:

maturity	STK	ATM	-1.5	-1.25	-1	-0.5	-0.25	0	0.25	0.5	1	2	3	5	10
1	-0.4					0.0014	0.0021								
1.5	-0.4	0.0007				0.0014	0.0015								
2	-0.4	0.0017				0.0011	0.0021	0.0025							
3	-0.3	0.0014			0.0024	0.0018	0.0019	0.0023	0.0027	0.0028					
4	-0.2	0.0012	0.0034	0.0031	0.0028	0.0022	0.0024	0.0027	0.0031	0.0034	0.0040	0.0055			
5	-0.2	0.0022	0.0037	0.0034	0.0031	0.0026	0.0027	0.0030	0.0033	0.0037	0.0043	0.0058	0.0071	0.0100	
6	-0.2	0.0031	0.0039	0.0037	0.0034	0.0029	0.0030	0.0033	0.0036	0.0039	0.0045	0.0058	0.0069	0.0088	
7	-0.1	0.0032	0.0041	0.0039	0.0037	0.0032	0.0033	0.0036	0.0039	0.0042	0.0047	0.0058	0.0068	0.0090	0.0147
8	0	0.0035	0.0042	0.0040	0.0039	0.0035	0.0036	0.0035	0.0041	0.0044	0.0049	0.0059	0.0069	0.0087	0.0135
9	0	0.0040	0.0043	0.0042	0.0040	0.0037	0.0038	0.0040	0.0043	0.0045	0.0050	0.0060	0.0069	0.0087	0.0134
10	0.04	0.0043	0.0044	0.0043	0.0042	0.0039	0.0040	0.0042	0.0045	0.0047	0.0052	0.0061	0.0070	0.0088	0.0130
12	0.16	0.0047	0.0047	0.0046	0.0045	0.0042	0.0043	0.0045	0.0047	0.0050	0.0054	0.0062	0.0071	0.0087	0.0127
15	0.29	0.0051	0.0050	0.0049	0.0048	0.0046	0.0047	0.0048	0.0051	0.0052	0.0055	0.0062	0.0070	0.0086	0.0123
20	0.41	0.0054	0.0053	0.0052	0.0051	0.0049	0.0050	0.0051	0.0053	0.0053	0.0056	0.0062	0.0069	0.0083	0.0118
25	0.45	0.0055	0.0052	0.0052	0.0051	0.0051	0.0051	0.0052	0.0053	0.0054	0.0057	0.0062	0.0068	0.0081	0.0113
30	0.44	0.0055	0.0052	0.0051	0.0051	0.0051	0.0051	0.0052	0.0053	0.0054	0.0056	0.0061	0.0067	0.0079	0.0108

Figure 6: Term implied normal volatility surface computed using Brent's method. Yellow cells correspond to volatilities derived from Floor prices, green ones from Cap prices. Missing values indicate algorithm failure.

The only case where the algorithm failed to converge was for the ATM volatility at the first maturity.

To address the convergence issue, we switched to the Newton-Raphson method, an iterative technique that updates an initial guess σ_0(fixed at 0.1) using the formula: $\sigma_{n+1} = \sigma_n - \frac{f(\sigma_n)}{f'(\sigma_n)}$ where $f(\sigma)$ is the difference between the market price and the Bachelier price, and $f'(\sigma)$ is its derivative. The iteration continues until either the change in σ or $f(\sigma)$ falls below a set tolerance (10^{-8}).

The volatility surfaces obtained from both root-finding methods are similar and both methods failed to compute the implied ATM volatility for the first maturity.

2.3 Challenges in the Numerical inversion of Bachelier's Formula

Different root-finding methods have their strengths and weaknesses in solving $MarketPrice - BachelierPrice(\sigma) = 0$. Newton-Raphson converges quickly but requires computing the derivative and can fail if the function is too flat. Bisection method guarantees convergence but is slow. Secant Method is faster than bisection but less stable (still uses an approximation of the derivative). Brent's Method is a combination that balances robustness and speed. The main numerical issues arise when the vega (the sensitivity of price to volatility) is very low, which happens when the option price is close to zero, leading to potential division by zero or when the price is insensitive to volatility, making the function locally flat and causing convergence problems. A critical term in the Bachelier formula is: $d = \frac{F-K}{\sigma\sqrt{T}}$ where F is the forward rate and K is the strike. Depending on their values, d can take extreme values:

- If $F \approx K$ numerical behavior should be stable.
- If $|F - K|$ is large then d is large and it's difficult to extract σ because $\Phi(d)$ and $\phi(d)$ reach extreme values (0 or 1).

Investigating the possible causes for ATM volatility failure at the first maturity, we note that at-the-money conditions imply $F = K$, which should lead to stable numerical behavior. However, both Newton-Raphson and Brent's method can struggle in this scenario due to the function's flatness.
Newton-Raphson relies on the derivative of the pricing function, and if the function is particularly flat in this region, division by small values can lead to numerical instability. Brent's method, while not explicitly requiring a derivative, depends on detecting a sign change in the objective function over the search interval. If the pricing function varies too little across the volatility range, the market premium might not be bracketed by the model prices at the endpoints, preventing root detection.
In the Bachelier model, the option price is given by:

$$C = (F - K)\Phi(d) + \sigma\sqrt{T}\phi(d)$$

For ATM options, where $F = K$ the price simplifies to: $C = \sigma\sqrt{T}\phi(0)$ and since $\phi(0) = \frac{1}{\sqrt{2\pi}}$, the price is primarily driven by $\sigma\sqrt{T}$. This term is nearly flat with respect to changes in σ, leading to very low vega (the sensitivity of price to volatility). When vega is close to zero, the pricing function becomes almost insensitive to volatility changes. This lack of sensitivity at ATM could explain the observed numerical failures in implied volatility computation for the first maturity.

2.4 Implied term shifted log-normal volatility surface with different shifts

The Black-Shifted formula allows pricing options when the forward rate F or the strike K can be negative. To do so, it introduces a shift parameter S ensuring all rates remain positive:

$$C = (F+S)\Phi(d_1) - (K+S)\Phi(d_2)$$

where $d_1 = \frac{ln(\frac{F+S}{K+S})+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$.
Given the market price, the market volatility is found by solving numerically the equation $MarketPrice - BlackShiftedPrice(\sigma) = 0$. For this inversion, we applied Brent's method using the same interval and iteration limits as before. To analyze the impact of the shift parameter, we computed the volatility surface for different values of S. We found that to obtain an implied volatility for each market-quoted premium, a shift of 3% was required.

maturity	STK	ATM	-1.5	-1.25	-1	-0.5	-0.25	0	0.25	0.5	1	2	3	5	10
1	-0.4	6.60%				5.73%	7.91%								
1.5	-0.4	2.81%				5.53%	5.82%								
2	-0.4	6.20%				4.29%	8.03%	8.98%							
3	-0.3	5.27%			10.51%	6.94%	7.20%	8.15%	8.97%	9.07%					
4	-0.2	4.44%	16.45%	14.23%	12.13%	8.47%	8.68%	9.42%	10.27%	11.09%	12.07%	14.59%			
5	-0.2	8.04%	17.71%	15.29%	13.25%	9.86%	9.67%	10.41%	11.06%	11.67%	12.73%	15.10%	16.82%	20.04%	
6	-0.2	11.04%	18.47%	16.31%	14.39%	11.03%	10.80%	11.35%	11.87%	12.38%	13.28%	14.89%	16.01%	17.37%	
7	-0.1	10.86%	19.03%	16.95%	15.11%	11.99%	11.73%	12.14%	12.57%	12.96%	13.62%	14.81%	15.61%	17.58%	21.52%
8	0	11.71%	19.50%	17.44%	15.79%	12.81%	12.50%	11.71%	13.10%	13.41%	13.89%	14.72%	15.61%	16.77%	19.34%
9	0	13.13%	19.66%	17.82%	16.18%	13.40%	13.07%	13.13%	13.55%	13.78%	14.19%	14.89%	15.47%	16.50%	18.99%
10	0.04	14.05%	20.01%	18.24%	16.67%	14.02%	13.66%	13.89%	13.94%	14.10%	14.45%	14.98%	15.57%	16.53%	18.30%
12	0.16	14.63%	20.87%	19.07%	17.52%	14.96%	14.52%	14.63%	14.55%	14.62%	14.73%	15.02%	15.35%	16.09%	17.58%
15	0.29	15.27%	21.76%	19.93%	18.41%	15.90%	15.35%	15.26%	15.26%	14.95%	14.82%	14.79%	14.95%	15.52%	16.78%
20	0.41	15.64%	22.55%	20.71%	19.20%	16.79%	16.13%	15.85%	15.69%	15.28%	14.95%	14.64%	14.65%	14.99%	16.02%
25	0.45	15.75%	22.64%	20.91%	19.47%	17.22%	16.55%	16.20%	15.95%	15.50%	15.09%	14.67%	14.57%	14.76%	15.54%
30	0.44	15.92%	22.62%	20.96%	19.58%	17.44%	16.77%	16.36%	16.07%	15.59%	15.13%	14.63%	14.46%	14.53%	15.09%

Figure 7: Term implied shifted (3%) lognormal volatility surface computed using Brent's method

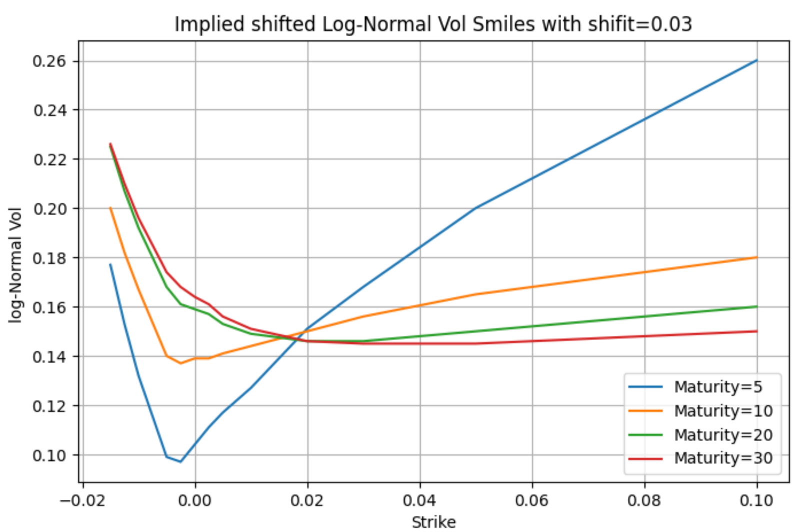

Figure 8: Shifted (3%) lognormal volatility smiles

The success of the numerical inversion depends on the values of F, K and S.

Specifically, the term d_1 contains a logarithm, which becomes undefined or unstable when $F + S$ is very small or negative. The formula remains valid as long as $K + S$ remains positive. Additionally, as we said before, when vega is very low or when the option price has minimal sensitivity to changes in σ, numerical methods might fail to converge. Lowering the shift will tend to decrease the number of volatilities that can be computed, this can be seen firstly for deep "in the money" Floors then to Floors closer to "at the money".
If we decrease the shift from 3%, certain volatilities may no longer be determinable. This occurs particularly when:

- $F + S \leq 0$ or $K + S \leq 0$, making the logarithm is undefined
- $K + S$ is small, causing $\frac{F+S}{K+S}$ to be divided by 0

In the case of deep in-the-money floors option prices become nearly insensitive to volatility changes. Since Brent's method relies on finding a volatility that matches the market price within a given search range $[10^{-9}, 5.0]$, the model price may never reach the observed market premium, leading to failed convergence.

2.4.1 Impact of changing shift

We analyzed the impact of the shift starting from 3% and lowering it to reach 0.5%. More precisely, we computed the surface with shift 3%, 2%, 1%, 0.75%, 0.5%. Below are shown the volatility surface with the highest shift used and the lowest one.

In vols
shift 0.03

maturity	STK	ATM	-1.5	-1.25	-1	-0.5	-0.25	0	0.25	0.5	1	2	3	5	10
1	-0.4	6.60%				5.73%	7.91%								
1.5	-0.4	2.81%				5.53%	5.82%								
2	-0.4	6.20%				4.29%	8.03%	8.98%							
3	-0.3	5.27%			10.51%	6.94%	7.20%	8.15%	8.97%	9.07%					
4	-0.2	4.44%	16.45%	14.23%	12.13%	8.47%	8.68%	9.42%	10.27%	11.09%	12.07%	14.59%			
5	-0.2	8.04%	17.71%	15.29%	13.25%	9.86%	9.67%	10.41%	11.06%	11.67%	12.73%	15.10%	16.82%	20.04%	
6	-0.2	11.04%	18.47%	16.31%	14.39%	11.03%	10.80%	11.35%	11.87%	12.38%	13.28%	14.89%	16.01%	17.37%	
7	-0.1	10.86%	19.03%	16.95%	15.11%	11.99%	11.73%	12.14%	12.57%	12.96%	13.62%	14.81%	15.61%	17.58%	21.52%
8	0	11.71%	19.50%	17.44%	15.79%	12.81%	12.50%	11.71%	13.10%	13.41%	13.89%	14.72%	15.61%	16.77%	19.34%
9	0	13.13%	19.66%	17.82%	16.18%	13.40%	13.07%	13.13%	13.55%	13.78%	14.19%	14.89%	15.47%	16.50%	18.99%
10	0.04	14.05%	20.01%	18.24%	16.67%	14.02%	13.66%	13.89%	13.94%	14.10%	14.45%	14.98%	15.57%	16.53%	18.30%
12	0.16	14.63%	20.87%	19.07%	17.52%	14.96%	14.52%	14.63%	14.55%	14.62%	14.73%	15.02%	15.35%	16.09%	17.58%
15	0.29	15.27%	21.76%	19.93%	18.41%	15.90%	15.35%	15.26%	15.26%	14.95%	14.82%	14.79%	14.95%	15.52%	16.78%
20	0.41	15.64%	22.55%	20.71%	19.20%	16.79%	16.13%	15.85%	15.69%	15.28%	14.95%	14.64%	14.65%	14.99%	16.02%
25	0.45	15.75%	22.64%	20.91%	19.47%	17.22%	16.55%	16.20%	15.95%	15.50%	15.09%	14.67%	14.57%	14.76%	15.54%
30	0.44	15.92%	22.62%	20.96%	19.58%	17.44%	16.77%	16.36%	16.07%	15.59%	15.13%	14.63%	14.46%	14.53%	15.09%

Figure 9: Shifted (3%) term implied lognormal volatility

In vols shift 0.005														
maturity	STK	ATM	-1.5	-1.25	-1	-0.5	-0.25	0	0.25	0.5	1	2	3	5	10
1	-0.4						178.24%								
1.5	-0.4	99.00%					127.83%								
2	-0.4														
3	-0.3	82.20%					104.06%	78.28%	68.15%	58.21%					
4	-0.2	47.50%					120.29%	84.36%	72.50%	66.80%	58.66%	56.33%			
5	-0.2	86.90%					128.16%	87.41%	72.73%	65.24%	57.57%	54.46%	53.12%	54.42%	
6	-0.2	126.30%					141.63%	90.57%	73.71%	65.20%	56.57%	50.27%	47.28%	43.84%	
7	-0.1	91.60%					155.50%	92.56%	74.29%	64.83%	54.98%	47.31%	43.63%	42.37%	44.00%
8	0	82.60%					170.35%	82.61%	73.86%	63.98%	53.41%	44.78%	41.70%	38.52%	37.67%
9	0	92.00%					186.37%	92.04%	73.51%	63.27%	52.56%	43.74%	39.90%	36.71%	35.99%
10	0.04	92.80%					235.02%	96.37%	73.45%	62.78%	52.02%	42.80%	39.18%	35.99%	33.91%
12	0.16	80.30%						98.49%	73.27%	62.16%	50.68%	41.17%	37.10%	33.77%	31.57%
15	0.29	72.50%						101.92%	74.84%	60.99%	48.92%	39.05%	34.94%	31.65%	29.46%
20	0.41	68.80%						123.69%	78.92%	62.50%	48.99%	38.38%	34.04%	30.51%	28.18%
25	0.45	72.00%							90.57%	67.43%	51.21%	39.39%	34.62%	30.67%	27.91%
30	0.44	83.50%							119.53%	75.05%	54.24%	40.68%	35.39%	31.00%	27.78%

Figure 10: Shifted (0.5%) term implied lognormal volatility

Interestingly, reducing the shift impacts Floors but not Caps. This difference does not originate from the pricing formula itself—both Caps and Floors use the same expression for d_1- but rather from the structure of market quotes.The key distinction lies in the region where d_1 is positioned after applying the shift. Since both caps and floors follow the same formula for d_1, reducing the shift by the same amount leads to similar increment in d_1 for both instruments. However, due to the nature of market quotations, Caps generally start with a d_1 value that is closer to zero or moderately positive. A reduction in the shift moves d_1 further to the right, but it remains in a region where the normal density function $\Phi(d_1)$ is relatively high. Conversely, quoted Floors typically start with a larger d_1. Lowering the shift moves this d_1 even further into the right tail of the distribution, where $\Phi(d_1)$ decays rapidly. Since the option price depends on $\Phi(d_1)$, this rapid decay makes Floor prices almost insensitive to volatility changes, leading to numerical failures in the inversion process.

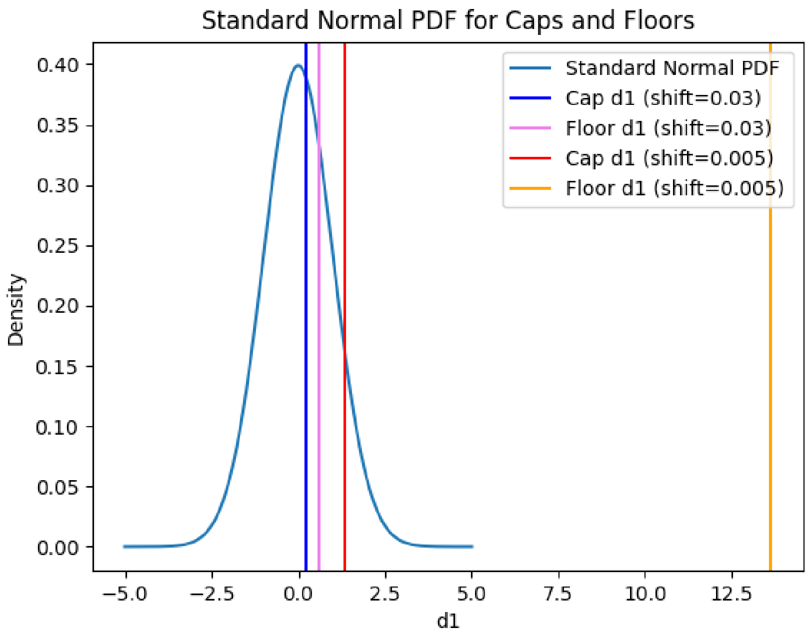

Figure 11: Effect of different shifts on d_1 values. Comparison of two Caps and two Floors with the same maturity, forward rate, and strike, evaluated with shifts of 3% and 0.005%. Parameters: $T = 30Y$, $F = 0.0044$, $K_{floor} = 0$,K$_{cap} = 1$.

This graph above was obtained by computing d_1 terms in four cases, in three of them d_1 was computed directly, as all inputs were available. However, for the Floor with a 0.005 shift, we used the final d_1 value from the Brent's method iteration routine, as it failed to converge.
Since vega is rouglhy proportional to d_1, a floor's sensitivity to volatility drops off much faster when the shift is reduced, making the model price almost insensitive to volatility changes.
The number of computed volatilities can be seen in the bar chart below, illustrating how lowering the shift results in a loss of volatilities.

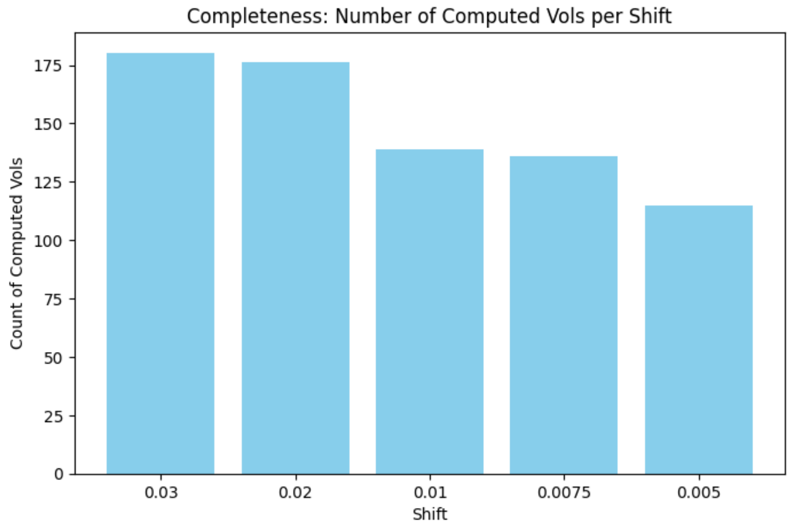

Figure 12: Number of computed volatilities at different shift levels

Another observed effect is the increase in computed volatility values when using a lower shift. Lower shifts yield higher volatility estimates, as shown in the box plot below:

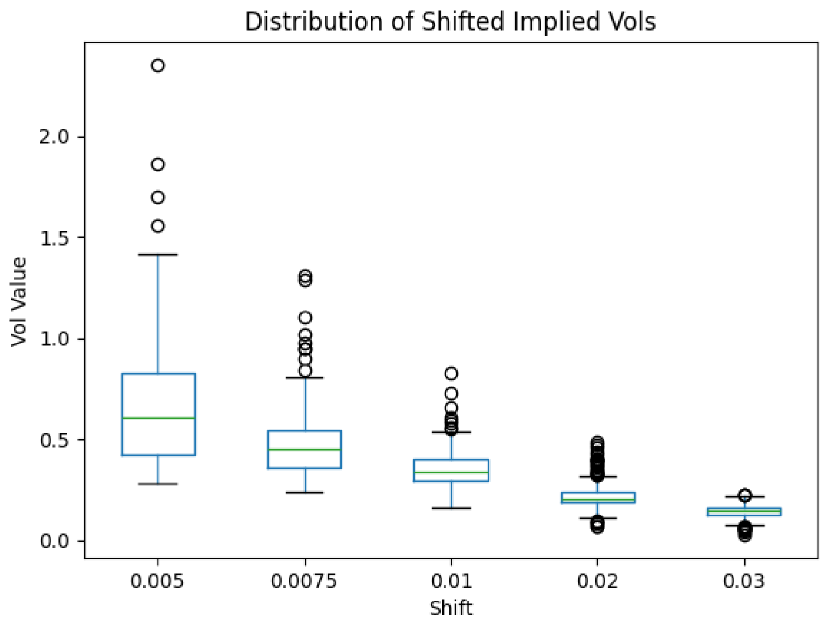

Figure 13: Box plot showing the distribution of computed volatilities across different shift levels

The box plot highlights how the median volatility increases as the shift decreases. The length of the box represents the spread of volatility values, which becomes wider with lower shifts. Additionally, outliers (represented by empty points) are more dispersed when the shift is reduced.

We further illustrate these differences by plotting volatility smiles for different shift values:

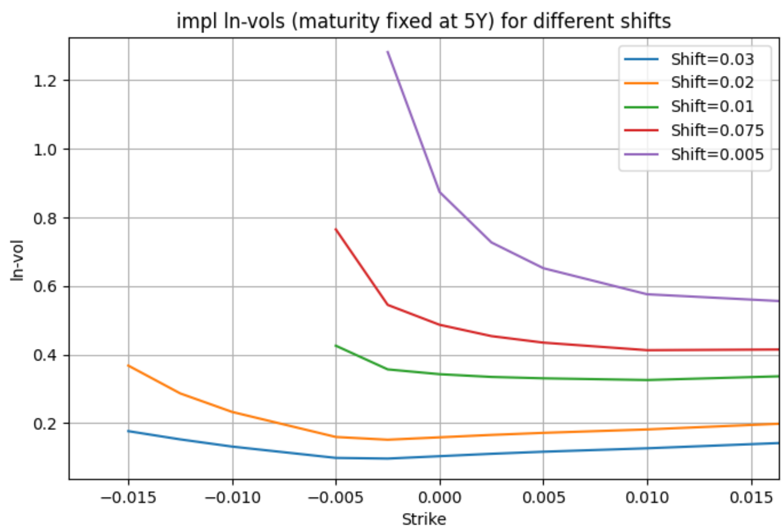

Figure 14: Impact of different shifts on ATM log-normal volatilities for 5Y tenor

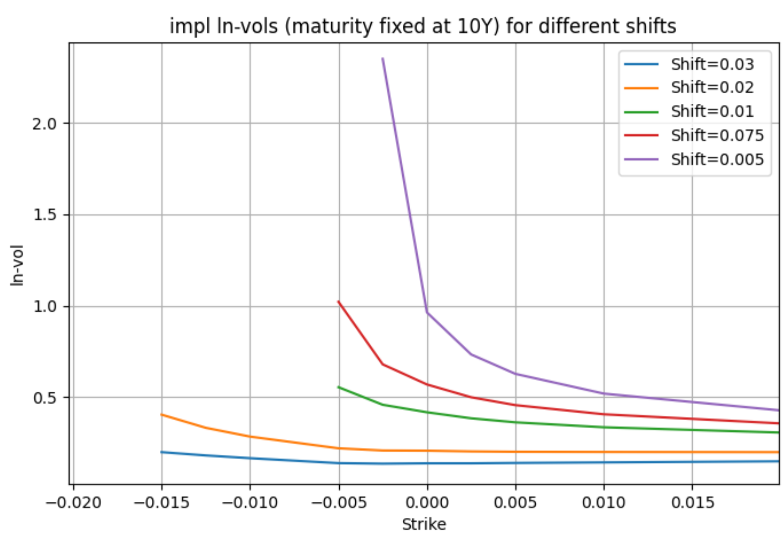

Figure 15: Impact of different shifts on ATM log-normal volatilities for 10Y tenor

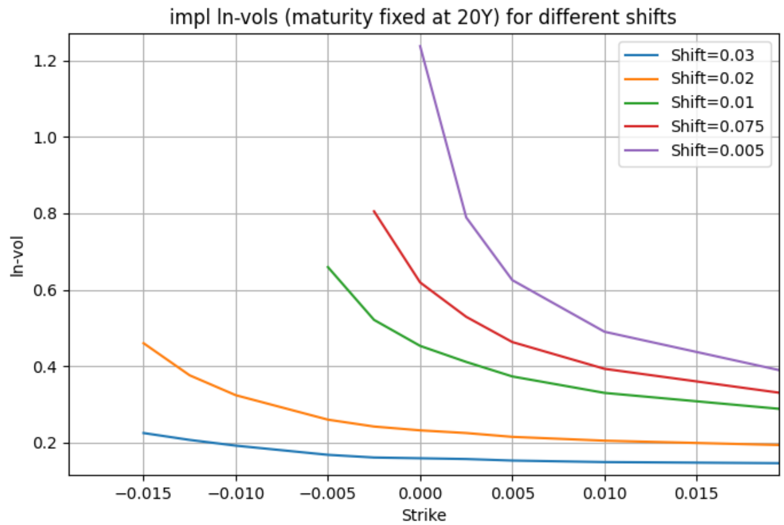

Figure 16: Impact of different shifts on ATM log-normal volatilities for 20Y tenor

These plots demonstrate how volatility values increase as the shift decreases. Additionally, increasing the shift causes the smile to shift slightly to the right, indicating that deep in-the-money Floors are not covered.

We also examined the impact of different shifts using an alternative root-finding algorithm. We compared the results obtained using Brent's method with those from the Newton-Raphson method. The same procedure was applied in both cases, allowing for a direct comparison of their performance and Newton-Raphson failed more frequently than Brent's method at the same shift level.

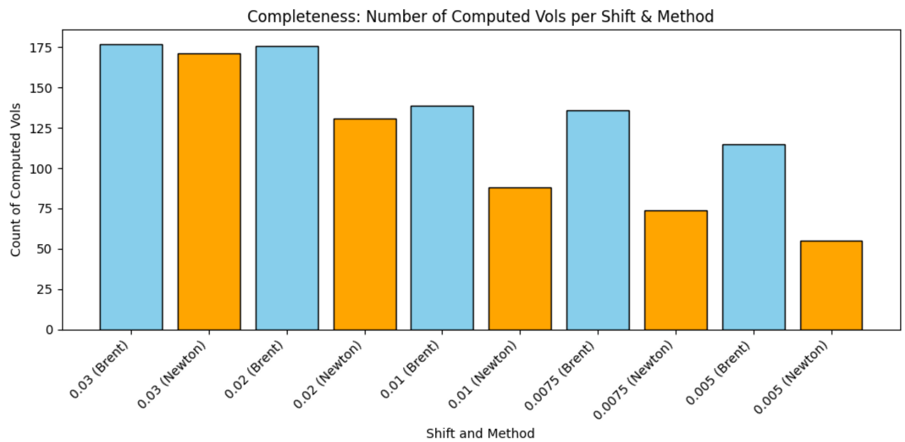

Figure 17: Bar chart comparing for each shift how many volatilities have been computed using Brent's method and Newton-Rapshon's method

For a given shift, the success rate difference highlights the varying robustness of the root-finding algorithms. Newton-Raphson can fail or diverge when the derivative is near zero, the initial guess is far from the root, or the local curvature is unfavorable. Brent's method, which combines bisection and interpolation, is less sensitive to these issues, resulting in a higher success rate and more computed volatilities for the same instruments.