

Thesis name



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Introduction

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Chapter 1

Valley Hall

1.1 Berry curvature in Gapped graphene

The Hamiltonian for the gapped graphene near the point K_1 and K_2 can be written as

$$H_{K_1} = H_{K_2}^\dagger = \begin{bmatrix} \Delta & \hbar v_F(k_x + ik_y) \\ \hbar v_F(k_x - ik_y) & \Delta \end{bmatrix} \quad (1.1)$$

Where Δ is the energy gap and v_F is the Fermi velocity. For ease of notation we are going to work with just H_{K_1} and drop the K_1 ,¹ and for ease of computation we define $\mathbf{q} = \hbar v_F \mathbf{k}$

$$H = \begin{bmatrix} \Delta & q_x + iq_y \\ q_x - iq_y & \Delta \end{bmatrix} = \sigma_x q_x + \sigma_y q_y + \sigma_z \Delta \equiv \boldsymbol{\sigma} \cdot \mathbf{E} \quad (1.2)$$

Here the energy vector \mathbf{E} is defined as $\mathbf{E} = (q_x, q_y, \Delta)$. The nice things about it is that $E = |\mathbf{E}| = \sqrt{q_x^2 + q_y^2 + \Delta^2}$ is the positive eigenvalue of the hamiltonian (the negative eigenvalue is just $-E$).

To calculate the Berry curvature we are first going to calculate the Berry connection 2.9, and to calculate the Berry connection we need the eigenvectors which are well known for the Hamiltonian of the form $\boldsymbol{\sigma} \cdot \mathbf{E}$.

$$|+; \theta, \phi\rangle = \begin{bmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{bmatrix} \quad |-; \theta, \phi\rangle = \begin{bmatrix} -e^{-i\phi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{bmatrix} \quad (1.3)$$

Where θ and ϕ are the coordinates of \mathbf{E} in the polar representation

Now we can calculate the Berry connection

$$A_\theta^+ = -A_\theta^- = 0 \quad A_\phi^+ = -A_\phi^- = \sin^2 \frac{\theta}{2} \quad (1.4)$$

¹Don't worry, I'll bring it back if when we'll need it

This means that the Berry curvature is

$$\Omega_{\theta\phi}^+ = -\Omega_{\theta\phi}^- = \partial_\theta A_\phi^+ = \frac{\sin \theta}{2} \quad (1.5)$$

From now on we are going to work with Ω^+ and we are going to drop the + sign to make the notation lighter.

We want to express Ω in terms of \mathbf{q} , however it's more convenient to write it in terms of $\cos \theta$ and ϕ , so we do a small coordinate transformation

$$\Omega_{\theta\phi} = \frac{\partial \cos \theta}{\partial \theta} \Omega_{\cos(\theta)\phi} \rightarrow \Omega_{\cos(\theta)\phi} = \frac{1}{2} \quad (1.6)$$

Now we can easily make the transformation to express Ω in terms of \mathbf{q} . The Berry curvature transforms like any other tensor under coordinate transformation, so

$$\Omega_{q_x q_y} = \frac{\partial \cos \theta}{\partial q_x} \frac{\partial \phi}{\partial q_y} \Omega_{\cos(\theta)\phi} + \frac{\partial \phi}{\partial q_x} \frac{\partial \cos \theta}{\partial q_y} \Omega_{\phi \cos(\theta)} \quad (1.7)$$

That can be rewritten as

$$\Omega_{q_x q_y} = \frac{1}{2} \det \left[\frac{\partial(\cos \theta, \phi)}{\partial(q_x, q_y)} \right] = \frac{1}{2} \frac{\Delta^2}{q^2 E^3} (q_x + q_y - 2q) \quad (1.8)$$

And finally we can express it in terms of \mathbf{k}

$$\Omega_{k_x k_y} = (\hbar v_F)^2 \Omega_{q_x q_y} = \frac{\hbar v_F}{2} \frac{\Delta^2}{k^2 E^3} (k_x + k_y - 2k) \quad (1.9)$$

Up until now we have worked with the Hamiltonian H_{K_1} , but with the K_1 hidden. The Berry curvature around K_2 is equal, but with opposite sign (figure 1.1) ²

1.2 Valley-Hall effect

The Hall conductivity σ_{xy} is

$$\sigma_{xy} = \frac{e^2}{\hbar} \int_{\mathbb{R}^2} f[E^+(k)] \Omega_{k_x k_y}^+ + f[E^-(k)] \Omega_{k_x k_y}^- \frac{d^2 \mathbf{k}}{2\pi} \quad (1.10)$$

Where $f(E) = [e^{\beta(E-\mu)} + 1]^{-1}$ is the Fermi-Dirac distribution, it is applied once for the states with positive energy and once for the states with negative energy.

²A short proof for it can be the following: If we send $k_y \rightarrow -k_y$ we effectively send $H_{K_1} \rightarrow H_{K_2}$.

The berry curvature can be written as $\Omega_{k_x k_y} = i \langle \partial_{k_x} n | \wedge | \partial_{k_y} n \rangle$. By sending $k_y \rightarrow -k_y$ we have that $\Omega \rightarrow -\Omega$.

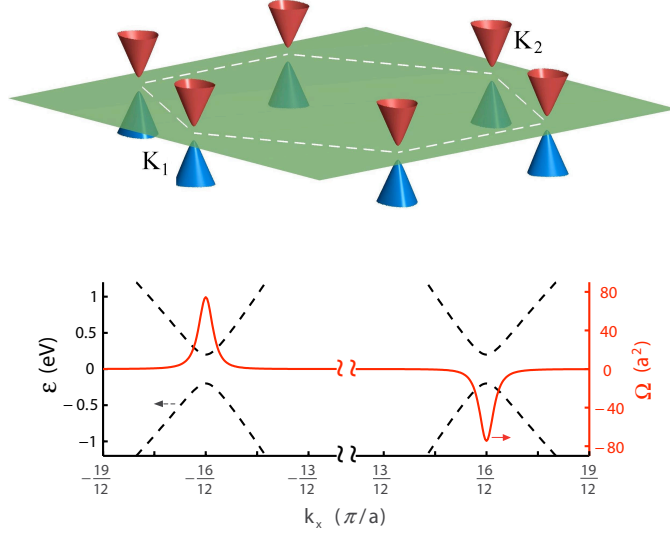


Figure 1.1: In the top panel are displayed the Energy bands in 2D. In the bottom panel with the dotted line are displayed a section of the energy bands, and with the continuous red line the Berry curvature.

We are going to analyze the system at low temperatures ($k_B T \ll 1$), so our Fermi-Dirac distribution can be considered like a step-function.

First let's integrate the conductivity for the positive energies and drop the + sign to make the notation lighter.

$$\begin{aligned}
 \int_{\mathbb{R}^2} f[E(k)] \Omega_{k_x k_y} dk_x dk_y &= \int_{\mathbb{R}^2} f[E(q)] \Omega_{q_x q_y} dq_x dq_y \approx \\
 &\approx \int_0^{2\pi} \int_0^{q_F} \frac{1}{2} \frac{\Delta^2}{q^2 E^3} (q_x + q_y - 2q) q dq d\theta = \\
 &= -2\pi \Delta^2 \int_0^{q_F} \frac{dq}{E^2} = -2\pi \Delta^2 \int_0^{q_F} \frac{dq}{(\Delta^2 + q^2)^{3/2}} = -\frac{2\pi q_F}{\sqrt{\Delta^2 + q_F^2}}
 \end{aligned}$$

And now we express it in terms of the chemical potential μ ³

$$\int_{\mathbb{R}^2} f[E(k)] \Omega_{k_x k_y} dk_x dk_y \approx -2\pi \frac{\sqrt{\mu^2 - \Delta^2}}{\mu} \theta(\mu - \Delta) \quad (1.11)$$

The $\theta(\mu - \Delta)$ is there to make sure that if no states are inside the Fermi-Dirac the integral is zero. One thing to notice is that if you have $\mu \gg \Delta$ (aka. all states in the band are occupied) then the integral is equal to -2π .

³Here we can use interchangeably μ and E_F

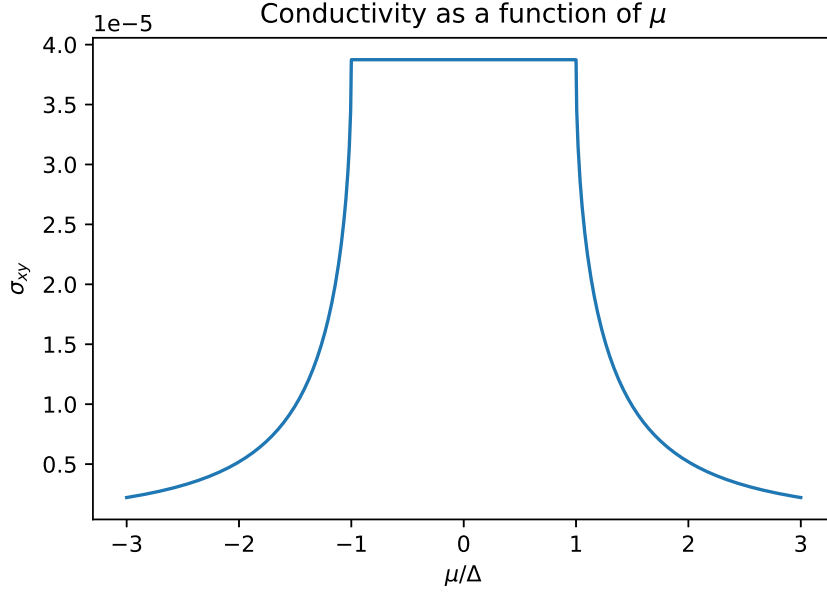


Figure 1.2: Here is shown $\sigma_{xy}(\mu)$ (eq. 1.12). Notice how, when $\mu \in [-\Delta, \Delta]$ then $\sigma_{xy} = \frac{e^2}{2\pi\hbar}$

The integral of the lower band is very similar. By the end equation of the conductivity 1.10 becomes

$$\sigma_{xy}(\mu) = -\frac{e^2}{2\pi\hbar} \left[\frac{\sqrt{\mu^2 - \Delta^2}}{\mu} \theta(\mu^2 - \Delta^2) - \theta(\mu - \Delta) \right] \quad (1.12)$$

To be fair we only calculated σ_{xy} for the electrons in the valley K_1 , the conductivity for the other valley is just $-\sigma_{xy}$. So, putting it all together, we have

$$\sigma_{K_i,xy}(\mu) = (-1)^i \frac{e^2}{2\pi\hbar} \left[\frac{\sqrt{\mu^2 - \Delta^2}}{\mu} \theta(\mu^2 - \Delta^2) - \theta(\mu - \Delta) \right] \quad (1.13)$$

However in most cases it's safe to assume that the chemical potential is inside the energy gap, so equation 1.13 becomes

$$\sigma_{K_i,xy} = (-1)^{i+1} \frac{e^2}{2\pi\hbar} \quad (1.14)$$

1.3 Non-local Charge transport

If we apply a voltage V in two opposite points of a strip of a ohmic material of width W and infinite length, and we see a current that flows from one

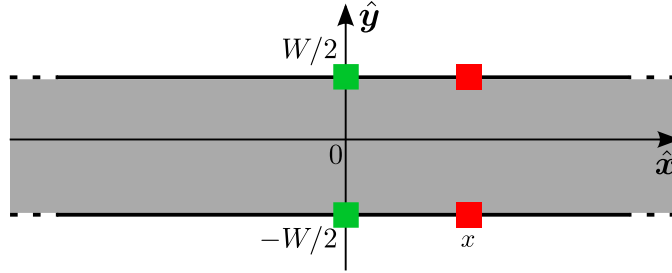


Figure 1.3: Representation of the strip

point to another figure 1.3.

Clearly the current isn't completely localized along the axis that unites the two injection points, and so does the voltage difference.

If we probe the voltage from two different points with an offset of x from the injection points and we divide it by the total current between the contacts we see that

$$\frac{V(x)}{I} = \frac{2\rho}{\pi} \ln \left| \coth \left(\frac{\pi x}{2W} \right) \right| \quad (1.15)$$

Where ρ is the resistivity. Don't worry later on there is the proof of this equation.

However, two-dimensional material like gapped graphene [1–3] and transition metal dichalcogenides [4–6], don't obey this equation. This is because these materials display the Valley Hall effect we talked about previously (inserire reference a sezione).

Non-local transport can be a useful tool to probe the existence of anomalous Hall effect [7–12]

1.4 Theory of non local charge transport

The charges inside the material get pushed around from the electrochemical potential ψ_K

$$\psi_K(\mathbf{r}) = V(\mathbf{r}) - \frac{1}{e} \mu_K[n_{K_1}(\mathbf{r}), n_{K_2}(\mathbf{r}), T] \quad (1.16)$$

Where ϕ is the electrical potential, and $\mu_K = \frac{\partial}{\partial n_K} F[n_{K_1}(\mathbf{r}), n_{K_2}(\mathbf{r}), T]$ is the chemical potential of the material and F is the free energy.

The current generated from this potential in the valley K_α in the i -th direction is

$$-eJ_{K_\alpha,i}(\mathbf{r}) = \sum_{j,b} \underbrace{-\sigma_{K_\alpha K_\beta,ij}}_{\text{conductivity}} \partial_j \psi_{K_\beta}(\mathbf{r}) \quad (1.17)$$

From now we are going to set $T \approx 0^4$ and ignore intervalley scattering, so if $K_\alpha \neq K_\beta$ $\sigma_{K_\alpha K_\beta,ij} = 0$, also because of this the free energy can be written as the sum of the two Free energies

$$F(n_{K_1}, n_{K_2}) = F_1(n_{K_1}(\mathbf{r})) + F_2(n_{K_2}(\mathbf{r})) \quad (1.18)$$

And so the chemical potential of a given valley depend only on the number of electron in the same valley

$$\mu_\alpha(n_{K_\alpha}(\mathbf{r})) = \frac{\partial}{\partial n_{K_\alpha}} F(n_{K_0}, n_{K_1}) = \frac{\partial}{\partial n_{K_\alpha}} F_\alpha(n_{K_\alpha}(\mathbf{r})) \quad (1.19)$$

This simplifies the trasport equation in

$$-e\mathbf{J}_{K_\alpha}(\mathbf{r}) = \sigma_{K_\alpha}(\mathbf{r}) \nabla \psi_{K_\alpha}(\mathbf{r}) \quad (1.20)$$

Where σ_{K_α} is the following matrix

$$\sigma_{K_\alpha} = \begin{bmatrix} \sigma_{K_\alpha K_\alpha,xx} & \sigma_{K_\alpha K_\alpha,xy} \\ -\sigma_{K_\alpha K_\alpha,xy}^* & \sigma_{K_\alpha K_\alpha,xx} \end{bmatrix}$$

Now we need to write the gradient electrochemical potential $\nabla \psi(\mathbf{r})$

$$\nabla \psi_{K_\alpha}(\mathbf{r}) = \nabla V(\mathbf{r}) - \frac{1}{e} \frac{\partial}{\partial n_{K_\alpha}} \mu_\alpha(n_{K_\alpha}(\mathbf{r})) \nabla n_{K_\alpha} \quad (1.21)$$

From equation INSERTIRE REFERENCE A EQUAZIONE we can write for gapped Dirac hamiltonians that VERIFICARE SE VALE ANCHE PER BILAYER GRAPHENE

$$\frac{\partial \mu_{K_\alpha}}{\partial n_{K_\alpha}} = \frac{\pi}{\sqrt{2\pi|n| + \Delta^2}} + \Delta \delta(n) \approx \frac{\pi}{\Delta} + \Delta \delta(n) \quad \forall \alpha$$

In this equation we assumed that there are very few charge carries, so $\frac{n}{\Delta^2} \approx 0$. We can shorten the equation 1.21 by defining

$$e^2 D_{K_\alpha,ij} = \sigma_{K_\alpha,ij} \frac{\partial \mu_\alpha}{\partial n_{K_\alpha}} [n_{K_\alpha}(\mathbf{r})] \quad (1.22)$$

So equation 1.20 becomes

$$-eJ_{K_\alpha,i}(\mathbf{r}) = \sigma_{K_\alpha,ij} E_j(\mathbf{r}) - eD_{K_\alpha,ij} \partial_j n_{K_\alpha}(\mathbf{r}) \quad (1.23)$$

or, written in matrix form

$$-e\mathbf{J}_{K_\alpha}(\mathbf{r}) = \sigma_{K_\alpha} \mathbf{E}(\mathbf{r}) - eD_{K_\alpha} \nabla n_{K_\alpha}(\mathbf{r}) \quad (1.24)$$

Where σ_{K_α} and $-eD_{K_\alpha}$ are matrices.

⁴A more precise statement is that the thermal De Broglie wavelenght λ_T must be much larger than the average distance between the electrons. We are not going into the math here, but if you want to calculate it, keep in mind that the dispersion relation is relativistic, so the formula of λ_T is going to be a bit different

1.4.1 Re-writing the equations in terms of charge current and valley current

Measuring the currents in different valley can be cumbersome, however measuring the charge current $\mathbf{J}_c = \mathbf{J}_{K_1} + \mathbf{J}_{K_2}$ is straightforward, and for mathematical convenience we also define the valley current $\mathbf{J}_v = \mathbf{J}_{K_1} - \mathbf{J}_{K_2}$.

Since we no longer describe the currents in terms of their valley index, but on the sum and the difference of what happens at the different valleys, we are going to reparametrize also the other quantities in the same fashion.

$$\begin{cases} \sigma_c = \sigma_{K_1} + \sigma_{K_2} = 2\sigma_{xx}\delta_{ij} \\ \sigma_v = \sigma_{K_1} - \sigma_{K_2} = \sigma_v = 2\sigma_{xy}\epsilon_{ij} \end{cases} \quad (1.25)$$

The term $-eD_{K_\alpha}\nabla n_{K_\alpha}(\mathbf{r})$ is a little harder to translate. First off we are going to impose the local charge conservation

$$\begin{aligned} n_c(\mathbf{r}) &= n_{K_0} + n_{K_1} \approx 0, \text{ and so} \\ n_v(\mathbf{r}) &= n_{K_0} - n_{K_1} = 2n_{K_0} = -2n_{K_1} \end{aligned}$$

Chapter 2

Berry phase and Berry curvature

2.1 Introduction

Berry phase is the simplest demonstration of how geometry and topology can emerge from quantum mechanics and it is in the heart of the quantum Hall effect

Let us consider a physical system described by a Hamiltonian that depends on a set of parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$. These parameters do not represent the degrees of freedom of the system like position and momentum, rather they describe things such as the mass of a particle, the strength of a potential and so on.

For each $H(\boldsymbol{\lambda})$ there exists a set of eigenstates such that

$$H(\boldsymbol{\lambda}) |n, \boldsymbol{\lambda}\rangle = E_n(\boldsymbol{\lambda}) |n, \boldsymbol{\lambda}\rangle \quad (2.1)$$

However the equation above does not completely determine the basis function $|n, \boldsymbol{\lambda}\rangle$; We can change arbitrarily the phase $\gamma_n(\boldsymbol{\lambda})$ of any eigenstate which is called *Berry phase*

$$|n, \boldsymbol{\lambda}\rangle \rightarrow \underbrace{e^{i\gamma_n(\boldsymbol{\lambda}(t))}}_{\text{Berry phase}} |n, \boldsymbol{\lambda}\rangle \quad (2.2)$$

Suppose we start off with a hamiltonian and then we slowly change the parameters for a time T until it reaches a different hamiltonian, this means that $\boldsymbol{\lambda} = \boldsymbol{\lambda}(T)$. For the adiabatic theorem we can say that if we start on an energy eigenstate, and the system changes slowly enough ¹, and has no

¹How slow you have to be in changing the parameters depends on the energy gap from the state you're in to the nearest other state. The smaller the gap, the slower you have to change the parameters. A way of showing this without doing long calculations is the following:

We know from the Heisenberg uncertainty principle that $T\Delta E \geq \hbar/2$. We want the uncertainty in the Energy to be way smaller than the energy gap $E_g \gg \Delta E$, so $E_g \gg \frac{\hbar}{2T}$, so if we make T big enough it can be achieved

degeneracies, then the system will cling on that energy eigenstate. This means that the equation of motion of a particle that for time $t = 0$ is equal to $|\psi_n(t = 0)\rangle = |n, \boldsymbol{\lambda}(0)\rangle$ is

$$|\psi_n(t)\rangle = \underbrace{e^{i\gamma_n(\boldsymbol{\lambda}(t))}}_{\text{Berry phase}} \cdot \underbrace{e^{-\frac{i}{\hbar} \int_0^t E_n(\boldsymbol{\lambda}(t')) dt'}}_{\text{dynamical phase}} |n, \boldsymbol{\lambda}(t)\rangle \quad (2.3)$$

Where the first exponent comes from eq. 2.2. We now insert the equation above into the time-dependent Shrodinger equation

$$i\hbar\partial_t|\psi_n(t)\rangle = H(\boldsymbol{\lambda}(t))|\psi_n(t)\rangle \quad (2.4)$$

By plugging equation 2.3 into the *right* term term of equation 2.4 we get we get that

$$H(\boldsymbol{\lambda}(t))|\psi_n(t)\rangle = E_n(t) |\psi_n(t)\rangle \quad (2.5)$$

Andy By plugging equation 2.3 into the *left* term term of equation 2.4 we get we get that

$$i\hbar\partial_t|\psi_n(t)\rangle = -\hbar\dot{\gamma}_n(t)|\psi_n(t)\rangle + E_n(t)|\psi_n(t)\rangle + e^{i\phi_n(t)}\partial_t|n, t\rangle \quad (2.6)$$

where we have defined $e^{i\phi_n(t)} \equiv e^{i\gamma_n(\boldsymbol{\lambda}(t))} e^{-\frac{i}{\hbar} \int_0^t E_n(\boldsymbol{\lambda}(t')) dt'}$

By equating the right terms in equations 2.5 and 2.6 we get that

$$i\hbar e^{i\phi_n(t)}\partial_t|n, t\rangle = \hbar\dot{\gamma}_n(t)|\psi_n(t)\rangle = \hbar\dot{\gamma}_n(t)e^{i\phi_n(t)}|n, t\rangle \quad (2.7)$$

now we multiply the term on the left and on the right of equation 2.7 by $\hbar^{-1}e^{-i\phi_n(t)}\langle n, t|$

$$\dot{\gamma}_n(t) = i\langle n, t|\partial_t|n, t\rangle \quad (2.8)$$

We can re-express it in terms of $\boldsymbol{\lambda}$

$$\dot{\gamma}_n(t) = \dot{\boldsymbol{\lambda}} \cdot \underbrace{i\langle n, t|\partial_{\boldsymbol{\lambda}}|n, t\rangle}_{\equiv \mathbf{A}_n(\boldsymbol{\lambda})} \quad (2.9)$$

Where $\mathbf{A}_n(\boldsymbol{\lambda})$ called the **Berry connection** This means that we can calculate the total change in $\gamma_n(t)$ can be obtained by doing a line integral in the space of parameters $\boldsymbol{\lambda}$ over the path \mathcal{P} of values that $\boldsymbol{\lambda}$ assumes during the time evolution

$$\gamma_n = \int_{\mathcal{P}} \mathbf{A}_n(\boldsymbol{\lambda}) \cdot d\boldsymbol{\lambda} \quad (2.10)$$

$$|n, \boldsymbol{\lambda}\rangle \rightarrow e^{if_n(\boldsymbol{\lambda})} |n, \boldsymbol{\lambda}\rangle \quad (2.11)$$

Keep in mind however that the eigenstates are defined up to a phase, meaning that we can re-define the base vectors like so (equation 2.11). If we apply this substitution into the formula of \mathbf{A}_n we have that

$$\mathbf{A}_n(\boldsymbol{\lambda}) = i \langle n, t | \partial_{\boldsymbol{\lambda}} | n, t \rangle \rightarrow i \langle n, t | \partial_{\boldsymbol{\lambda}} | n, t \rangle - \partial_{\boldsymbol{\lambda}} f_n(\boldsymbol{\lambda})$$

$$\mathbf{A}_n \rightarrow \mathbf{A}_n - \partial_{\boldsymbol{\lambda}} f_n \quad (2.12)$$

So the system is invariant under the gauge transformation in equation 2.12. If we do this transformation to equation 2.10 we have that

$$\gamma_n = \int_{\mathcal{P}} \mathbf{A}_n(\boldsymbol{\lambda}) \cdot d\boldsymbol{\lambda} - \int_{\mathcal{P}} \partial_{\boldsymbol{\lambda}} f_n(\boldsymbol{\lambda}) \cdot d\boldsymbol{\lambda} = \int_{\mathcal{P}} \mathbf{A}_n(\boldsymbol{\lambda}) \cdot d\boldsymbol{\lambda} + f(\boldsymbol{\lambda}(0)) - f(\boldsymbol{\lambda}(T))$$

This means that if the path \mathcal{P} is open we can always choose a function f_n such that $f(\boldsymbol{\lambda}(0)) - f(\boldsymbol{\lambda}(T)) = \int_{\mathcal{P}} \mathbf{A}_n(\boldsymbol{\lambda}) \cdot d\boldsymbol{\lambda}$, thus we can conclude that one can always choose a suitable $f(\boldsymbol{\lambda})$ such that γ_n accumulated along the path \mathcal{P} is canceled out leaving equation 2.3 with only the dynamical phase. However if the path is closed $\boldsymbol{\lambda}(0) = \boldsymbol{\lambda}(T)$, in order to make the phase change in equation 2.11 single value we must have that

$$e^{f(\boldsymbol{\lambda}(0)) - f(\boldsymbol{\lambda}(T))} = 1$$

so

$$f(\boldsymbol{\lambda}(0)) - f(\boldsymbol{\lambda}(T)) = 2n\pi \quad n \in \mathbb{R}$$

This leads us to the important result that

$$\gamma_n = \oint_{\mathcal{P}} \mathbf{A}_n(\boldsymbol{\lambda}) \cdot d\boldsymbol{\lambda} + 2n\pi \quad (2.13)$$

This time, if the line integral is not a multiple of 2π (and there is no reason why it should) there is no way of choosing a suitable f_n to cancel it out and the Berry phase in equation 2.3 is there to stay

2.2 Berry curvature

In EM the field tensor $F_{\mu\nu}$ is defined as in equation 2.14. Since Berry connection has the same Gauge invariance as the one of the EM vector potential it is useful to define, a gauge field tensor derived from the Berry connection:

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \quad (2.14)$$

$$\Omega_{\mu\nu}^n = \partial_{\mu} A_{\nu}^n(\boldsymbol{\lambda}) - \partial_{\nu} A_{\mu}^n(\boldsymbol{\lambda}) \quad (2.15)$$

This new field tensor is defined as **Berry curvature** and it is gauge independent just like $F_{\mu\nu}$.²

²The notation has changed a bit, now $A_{\mu}^n \equiv (\mathbf{A}_n)_{\mu}$

2.2.1 Other formulas for $\Omega_{\mu\nu}$

With a few mathematical steps it is possible to re cast the Berry curvature into a different form that might be useful later

$$\partial_\mu A_\mu^n = i \partial_\mu \langle n, \boldsymbol{\lambda} | \partial_\nu n, \boldsymbol{\lambda} \rangle = i \langle \partial_\mu n, \boldsymbol{\lambda} | \partial_\nu n, \boldsymbol{\lambda} \rangle + i \langle n, \boldsymbol{\lambda} | \partial_\mu \partial_\nu n, \boldsymbol{\lambda} \rangle$$

$$\boxed{\Omega_{\mu\nu}^n = i \langle \partial_\mu n | \partial_\nu n \rangle - i \langle \partial_\nu n | \partial_\mu n \rangle} \quad (2.16)$$

It is also possible to express Ω in terms of the eigenstates of the Hamiltonian with some mathematical manipulation

$$\begin{aligned} \langle n' | H | n \rangle &= \delta_{n'n} \rightarrow \partial_\mu \langle n' | H | n \rangle = 0 \\ \partial_\mu \langle n' | H | n \rangle &= \langle \partial_\mu n' | H | n \rangle + \langle n' | H | \partial_\mu n \rangle + \langle n' | \partial_\mu H | n \rangle \\ E_n \langle \partial_\mu n' | n \rangle + E_{n'} \langle n' | \partial_\mu n \rangle &= \langle n' | \partial_\mu H | n \rangle \\ (E_{n'} - E_n) \langle n' | \partial_\mu n \rangle &= \langle n' | \partial_\mu H | n \rangle \\ \langle n' | \partial_\mu n \rangle &= \frac{\langle n' | \partial_\mu H | n \rangle}{E_{n'} - E_n} \end{aligned} \quad (2.17)$$

Now we write equation 2.16 like so

$$\Omega_{\mu\nu}^n = i \langle \partial_\mu n | \partial_\nu n \rangle - (\mu \leftrightarrow \nu) = i \sum_{n' \neq n} \langle \partial_\mu n | n' \rangle \langle n' | \partial_\nu n \rangle - (\mu \leftrightarrow \nu)$$

By plugging in above equation 2.17 we get

$$\boxed{\Omega_{\mu\nu}^n = i \sum_{n' \neq n} \frac{\langle n | \partial_\mu H | n' \rangle \langle n' | \partial_\nu H | n \rangle}{(E_{n'} - E_n)^2} - (\mu \leftrightarrow \nu)} \quad (2.18)$$

This last form of the Berry curvature has the advantage that no differentiation of the wavefunction is needed. This equation also tells us that

$$\sum_n \Omega_{\mu\nu}^n(\boldsymbol{\lambda}) = 0$$

2.3 Stokes' Theorem

From the Stokes theorem we have that

$$\gamma_n = \oint_{\mathcal{P}} A_\mu^n d\lambda^\mu = \frac{1}{2} \int_{\Sigma} \Omega_{\mu\nu}^n d\lambda^\mu \wedge d\lambda^\nu \quad (2.19)$$

where we have used the Einstein convention of summation and the \wedge operator represents the exterior product

There is a subtlety in this last equation, as we know the Berry curvature

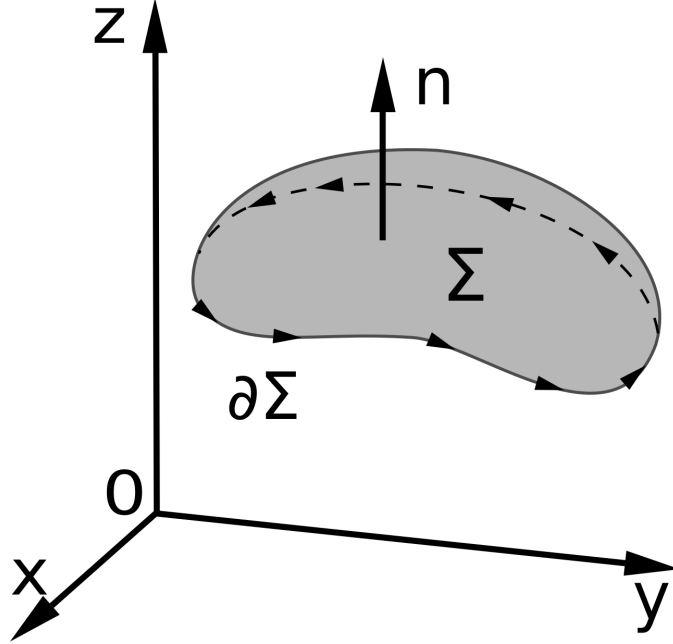


Figure 2.1: Here we divide the surface of the sphere in two different surfaces \mathcal{A} and \mathcal{B} that share the edge \mathcal{P}

tensor is Gauge-invariant, so the integral over the surface is too, but the integral over the closed path of the Berry connection is defined up to a factor $2n\pi$ that is gauge dependant. So is there a modulo 2π ambiguity or not?

The answer is that if γ_n is to be determined using the knowledge of $|n, \boldsymbol{\lambda}\rangle$ only on the curve \mathcal{P} then it is really well defined modulo 2π . In this case we can re-write equation 2.19 as

$$\frac{1}{2} \int_{\Sigma} \Omega_{\mu\nu}^n d\lambda^\mu \wedge d\lambda^\nu := \oint_{\mathcal{P}} A_\mu^n d\lambda^\mu$$

Meaning that the integral over the surface \pm is equal to *one of the values of* the integrals along the closed path \mathcal{P}

But what kind of Gauge gives the "correct" answer? If we choose a gauge that is continuous and smooth everywhere along the surface Σ including on its boundary \mathcal{P} then equation 2.19 becomes unambiguous.

While it is possible to make a radical gauge transformation that shifts γ_n by 2π when regarding $|n, \boldsymbol{\lambda}\rangle$ as a function defined only in the neighborhood of \mathcal{P} , such a gauge change cannot be smoothly continued into the interior \mathcal{S} without creating a vortex-like singularity of $\gamma_n(\boldsymbol{\lambda})$.

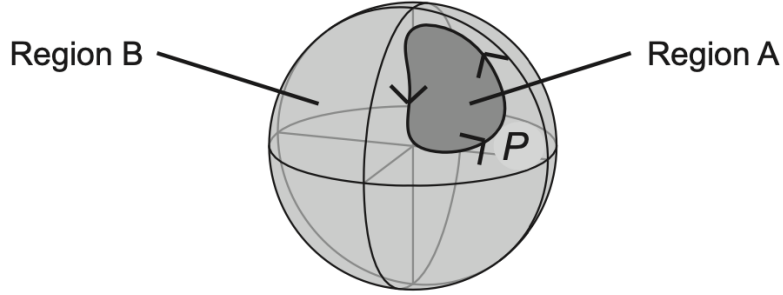


Figure 2.2: Here we divide the surface of the sphere in two different surfaces \mathcal{A} and \mathcal{B} that share the edge \mathcal{P}

2.4 Chern Theorem

Let's take as an example Gauss's theorem. It tells us that the flux of the field through a closed surface is equal to the charges inside.

Now let's calculate the flux of the Berry curvature through a closed surface. We can divide the closed surface as two different open surfaces that share the same edge \mathcal{P} .

Thanks to Stokes theorem the flux through the surface \mathcal{A} is $\oint_{\mathcal{P}} \mathbf{A} \cdot d\boldsymbol{\lambda}$, but the flux through the surface \mathcal{B} is $-\oint_{\mathcal{P}} \mathbf{A} \cdot d\boldsymbol{\lambda}$.

These two integrals must be equal modulo 2π , so

$$\oint_{\mathcal{S}} \Omega_{\mu\nu}^n d\lambda^\mu \wedge d\lambda^\nu = 2\pi C \quad C \in \mathbb{Z} \quad (2.20)$$

This means that the flux through a closed surface of the Berry curvature is quantized

The constant C is known as the Chern number. Note that when the Chern index is nonzero, it is impossible to construct a smooth and continuous gauge over the entire surface \mathcal{S} . If such a gauge did exist, then we could apply Stokes' theorem directly to the entire surface and conclude that the Chern number vanishes, in contradiction with the assumption.

But what are these "pseudo-charges" inside the closed surface that generate the flux?

In E.M. a simple way to spot charges (or monopoles) is to look at the field tensor and see if at some point it diverges as $1/(\mathbf{r} - \mathbf{r}_0)^2$. Let's take a look

at $\Omega_{\mu\nu}$ (eq. 2.18) and see if we can spot anything similar ³

$$\Omega_{\mu\nu}^n = i \sum_{n' \neq n} \frac{\langle n | \partial_\mu H | n' \rangle \wedge \langle n' | \partial_\nu H | n \rangle}{\underbrace{[E_{n'}(\boldsymbol{\lambda}) - E_n(\boldsymbol{\lambda})]^2}_{\substack{\text{what happens if for some } \boldsymbol{\lambda}=\boldsymbol{\lambda}_d \\ \text{the two energies are the same?}}}} \quad (2.21)$$

So, suppose that for some $\boldsymbol{\lambda} = \boldsymbol{\lambda}_d$ we have that $E_n(\boldsymbol{\lambda}_d) = E_m(\boldsymbol{\lambda}_d)$, now we expand the energies near $\boldsymbol{\lambda}_d$ at first order

$$\begin{cases} E_n(\boldsymbol{\lambda}) \approx E_n(\boldsymbol{\lambda}_d) + \partial_{\boldsymbol{\lambda}} E_n|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}_d} \cdot (\boldsymbol{\lambda} - \boldsymbol{\lambda}_d) \\ E_m(\boldsymbol{\lambda}) \approx E_m(\boldsymbol{\lambda}_d) + \partial_{\boldsymbol{\lambda}} E_m|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}_d} \cdot (\boldsymbol{\lambda} - \boldsymbol{\lambda}_d) \end{cases}$$

This means that

$$E_n(\boldsymbol{\lambda}) - E_m(\boldsymbol{\lambda}) \approx \partial_{\boldsymbol{\lambda}}(E_n - E_m)|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}_d} \cdot (\boldsymbol{\lambda} - \boldsymbol{\lambda}_d)$$

so the denominator of the berry curvature near $\boldsymbol{\lambda}_d$ goes like $1/(\boldsymbol{\lambda} - \boldsymbol{\lambda}_d)^2$. This means that there are "charges" or "monopoles" that induce the flux through the closed surface, and they are localized where 2 (or more) energy levels cross

³In the equation below I expressed explicitly the $\boldsymbol{\lambda}$ dependence in the denominator and condensed the formula using the wedge product \wedge

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