Proof that the derivative of e^x is itself

Francesco Sacco

24 Aprile 2019

In this document i'm going to use just the definition of derivative and basic algebraic rules to prove that the solution to $f'(x) = \tau f(x)$ is $e^{\tau x}$.

Proof

let's start by writing $f'(x)$ in terms of the limit.

$$f'(x) = \lim_{dx\to 0} f'(x - dx) = \lim_{dx\to 0} \frac{f(x) - f(x - dx)}{dx} = \lim_{dx\to 0} \tau f(x - dx) \tag{1}$$

I've already used the fact that the derivative must be continuos[1].
Now we can isolate $f(x)$ by bringing on the right side everything else.

$$f(x) = \lim_{dx\to 0} f(x - dx)(\tau dx + 1) \tag{2}$$

that means that

$$\lim_{dx\to 0} f(x - dx) = \lim_{dx\to 0} f(x - 2dx)(\tau dx + 1) \to f(x) = \lim_{dx\to 0} f(x - 2dx)(\tau dx + 1)^2$$

and

$$\lim_{dx\to 0} f(x - 2dx) = \lim_{dx\to 0} f(x - 3dx)(\tau dx + 1) \to f(x) = \lim_{dx\to 0} f(x - 3dx)(\tau dx + 1)^3$$

continuing this thing n times we get

$$f(x) = \lim_{dx\to 0} f(x - ndx)(\tau dx + 1)^n \tag{3}$$

that is true for every n[2], if we choose to set $n = \frac{x}{dx}$ we get that

$$f(x) = \lim_{dx\to 0} f(0)(\tau dx + 1)^{\frac{x}{dx}} \tag{4}$$

if we multiply and divide the exponent by τ e define $k = \tau dx$ we get

$$f(x) = f(0)\big[\lim_{k\to 0}(k + 1)^{1/k}\big]^{\tau x} \tag{5}$$

if we define e the number to witch the limit $\lim_{k\to 0}(k + 1)^{1/k}$ converges we can write[3]

$$f(x) = f(0)e^{\tau x} \tag{6}$$

Appendix

Scrivi quelle cose

[1] Look at the appendix to see why that must be true, but for now i think you can survive by assuming it to be true

[2] To be fear n should be an integer, but we can choose either dx to be a integer divisor of x or we can do some trickery whidt some ϵs and δs. For now i don't think this is really important

[3] How can we be so sure that the limit converges? Look at the appendix.