Optimal Public Good Provision and Taxation with Heterogeneous Risk Preferences*

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Abstract

This paper develops a theory of optimal public good provision when individuals have heterogeneous risk preferences, and derives optimal tax conditions in terms of sufficient statistics. When the public good is privately provided, the risk allocation between private and public consumption is inefficient because people do not internalise the insurance gains that the public good provides to other agents who have different preferences for risk. Then, I study the optimal taxation of earnings, as well as risky and riskless capital income, when the public good is publicly provided, and agents also have heterogeneous labour productivities. The public good risk profile depends on a weighted average of agents' consumption volatility, as the different tastes for risk are balanced at the societal level. The progressivity of the labour income tax schedule and the desirability of taxing riskless savings crucially depend on whether the government prioritises redistribution towards low-income individuals or highly risk-averse individuals. Unless aversion to risk is negatively related to labour productivity and income, a tension between equity and insurance motives arises.

JEL classification: H21, H41.

Keywords: public good provision, heterogeneous risk preferences, optimal taxation, insurance, equity.

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1 Introduction

Individual characteristics drive individual as well as social preferences. Age, gender, economic background, educational attainment and many other factors shape people's view concerning what the government should invest in and how progressive the tax system should be (Stantcheva, 2021). One area of government's policy that often polarises the public opinion involves large public projects. For instance, infrastructure investments are sometimes perceived by some people as too risky, either because these projects might take longer to be completed, or because the net benefits of the projects might fail to materialise.[1] Similarly, attitudes towards environmental policies are heterogeneous. While most people agree climate change is a problem and measures must be taken to mitigate or stop it, different views on the scale of the actions to implement emerge.[2] While differences across countries can be explained by different exposures to climate risks (Dechezleprêtre et al., 2022), differences within countries could partly reflect different attitudes to (climate change) risk.

As the examples described earlier suggest, people with different attitudes to risk not only make different individual decisions throughout their lives, but also have different views on the extent to which the government should invest in certain (risky) public goods, like infrastructures or the environment. There is increasing evidence of heterogeneous risk preferences in the population (Bach et al., 2020; Fagereng et al., 2020; Falk et al., 2018; Von Gaudecker et al., 2011), of how these preferences change over the course of life (Dohmen et al., 2017), and of how they correlate with other individual characteristics such as cognitive ability (Dohmen et al., 2010, 2018). However, the normative issue of public provision of public goods in presence of heterogeneous risk preferences, and the optimal tax system needed to implement, has not received

[1]For instance, a survey conducted in 2018 on the high speed rail project "HS2", that is currently under construction in the UK, shows that 59% of those who oppose the project are worried that "costs are or potentially will be too high", possibly reflecting different preferences for risk associated with skyrocketing costs of public projects.

[2]According to the 2021 Eurobarometer survey, 78% of the respondents consider climate change "a very serious problem", while 15% see it as "a fairly serious problem". Moreover, 80% of respondents from Sweden agree that "reducing fossil fuel imports from outside the EU can increase energy security and benefit the EU economically", while only 59% in France think so. Moreover, the perceived costs of climate change are also heterogeneous. In Portugal, 52% of respondents totally agree with the statement that the cost of damage due to climate change is much higher than the cost of investment needed for a green transition, but only 28% do so in Poland and Finland.

much attention. This paper addresses this question by providing optimality conditions and tax formulas that are based on appropriately defined elasticities and social marginal welfare weights.

In the first part of the paper (sections 2-3), I consider a two-period framework in which agents differ only by their risk preferences, and labour supply is exogenous. I analyse the Nash equilibrium outcome when the public good is offered on the basis of individual contributions (section 2), and the solution of a social planner maximising a general social welfare function (section 3) to derive the Samuelson rule. Then, I study the role of labour and capital income taxes to finance a public good in a more complex environment. Agent-types are characterised by their level of risk aversion and labour productivity (wage), and make decisions on labour supply, consumption and savings, with constant relative risk aversion utility. They also make portfolio decisions, choosing between two types of assets: one is risk-free, while the other is subject to aggregate risk, but with a positive expected excess return.[3] The government raises revenues with a piece-wise linear tax schedule on labour earnings and with two distinct proportional taxes on the riskless (normal) return and the risky excess return (with symmetric loss offsets), along with a lump-sum tax that is equal for all agents. In this environment, agents with equal labour productivities but different risk preferences make different labour supply, savings and portfolio decisions, which is why the zero capital income tax result from Atkinson and Stiglitz (1976) does not hold.[4]

In the model, the public good plays an insurance role, as it allows individuals to shift risk from private to public consumption, as in the work of Christiansen (1993), Schindler (2008) and Boadway and Spiritus (2024). The public good is itself risky since tax revenues depend on the state of the economy. While the tax parameters are chosen optimally ex-ante, the policy is implemented in the second period, after the realization of the state of the economy, such that the budget is balanced. This captures the idea that funds for a specific project might be limited in the short run,

[3] Following the optimal tax literature, I distinguish two different components of the rate of return on savings: the (riskless) normal return and the excess return. The normal return is the price for forgoing present-time-consumption. The excess return reflects the presence of aggregate risk in the economy and drives returns heterogeneity.

[4] The Atkinson and Stiglitz (1976) theorem would hold if the labour income tax schedule were risk-aversion-specific.

so that the project gets paused or reduced in scale if a negative shock occurs.[5]

This paper presents three set of findings. First, the market would generally provide an inefficient probability distribution of the public good in the case of heterogeneous risk preferences. This is because agents do not internalise the fact that their contribution to the public good affects the distribution of risk between private and public consumption also for the other agents who have different risk preferences. Hence, agents underestimate the insurance effect that the public good provides to the other individuals. Given a general social welfare function, the (ex-ante) first best allocation is achieved with (risk-aversion) type-specific lump-sum taxation.[6] Then, when different types cannot be targeted without efficient costs, the expected level of the public good will be generally suboptimal, i.e. second best.

Second, I analyse the second-best policy in the case in which the public good is financed by a piece-wise linear tax schedule on labour earnings and proportional taxes on risky and safe capital income. I study the optimal tax system under two alternative types of welfare criteria, prioritising either redistribution towards low-income agents (equity-driven redistribution) or high-risk-aversion agents (insurance-driven redistribution). Beside the standard equity-efficiency trade-off, there is a tension between the equity and insurance motives that affects the optimal tax system. Under a more equity-oriented social welfare criterion, the labour income tax system is progressive and positive taxation of riskless capital income can be optimal. On the contrary, under an insurance-oriented welfare criterion, the optimal degree of progressivity of the labour income tax schedule declines, and the taxation of riskless capital income can become redundant or negative at the optimum.

Third, I investigate the optimal allocation of risk between private and public consumption. Individuals with different preferences for risk have different benefits from the tax/insurance policy at the margin, meaning their willingness to shift risk from the individual to the societal level depends on their risk aversion. I find that the excess return tax plays a key role to find the optimal balance, aligning the volatility of

[5] Alternatively, if we consider the environment as the public good, the risk in the economy is given by the climate risk associated with the range of possible future scenarios.

[6] This differs from ex-post first best that would require the type-specific lump-sum taxes to be state-contingent. From now on, first best will simply indicate the ex-ante First Best.

public consumption with a weighted average of private consumption volatility across different agent-types. While the taxation of the risky excess return is key to set the risk profile of the public good, it cannot be directly used to redistribute across agents with different productivity and/or risk preferences.

This paper contributes to two main strands of literature. First, this paper is related to the literature that analyses public good provision and optimal taxes in a risky environment. For instance, Christiansen (1993), Schindler (2008) and Boadway and Spiritus (2024) examine the role of capital income taxes when the economy faces aggregate risk. The contribution of this paper is to develop a theory of public good provision in a risky environment in presence of heterogeneous agents that possess different attitudes to risk and different labour productivities. Using a general social welfare function, the appropriate Samuelson rule is derived. Coherently with the work of Schindler (2008) and Boadway and Spiritus (2024), I show that the taxation of risky excess returns is key to set the risk-profile of the public good policy. The key novelty with respect to this literature is that the marginal benefit from the public good policy differs across types due to different preferences for risk. Hence, the optimal policy requires not just balancing the volatility of the public good with that of private consumption, as in the cases of Schindler (2008) and Boadway and Spiritus (2024), but also requires to balance the different "tastes" for the public good.

Second, this paper is related to the growing literature on optimal taxation with heterogeneous returns. Previous papers have concentrated on the concepts of heterogeneous "investment ability" and/or scale effects (Boadway and Spiritus, 2024; Gahvari and Micheletto, 2016; Gerritsen et al., 2025; Kristjánsson, 2024; Saez and Stantcheva, 2018) as drivers of heterogeneous returns. This paper, in line with new empirical evidence (Bach et al., 2020), considers different preferences for risk as an alternative driver of heterogeneous returns. This creates a connection between savings and returns heterogeneity, and preference heterogeneity, as returns stem endogenously from preferences for risk through type-specific portfolio choices. Most of the above contributions argue in favour of taxing the riskless (normal) return on grounds of equity.[7]

[7] In Boadway and Spiritus (2024); Gerritsen et al. (2025); Gahvari and Micheletto (2016), the taxation of safe capital income complements the redistributive role of earnings taxation.

This paper shows that it can still play a role on the grounds of insurance, complementary to labour earnings taxes, when risk preferences are heterogeneous. The reason for this is that taxing (or subsiding) the riskless return from savings allows to redistribute across different risk-aversion types of agent by exploiting their different savings and portfolio decisions. Finally, the implicit tax formulas for earnings and riskless capital income are derived in terms of sufficient statistics, with appropriately defined elasticities and social marginal welfare weights. Beside the standard equity-efficiency trade-off, these formulas highlight the tension between insurance and equity motives when individuals have heterogeneous labour productivities and risk preferences.

The rest of the paper is organised as follows. Section 2 revisits the theory of private provision of public good in the case of heterogeneous risk preferences. Section 3 presents the first and second best allocations with a general social welfare function. Section 4 presents the full model in which agents have heterogeneous productivities and risk preferences, and the government funds the public good with the taxation of labour and capital income. Section 5 discusses the optimal tax system under alternative social welfare criteria and distribution of agents' characteristics in the population. Section 6 concludes.

2 Public Good Provision in a Risky Environment

From the textbook theory of public goods, we know that public goods will be undersupplied by the market when provided on the basis of individual voluntary contributions (Stiglitz, 1980). In this section, I revisit the standard theory and show that the private provision of public goods is still inefficient in a risky environment when agents possess different risk preferences. Then, after presenting the Pareto optimal case – given by the appropriate Samuelson rule – I discuss the type of inefficiency that private provision produces in a risky environment.

Let us consider a set-up in which each agent, $j = 1, \cdots, n$, allocates a share of her endowment z to consumption in the first period c_0, and saves the residual $z - c_0$ investing in a risky portfolio with gross return $\tilde{R}_p$, given their level of risk aversion θ_j. In the second period, each agent enjoys private consumption $\tilde{c}_1$ and public consumption $\tilde{G}$, that are both subject to risk. Without loss of generality, I impose no discounting of future consumption. Moreover, individuals evaluate private and public consumption with the same sub-utility function $u(\cdot)$, meaning that no specific difference in taste between private and public goods is modelled.

$$
\begin{aligned}
\max_{c_0^j,\, g_j} \quad & U^j = u_j(c_0^j) + \mathbb{E}\left[u_j(\tilde{c}_1^j)\right] + \mathbb{E}\left[u_j\left(\tilde{G}\right)\right] \\
\text{s.t.} \quad & \tilde{c}_1 = \tilde{R}_p^j\left(z - c_0^j\right) - g_j, \\
& \tilde{G} = f\big(\textstyle\sum_j g_j, \tilde{x}\big)
\end{aligned}
$$

Under private provision, each individual j chooses how much to contribute (g_j) to the public good $\tilde{G}$ in the first period by maximising lifetime utility U taking the contributions of the other agents (g_{-j}) as given. In a no-risk situation, the level of the public good will be given by the sum of individual contributions. With an underlying source of risk in the economy $\tilde{x}$, the public good is risky as it also depends on the state of the economy in the second period, i.e. $\tilde{G} = f\big(\sum_j g_j, \tilde{x}\big)$.[8] Hence, even if the marginal rate of transformation is one, meaning one unit of private consumption buys one unit of public consumption in a riskless economy, aggregate risk can either increase or decrease the actual value of the public good relative to the value of private consumption that was initially sacrificed. The optimal private contribution rule g_j^* satisfies

$$
g_j^* : MRS_{\tilde{G},\tilde{c}_1^j} = \frac{\mathbb{E}[u_j'(\tilde{G})]}{\mathbb{E}[u_j'(\tilde{c}_1^j)]} = 1 \;\; \forall j, \tag{1}
$$

where the size of the public good $\tilde{G}$ depends on the sum of the individual contributions and aggregate risk in the economy $\tilde{x}$ i.e. $\tilde{G} = f\big(\sum_j g_j, \tilde{x}\big)$. Each agent j contributes until the risk-adjusted marginal rate of substitution between the public and private good $MRS_{\tilde{G},\tilde{c}_1^j}$ equates the marginal cost (exp. 1). At the Nash equilibrium, the set of individual contribution rules $\{g_j, g_{-j}\}$ will satisfy (1) for each type.

[8]The underlying source of risk should be seen as a risk for the overall economy, affecting both private consumption through portfolio returns, as well as public good consumption through aggregate shocks to public finances.

Summing up (1) across agents $j = 1, \cdots, n$ gives $\sum_j MRS_{\tilde{G},\tilde{c}_1^j} = n$. By comparing the Nash equilibrium outcome with the Pareto efficient allocation,[9] given by the appropriate Samuelson rule

$$\sum_j MRS^j_{\tilde{G},\tilde{c}_1^j} = \sum_j \frac{\mathbb{E}[u_j'(\tilde{G})]}{\mathbb{E}[u_j'(\tilde{c}_1^j)]} = 1 < n, \tag{2}$$

we can state the following lemma.

Lemma 1. *When agents face aggregate risk in the economy, private provision of public goods is inefficient as agents fail to internalise the insurance effect that the public good provides to agents who have different preferences for risk.*

This lemma has two implications. First, the public good will be under-supplied in expectation. Second, private provision of public good generates an inefficient allocation of risk between private and public consumption, meaning that the probability distribution of the public good over different states of nature is suboptimal.

3 Samuelson Rule with a Social Welfare Function

The concept of Pareto optimality can be quite restricting when preferences are heterogeneous. Policy changes often benefits some agents while hurting others. Hence, we proceed by considering the optimal allocation that stems from the government's maximisation of a general Social Welfare (SW) function

$$SW = \sum_j \phi_j\left(U_j\right),$$

where $\phi_j\left(U_j\right)$ is a weakly concave function of individual expected utility U_j, and $\eta_j = \phi_j'\left(U_j\right)$ is the corresponding individual welfare weight.[10] Then, the optimal rules for public good provision are derived. Two benchmark cases are presented: 1) (Ex-

[9] Each Pareto-efficient allocation can be achieved by private, decentralised optimisation when the public good is financed by (risk-aversion) type-specific lump-sum taxes such that (2) is satisfied.

[10] This transformation is necessary to accommodate different individual risk preferences. The issue of choosing the appropriate functions ϕ_j for utility functions with different curvatures is widely debated in the welfare theory literature. For instance, Grant et al. (2010) shows how it is possible to accommodate concerns about different individuals' risk attitudes and concerns about fairness; Eden (2020) provides a practical method to perform welfare analysis.

ante) first best: the government can levy type-specific lump-sum taxes that depend on individual risk aversion (section 3.1); 2) Second best: the government sets one unique lump-sum tax for all types of agent (section 3.2).

3.1 First Best: Type Specific Lump-sum Taxes

The government chooses t_i for each agent i, therefore perfectly targeting agents with different risk preferences. Expression 3 states that the optimal t_i equates the marginal private costs of agent i from contributing to the public good, expressed in terms of private consumption, to the social benefit of a marginal individual contribution:

$$\eta_i \, \mathbb{E}[u_i'(\tilde{c}_1^i)] = \sum_j \eta_j \, \mathbb{E}[u_j'(\tilde{G})] \quad \forall \, i. \tag{3}$$

At the optimum, expression (3) is satisfied for each agent i and gives the set of optimal lump-sum taxes t_j for all types of agent. The government sets t_i for each agent i such that the weighted expected marginal utility of private consumption is equalised across types to a certain level $\overline{k}$.

$$\{t_i, i = 1, \cdots, n\} \;\; \rightarrow \;\; \eta_i \, \mathbb{E}[u_i'(\tilde{c}_1^i)] = \overline{k} \quad \forall \, i = 1, \cdots, n. \tag{4}$$

By summing up condition (3) across types $i = 1, \cdots, n$, and using (4), we can obtain a modified Samuelson rule that is expressed in terms of the sum of socially-evaluated Marginal Rates of Substitutions (MRS^s).

Proposition 1 (First Best)**.** *At the first best optimum, the sum of the MRS^s is equal to the marginal rate of transformation between the public and private good.*

$$\sum_j MRS_j^s = \sum_j \frac{\eta_j \, \mathbb{E}[u_j'(\tilde{G})]}{\eta_j \, \mathbb{E}[u_j'(\tilde{c}_1^j)]} = 1. \tag{5}$$

Condition (5) is the Samuelson rule that reflects the social preferences that are being maximised, where

$$MRS_j^s = \frac{\eta_j \, \mathbb{E}[u_j'(\tilde{G})]}{\eta_j \, \mathbb{E}[u_j'(\tilde{c}_1^j)]}$$

is defined as the ratio between the marginal social welfare associated with an additional unit of the public good and the marginal social welfare associated with an additional unit of private consumption for an agent j.

While type specific taxation is a useful theoretical benchmark, it is unlikely to be feasible in practise. The next section considers an alternative case of public provision in which the lump-sum tax is set equal for all agents.

3.2 Second Best: Uniform Lump-sum Taxes

If the government cannot screen agents with different risk preferences, and therefore cannot set (risk-aversion) type specific lump-sum taxes, a uniform lump-sum tax t for all agents apply. At the second best optimum, marginal social costs, expressed in terms of private consumption, are balanced with the social benefits:

$$\sum_j \eta_j \, \mathbb{E}[u'_j(\tilde{c}_1^j)] = n \sum_j \eta_j \, \mathbb{E}[u'_j(\tilde{G})]. \tag{6}$$

Condition (6) can be rewritten in terms of $\widetilde{MRS}_j^s$, where

$$\widetilde{MRS}_j^s = \frac{\eta_j \, \mathbb{E}[u'_j(\tilde{G})]}{\frac{1}{n}\sum_j \eta_j \, \mathbb{E}[u'_j(\tilde{c}_1^j)]},$$

that is defined as the ratio between the marginal social welfare associated with an additional unit of the public good for an agent j and the average (across agents $j = 1, \cdots, n$) marginal social welfare associated with an additional unit of private consumption. This means that when type specific taxes are not available, the government evaluates the individual willingness to trade private with public consumption on the basis of the average sacrifice in terms of private consumption, e.g. $\widetilde{MRS}_j^s$ rather than MRS_j^s. Hence, condition (7) is the modified Samuelson rule in the case of heterogeneous risk preferences when the government does not discriminate the different risk-aversion-types.

Proposition 2 (Second Best)**.** *At the second best optimum, the sum of* $\widetilde{MRS}_j^s$ *equates* *the marginal rate of transformation.*

$$\sum_j \widetilde{MRS}_j^s = \sum_j \left(\frac{\eta_j \, \mathbb{E}[u_j'(\tilde{G})]}{\frac{1}{n}\sum_j \eta_j \, \mathbb{E}[u_j'(\tilde{c}_1^j)]} \right) = 1. \tag{7}$$

The Second Best outcome will differ from the First Best benchmark, both in terms of public good provision and private consumption/savings, and therefore social welfare, when $\sum \widetilde{MRS}_j^s \neq \sum MRS_j^s$, namely when condition (7) differs from (5). We can now state the following corollary.

Corollary 1. *The inability of the government to discriminate different risk-aversion-types in the economy leads to a suboptimal provision of the public good.*

Proof. Suppose corollary 1 is false, and $\tilde{G}^{SB} \equiv \tilde{G}^{FB}$, then it must be that (5) coincides with (7) and that

$$\frac{1}{n}\sum_j \eta_j \, \mathbb{E}[u_j'(\tilde{c}_1^{SB})] = \eta_i \, \mathbb{E}[u_i'(\tilde{c}_1^{FB})] \;\; \forall i = 1, \cdots, n.$$

In order for this to be true, it should follow that $t_i^{FB} = t^{SB}$ for $i = 1, \cdots, n$, meaning the individual private costs of raising a unit of consumption are equal across different types to begin with. Unless specific welfare weights are chosen to obtain this result or the equality happens to hold given the utility functions being used in the first place, corollary 1 will be true in the case of heterogeneous preferences. □

While risk-preference-specific lump-sum taxes might be unfeasible, let alone lump-sum taxes, we know that agents with different risk preferences will make different consumption, savings, portfolio and labour supply choices. These differences can therefore be exploited to increase social welfare. In the next section, I examine an optimal tax system on labour and capital income that finances a public good, within the second-best framework that incorporates heterogeneous risk preferences and labour productivities.

4 Heterogeneous Productivity and Risk Aversion

I consider an environment in which each type of agent is characterised by their labour productivity and risk aversion, and the government taxes labour earnings with a piecewise linear schedule and capital income linearly, distinguishing the types of return from investment, i.e. riskless (normal) return versus risky excess return. Agents with different risk preferences choose different amount of hours worked, conditional on their productivity (wage). This section aims to investigate the roles of the different tax instruments in this setting and the interplay between the insurance mechanism and the standard equity-efficiency trade-off.

4.1 Set-up

The set of agent-types is a $m \times n$ matrix Θ, where each element is given by the pair (w_i, θ_j) that represents an individual-type who has productivity (wage) w_i and relative risk aversion θ_j. F is the $m \times n$ frequency matrix of types in the population, with each type having frequency f_{ij} in the population. I also define a risk aversion group j as the j-th column of Θ: $(\cdot, \theta_j) := \{(w_1, \theta_j), \cdots, (w_m, \theta_j)\}$, and a productivity group i as the i-th row of Θ: $(w_i, \cdot) := \{(w_i, \theta_1), \cdots, (w_i, \theta_n)\}$. Thus, $(\cdot, \theta_j)$ is the group of agents that have different productivities $w_1, \cdots, w_m$, but equal risk aversion parameter θ_j, and $(w_i, \cdot)$ is the group of agents with productivity w_i and different risk aversion parameters $\theta_1, \cdots, \theta_n$.

$$\Theta_{m\times n} = \begin{pmatrix} w_1, \theta_1 & \cdots & w_1, \theta_n \\ \vdots & \ddots & \vdots \\ w_m, \theta_1 & \cdots & w_m, \theta_n \end{pmatrix}, \quad F_{m\times n} = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{mn} \end{pmatrix}.$$

Agents choose their labour supply l, first period consumption c_0, and the portfolio composition of their savings. Agents invest their savings $a = z - c_0$ choosing the portfolio share of risky assets s with the risky return $\tilde{r}_e$ being normally distributed: $\tilde{r}_e \sim \mathcal{N}(\bar{r}_e - \sigma_r^2/2, \sigma_r^2)$. The resulting portfolio return $\tilde{r}_p$ will also be normally distributed: $\tilde{r}_p \sim \mathcal{N}(\bar{r}_p - \sigma_p^2/2, \sigma_p^2)$. The (risky) excess return is defined as the difference between the risky return and the riskless return: $\tilde{r}^{exc} := \tilde{r}^{e} - r$. Without loss of generality, I impose no time discounting of future consumption.

Each component of capital income is taxed linearly: τ_k is the tax on the excess return, τ_r is the tax rate on the safe return. Labour earnings z are subject to a piece-wise linear schedule $\tau_l(z^{ij})$. With a high number of types and individual preferences, this approximates a non-linear tax schedule on labour earnings. The social marginal welfare weight of an agent-type, normalised with respect to the population, is defined as: $g_{ij} = \frac{f_{ij}\eta_{ij}U_c^{ij}}{\sum_{i,j} f_{ij}\eta_{ij}U_c^{ij}}$. Each agent-type will "count more" in the SW function, the more frequent its type is in the population (higher f_{ij}), the higher the weight η_{ij} which depends on the concavity of the SW function, the higher the expected marginal utility of second-period consumption U_c^{ij}.

Then, define the uncompensated elasticities of labour earnings z^{ij} and savings a^{ij} with respect to net-of-tax rate on labour earnings, $\epsilon_{z,l}^{ij}$, $\epsilon_{a,l}^{ij}$, and with respect to net-of-tax rate on safe capital income, $\epsilon_{z,r}^{ij}$, $\epsilon_{a,r}^{ij}$ as follows:

$$\epsilon_{z,l}^{ij} = \frac{\partial z^{ij}}{z^{ij}} \cdot \frac{1-\tau_l^{ij}}{\partial(1-\tau_l^{ij})}; \qquad \epsilon_{z,r}^{ij} = \frac{\partial z^{ij}}{z^{ij}} \cdot \frac{1-\tau_r}{\partial(1-\tau_r)};$$

$$\epsilon_{a,l}^{ij} = \frac{\partial a^{ij}}{a^{ij}} \cdot \frac{1-\tau_l^{ij}}{\partial(1-\tau_l^{ij})}; \qquad \epsilon_{a,r}^{ij} = \frac{\partial a^{ij}}{a^{ij}} \cdot \frac{1-\tau_r}{\partial(1-\tau_r)}.$$

These elasticities are all positive when the substitution effect outweighs the income effect: $\epsilon_{z,l}^{ij}; \epsilon_{a,l}^{ij}; \epsilon_{z,r}^{ij}; \epsilon_{a,r}^{ij} > 0$.[11] I will assume this is the case throughout the next sections.

4.2 Agent's Problem

Agent's utility is separable over time, and separable between private and public consumption. Agents choose $\{c_0^{ij}, s_j, l^{ij}\}$ to maximise

$$U_{ij} = u_{ij}(c_0^{ij}) + \mathbb{E}\left[u_{ij}(\tilde{c}_1^{ij}) + u_{ij}(\tilde{G})\right] - v(l^{ij})$$

[11]This is the case, for instance, when we consider CRRA utility with relative risk aversion parameters θ_{ij} lower than 1 for all agents.

subject to $\tilde{c}_{ij} = \tilde{R}_p^j a^{ij} - t$, where $a^{ij} = z^{ij}(1-\tau_l^{ij}) - c_0^{ij}$ is the value of assets, given labour earnings $z^{ij} = w_i z^{ij}$; $\tilde{R}_p$ is the gross portfolio return defined as

$$\tilde{R}_p^j = \left[1 + r\left(1-\tau_r\right) + s_j\left(\tilde{r}^{exc}\right)\left(1-\tau_k\right)\right],$$

and t is a lump-sum tax or transfer, equal for all types. As each agent is infinitesimal compared to the size of the economy, the effects of individual choices on the level of the public good are not taken into account.

$$c_0^* : \qquad u'_{ij}(c_0^{ij}) = \mathbb{E}\left[u'_{ij}(\tilde{c}_1^{ij})\tilde{R}_p\right] \tag{8}$$

$$l^* : \qquad 0 = \mathbb{E}\left[u'_{ij}(\tilde{c}_1^{ij})\tilde{R}_p\right] w_i(1-\tau_l(z^{ij})) - v'(z^{ij}/w_i) \tag{9}$$

$$s^* : \qquad 0 = \mathbb{E}\left[u'_{ij}(\tilde{c}_1^{ij})\tilde{r}^{exc}\right] \tag{10}$$

Conditions (8)-(9) are the standard first order conditions (FOCs) for first-period consumption c_0, and labour supply l. Expression (10) is the FOC for the share of risky assets s, determining the portfolio allocation with risky and riskless assets. After applying the covariance identity to condition (10), we can derive the expression for the risk premium with respect to the expected excess return.

$$\mathbb{E}[R^e - R] = -\frac{\text{cov}\left[u'_{ij}(\tilde{c}_1^{ij}), \tilde{r}^{exc}\right]}{\mathbb{E}\left[u'_{ij}(\tilde{c}_1^{ij})\right]}. \tag{11}$$

For each agent with productivity w_i and relative risk aversion parameter θ_j, expression (11) maps the covariance term to the expected marginal utility of second-period consumption $\tilde{c}_1$. Under constant relative risk aversion utility, a close form solution for the share of risky assets can also be derived:[12]

$$s_j^* = \frac{\bar{r}^e - r}{\theta_j \sigma_r^2} \cdot \frac{(R - r\tau_r)}{R\left(1-\tau_k\right)} \tag{12}$$

While the first term relates to the original formula by Merton (1969) and Samuelson

[12]The derivation is provided in the appendix B.1.1.

(1969),[13] the second term captures the effect of the different taxes on capital income. In particular, increasing the tax on safe capital income reduces the optimal share of risky assets: $\frac{\partial s^*}{\partial \tau_r} < 0$. On the contrary, increasing the excess return tax, while keeping the loss offsets, increases s_j^*: $\frac{\partial s^*}{\partial \tau_k} > 0$ (Domar and Musgrave, 1944). Moreover, while taxation of the riskless return changes the resource allocation (first period consumption, savings), taxing the excess returns with loss offsets does not. Agents will adjust their portfolio shares to get the same pre-tax expected portfolio return (Boadway and Spiritus, 2024; Domar and Musgrave, 1944).

4.3 The Government

The government's objective is to maximise social welfare (SW) that is defined as a weighted sum of agents' lifetime expected utilities U_{ij}, given population weights f_{ij}, and individual welfare weights η_{ij}:

$$\max_{\tau_k, \tau_r, \tau_l(z^{ij}), t} SW = \sum_{i,j} f_{ij} \eta_{ij} \Big(U_{ij} \Big(c_0^{ij*}, \underbrace{a^{ij} \tilde{R}_p^j - t}_{\tilde{c}_1^{j*}(s_j^*, l^{ij*})}, l^{ij*}, \tilde{G} \Big) \Big).$$

The government provides a public good $\tilde{G}$ that enters the utility function separately from consumption. The public good is financed by taxation of labour income through the schedule $\tau_l(z^{ij})$, capital income taxation on the excess return at rate τ_k and on the safe return at rate τ_r, and a uniform lump-sum tax t. The public good is given by the following expression.

$$\tilde{G} = \sum_{i,j} f_{ij} \left[\left(\tau_k s_j^* \left(\tilde{r}^{exs} \right) + \tau_r r \right) a^{ij*} + \tau_l(z^{ij}) z^{ij*} + t \right] \tag{13}$$

Tax rates are set by the government in the first period anticipating agents' optimal behaviour. After the state of the economy is realised, the government implements the public good policy and balances the budget. As a result, the provision of the public

[13] Merton (1969) and Samuelson (1969) study the optimal portfolio choice of a consumer with CRRA utility with a multi-period horizon and multiple risky assets.

good is stochastic and depends on the state of the economy in the second period, determining the value of revenues from the taxation of excess returns on savings. I substitute the expression for the public good directly in the social welfare function, as in Schindler (2008). It ensures that the budget is balanced for any state of the world.[14]

4.4 Mechanisms

In this economy, the provision of the public good and its funding through taxation has two possible redistributive objectives: (i) insurance across risk-aversion agent-types; (ii) redistribution across productivity agent-types motivated by equity concerns. Figure I represents these two mechanisms at play in a 4-types version of the economy in which each agent-type is characterised by either high or low productivity $\{w_h; w_l\}$, and high or low risk aversion $\{\theta_h; \theta_l\}$: (w_l, θ_l); (w_l, θ_h); (w_h, θ_l); (w_h, θ_h). The red-coloured arrows represent transfers that are motivated by equity concerns, which benefits some productivity (income) people at the expense of others, e.g. redistribution from high productivity to low productivity types. Then, the blue-coloured arrows represent transfers that are motivated by insurance purposes which may benefit either high or low risk aversion people, depending on the specific social preferences. Finally, the gray-coloured arrows represent transfers between agent-types that differ by both characteristics – productivity and risk aversion – and are motivated by a mix of equity and insurance concerns. The relative importance of these two channels will determine which agent-types will benefit the most from the public good policy.

Hence, the optimal tax mix to fund the public good will depend on social preferences, agent-types' characteristics, their distribution in the economy, and on the ability of the government to observe them. Given these factors, a tension between insurance and equity arises, as some policies targeted to high risk averse individuals might benefit high income people as well. In the following subsections, we explore how the the tax system would look like under two scenarios: i) observable types; ii) unobservable risk preferences. In each of these two scenarios, we can explore how

[14] This is a stricter requirement than balancing the budget in expectation, that would instead imply transferring resources from good states to bad states of the world.

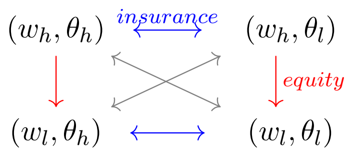

Figure I: Mechanisms at play in a 4-types economy: insurance (blue arrows); redistribution (red arrows); mix of insurance and redistribution (gray arrows). Each agent-type is characterised by either high or low productivity $\{w_h; w_l\}$, and high or low risk aversion $\{\theta_h; \theta_l\}$. The set of agent-types is therefore: $(w_l, \theta_l); (w_l, \theta_h); (w_h, \theta_l); (w_h, \theta_h)$.

the equity and the insurance motives affect the tax system under alternative welfare criteria.

I will consider two types of criteria. The first one, named insurance-driven redistribution, gives more weight to individuals with high risk aversion, and to low income people conditional on risk preferences.

Criterion 1 (Insurance-driven redistribution)**.** *High risk averse agents count more than low risk averse agents, but low income people count more than high income people conditional on risk preference:* $g_{lh} > g_{hh} > g_{ll} > g_{hl}$.

Alternatively, the government could prioritise equity to the insurance motive through the equity-driven redistribution criterion.

Criterion 2 (Equity-driven redistribution)**.** *Low income people count more than high income people, but high risk averse people more than low risk averse people conditional on productivity:* $g_{lh} > g_{ll} > g_{hh} > g_{hl}$.

With heterogeneity on two dimensions – labour productivity and risk preferences – these two criteria can be partially conflicting. The choice of the welfare criterion will have an impact on the optimal tax system.

4.5 Optimality Conditions: Observable Types

In this section, I analyse the optimality conditions for a piecewise linear tax schedule on labour earnings and linear taxation of riskless and risky capital income when the government has complete information on the different types of agent in the economy. Hence, individual productivities, labour supply decisions, the risk type (risky versus safe) and amount of capital income are known. The whole set of optimality conditions is provided in the appendix B.

Under observable types, the government is able to set agent-type-specific tax rates on labour income. However, capital income taxes remain uniform across types of agent. Hence, any redistribution on the grounds of equity or insurance can be made through the labour income tax schedule. Therefore, consistently with the Atkinson and Stiglitz (1976) theorem, there is no need for distorting the intertemporal consumption decisions by taxing the riskless rate of return. Hence, we can analyse optimal agent-type-specific labour income taxes and the risky capital income tax τ_k on excess returns, under zero tax on safe capital income.

Proposition 3. *Under observable types, the optimal tax system on labour income and risky capital income $\{\tau_l^{pq}, \tau_k\}$ for each agent-type (w_p, θ_q) is given by the following optimality conditions:*

1. *taxation of labour income*

$$\tau_l^{pq} = \frac{1 - g_{pq}}{1 - g_{pq} + \epsilon_{z,l}^{pq}}, \tag{14}$$

2. *taxation of risky capital income*

$$\tau_k : \mathbb{E}[\tilde{r}^{exc}] = -\frac{\sum_{i,j} f_{ij}\eta_{ij} \operatorname{cov}\left[u'_{ij}(\tilde{G}), \tilde{r}^{exc}\right]}{\sum_{i,j} f_{ij}\eta_{ij}\, \mathbb{E}[u'_{ij}(\tilde{G})]} \tag{15}$$

where $g_{pq} = \frac{\eta_{pq}U_c^{pq}}{\sum_{i,j} f_{ij}\eta_{ij}U_c^{ij}}$ is the social marginal welfare weight of an agent-type (w_p, θ_q), normalised with respect to the population.

Conditions (14) is an implicit tax formula capturing the standard equity-efficiency trade-off, as well as the new insurance channel. While the efficiency channel is represented by the elasticity of $\epsilon_{z,l}^{pq}$, the social marginal welfare weight g_{pq} captures both the insurance and the equity concerns. This can be seen easily by considering that g_{pq} can be rewritten in terms of covariances: $g_{pq} = \frac{f_{pq}\eta_{pq} \operatorname{cov}(U_c^{pq}, \tilde{r}^{exc})}{\sum_{i,j} f_{ij}\eta_{ij} \operatorname{cov}(U_c^{ij}, \tilde{r}^{exc})}$ for agent-type (w_p, θ_q). Hence, the higher the covariance term, the more an agent-type will weight in the social welfare function, therefore lowering the optimal tax rate on labour earnings.

For each agent-type, the sign of the labour income tax rate depends on the value of the welfare weight g_{pq}. For instance, under insurance-based redistribution (criterion

1), agents with low risk aversion will face higher tax rates than high risk aversion agents conditional on income. This will ensure that resources are redistributed from low risk aversion agents towards high risk aversion agents. However, the resulting tax system would not be progressive as some high income people would be subsidised, while some low income people would be taxed at a positive rate. This is because criterion 1 prioritises insurance over equity concerns.

Alternatively, under equity-based redistribution (criterion 2), low income individuals are subsidized while high income agents face a positive tax rate. While the resulting tax schedule is progressive, some of the high income individuals facing a positive tax rate have high risk aversion, while some low risk aversion agents are subsidised. This is because criterion 2 prioritises the equity channel over the insurance channel. These two examples show that, in general, there is a tension between the insurance and the equity concerns when individuals differ by their productivity and risk preferences.

Then, the optimal tax rate on excess returns τ_k is set by condition (15). The revenues collected from taxing the excess return are risky as the realisation of the excess return depends on the state of the economy. Hence, the public good will have a certain probability distribution which entails a certain allocation of risk between private and public consumption. Condition (15) determines the variance of the public good policy and the allocation of risk between private and public consumption that maximises social welfare.[15] In other terms, the government is choosing the optimal risky share of the budget (tax revenues), given agents' preferences and social welfare weights.

The key novelty of this setting with respect to the literature, specifically the work by Schindler (2008) and Boadway and Spiritus (2024), is that the "benefits" of the public good differ among agents because of heterogeneous attitudes to risk. Since the public good is itself risky, the welfare gain from increasing the tax rate τ_k varies across different types of agent. Hence, the government aims to balance the different "tastes" for the public good, providing insurance against aggregate risk. As individuals make

[15]When a bad (good) state of economy realises in the second period, losses (gains) due to negative (positive) excess returns will be spread over private and public consumption, so that the utility loss (gain) is minimised (maximised).

portfolio choices on the basis of their risk preferences according to condition (11), similarly the government decides the distribution of tax revenues financing the public good over states of nature on the basis of (15). In doing so, the government takes into account the agents' willingness to shift risk to the societal level. Condition (16) in corollary 2 reformulates (15) and better represents this concept. Private and public consumption volatility are represented by the covariance between marginal utility of private and public consumption respectively with the risky excess return and jointly govern the individual willingness to pay for the public good with an extra euro of risky capital income.

Corollary 2. *When agents with different risk preferences cannot be discriminated with zero efficiency costs (second best), τ_k^* equalizes public consumption volatility with (average) private consumption volatility. The optimal tax rate is positive but strictly lower than 100%.*

$$\sum_{i,j} \frac{f_{ij}\eta_{ij} \operatorname{cov}\left(u'_{ij}(\tilde{G}), \tilde{r}^{exc}\right)}{\sum_{i,j} f_{ij}\eta_{ij} \operatorname{cov}\left(u'_{ij}\left(\tilde{c}_1\right), \tilde{r}^{exc}\right)} = 1 \tag{16}$$

Proof. See appendix A.1.

Corollary 2 says that τ_k^* is chosen on the basis of a weighted average of private consumption volatility, i.e. $\sum_{i,j} f_{ij}\eta_{ij} \operatorname{cov}\left(u'_{ij}\left(\tilde{c}_1\right), \tilde{r}^{exc}\right)$. This is because the government cannot target individual risk-aversion types at no efficiency cost. This creates an inefficiency: an agent with relatively (more) risky private consumption c_1, compared to other agents, would be better-off by shifting more risk to the public good. For that to be the case, a higher tax rate τ_k should be implemented, so that private consumption volatility is traded with public consumption volatility.[16] Moreover, it can be shown that the optimal tax rate is positive but strictly lower than 100%.

Another aspect to investigate is whether the taxation of excess returns can have any redistributive role across types of agent. This is clarified by corollary 3.

[16] If the government were able to target different types at no efficiency costs, then it would be possible to compensate agents that would prefer a different distribution of the public good from the optimal one given by (16). For a short proof, see the *addendum* to appendix A.1.

Corollary 3. *The optimal taxation of the risky excess returns does not serve a redistributive role across different types of agent, either across different levels of productivity or different levels of risk aversion.*

Proof. Suppose it is possible for the government to condition τ_k on agents' characteristics by setting a set of taxes τ_k^{pq}, one for each type (w_p, θ_q). Then, it is possible to show that the optimality conditions would be equal across types and equal to condition (15). Hence, we would have $\tau_k^{pq} = \tau_k$ for any agent-type (w_p, θ_q). The derivation is provided in appendix A.2.

Hence, the role of the risky excess return is solely to set the variability of tax revenues and the public good on the aggregate level. This extends the finding by Boadway and Spiritus (2024) that such tax cannot enhance redistribution on equity grounds when agents differ by their labour and investment productivity.

4.6 Optimality Conditions: Unobservable Risk Preferences

When the government does not observe risk preferences, labour income taxes only depend on the productivity level, while capital income tax rates still apply uniformly to all agents. Condition 15 for the risky excess return and its intuition remain valid. Then, introducing a tax or subsidy on the normal return (riskless capital income) is welfare enhancing when the following condition is satisfied:

$$\left.\frac{\partial SW}{\partial \tau_r}\right|_{\tau_r=0} = 1 - \overline{g}_a - \frac{R}{A} \cdot \left[\sum_{i,j} f_{ij} \epsilon_{z,r}^{ij} z^{ij} \tau_l^i\right] \neq 0. \tag{17}$$

Condition (17) is the social welfare effect of introducing τ_r, where the term $\overline{g}_a = \frac{\sum_{i,j} f_{ij} g_{ij} a_{ij}}{A}$ captures the social concerns for equity and insurance, and it is defined as the ratio between average savings within productivity group $(w_i, \cdot)$ weighted by individual welfare weights g_{ij}, and average savings $A = \sum_{i,j} f_{ij} a^{ij}$. The last term relates to the indirect effect of capital income taxation on revenues from labour income taxes. When condition (17) is satisfied, taxing (or subsidising) the normal (safe) return on savings complements the role of labour income taxation, and the optimal tax system will be characterised by proposition 4.

Proposition 4. *Under unobservable risk preferences, the following optimal tax formulas and conditions apply:*

1. *taxation of labour income for each productivity group* $(w_i, \cdot)$

$$\tau_l^i = \frac{1 - \overline{g}_z^i - \tau_r \cdot \overline{\epsilon_{a,l}^i}/p_0}{1 - \overline{g}_z^i + \overline{\epsilon_{z,l}^i}}, \tag{18}$$

$$\textit{with} \quad \overline{g}_z^i = \frac{1}{Z^i}\sum_j f_{ij} g_{ij} z_{ij}; \qquad Z^i = \sum_j f_{ij} z^{ij};$$

$$\overline{\epsilon_{z,l}^i} = \frac{1}{Z^i}\sum_j f_{ij}(\epsilon_{z,l}^{ij} z_{ij}); \qquad \overline{\epsilon_{a,l}^i} = \frac{1}{Z^i}\sum_j f_{ij}(\epsilon_{a,l}^{ij} a_{ij}).$$

2. *taxation of normal returns (riskless capital income)*

$$\tau_r = \frac{1 - \overline{g}_a - (p_0/A) \cdot \sum_{i,j} f_{ij} \epsilon_{z,r}^{ij} z^{ij} \tau_l^i}{1 - \overline{g}_a + \overline{\epsilon}_{a,r}}, \tag{19}$$

$$\textit{with} \quad \overline{g}_a = \frac{\sum_{i,j} f_{ij} g_{ij} a_{ij}}{A}; \quad \overline{\epsilon}_{a,r} = \frac{\sum_{i,j} f_{ij} \epsilon_{a,r}^{ij} a_{ij}}{A}; \quad A = \sum_{i,j} f_{ij} a^{ij}.$$

3. *taxation of risky excess returns: condition (15) applies.*

Formula (18) gives the optimal tax rate on labour earnings for agents in the productivity group $(w_i, \cdot)$. The redistributive motives due to equity and insurance concerns are captured by $\overline{g}_z^i = \sum_j f_{ij} g_{ij} z_{ij}/Z^i$ that is the ratio between the average pre-tax income within productivity group $(w_i, \cdot)$ weighted by welfare weights g_{ij}, and average pre-tax earnings Z^i. The efficiency channel is represented by two elasticity terms: (i) the normalised weighted average of earnings elasticities $\overline{\epsilon_{z,l}^i} = \sum_j f_{ij}(\epsilon_{z,l}^{ij} z_{ij})/Z^i$; (ii) the normalised weighted average of savings elasticities $\overline{\epsilon_{a,l}^i} = \sum_j f_{ij}(\epsilon_{a,l}^{ij} a_{ij})/Z^i$. The first term relates to the direct efficiency costs of taxation due to its discouraging effect on labour supply. The second term relates to the indirect effect on revenues from the taxation of safe capital income: discouraging labour supply has a negative impact on savings that in turn lowers revenues from capital income taxation.

When the government has strong equity concerns (equity-driven redistribution criterion), meaning that low income people count more than high income people, formula (18) will entail a progressive tax schedule. However, as the government priorities more the insurance concerns (insurance-driven redistribution criterion), giving more weight to high-risk aversion agents relative to low-risk aversion ones, the degree of progressivity of the optimal labour income tax schedule will decline. The reason is that those agents with relatively high income as well as high risk aversion could count more in the social welfare function than some low-income people with low risk aversion, which makes the optimal policy less redistributive in terms of equity. This is summarised by the following corollary.

Corollary 4. *The optimal tax schedule on labour income based on insurance-based welfare criteria (e.g. criterion 1) tends to be less progressive than under equity-based welfare criteria (e.g. criterion 2).*

As for capital income taxes, formula (19) gives the optimal rate on normal returns, which depends on redistributive concerns along the distribution of savings (value of $\overline{g}_a$), and efficiency considerations, given by the elasticity terms. When the government has strong equity concerns, condition (17) is positive, provided that the indirect effect of capital income taxation on revenues on labour income tax revenues is small enough. This would imply that $\tau_r > 0$ is welfare improving. On the contrary, if the government prioritises insurance concerns (insurance-driven redistribution criterion), then the sign becomes uncertain. Subsidising riskless capital income ($\tau_r < 0$) can be optimal when high risk averse agents have higher savings on average relative to the population. Hence, the desirability of positive taxation of safe capital income will also depend on the importance of equity relative to the insurance motive.

5 Discussion

The implicit tax formulas from section 4 are derived using appropriately defined elasticities, and social marginal welfare weights. We can draw implications for the overall optimal tax system and the role of each tax instrument under alternative social welfare criteria.

If social preferences are driven by equity considerations (e.g. criterion 1), the optimal tax system from proposition 4 will feature a progressive labour income tax schedule, and a positive taxation of the risky and the riskless returns. In this case, the taxation of riskless capital income complements the redistribution role of labour income taxes as transfers along the distribution of savings complement transfers across the labour income distribution. Then, the taxation of risky capital income ensures a welfare-maximising allocation of risk between private and public consumption.

Alternatively, if social preferences are driven by insurance considerations, with high risk aversion agents weighting relatively more in the social welfare function, the redistribution role of the labour income tax schedule along the income distribution would be mitigated by subsidies on riskless capital income, provided that the high risk averse agents save more than the average in the population. Moreover, the optimal level of taxation on the risky excess return, and the risk associated with the public good investment, would be relatively lower. One practical example can be made with regard to age. While age is not explicitly modelled in this paper, there is evidence that it is correlated to risk aversion. Dohmen et al. (2017) find that older individuals tend to have higher aversion to risk than younger individuals. Hence, social welfare criteria that prioritise agents with high risk aversion would benefit older people relatively more, and *vice versa*.

Another factor influencing the tax schedule and its progressivity is the frequency distribution of agent-types, i.e. how labour productivities and risk preferences are jointly distributed in the population. For example, Dohmen et al. (2010, 2018) find evidence of a negative correlation between cognitive ability and risk aversion, meaning that individuals with higher cognitive ability tend to have lower aversion to risk. A direct positive mapping between cognitive ability and labour productivity would imply that higher income people are on average less risk averse than low income people. Under these circumstances, the optimal tax system featured by proposition 4 entails a progressive labour income tax schedule, and a positive taxation of the riskless return under both equity-driven and insurance-driven welfare criteria. This is because in most cases transfers from high income to low income individuals would coincide with transfers from low-risk aversion and high-savings individuals to high-risk aversion and low-savings individuals.

6 Concluding Remarks

People with different attitudes to risk have different views on the extent to which society should invest in certain risky projects. This paper presents a theory of optimal provision and funding of a public good when individuals have heterogeneous preferences for risk. In this framework, the public good is a tool to shift risk from private to public consumption, as agents face aggregate risk in the economy. Private provision is inefficient as agents fail to internalize the insurance gains that the public good provides to other agents who have different risk preferences.

Then, I consider the case of public provision. Within a model with bi-dimensional heterogeneity – labour productivity and risk preferences – I characterise the optimal piece-wise linear tax schedule for labour earnings and proportional taxation of risky and safe capital income. I show that the excess return tax is key to balance the individual savings and portfolio decisions with the variance of the public good policy. By setting this tax optimally, the government finds a balance among the different preferences for risk of the different agents. However, the excess return tax cannot be used to directly pursue redistribution across different productivity and/or risk-aversion types of agent.

The taxation of the riskless capital income can play a complementary role to labour earnings taxation when risk preferences are unobserved by the government. When social preferences prioritise low-income individuals, the labour-income tax schedule is progressive and positive taxation of riskless capital income tends to be optimal. However, when the government prioritises redistribution towards the high risk averse agents, the labour income tax schedule is less progressive and safe capital income taxation can become redundant or negative. Hence, in general, unless aversion to risk is negatively correlated to labour productivity and income in the population, a tension between equity and insurance motives arises.

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Appendix A Proofs

A.1 Corollary 2

The optimality condition for the excess return tax τ_k reads as follows.

$$\sum_{i,j} f_{ij}\eta_{ij}\mathbb{E}\Big[u'_{ij}(\tilde{G})\sum_{i,j} f_{ij}s_j\left(z^{ij}-c_0^{ij}\right)\tilde{r}^{exc}\Big]+\tau_k\sum_{i,j} f_{ij}\eta_{ij}\mathbb{E}\left[u'_{ij}(\tilde{G})\sum_{i,j} f_{ij}\frac{\partial s_j}{\partial \tau_k}\left(z^{ij}-c_0^{ij}\right)\tilde{r}^{exc}\right]=0.$$

Using the fact that $\frac{\partial s_j}{\partial \tau_k}=\frac{s_j}{(1-\tau_k)}$, we can collect terms.

$$\sum_{i,j} f_{ij}\eta_{ij}\mathbb{E}\Big[u'_{ij}(\tilde{G})\sum_{i,j} f_{ij}s_j\left(z^{ij}-c_0^{ij}\right)\tilde{r}^{exc}\Big]\left(1+\frac{\tau_k}{1-\tau_k}\right)=0.$$

Then, as $\sum_{i,j} f_{ij}s_j\left(z^{ij}-c_0^{ij}\right)\left(1+\frac{\tau_k}{1-\tau_k}\right)\neq 0$, we get the following expression

$$\sum_{i,j} f_{ij}\eta_{ij}\mathbb{E}\left[u'_{ij}(\tilde{G})\tilde{r}^{exc}\right]=0.$$

By further manipulating the above expression using the covariance identity, and conditions (7) and (11), we can rewrite the optimality condition for τ_k as follows

$$\sum_{i,j} f_{ij}\eta_{ij}\,\mathrm{cov}\left(u'_{ij}\left(\tilde{c}_1^{ij}\right),\tilde{r}^{exc}\right)=\sum_{i,j} f_{ij}\eta_{ij}\,\mathrm{cov}\left(u'_{ij}(\tilde{G}),\tilde{r}^{exc}\right).$$

or

$$\sum_{i,j}\frac{f_{ij}\eta_{ij}\,\mathrm{cov}\left(u'_{ij}(\tilde{G}),\tilde{r}^{exc}\right)}{\sum_{i,j} f_{ij}\eta_{ij}\,\mathrm{cov}\left(u'_{ij}\left(\tilde{c}_1\right),\tilde{r}^{exc}\right)}=1. \tag{20}$$

Moreover, the optimal tax rate is positive but strictly lower than 100%. Only $0\% < \tau_k < 100\%$ ensures that the numerator and the denominator on the left hand side of the optimality condition are non-zero terms with the same sign. □

Addendum to Corollary 2. Suppose the government can target different risk aversion types at no efficiency costs using type-specific lump-sum taxes, then condition

20 simplifies to

$$\sum_{i,j} \frac{f_{ij}\eta_{ij} \operatorname{cov}\left(u'_{ij}(\tilde{G}), \tilde{r}^{exc}\right)}{f_{ij}\eta_{ij} \operatorname{cov}\left(u'_{ij}\left(\tilde{c}_1\right), \tilde{r}^{exc}\right)} = 1 \tag{21}$$

and the allocation of risk between private and public consumption relates to individual willingness to shift risk from private to public consumption $\frac{f_{ij}\eta_{ij} \operatorname{cov}\left(u'_{ij}(\tilde{G}), \tilde{r}^{exc}\right)}{f_{ij}\eta_{ij} \operatorname{cov}\left(u'_{ij}(\tilde{c}_1), \tilde{r}^{exc}\right)}$.

This formula is obtained by noticing that the conditions (5), (11), and the covariance identity imply that the term

$$f_{ij}\eta_{ij} \operatorname{cov}\left(u'_{ij}\left(\tilde{c}_1^{ij}\right), \tilde{r}^{exc}\right)$$

is equal across agent-types. Hence, expression (16) can be reformulated as (20).

A.2 Corollary 3

The optimality condition of τ_k^{pq} for agent type (w_p, θ_q) reads:

$$\begin{aligned}
&- f_{pq}\eta_{pq}\, \mathbb{E}\left[u'_{pq}(\tilde{c}_1^{pq})\tilde{r}^{exc}\right] s_q a^{pq} \\
&+ \sum_{i,j} f_{ij}\eta_{ij}\mathbb{E}\Big[u'_{ij}(\tilde{G}) f_{pq} s_q \left(a^{pq}\right) \tilde{r}^{exc}\Big] + \tau_k \sum_{i,j} f_{ij}\eta_{ij}\mathbb{E}\left[u'_{ij}(\tilde{G}) f_{pq} \frac{\partial s_q}{\partial \tau_k^{pq}} \left(a^{pq}\right) \tilde{r}^{exc}\right] = 0.
\end{aligned}$$

First, notice that the first term is zero, given FOC for the share of risky assets (10). Then, recall that $\frac{\partial s}{\partial \tau_k} = \frac{s}{(1-\tau_k)}$, so that the condition can be rewritten as follows:

$$\sum_{i,j} f_{ij}\eta_{ij}\mathbb{E}\Big[u'_{ij}(\tilde{G}) f_{pq} s_q \left(a^{pq}\right) \tilde{r}^{exc}\Big] \left(1 + \frac{\tau_k^{pq}}{1-\tau_k^{pq}}\right) = 0,$$

that is satisfied if and only if $\sum_{i,j} f_{ij}\eta_{ij}\mathbb{E}\Big[u'_{ij}(\tilde{G})\tilde{r}^{exc}\Big] = 0$, which coincides with condition (15). If the condition for τ_k^{pq} coincides with condition (15) for any agent-type, then $\tau_k^{pq} = \tau_k$ for any agent-type (w_p, θ_q). $\square$

Appendix B

B.1 Individual Maximisation Problem and First Order Conditions

The individual maximisation problem for agent-type (w^i, θ_j) is:

$$\begin{aligned}
\max_{c_0^{ij}, s_j, l^{ij}} \quad & U_{ij} = u_{ij}(c_0^{ij}) + \beta\,\mathbb{E}\left[u_{ij}(\tilde{c}_1^{ij}) + u_{ij}(\tilde{G})\right] - v(z^{ij}/w_i) \\
\text{s.t.} \quad & \tilde{c}_{ij} = \tilde{R}_p^j \left(z^{ij}(1-\tau_l(z^{ij})) - c_0^{ij}\right) - t, \\
& \tilde{R}_p^j = \left[1 + r\,(1-\tau_r) + s_j\,(\tilde{r}^{exc})\,(1-\tau_k)\right]
\end{aligned}$$

where $\tilde{R}_p$ is the gross portfolio return; t is a lump-sum tax or transfer, equal for all types; τ_r and τ_k are the tax rates on the riskless and excess return respectively. As each agent is infinitesimal compared to the size of the economy, the effects of individual choices on the level of the public good are not taken into account.

The FOCs for first period consumption c_0, share of risky assets s, and labour supply l read:

$$\begin{aligned}
c_0^* : \quad & u'_{ij}(c_0^{ij}) = \mathbb{E}\left[u'_{ij}(\tilde{c}_1^{ij})\tilde{R}_p\right] \\
s^* : \quad & 0 = \mathbb{E}\left[u'_{ij}(\tilde{c}_1^{ij})\tilde{r}^{exc}\right] \\
l^* : \quad & 0 = \mathbb{E}\left[u'_{ij}(\tilde{c}_1^{ij})\tilde{R}_p\right] w_i(1-\tau_l(z^{ij})) - v'(z^{ij}/w_i)
\end{aligned}$$

B.1.1 Optimal Portfolio Share of Risky Assets

To derive formula 12, I follow Campbell and Viceira (2002). The portfolio return factor is $\tilde{R}_p = (1 + r(1-\tau_r)) + s(R^e - R)(1-\tau_k)$, or

$$\frac{\tilde{R}_p}{R} = 1 - \frac{r}{R}\tau_r + s\left(\frac{R^e}{R} - 1\right)(1-\tau_k).$$

Taking logs we get

$$\tilde{r}_p - r = f(r^e - r) = \log\left(1 - \frac{r}{R}\tau_r + s(e^{r^e - r} - 1)(1-\tau_k)\right). \tag{22}$$

It's possible to derive an approximation of the portfolio excess return using a second-degree Taylor's expansion around the point $r^e - r = 0$:

$$\tilde{r}_p - r = \log\Big(1 - \frac{r}{R}\tau_r\Big) + \frac{s(1-\tau_k)R}{R - r\tau_r}\varphi_1 + \frac{1}{2}\frac{s(1-\tau_k)R}{R - r\tau_r}\Big(1 - \frac{s(1-\tau_k)R}{R - r\tau_r}\Big)\sigma_r^2. \quad (23)$$

where $\varphi_1 := r^e - r$. Then, we can write the expected utility at time 0 from consumption in period 1 as follows:

$$\begin{aligned}
\mathbb{E}_0(u(c_1)) &\approx (1-\theta)^{-1}\,\mathbb{E}_0\left[\left(a_0 e^r e^{\tilde{r}_p - r}\right)\right] && (24)\\
&\approx (1-\theta)^{-1}\,\mathbb{E}_0\left[\left(a_0 e^r e^{\log\left(1-\frac{r}{R}\tau_r\right)+\frac{s(1-\tau_k)R}{R-r\tau_r}\varphi_1+\frac{s(1-\tau_k)R}{R-r\tau_r}\left(1-\frac{s(1-\tau_k)R}{R-r\tau_r}\right)\frac{\sigma_r^2}{2}}\right)^{1-\theta}\right]\\
&\approx (1-\theta)^{-1}\Big[a_t e^r\Big(1 - \frac{r}{R}\tau_r\Big)\Big]^{1-\theta} \underbrace{e^{(1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}\left(1-\frac{s(1-\tau_k)R}{R-r\tau_r}\right)\frac{\sigma_r^2}{2}}\,\mathbb{E}_0\left[e^{(1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}\varphi_1}\right]}_{\text{Excess return component (Exc)}}
\end{aligned}$$

where the term $(1-\theta)^{-1}\Big[a_t e^r\Big(1 - \frac{r}{R}\tau_r\Big)\Big]^{1-\theta}$ is constant and positive when $\theta < 1$. Hence, maximising $\mathbb{E}_0(u(c_1))$ is equivalent to maximising the "excess return component" (Exc). Notice that the term inside the expectation can be rewritten as follows when the return is log-normally distributed:

$$(1-\theta)\frac{s(1-\tau_k)R}{R - r\tau_r}\varphi_1 \sim \mathcal{N}\Big((1-\theta)\frac{s(1-\tau_k)R}{R - r\tau_r}(\varphi - \sigma_r^2/2), (1-\theta)^2\Big(\frac{s(1-\tau_k)R}{R - r\tau_r}\Big)^2\sigma_r^2\Big).$$

where $\varphi = \bar{r}^e - r$. Hence, taking the log of $\mathbb{E}_0\left[e^{(1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}\varphi_1}\right]$, we get:

$$\log \mathbb{E}_0\left[e^{(1-\theta)\frac{s(1-\tau_k)R}{R-r\tau_r}\varphi_1}\right] = (1-\theta)\frac{s(1-\tau_k)R}{R - r\tau_r}(\varphi - \sigma_r^2/2) + \Big(\frac{s(1-\tau_k)R}{R - r\tau_r}\Big)^2\frac{\sigma_r^2}{2}(1-\theta)^2.$$

Then, we can compute the log of the excess return component:

$$\begin{aligned}
\log(\text{Exc}) &= (1-\theta)\frac{s(1-\tau_k)R}{R - r\tau_r}\Big(1 - \frac{s(1-\tau_k)R}{R - r\tau_r}\Big)\frac{\sigma_r^2}{2} + (1-\theta)\frac{s(1-\tau_k)R}{R - r\tau_r}\Big(\varphi - \frac{\sigma_r^2}{2}\Big)\\
&\quad + (1-\theta)^2\left(\frac{s(1-\tau_k)R}{R - r\tau_r}\right)^2\frac{\sigma_r^2}{2}\\
&= (1-\theta)\left[\frac{s(1-\tau_k)R}{R - r\tau_r}\varphi - \theta\left(\frac{s(1-\tau_k)R}{R - r\tau_r}\right)^2\frac{\sigma_r^2}{2}\right] && (25)
\end{aligned}$$

Then, with $\theta < 1$, maximising (24) is equivalent to maximising the terms multiplying $(1-\theta)$ in (25):

$$\max_{s} \frac{s(1-\tau_k)R}{R-r\tau_r}\varphi - \theta\Big(\frac{s(1-\tau_k)R}{R-r\tau_r}\Big)^2\frac{\sigma_r^2}{2},$$

with first order condition:

$$s_j^* = \frac{\varphi}{\theta_j\sigma_r^2}\cdot\frac{(R-r\tau_r)}{R\,(1-\tau_k)},$$

where $\varphi = \bar{r}^e - r$.

B.2 The Government's problem

$$\max_{\tau_k,\tau_r,\tau_l^{ij},t} SW = \sum_{i,j} f_{ij}\eta_{ij}\Big(U_{ij}\Big(c_0^{ij*}, \underbrace{a^{ij}\tilde{R}_p^j - t}_{\tilde{c}_1^{j*}(s_j^*, l^{ij*})}, l^{ij*}, \tilde{G}\Big)\Big)$$

$$\text{s.t. } \tilde{G} = \sum_{i,j} f_{ij}\left[\big(\tau_k s_j^*\left(\tilde{r}^{exs}\right) + \tau_r r\big)\Big(z^{ij*}(1-\tau_l(z^{ij}) - c_0^{ij*}\Big) + \tau_l(z^{ij})z^{ij*} + t\right]$$

$$\text{with } \tilde{R}_p^j = \big[1 + \underbrace{r(1-\tau_r)}_{\substack{\text{net-of-tax}\\ \text{safe return}}} + s_j^*\underbrace{\left(\tilde{r}^{exs}\right)(1-\tau_k)}_{\substack{\text{net-of-tax}\\ \text{excess return}}}\big]$$

B.3 Endogenous Labour Supply with Observable Types

First, the conditions for transfer t and excess return tax τ_k:

$$t^* : \sum_{i,j} f_{ij}\eta_{ij}\,\mathbb{E}[u'_{ij}(\tilde{c}_1^{ij})] = \sum_{i,j} f_{ij}\eta_{ij}\,\mathbb{E}[u'_{ij}(\tilde{G})] \tag{26}$$

$$\tau_k^* : \sum_{i,j} f_{ij}\eta_{ij}\,\mathbb{E}\left[u'_{ij}(\tilde{G})r^{exc}\right] = 0 \tag{27}$$

Then, under optimally set taxes/transfer t^* and τ_k^* the conditions for a labour income tax on each ability and risk aversion type (w_p, θ_q), and the safe return tax equal for all types are:

$$\tau_l^{pq*} : f_{pq}\eta_{pq}\,\mathbb{E}\left[u'_{pq}(\tilde{c}_1^{pq})\right] z_{pq}p_0 = \sum_{i,j} f_{ij}\eta_{ij}\,\mathbb{E}\left[u'_{ij}(\tilde{G})\right] \tag{28}$$

$$\times f_{pq}\left[z_{pq}p_0 - \frac{\tau_r r}{(1-\tau_l^{pq})}\cdot\left(\epsilon_{a,l}^{pq}a^{pq}\right) - \frac{\tau_l^{pq}}{1-\tau_l^{pq}}\epsilon_{z,l}^{pq}z_{pq}p_0\right];$$

$$\tau_r^* : \sum_{i,j} f_{ij}\eta_{ij}\,\mathbb{E}\left[u'_{ij}(\tilde{c}_1^{i})\right]a^{ij}r = \sum_{i,j} f_{ij}\eta_{ij}\,\mathbb{E}\left[u'_{ij}(\tilde{G})\right] \tag{29}$$

$$\times \sum_{i,j} f_{ij}\left[a^{ij}r - \frac{\tau_r\cdot r}{(1-\tau_r)}\left(\epsilon_{a,r}^{ij}a^{ij}\right) - \frac{\tau_l^{ij}}{1-\tau_r}z^{ij}p_0\epsilon_{z,r}^{ij}\right].$$

To derive formula 14 from condition 28, define $g_{pq} = \frac{\eta_{pq}U_c^{pq}}{\sum_{i,j} f_{ij}\eta_{ij}U_c^{ij}}$ as the social marginal welfare weight of an agent-type (w_p, θ_q), normalised with respect to the population. Set $\tau_r = 0$, and rework the condition as

$$(1-g_{pq})(1-\tau_l^{pq}) - \tau_l^{pq}\epsilon_{z,l}^{pq} = 0.$$

Finally, solve for τ_l^{pq}.

B.4 Endogenous Labour Supply with Unobservable Risk Aversion

Suppose that risk aversion is unobservable such that a risk-aversion-specific earnings tax schedule is not feasible. The government optimally chooses τ_l^i for each productivity group $(w_i, \cdot)$, composed by agents with productivity (w_i) across different levels of risk aversion.. The condition for a productivity-specific labour earnings tax reads:

$$\begin{aligned}\tau_l^{i*}: \sum_j f_{ij}\eta_{ij}\,\mathbb{E}\left[u'_{ij}(\tilde{c}_1^{ij})\right] z_{ij}p_0 = &\sum_{i,j} f_{ij}\eta_{ij}\,\mathbb{E}\left[u'_{ij}(\tilde{G})\right] \\ &\times \sum_j f_{ij}\left[z_{ij}p_0 - \frac{\tau_r r}{(1-\tau_l^i)}\cdot\left(\epsilon_{a,l}^{ij}a^{ij}\right) - \frac{\tau_l^i}{1-\tau_l^i}\epsilon_{z,l}^{ij}z_{ij}p_0\right]\end{aligned} \tag{30}$$

Rewrite the condition using the definition of the relative marginal social welfare weight of type ij: $g_{ij} = \frac{\eta_{ij}U_c^{ij}}{\sum_{i,j} f_{ij}\eta_{ij}U_c^{ij}}$ and aggregate earnings for productivity level i, $Z^i = \sum_j f_{ij}z_{ij}$:

$$\sum_j f_{ij}g_{ij}z_{ij} - \left(\sum_j f_{ij}z_{ij} - \frac{\tau_r/p_0}{1-\tau_l^i}\sum_j f_{ij}\left(\epsilon_{a,l}^{ij}a_{ij}\right) - \frac{\tau_l^i}{1-\tau_l^i}\sum_j f_{ij}\left(\epsilon_{z,l}^{ij}z_{ij}\right)\right) = 0. \tag{31}$$

Then, define $\overline{g}_z^i = \sum_j f_{ij}g_{ij}z_{ij}/Z^i$ as the average pre-tax income within productivity group $(w_i, \cdot)$ weighted by individual welfare weights g_{ij}, and average pre-tax earnings Z^i; $\epsilon_{a,l}^i = \sum_j f_{ij}(\epsilon_{a,l}^{ij}a_{ij})/Z^i$, namely the normalised weighted average of savings elasticities; and $\overline{\epsilon_{z,l}^i} = \sum_j f_{ij}(\epsilon_{z,l}^{ij}z_{ij})/Z^i$: normalised weighted average of earnings elasticities. Now, the condition can be rewritten as

$$\overline{g}_z^i - \left(1 - \frac{\tau_r/p_0}{1-\tau_l^i}\overline{\epsilon_{a,l}^i} - \frac{\tau_l^i}{1-\tau_l^i}\overline{\epsilon_{z,l}^i}\right) = 0. \tag{32}$$

Solving for the tax rate gives:

$$\tau_l^i = \frac{1-\overline{g}_z^i - \tau_r\cdot\overline{\epsilon_{a,l}^i}/p_0}{1-\overline{g}_z^i+\overline{\epsilon_{z,l}^i}}.$$

Then, the safe capital income tax condition reads as follows:

$$\tau_r^* : \sum_{i,j} f_{ij}\eta_{ij}\,\mathbb{E}\left[u'_{ij}(\tilde{c}_1^i)\right]a^{ij}r = \sum_{i,j} f_{ij}\eta_{ij}\,\mathbb{E}\left[u'_{ij}(\tilde{G})\right] \qquad (33)$$
$$\times \sum_{i,j} f_{ij}\left[a^{ij}r - \frac{\tau_r \cdot r}{(1-\tau_r)}\left(\epsilon_{a,r}^{ij}a^{ij}\right) - \frac{\tau_l^i}{1-\tau_r}z^{ij}p_0\epsilon_{z,r}^{ij}\right].$$

To derive condition 17, rewrite 33 using $\overline{g}_a = \frac{\sum_{i,j} f_{ij}g_{ij}a_{ij}}{A}$, and average savings as $A = \sum_{i,j} f_{ij}a^{ij}$.

$$1 - \overline{g}_a - \frac{1}{A}\cdot\left[\frac{\tau_r \cdot r}{(1-\tau_r)}\left(\sum_{i,j} f_{ij}\epsilon_{a,r}^{ij}a^{ij}\right) + \frac{p_0}{(1-\tau_r)}\left(\sum_{i,j} f_{ij}\epsilon_{z,r}^{ij}z^{ij}\tau_l^i\right)\right] = 0 \qquad (34)$$

This is the welfare effect of perturbating τ_r. By evaluating it at $\tau_r = 0$, we obtain condition 17. To obtain an implicit formula for τ_r, pre-multiply (34) by $(1-\tau_r)$ and solve for τ_r:

$$\tau_r = \frac{1 - \overline{g}_a - (p_0/A)\cdot\sum_{i,j} f_{ij}\epsilon_{z,r}^{ij}z^{ij}\tau_l^i}{1 - \overline{g}_a + \overline{\epsilon}_{a,r}}, \qquad (35)$$

with $\overline{\epsilon}_{a,r} = \frac{\sum_{i,j} f_{ij}\epsilon_{a,r}^{ij}a_{ij}}{A}$.