MAXIMA OF POLYNOMIALS FOR THE STUDY OF SMALL REGULATORS

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Abstract. Current methods for the classification of number fields with small regulator depend mainly on an upper bound for the discriminant, which can be improved by looking for the best possible upper bound of a specific polynomial function over an hypercube. In this paper, we provide new and effective upper bounds for the case of fields with one complex embedding and degree between five and nine: this is done by adapting the strategy we have adopted to study the totally real case, but for this new setting several new computational issues had to be overcome. As a consequence, we detect the two number fields of signature $(6,1)$ with smallest regulator; we also expand current lists of number fields with small regulator in signatures $(3,1),(4,1)$ and $(5,1)$.

1. Introduction

A problem in Algebraic Number Theory consists in giving a complete classification, up to isomorphism, of number fields with prescribed properties: in particular, scholars have been interested in tabulate complete lists of number fields K such that some chosen invariant is less than some given upper bound. Typical examples are the classification of fields with small discriminant [2, 3, 18, 20, 21] and the classification of imaginary quadratic fields with bounded class number [13, 23, 25]. Both these investigations benefit from the crucial fact that we can mathematically prove that the desired fields occur in finite number, so that it is theoretically possible to describe them all in a finite time; furthermore, this permits the construction of databases gathering fields with small discriminant, like the LMFDB database [24], Klüners-Malle's database for fields with prescribed Galois group [15], PARIgp tables [16] and Jones-Roberts' database [14].

In this paper we are interested in the classification of number fields K of given signature (r_1, r_2), i.e. with r_1 real embeddings and r_2 couples of conjugated complex embeddings, and with small regulator R_K. For every $C > 0$, there exists only a finite number of such fields satisfying $R_K \leq C$, thanks to an inequality by Remak [22] derived from Geometry of Numbers which bounds the absolute value of the discriminant d_K in terms of a constant $R(r_1, r_2, C)$ depending on r_1, r_2 and C. However, the resulting upper bound is too big to also permit a real description of all the fields with discriminant within the output range. A procedure to avoid this difficulty was developed by Astudillo, Diaz y Diaz and Friedman [1], who combined Remak's inequality with a lower bound for R_K derived from explicit formulae of Dedekind Zeta functions [11]: the resulting method allowed the aforementioned authors to obtain the full lists of fields with smallest regulators for the following cases:

- fields of degree at most 6, in any signature,
- fields of degree 7 in any signature apart from $(5,1)$,
- fields of degree 8 with signature $(8,0)$ and $(0,4)$,
- fields of degree 9 with signature $(9,0)$.

The case of the signature $(5,1)$ was solved later by Friedman and Ramirez-Raposo [12] with an ad hoc argument taking advantage of the fact that for this signature there is one non-conjugated complex embedding. This, together with the success obtained for the totally real signatures $(8,0)$ and $(9,0)$, suggested that the signature of the considered fields should play a crucial role not only in the formulation of the problem, but also in the possible success of the specific procedure.

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In greater detail, the key ingredient in these procedures is an upper bound as sharp as possible for the function

$$P_{n,r_2}(y_1,\dots,y_n) := \prod_{1\le i<j\le n}\left|1-\frac{y_i}{y_j}\right|$$

where $n\geq 2$ and the n-ple $(y_1,\dots,y_n)$ satisfies the following two conditions:

A) The numbers y_i are ordered as $0<|y_1|\le|y_2|\le\dots\le|y_n|$,

B) Among the numbers y_i, there are r_2 couples of complex conjugated and non-real numbers, while all the remaining numbers are real.

A simple estimate for the considered object is $n^{n/2}$ [7] and is derived by looking at the function as the determinant of a $n\times n$ matrix with entries bounded by 1 in absolute value, and applying then Hadamard's inequality; however, this upper bound results to be not suitable for the procedure as the degree of the considered fields increases, and in fact this method applied to signature $(5,1)$ with the upper bound $7^{7/2}$ is not able to provide small regulators. The same phenomenon occurs if one applies the method to the signatures $(6,1),(4,2)$ and $(2,3)$ in degree 8 with the upper bound $8^{8/2}$. In [12] Friedman and Ramirez-Raposo succeeded to lower the upper bound of $P_{7,1}$ from $7^{7/2}=907.49\dots$ to $\exp(6)=403.42\dots$ and this improvement was enough for the procedure to actually classify fields with small regulator in this signature.
The results for the totally real cases in degree 8 and 9 were obtained earlier because the desired upper bound is consistently smaller whenever only real numbers are considered: in fact, Pohst [19] proved that the correct upper bound for the totally real case is $2^{\lfloor n/2\rfloor}$ when $n\le 11$ already in '77.
Recently we were able to extend this bound to every n, see [6]. This is achieved by developing in greater detail the idea (initially sketched by Pohst) of associating to the polynomial $P_{n,0}$ a graphical scheme, i.e. a triangular array such as

$$\begin{array}{|c|c|c|c|}\hline + & - & - & + \\ \hline \multicolumn{1}{c|}{} & - & - & + \\ \cline{2-4} \multicolumn{2}{c|}{} & + & - \\ \cline{3-4} \multicolumn{3}{c|}{} & - \\ \cline{4-4}\end{array} \tag{1}$$

where every square represents a factor of $P_{n,0}$ after the change of variables $x_i:=y_i/y_{i+1}$ and the sign at place (i,j) represents the sign of the product $\prod_{k=i}^{j}x_k$, and then to recognize patterns of signs in any possible graphical scheme that can be replaced with other patterns producing an upper bound for the overlaying functions. Eventually, we were able to prove that it is possible to reduce each graphical scheme into the one defined by signs "−" on the main diagonal. Finally, this special scheme can be covered by elementary block of signs which are bounded by known constants. In the example below, the scheme turns out to be estimated by 4 as effect of the fact that each triangular pattern is estimated by 2 and the square one is bounded by 1.

$$\begin{array}{|c|c|c|c|}\hline - & + & - & + \\ \hline \multicolumn{1}{c|}{} & - & + & - \\ \cline{2-4} \multicolumn{2}{c|}{} & - & + \\ \cline{3-4} \multicolumn{3}{c|}{} & - \\ \cline{4-4}\end{array} \le 2\cdot 1\cdot 2 = 4 \tag{2}$$

Motivated by this result, we decided to adapt the technique of graphical schemes to the study of the functions $P_{n,1}$ and so to the research of fields with one complex embedding and small regulator. The new setting emerged to be much more intricate: as we will explain in the next sections, more than 150 estimates were needed for the estimate of the new graphical schemes (contrary to the only nine inequalities needed for the totally real case), and the proof of several of these estimates required deeper analytic and computational work. The results we obtained are the following.

Theorem 1. *Let* $M:=3^{15/2}/(4\cdot 7^{7/2})$. *Then*

$$P_{5,1}\le 16\,M = 16.6965\dots$$

and this upper bound is the best possible one.

Theorem 2. *We have*

$$P_{6,1} \leq 34.89, \quad P_{7,1} \leq 65.81, \quad P_{8,1} \leq 83.90, \quad P_{9,1} \leq 233.1.$$

The results of Theorems 1 and 2 constitute meaningful improvements with respect to all their previous bounds. In particular, the estimate for $P_{8,1}$ allows to detect the fields of signature $(6,1)$ with smallest regulator.

Corollary 1. *Let K_1 and K_2 be the fields with signature $(6,1)$ defined respectively by the polynomials*

$$x^8 - 2x^7 - x^6 + x^5 - 2x^4 + 4x^3 + 3x^2 - 2x - 1$$

and

$$x^8 - 2x^7 - 3x^6 + 10x^5 - 2x^4 - 11x^3 + 5x^2 + 2x - 1.$$

Then K_1 and K_2 are the fields with signature $(6,1)$ and smallest possible regulator. We have $R_{K_1} = 7.135\ldots$ and $R_{K_2} = 7.38\ldots$

Corollary 2. *The non-primitive field with signature $(6,1)$ and smallest regulator is the field L_1 given by the polynomial*

$$x^8 - 9x^6 - 8x^5 + 11x^4 + 21x^3 + 17x^2 + 7x + 1.$$

The value of its regulator is $R_{L_1} = 7.414\ldots$

The new bound for $P_{5,1}$, $P_{6,1}$ and $P_{7,1}$ allow to improve the known lists of fields ordered according to their regulator in the corresponding signatures. The new lists are described in the next corollaries.

Corollary 3. *There exist 40 number fields K with signature $(3,1)$ satisfying $R_K \leq 2.15$ and they all have $|d_K| \leq 25679$.*

Corollary 4. *There exist 136 number fields K with signature $(4,1)$ satisfying $R_K \leq 4.6$ and they all have $|d_K| \leq 712603$.*

Corollary 5. *There exist 59 number fields K with signature $(5,1)$ satisfying $R_K \leq 6.1$ and they all have $|d_K| \leq 7495927$.*

The first author conjectured in [4] the maximum for the functions studied in Theorems 1 and 2. In particular Conjecture 3 appearing there states that it should satisfy an iterative behaviour as the degree n increases, similarly to what happens for the totally real case. For the $r_2 = 1$ case the conjecture states that

$$P_{n,1} \leq \begin{cases} 2^{\frac{n+3}{2}} M & n \text{ odd } \geq 5, \\ 2^{\frac{n+4}{2}} & n \text{ even } \geq 6. \end{cases}$$

Theorem 1 proves the conjecture for the case $P_{5,1}$ and the values in Theorem 2 are larger than but very close to the conjectural ones.

Here we give a brief description of the next sections of the paper. In Section 2 we derive the corollaries from the results of Theorems 1 and 2, and in doing so we recall the steps for the classification method of number fields with small regulator. Section 3 recalls the framework in which the result for $P_{n,0}$ was obtained, including the notion of graphical schemes and some basic estimates. Section 4 introduces the change of variables which allows to study the problem for the functions $P_{n,1}$ and the idea of ordering of these functions. Section 5 presents the proof of Theorem 1 and the resultant tree strategy we used for it. Section 6 illustrates how graphical schemes are adapted in the new context of fields with one complex embedding and the several results and difficulties are encountered whenever one tries to estimate them. Section 7 lists some technical and computational remarks regarding the procedures we used for the estimate of this graphical schemes and the dataset associated. Finally, Section 8 presents some considerations about the upper bounds we found and their possible improvement, including a discussion about the classification of number fields with signature $(7,1)$ and small regulator.

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2. Proof of the corollaries

2.1. **Proof of Corollaries 1 and 2.** We refer to Sections 2 and 3 of [1] for the proof of the propositions of this section.

Proposition 1. *Let $K = \mathbb{Q}(\varepsilon)$ be a number field of degree n and signature (r_1, r_2) with $\varepsilon \in \mathcal{O}_K^*$. Let $m_K(\varepsilon)$ be the length of ε in the logarithmic lattice of the units of K. Let $\varepsilon_1, \dots, \varepsilon_n$ be the conjugates of ε ordered in increasing absolute value. Then*

$$\log|d_K| \leq 2\log(P_{n,r_2}(\varepsilon_1, \dots, \varepsilon_n)) + m_K(\varepsilon) \cdot \sqrt{\frac{n^3 - n - 4r_2^3 - 2r_2}{3}} =: U_0.$$

Proposition 2. *Assume $K = \mathbb{Q}(\varepsilon)$ as above and that ε is such that $m_K(\varepsilon)$ is the minimum non-zero length in the logarithmic lattice. Write $r := r_1 + r_2 - 1$: then*

$$m_K(\varepsilon) \leq \left(\sqrt{r+1} R_K \gamma_r^{r/2}\right)^{1/r}$$

where γ_j is the Hermite constant of dimension j.

Let us consider now the family of fields K of degree 8 and signature $(6, 1)$ with $R_K \leq R_0 := 7.39$. The inequalities of Propositions 1 and 2, together with the upper bound for $P_{8,1}$ from Theorem 2 and the fact that $\gamma_6 = 2\sqrt[6]{3}$ [9, p.332], imply that, whenever K is generated by a unit with minimum logarithmic length, we have

$$\log|d_k| \leq 2\log(83.9) + \left(\sqrt{7} \cdot 7.39 \cdot \sqrt{64/3}\right)^{1/6} \cdot \sqrt{\frac{8^3 - 8 - 4 - 2}{3}} = 36.149522\dots \tag{3}$$

In particular, the hypotheses of Proposition 2 are satisfied whenever K is primitive, i.e. has no subfields different from $\mathbb{Q}$ and itself. Thus, the first conclusion we can get is that any field of signature $(6, 1)$ with $|d_K| > \exp(36.149522\dots)$ must have $R_K > 7.39$.

The role of the upper bound (3) is crucial in the next proposition.

Proposition 3. *Let $g_{r_1,r_2} : (0, +\infty) \to \mathbb{R}$ be the analytic function defined as*

$$g_{r_1,r_2}(x) := \frac{1}{2^{r_1} 4\pi i} \int_{2-i\infty}^{2+i\infty} (\pi^n 4^{r_2} x)^{-s/2} (2s-1) \Gamma(s/2)^{r_1} \Gamma(s)^{r_2} ds.$$

Let K be a field with signature (r_1, r_2) and such that $|d_K| < \delta$ where δ is a lower bound for the discriminant of a field of signature $(3r_1, 3r_2)$. Let d_1, d_2 be such that $0 < d_1 < |d_K| < d_2 < \delta$. If $4g(1/d_1) > R_0$ and $4g(1/d_2) > R_0$, then

$$R_K \geq 4g(1/|d_K|) > R_0. \tag{4}$$

The bound δ is important in order to have a factor 4 in the above inequalities, otherwise we would be forced to use a factor 2; we notice that, in our specific case, $\exp(36.149522\dots)$ is far less than 19^{24}, which is a lower bound for the fields of degree 24 and signature $(18, 3)$ (this value can be recovered from Diaz y Diaz' tables of discriminant lower bounds [10]). One then verifies that

$$4g_{6,1}(\exp(-36.149522\dots)) = 8.25523\dots > 7.39.$$

From [3] we have the list of all the fields K of signature $(6, 1)$ with $|d_K| \leq 79259702$, and one verifies that

$$4g_{6,1}(1/79259702) = 7.48749 > 7.39.$$

Hence from Proposition 3 it follows that every field K of signature $(6, 1)$ with $79259702 < |d_K| < \exp(36.149522\dots)$ must have $R_K > 7.39$, and so the only possible fields with regulator below this bound must be contained in the known list (which is formed of eight fields). For every such field we compute their regulator using PARI/GP: the computation is a priori conditional on GRH and the output is an integer multiple of the true value, but we can prove that every output is the correct value for R_K because they satisfy the following condition.

Proposition 4. *Let K be a field as in Proposition 3. Let $\tilde{R}_K = mR_K$ be an integer multiple of the regulator R_K. Let w_K be the number of roots of unity of K (which is 2 for every field with $r_2 > 1$). If*

$$0 < \frac{\tilde{R}_K/w_k}{2g_{r_1,r_2}(1/|d_K|)} < 2$$

then $\tilde{R}_K = R_K$.

The unique fields in the list satisfying $R_K \leq 7.39$ are the fields K_1 and K_2 described in the statement of Corollary 1. This proves that K_1 and K_2 are the primitive fields of signature (6,1) with smallest regulator, equal to 7.135... and 7.38... respectively. The discussion, however, is not concluded: we have to deal with the case of non-primitive fields, for which Inequality (3) is not guaranteed to be correct, since it may happen that the unit ε with smallest logarithmic length generates a proper subfield of K. If now $\mathbb{Q}(\varepsilon) \subset K$, we need a relative version of Proposition 1 and some lower bounds on the logarithmic lengths in tower of fields.

Proposition 5. *Let K, F be number fields with $K = F(\varepsilon)$ where $\varepsilon \in \mathcal{O}_K^*$. Then*

$$\log|d_K| \leq [K:F]\log|d_F| + [K:\mathbb{Q}]\log([K:F]) + m_K(\varepsilon)C(K/F)$$

with

$$C(K/F) := \sqrt{\frac{1}{3}\sum_{v\in\infty_F}([K:F]^3 - [K:F] - 4r_2(v)^3 - 2r_2(v))}$$

where $r_2(v)$ is the ramification index in K of an infinite place v of F.

Proposition 6. *If $F \subset K$ and $\eta \in \mathcal{O}_F^*$, then $m_K(\eta) \geq \sqrt{[K:F]}\cdot m_F(\eta)$. Moreover, if $F = \mathbb{Q}$ and K is totally real, then for $\varepsilon \in \mathcal{O}_K^*$ we have $m_K(\varepsilon) \geq \sqrt{[K:\mathbb{Q}]}\log((1+\sqrt{5})/2)$.*

Consider a field K of signature (6,1) such that its unit η_1 of minimum length generates a quadratic or quartic subfield, and let η_2 and η_3 be units of second smallest length and third smallest length, respectively. We have three possible cases to consider, which are the only possible ones due to the signature of the involved fields:

a) $F = \mathbb{Q}(\eta_1)$ is quadratic totally real, and $K = F(\eta_2)$;
b) $F = \mathbb{Q}(\eta_1)$ is quartic of signature $(4,0)$, and $K = F(\eta_2)$;
c) $F = \mathbb{Q}(\eta_1)$ is quadratic totally real, $L = F(\eta_2)$ is quartic of signature $(4,0)$ and $K = L(\eta_3)$.

For every such possible case we employ Propositions 5 and 6 to obtain an estimate of the form $\log|d_K| \leq A_0 + \sum_{i=1}^{s} A_i m_K(\eta_i)$. We have then

$$\text{a) } \log|d_K| \leq 24\log 2 + 2\sqrt{2}\cdot m_K(\eta_1) + \sqrt{38}\cdot m_K(\eta_2),$$

$$\text{b) } \log|d_K| \leq 24\log 2 + 2\sqrt{5}\cdot m_K(\eta_1) + \sqrt{6}\cdot m_K(\eta_2),$$

$$\text{c) } \log|d_K| \leq 24\log 2 + 2\sqrt{2}\cdot m_K(\eta_1) + 2\sqrt{2}\cdot m_K(\eta_2) + \sqrt{6}m_K(\eta_3). \tag{5}$$

This upper bound can then be properly optimized thanks to Lemma 3 of [1], giving

$$\log|d_K| \leq \begin{cases} 35.724649\ldots & \text{in case a),} \\ 31.328077\ldots & \text{in case b),} \\ 33.842880\ldots & \text{in case c).} \end{cases}$$

Therefore a non-primitive field of signature $(6,1)$ with $R_K \leq 7.39$ satisfies $\log|d_K| \leq 35.724649\ldots$: we have $4\cdot g_{6,1}(\exp(-35.724649)) > 91 > 7.39$ and thus (applying again the previous propositions) we return to the aforementioned list of eight fields, where we see that there are no non-primitive fields with regulator below 7.39. Thus we can conclude that K_1 and K_2 are indeed the fields of signature (6,1) with smallest regulator. This completes the proof of Corollary 1.

For Corollary 2 we apply the same method described above assuming $R_K \leq 7.48$ but using only the inequalities in (5). these produce the following upper bounds in the three non-primitive cases:

$$\log|d_K| \leq \begin{cases} 35.724649\ldots & \text{in case a),} \\ 31.432169\ldots & \text{in case b),} \\ 33.842880\ldots & \text{in case c).} \end{cases}$$

Notice that these upper bounds are very similar to the previous ones, with the differences in cases a) and c) found only after several decimal digits.
Thus a non-primitive field of signature $(6,1)$ and $R_K \leq 7.81$ satisfies $\log|d_K| \leq 35.724649\ldots$. One verifies that $4\cdot g_{6,1}(\exp(-35.724649)) > 91 > 7.81$: hence we reduce again to the list of 8 fields with smallest discriminant. Inside this list, the field presented in the statement of the corollary is the unique one which is non-primitive and satisfies the desired inequality on the regulator (the PARI output for the regulators is again unconditional because Proposition 4 is satisfied).

2.2. **Proof of Corollaries 3, 4 and 5.** The proof of the last three results is similar to the one above in the degree 8 case. Notice that, since 5 and 7 are primes, the hypothesis of Proposition 1 are satisfied and thus only the corresponding inequality must be used.

In the degree 5 case, assume $R_0 = 2.15$: if K is such that $R_K \leq R_0$, then $P_{5,1} \leq 16\,M$ implies $|d_K| \leq \exp(16.882140\ldots)$. This number is bigger than $\exp(12.876782\ldots)$ which is a lower bound for the discriminant of fields of degree 15 and signature $(9,3)$, hence we have to verify the condition (4) with a factor 2 instead of 4. This yields $2g_{3,1}(\exp(-16.882140\ldots)) = 3.406994\ldots > 2.15$ and $2g_{3,1}(\exp(-12.876782\ldots)) = 2.158866\ldots > 2.15$. Since $48000 < \exp(12.876782\ldots)$, we verify that $4g_{3,1}(1/48000) = 2.157100\ldots > 2.15$ and so the only fields with signature $(3,1)$ and $R_K \leq 2.15$ are among the 145 fields in this signature with $|d_K| \leq 48000$. We download the lists from LMFDB (which are complete up to this bound) and we put them in PARI: a computation of the regulator shows that only 40 fields among them satisfy $R_K \leq 2.15$, and they all have $|d_K| \leq 25679$. The output regulators are the true values since the conditions described in Proposition 4 are always satisfied.

Now, assume the degree is 6 and $R_0 = 4.6$: since $P_{6,1} \leq 34.89$, assuming $R_K \leq R_0$ and mimicking the computations in [1, Section 5.2] we have that the several relations between units of minimum length and subfields of K imply

$$|d_K| \leq \begin{cases} \exp(24.666347\ldots) & K = \mathbb{Q}(\eta_1), \\ \exp(23.942996\ldots) & \mathbb{Q}(\eta_1) \text{ is a real quadratic subfield of } K, \\ \exp(19.716604\ldots) & \mathbb{Q}(\eta_1) \text{ is a totally real cubic subfield of } K. \end{cases}$$

Hence, in any case we have $|d_K| \leq \exp(24.666347\ldots)$ and this upper bound is bigger than $\exp(16.458959\ldots)$, which is a lower bound for the discriminant of fields with degree 18 and signature $(12,3)$. We have $2g_{4,1}(\exp(-24.666347\ldots)) = 5.083637\ldots > 4.6$ and $2g_{4,1}(\exp(-16.458959\ldots)) = 5.266992\ldots > 4.6$. Since $1300000 < \exp(16.458959\ldots)$, we verify that $4g_{4,1}(1/130000) = 4.631338\ldots > 4.61$: therefore, the only possible fields with signature $(4,1)$ and $R_K \leq 4.61$ must be among the 613 ones with $|d_K| \leq 1300000$. A GP computation as above shows that 136 among them have the desired regulator, and all the output values are the true values because Proposition 4 is satisfied.

Finally, assume the degree is 7 and $R_0 = 6.1$: then $P_{7,1} \leq 65.81$ and any field with signature $(5,1)$ and $R_K \leq R_0$ has $|d_K| \leq \exp(30.549708\ldots)$, which is an upper bound above the number $\exp(20.109413\ldots)$, which is a lower bound for the discriminant a field of degree 21 and signature $(15,3)$. One has $2g_{5,1}(\exp(-30.549708\ldots)) = 7.409746\ldots > 6.1$ and $2g_{5,1}(\exp(-20.109413\ldots)) = 13.744146\ldots > 6.1$. Since $14500000 < \exp(20.109413\ldots)$, we verify that $4g_{5,1}(1/14500000) = 7.060629\ldots > 6.1$, so that one only has to check if among the 294 fields with signature $(5,1)$ and $|d_K| \leq 14500000$ there exist some with $R_K \leq 6.1$. The usual GP computation on the complete LMFDB list find 59 of them, and they all satisfy $|d_K| \leq 7495927$.

3. Recalls on the totally real case

In this section we briefly recall how the correct supremum for $P_{n,0}$ was obtained: this discussion will set the foundation for the work on the function $P_{n,1}$. All the results in this section are proved in [6].

In the totally real case all the numbers y_i satisfying A) and B) that we consider are real. Setting the change of variables

$$x_i := \frac{y_i}{y_{i+1}}, \qquad i = 1, \ldots, n-1 \tag{6}$$

the function $P_{n,0}$ becomes the polynomial in $n-1$ variables (each one in $[-1,1]$)

$$Q_{n-1}(x_1,\ldots,x_{n-1}) = \prod_{i=1}^{n-1}\prod_{j=i}^{n-1}\left(1-\prod_{k=i}^{j} x_k\right)$$

and looking for the correct supremum of $P_{n,0}$ is equivalent to looking for the maximum of Q_{n-1} over $[-1,1]^{n-1}$. Instead of directly studying the function Q_{n-1}, one considers all the 2^{n-1} subcases defined by a choice for the signs of the variables; if $\varepsilon := (\varepsilon_1,\ldots,\varepsilon_{n-1})$ is a vector formed by elements $\varepsilon_i \in \{\pm 1\}$, we consider the 2^{n-1} configurations

$$Q_{n-1,\varepsilon}(x_1,\ldots,x_{n-1}) = \prod_{i=1}^{n-1}\prod_{j=i}^{n-1}\left(1-\prod_{k=i}^{j}\varepsilon_k\prod_{k=i}^{j} x_k\right) \tag{7}$$

where now each variable is assumed to be in $[0,1]$.

Using the notation of [6], we recall the notion of *graphical scheme*, i.e. a triangular $n\times n$ array C which has only symbols "+" or "-" in its entries $C_{i,j}$ and $1 \le i \le j \le n-1$, as (1) and (2), are graphical schemes. A function F_C defined over $[0,1]^{n-1}$ can be associated to every graphical scheme C, and it has the form:

$$F_C(x_1,\ldots,x_{n-1}) = \prod_{i=1}^{n-1}\prod_{j=i}^{n-1}\left(1-C_{i,j}\prod_{k=i}^{j} x_k\right).$$

The configuration (7) is thereby the function F_C for the graphical scheme C with entries

$$C_{i,j} = \begin{cases} + & \text{if } \prod_{k=i}^{j}\varepsilon_k = 1, \\ - & \text{if } \prod_{k=i}^{j}\varepsilon_k = -1. \end{cases}$$

Given two graphical schemes C and C', we say that $C \le C'$ if $F_C \le F_{C'}$. Estimates of graphical schemes can be obtained by recognizing patterns in the starting scheme C and replacing them with a new pattern such that the replacement corresponds to an estimate between the corresponding factors: the resulting scheme C' will then satisfy $C \le C'$. We denote these estimates as *dynamical estimates*. We recall four dynamical estimates.

Lemma 1. *Let C be a graphical scheme.*

P) Let C' be the graphical scheme defined changing the element $\boxed{+}$ into $\boxed{-}$ and keeping everything else unchanged. Then $C \le C'$.

H) Let C' be the graphical scheme defined changing the two elements $\begin{array}{|c|c|}\hline + & - \\ \hline\end{array}$ into $\begin{array}{|c|c|}\hline - & + \\ \hline\end{array}$ and keeping everything else unchanged. Then $C \le C'$.

V) Let C' be the graphical scheme defined changing the two elements $\begin{array}{|c|}\hline - \\ \hline + \\ \hline\end{array}$ into $\begin{array}{|c|}\hline + \\ \hline - \\ \hline\end{array}$ and keeping everything else unchanged. Then $C \le C'$.

S) Let C' be the graphical scheme defined changing the four elements $\begin{array}{|c|c|}\hline - & + \\ \hline + & - \\ \hline\end{array}$ into $\begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline\end{array}$ and keeping everything else unchanged. Then $C \le C'$.

These four estimates are sufficient, as the following result states.

Theorem 3 (Th.2 in [6]). *Let $C_{n-1,\varepsilon}$ be the graphical scheme of a configuration $Q_{n-1,\varepsilon}$. Let $C_{n-1,-}$ be the graphical scheme of the configuration Q_{n-1,ε_-} which has vector of signs $\varepsilon_- := (-1,-1,\ldots,-1)$. Then $C_{n-1,\varepsilon} \le C_{n-1,-}$: the estimate is obtained by applying only estimates of the form P, H, V and S to the entries of $C_{n-1,\varepsilon}$.*

The desired maximum for our configurations can thus be obtained by just studying the graphical scheme $C_{n-1,-}$, which is of the form

$$\begin{array}{|c|c|c|c|l}
\hline - & + & - & + & \cdots \\
\hline \multicolumn{1}{c|}{} & - & + & - & \cdots \\
\cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & - & + & \cdots \\
\cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & - & \cdots \\
\cline{4-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \cdots
\end{array}$$

with every line made of alternating signs. Patterns in the scheme correspond to factors of the corresponding function, and each such factor may be bounded by proper constants.

Lemma 2. *Let C be a graphical scheme. The following patterns, if contained in C, can be estimated with the following constants:*

- $\begin{array}{cc} & j \\ i & \boxed{+} \end{array} \leq 1$, *i.e.* $F_{C_{i,j}} \leq 1$,
- $\begin{array}{ccc} & j & j' \\ i & \boxed{+} & \boxed{-} \end{array} \leq 1$, *i.e.* $F_{C_{i,j}} \cdot F_{C_{i,j'}} \leq 1$,
- $\begin{array}{cc} & j \\ i & \boxed{-} \\ i' & \boxed{+} \end{array} \leq 1$, *i.e.* $F_{C_{i,j}} \cdot F_{C_{i',j}} \leq 1$,
- $\begin{array}{ccc} & j & j' \\ i & \boxed{-} & \boxed{+} \\ i' & \boxed{+} & \boxed{-} \end{array} \leq 1$, *i.e.* $F_{C_{i,j}} \cdot F_{C_{i',j}} \cdot F_{C_{i,j'}} \cdot F_{C_{i',j'}} \leq 1$,
- *Assume $j' = j + 1$. Then*
  $\begin{array}{ccc} & j & j' \\ i & \boxed{-} & \boxed{+} \\ i' & & \boxed{-} \end{array}$, $\begin{array}{ccc} & j & j' \\ i & \boxed{-} & \boxed{-} \\ i' & & \boxed{+} \end{array}$, $\begin{array}{ccc} & j & j' \\ i & \boxed{+} & \boxed{-} \\ i' & & \boxed{-} \end{array}$ *are all ≤ 1 i.e.* $F_{C_{i,j}} \cdot F_{C_{i,j'}} \cdot F_{C_{i',j'}} \leq 1$,
- *Assume $i' = i + 1$. Then*
  $\begin{array}{ccc} & j & j' \\ i & \boxed{-} & \boxed{+} \\ i' & & \boxed{-} \end{array}$, $\begin{array}{ccc} & j & j' \\ i & \boxed{-} & \boxed{-} \\ i' & & \boxed{+} \end{array}$, $\begin{array}{ccc} & j & j' \\ i & \boxed{+} & \boxed{-} \\ i' & & \boxed{-} \end{array}$ *are all ≤ 1 i.e.* $F_{C_{i,j}} \cdot F_{C_{i,j'}} \cdot F_{C_{i',j'}} \leq 1$.

These *static estimates* are what is needed in order to recover the result for $C_{n-1,-}$.

Theorem 4 (Lemma 3 and Th. 1 in [6])**.** *One has $C_{n-1,-} \leq 2^{\lfloor n/2 \rfloor}$ so that this is the maximum of Q_{n-1} on $[-1, 1]^{n-1}$. This bound cannot be improved since it is attained at the point $(-1, 0, -1, 0, \ldots)$, so that the correct supremum of $P_{n,0}$ is thus $2^{\lfloor n/2 \rfloor}$.*

4. The change of variables for $r_2 = 1$

Starting from the previous considerations for the totally real case, we begin the investigation for the function $P_{n,1}$. As before, we would like to transform it into a polynomial function via a change of variables and to look for the maximum of this new function over a proper domain. However, this transformation will no longer be immediate due to the presence of a couple of complex conjugated numbers among the y_i.

In fact, we have to consider an n-ple $(y_1, y_2, \ldots, y_k, \overline{y_k}, y_{k+1}, \ldots, y_{n-1})$ which satisfies condition A). The change of variables (6) cannot be applied directly, since quotients y_{k-1}/y_k and $\overline{y_k}/y_{k+1}$ are not real. A more suitable change of variables is instead the following:

$$x_i := \begin{cases} y_i/y_{i+1}, & i \neq k-1, k, \\ y_{k-1}/|y_k|, & i = k-1 \\ |\overline{y_k}|/y_{k+1}, & i = k \end{cases}, \quad g := \cos(\arg y_k) \tag{8}$$

Several things have to be remarked about this transformation: first of all, the resulting function will no longer be purely polynomial, since it will always contain the term $2\sqrt{1-g^2}$, corresponding to the factor $|1 - y_k/\overline{y_k}|$; moreover, and most importantly, different functions will arise as different indexes k are chosen for the position of the complex conjugated couple $(y_k, \overline{y_k})$.

We define *k-th ordering of $P_{n,1}$* the function $L_{n,k}$ resulting from the change of variables (8) applied to $P_{n,1}$ with the complex couple $(y_{n-k}, \overline{y_{n-k}})$. Here are the first and the second ordering for $n = 5$, corresponding to indexes $k = 4$ and $k = 3$ for the complex couple.

$$\begin{aligned} L_{5,1} =& (1 - x_1)(1 - x_1x_2)(1 - 2x_1x_2x_3g + (x_1x_2x_3)^2) \\ & \qquad (1 - x_2) \quad (1 - 2x_2x_3g + (x_2x_3)^2) \\ & \qquad\qquad (1 - 2x_3g + x_3^2) \cdot 2\sqrt{1 - g^2}, \end{aligned}$$

$$\begin{aligned} L_{5,2} =& (1 - x_1)(1 - 2x_1x_2g + (x_1x_2)^2)(1 - x_1x_2x_3) \\ & \qquad (1 - 2x_2g + x_2^2) \qquad (1 - x_2x_3) \\ & \qquad\qquad (1 - 2x_3g + x_3^2) \cdot 2\sqrt{1 - g^2}. \end{aligned}$$

Notice that the polynomial factors containing g and different from the square root correspond to either products of the form $|1 - y_i/y_k| \cdot |1 - y_i/\overline{y_k}| = |1 - y_i/y_k|^2$ or $|1 - y_k/y_j| \cdot |1 - \overline{y_k}/y_j| = |1 - y_k/y_j|^2$.

A priori we should then study all the $n - 1$ possible orderings resulting from this change of variables, since every time we end up with a different function over $[-1, 1]^{n-1}$. Fortunately, there is a symmetry between the orderings which allows us to discard half of the cases.

Lemma 3. *Let $n \geq 3$ and $1 \leq k \leq n - 1$. Then*

$$L_{n,k}(x_1, x_2, \ldots, x_{n-2}, g) = L_{n,n-k}(x_{n-2}, x_{n-3}, \ldots, x_1, g).$$

Proof. The sequence

$$0 < |y_1| \leq |y_2| \leq \cdots \leq |y_k| = |\overline{y_k}| \leq \cdots \leq |y_{n-1}|$$

is equivalent to the sequence

$$0 < \left|\frac{1}{y_{n-1}}\right| \leq \cdots \leq \left|\frac{1}{y_k}\right| \leq \left|\frac{1}{\overline{y_k}}\right| \leq \cdots \leq \left|\frac{1}{y_1}\right|.$$

The claim follows by constructing the function $L_{n,n-k}$ using the numbers $z_j := 1/y_{n-j}$. □

Remark: the quantity $P_{n,1}$ can be seen to be estimated by the quantity $2^{\lfloor (5n-8)/2 \rfloor}$: in fact, if y_k and $\overline{y_k}$ form the complex conjugated couple, there are $n - 2$ factors of the form $|1 - y_j/y_k|^2$ or $|1 - y_k/y_j|^2$ with $j \neq k$, each term bounded by 4. The term $|1 - y_j/\overline{y_j}|$ is bounded by 2 and removing all these terms from $P_{n,1}$ we obtain a function which becomes $P_{n-2,0}$ via an additional change of variable and which is at most $2^{\lfloor (n-2)/2 \rfloor}$. The product $P_{n,1}$ is thus estimated by $2 \cdot 4^{n-2} \cdot 2^{\lfloor (n-2)/2 \rfloor} = 2^{\lfloor (5n-8)/2 \rfloor}$. However, this estimate turns out to be better than the trivial bound $n^{n/2}$ only for $n \geq 25$, so it is of no use for the applications we study in Theorems 1 and 2.

5. Proof of Theorem 1

In order to prove an estimate for $P_{5,1}$, we consider the only two possible orderings for its transformation via the change of variables (8), which are the examples $L_{5,1}$ and $L_{5,2}$ shown in the previous section. The study of the two functions will be quite similar, but we consider them separately since some meaningful differences will occur; moreover, the techniques employed in this section will be later employed for the study of functions $P_{n,1}$ with $n \geq 6$.

Remark: in the following lines, we shall often factorize polynomials in several variables with rational coefficients. The factorization results turned to be available thanks to the computer algebra MAGMA [8]. Moreover, real roots of rational polynomials shall be computed: this have been made by employing the computer algebra PARI/GP [17]. The MAGMA and PARI/GP files describing the computations in detail can be found in [5].

5.1. Estimate for $\mathbf{L_{5,1}}$.

The first ordering was partially studied in [4] as a toy model for the formulation of the conjectures about upper bounds for $P_{n,1}$: there, a partial result about the maximum of $L_{5,1}$ is proved.

Lemma 4. *The function $L_{5,1}(x_1, x_2, x_3, g)$ assumes its maximum over $[-1, 1]^4$ for $x_3 = 1$ and $g \neq \pm 1$.*

Under this assumption, $L_{5,1}$ reduces to the simpler expression

$$\begin{aligned}(1-x_1)(1-x_1x_2)&(1-2x_1x_2g+(x_1x_2)^2)\\ (1-x_2)\quad&(1-2x_2g+x_2^2)\\ &\quad 4\cdot(1-g)\sqrt{1-g^2}.\end{aligned}$$

The research of the maximum for this function over $[-1,1]^2\times(-1,1)$ is carried through various steps: first of all, we study the behaviour of the function whenever one of the variables is equal to 0 (this will turn useful for next computations). We have:

- $x_1 = 0$: the function is $L_{5,1}(0,x_2,1,g) = L_{4,1}(x_2,1,g)$ and we know this function is at most 16 (see the remark in Section 1). This value is attained at $x_2 = -1$ and $g = 0$.
- $x_2 = 0$: the function is $4\cdot(1-x_1)(1-g)\sqrt{1-g^2}$ which is maximized at $x_1 = -1$ and $g = -1/2$ giving $6\sqrt{3} = 10.392\ldots$(this is verified by studying the partial derivatives of the obtained function)
- $g = 0$: the function becomes

  $$L = 4(1-x_1)(1-x_1x_2)(1+(x_1x_2)^2)(1-x_2)(1+x_2^2).$$

  We look for the maximum of L over $[-1,1]^2$. First of all, we determine whether there are stationary points in the interior $(-1,1)^2$: we derive L with respect to x_1 and denote by L_{x_1} the only factor of the derivative (which is a polynomial) which is not zero on the boundary. This gives

  $$L_{x_1} = 4x_2^3x_1^3 + (-3x_2^3-3x_2^2)x_1^2 + (2x_2^2+2x_2)x_1 + (-x_1-1).$$

  We define L_{x_2} in the same way but from the derivative with respect to x_2 and we obtain

  $$\begin{aligned}L_{x_2} =&(6x_2^5-5x_2^4+4x_2^3-3x_2^2)x_1^3 + (-5x_2^4+4x_2^3-3x_2^2+2x_2)x_1^2\\ &+(4x_2^3-3x_2^2+2x_2-1)x_1 + (-3x_2^2+2x_2-1).\end{aligned}$$

  An eventual stationary point must be a common root (α,β) of L_{x_1} and L_{x_2}, thus its coordinate α must be a root of the resultant between the two polynomials with respect to the variable x_2. This resultant is equal to

  $$32x_1^9 + 24x_1^8 + 20x_1^7 + 7x_1^6 + 24x_1^5 + 67x_1^4 + 15x_1^2 + 27.$$

  However, this polynomial has no real roots between -1 and 1, so this means that L has no stationary points in the interior.

  One then studies the behaviour of L on the boundaries: clearly L is 0 when $x_1 = 1$ or $x_2 = 1$, while at $x_1 = -1$ we have

  $$L = 8(1-x_2^4)(1+x_2^2) \le 8\cdot\frac{32}{27} = 9.481\ldots$$

  (the upper bound is obtained at $x_2 = \pm\frac{1}{\sqrt{3}}$) and at $x_2 = -1$ we have

  $$L = 16(1-x_1^4) \le 16.$$

Thus we have proved that, whenever one of the three variables is equal to 0, the function $L_{5,1}$ is at most 16: this information will be useful for the next computations. After this preliminary discussion, we have two things to consider in order to optimize $L_{5,1}$: we must study the behaviour of the function on the boundaries of $[-1,1]^3$ and then we must look for possible stationary points in the interior of this cube. Let us begin with the boundary investigation.

- $x_1 = 1$: the function $L_{5,1}$ assumes the value 0 under this condition. The same holds for the boundaries $x_2 = 1$ and $g = \pm 1$.
- $x_1 = -1$: the function $L_{5,1}$ becomes

  $$S = 8\cdot(1-x_2^2)((1+x_2^2)^2-4x_2^2g^2)(1-g)\cdot\sqrt{1-g^2}.$$

  Again, we would like to consider the partial derivatives of S with respect to x_2 and g in order to find an eventual maximum point (notice that in this case S becomes 0 whenever a boundary condition is satisfied by either x_2 or g). However, we do not compute directly the derivatives of S since we do not want to carry the square root term in the research of the points: we consider thus only the term

$$L = 8 \cdot (1 - x_2^2)((1 + x_2^2)^2 - 4x_2^2 g^2)(1 - g)$$

and we study the system

$$\begin{cases} \frac{\partial L}{\partial x_2} = 0, \\ \frac{\partial L}{\partial g}(1 - g^2) - Lg = 0, \end{cases}$$

where the second quantity is obtained from the derivative rule for S with respect to g. We factorize the left hand sides of these equations and we discard any factor that either has roots only on the boundary or has roots only when some variable is zero: this is because the necessary considerations have been already made or will be done in further boundary study. We are thus left with the factors

$$L_{x_2} = 3x_2^4 + (-8g^2 + 2)x_2^2 + (4g^2 - 1)$$

and

$$L_g = (2g + 1)x_2^4 + (-16g^3 - 4g^2 + 12g + 2)x_2^2 + (2g + 1).$$

and we study the system $L_{x_2} = L_g = 0$. A stationary point for S must be then a common root (β, γ) of L_{x_2} and L_g, hence β must be a root of their resultant with respect to g, which is $8x_2^4 + x_2^2 - 1$. This polynomial has two roots between -1 and 1, and substitution of these roots in L_{x_2} provides four stationary points $(\pm\beta, \pm\gamma)$ where $\beta = 0.54455\ldots$ and $\gamma = 0.29653$. One then verifies that $S(\pm\beta, \gamma) = 5.9612\ldots$ and $S(\beta, \pm\gamma) = 10.9870\ldots$ (so we are below the value 16 we have found before).

- $x_2 = -1$: this case has been studied in [4, Conjecture 1], where it was proved that under the assumption $x_2 = -1$ and $x_3 = 1$ the function $L_{5,1}$ has a maximum at the point $(x_1, x_2, x_3, g) = (1/\sqrt{7}, -1, 1, 1/(2\sqrt{7}))$ and the maximum value is exactly $16M$ (the first author stated in [4] that the determination of this maximum for $L_{5,1}$ was still an open problem because of the lack of techniques for a rigorous optimization over the interior of $[-1, 1]^3$).

Finally, we discuss the possible existence of maximum points for $L_{5,1}$ in the open set $(-1, 1)^3$. First of all, we compute the derivatives

$$\frac{\partial L}{\partial x_1}, \quad \frac{\partial L}{\partial x_2}, \quad \frac{\partial L}{\partial g}(1 - g^2) - Lg.$$

We factorize them and then we denote by L_{x_1}, L_{x_2} and L_g the unique factors of each derivative which has not roots in the boundary or for some variable equal to 0. We study then the system $L_{x_1} = L_{x_2} = L_g = 0$: common zeros (α, β, γ) of these three objects must give also common zeros (α, β) of the resultants

$$\mathrm{Res}(L_{x_2}, L_{x_1}; g), \quad \mathrm{Res}(L_g, L_{x_1}; g).$$

We denote by res1g and res2g the unique factors of these resultants which again have no roots on the boundary or for some variable equal to zero, and we study the system res1g = res2g = 0. Finally, we look for the x_1-coordinate α of our stationary points by studying the factors of Res(res1g, res2g; x_2): one verifies that the only acceptable values for α (i.e. roots of this resultant which are in $(-1, 1)$) are $-1/2$ and $3/5$. But if we substitute $x_1 = -1/2$ in res1g, we notice that the only roots of the new polynomial (i.e. the only acceptable values for β) are $x_2 = 0$ or $x_2 = -1$, which constitute boundary cases which we already studied. If instead we substitute $x_1 = 3/5$ in res1g, we find a unique admissible value $\beta = 0.7373\ldots$ but substitution of both $x_1 = \alpha$ and $x_2 = \beta$ in L_x gives no admissible values for g.

This means that we do not find stationary points in $(-1, 1)^3$ for the function $L_{5,1}$: gathering all the results, this proves that $L_{5,1} \leq 16\,M$ over $[-1, 1]^4$ and this upper bound is attained at the point $(x_1, x_2, x_3, g) = (1/\sqrt{7}, -1, 1, 1/(2\sqrt{7}))$.

5.2. **Estimate for $\mathbf{L_{5,2}}$.** The main difference with the previous case is that we do not have a result similar to Lemma 4 allowing to immediately remove one variable from the optimization. However, it is straightforward to verify that

$$L_{5,2}(x_1,x_2,1,g)=L_{5,1}(x_1,x_2,1,g)=L_{5,2}(x_1,-x_2,-1,-g)$$

and thus anytime we are reduced to a case in our computation where $x_3=\pm 1$, we already know that $L_{5,2}$ is at most $16\,M$. We optimize this function by following the steps we presented for the function $L_{5,1}$: so we study the cases where one variable is zero as the first thing.

- $x_1=0$: just like for $L_{5,1}$, we obtain a function which is a transformation of $P_{4,1}$ via a change of variables (8), so we know it is ≤ 16 (and the value is attained at $x_2=-1,x_3=1,g=0$).
- $x_2=0$: this is the same as the case $x_2=0$ for $L_{5,1}$, so we already know it is at most $6\sqrt{3}=10.392\ldots$.
- $x_3=0$: just like $x_1=0$, this case provides a transformation of $P_{4,1}$, and so the function under this condition is at most 16.
- $g=0$: the function becomes

$$L=2\cdot(1-x_1)(1+(x_1x_2)^2)(1-x_1x_2x_3)(1+x_2^2)(1-x_2x_3)(1+x_3^2).$$

Again, we first look for stationary points in the interior $(-1,1)^3$ for the variables (x_1,x_2,x_3) and then we look at the behaviour on the boundaries of the three-dimensional cube.
For the stationary points, we need common roots of the derivatives

$$\frac{\partial L}{\partial x_1},\quad \frac{\partial L}{\partial x_2},\quad \frac{\partial L}{\partial x_3}$$

and we denote by L_{x_1}, L_{x_2} and L_{x_3} the unique factors of each derivative which is not always positive and has not roots only in the boundary or for some variable equal to 0. A common zero (α,β,γ) of these objects must give a common zero (α,β) for the resultants

$$\mathrm{Res}(L_{x_1},L_{x_2};x_3),\quad \mathrm{Res}(L_{x_3},L_{x_2};x_3).$$

We factorize the first resultant and we keep only the factor without roots on the boundary or for a variable equal to 0: we call it res$1x_3$. We do the same with the second resultant, with the remark that the non-trivial factors are now 2 (both different from res$1x_3$): we denote their product as res$2x_3$. Finally, the common zero for these two resultants must give a root α of their resultant with respect to the variable x_2, which is

$$\begin{aligned}21840x_1^{11}&+107944x_1^{10}+280513x_1^9+529240x_1^8+728752x_1^7+802042x_1^6\\&+728398x_1^5+511918x_1^4+311680x_1^3+126574x_1^2+44721x_1+5418.\end{aligned}$$

This polynomial has only one root $\alpha\in(-1,1)$: substitution of $x_1=\alpha$ in res$2x_3$ gives two possible values for β, and substitution of $x_1=\alpha$ and $x_2=\beta$ in L_{x_3} gives two values for γ. We end with two stationary points, and the function L assumes at both the value $3.0285\ldots$

We see now what happens to L if we assume boundary conditions on x_1 and x_2: at $x_1=1$ the function is trivially zero, while at $x_1=-1$ it becomes

$$4\cdot(1+x_2^2)(1+x_2^2)(1-(x_2x_3)^2)(1+x_3^2).$$

The first factor $(1+x_2^2)$ is trivially ≤ 2, while the remaining three factors are estimated by 2 thanks to Lemma (2). Hence the function L is at most 16 in this case.

If we assume $x_2=1$ the function L becomes

$$4\cdot(1-x_1)(1+x_1^2)(1-x_1x_3)(1-x_3)(1+x_3^2)=4\cdot F.$$

By using partial derivatives and standard boundary optimization in 2 variables, the factor F results to be at most 4, so that also in this case we obtain $L\leq 16$. Finally, we notice that for $x_2=-1$ there is nothing to prove, since $L(x_1,-1,x_3)=L(x_1,1,-x_3)$.

Once this preliminary case for the variables equal to zero is discussed, we begin by studying what happens in the more general cases of either x_1 or x_2 being equal to ± 1.

- $x_1=1$: the function is trivially zero.

- $x_2 = 1$: the function becomes

$$L = 4 \cdot (1-x_1)(1-2x_1g+x_1^2)(1-x_1x_3)(1-g)(1-x_3)(1-2x_3g+x_3^2) \cdot \sqrt{1-g^2}.$$

  The procedure of studying successive resultants employed before and applied to the quantities

$$\frac{\partial L}{\partial x_1}, \quad \frac{\partial L}{\partial x_3}, \quad \frac{\partial L}{\partial g}(1-g^2) - Lg$$

  finds no stationary points in the interior of $[-1,1]^3$. Assuming then $x_1 = -1$ (which is the only meaningful boundary condition we can impose on this case) we obtain

$$16 \cdot (1-x_3^2)(1-2x_3g+x_3^2)(1-g^2) \cdot \sqrt{1-g^2}.$$

  This is exactly the function with maximum equal to $16 \cdot M$ (attained at $x_3 = 1/\sqrt{7}$ and $g = 1/(2\sqrt{7})$).
- $x_2 = -1$: this case reduces to the previous one since it is immediate to verify that

$$L_{5,2}(x_1,-1,x_3,g) = L_{5,2}(x_1,1,-x_3,-g).$$

- $x_1 = -1$: the function becomes

$$L = 4 \cdot (1+2x_2g+x_2^2)(1-2x_2g+x_2^2)(1-(x_2x_3)^2)(1-2x_3g+x_3^2) \cdot \sqrt{1-g^2}.$$

  The procedure of studying successive resultants applied to the quantities

$$\frac{\partial L}{\partial x_2}, \quad \frac{\partial L}{\partial x_3}, \quad \frac{\partial L}{\partial g}(1-g^2) - Lg$$

  finds no stationary points in the interior of $[-1,1]^3$. There are no other meaningful boundary conditions to impose.

We are thus left with the research of stationary points for $L_{5,2}$ in the open set $(-1,1)^4$. We have four quantities to consider, which are

$$\frac{\partial L}{\partial x_1}, \quad \frac{\partial L}{\partial x_2}, \quad \frac{\partial L}{\partial x_3}, \quad \frac{\partial L}{\partial g}(1-g^2) - Lg.$$

We factorize each one of these quantities and we keep only the non-trivial factors (i.e. factors which do not have roots on the boundary or for some variable equal to zero): we call these factors L_{x_1}, L_{x_2}, L_{x_3} and L_g and we study the system $L_{x_1} = L_{x_2} = L_{x_3} = L_g = 0$. Then, as in previous cases, we consider their resultants in order to look for common zeros: we compute then

$$\mathrm{Res}(L_{x_2}, L_{x_1}; g), \quad \mathrm{Res}(L_{x_3}, L_{x_1}; g), \quad \mathrm{Res}(L_g, L_{x_1}; g),$$

we factorize them and we keep the non-trivial factors, denoting them as $\mathrm{res}1g$, $\mathrm{res}2g$ and $\mathrm{res}3g$. We study the system $\mathrm{res}1g = \mathrm{res}2g = \mathrm{res}3g = 0$ and we compute then the resultants

$$\mathrm{Res}(\mathrm{res}2g, \mathrm{res}1g; x_3), \quad \mathrm{Res}(\mathrm{res}3g, \mathrm{res}1g; x_3).$$

Now, a particular phenomenon occurs: if we factorize both resultants, it results that they share a common factor of the form $x_1x_2^2 - 1/4x_2^2 - 3/4$ which may have roots in the interior. We denote this common factor as $\mathrm{res}C$. Together with this, each one of these two resultants possesses a specific non-trivial factor which is not shared by the other resultant: we denote this factors as $\mathrm{res}1x_3$ and $\mathrm{res}2x_3$. We must then proceed considering two distinct subcases.

CASE 1: We focus on $\mathrm{res}C = 0$. This means that we have to extract common roots of the four derivatives above from the curve $x_1x_2^2 - \frac{1}{4}x_2^2 - \frac{3}{4} = 0$. Luckily, we can recover a relation between x_1 and x_2 from this equation of the form

$$x_1 = \frac{3+x_2^2}{4x_2^2}. \tag{9}$$

Substitution of (9) in $\mathrm{res}1g$ and $\mathrm{res}2g$ allows to eliminate the variable x_1 and obtain two new expressions (after factorization) $\mathrm{res}1gS$ and $\mathrm{res}2gS$, which are the only non-trivial factors arising from this substitution. We can then compute the resultant

$$\mathrm{Res}(\mathrm{res}1gS, \mathrm{res}2gS; x_3)$$

whose only non-trivial factor has the form $x_2^8 - 4x_2^6 - \frac{17}{16}x_2^4 - \frac{45}{8}x_2^2 + \frac{27}{16}$. We obtain then the possible stationary points of this case by looking at the system

$$\begin{cases} x_2^8 - 4x_2^6 - \frac{17}{16}x_2^4 - \frac{45}{8}x_2^2 + \frac{27}{16} = 0 \\ \mathrm{res1}gS = 0 \\ \mathrm{resC} = 0 \\ L_{x_1} = 0. \end{cases}$$

Studying the first equation, one verifies that substitution of the roots of this equation in the second one give values for x_1 which are bigger than 1 in absolute value: hence Case 1 does not yield any stationary point in our domain.

CASE 2: We proceed with the usual resultant tree by computing

$$\mathrm{Res}(\mathrm{res1}x_3, \mathrm{res1}x_3; x_2).$$

This gives several non-trivial factors which we can multiply with each other to obtain a unique polynomial $\mathrm{res}x_2$ in the variable x_1. We can then proceed backwards as before and study the system

$$\begin{cases} \mathrm{res}x_2 = 0 \\ \mathrm{res1}x_3 = 0 \\ \mathrm{res1}g = 0 \\ L_{x_1} = 0. \end{cases}$$

The first equation has only x_1 as a variable, the second one has x_1 and x_2, and so on. This makes the discussion of the system easier and it allows to show that it has no solutions. Hence, the function $L_{5,2}$ does not have stationary points in the open set $(-1,1)^4$, and from the previous discussion we can conclude that the maximum of $L_{5,2}$ over $[-1,1]^4$ is $16\,M$.

6. Graphical schemes for the proof of Theorem 2

Unfortunately the resultant tree procedure employed in the previous section cannot be applied to the optimization of $P_{n,1}$ when $n \geq 6$. This happens because the increased degree of $P_{n,1}$ not only makes more complicated the factorization of the resultants in the tree, but also because those resultants share more and more factors that should be addressed with ad hoc arguments that however are possible only when their degree is small, and this is no more the case: the lucky argument we have used to deal Case 2 described before is essentially unavailable in general. Thus, we renounce to compute the exact maximum and we content to compute an upper bound by splitting $P_{n,1}$ into several blocks and producing upper bounds for each block. However, the splitting is strongly influenced by the signs of the variables and in order to improve the final result we have to treat separately the 2^{n-1} subcases corresponding to a well determined choice for the signs of each variable. The next result shows that we can restrict the range for g to $[0,1]$: this halves the total number of cases.

Lemma 5. *Let $L_{n,k}(x_1,\dots,x_{n-2},g)$ be the k-th ordering of $P_{n,1}$. Then the maximum of $L_{n,k}$ over $[-1,1]^{n-1}$ is assumed at $g \geq 0$.*

Proof. Remember that the k-th ordering is defined by the couple of complex conjugated numbers $(y_{n-k}, \overline{y_{n-k}})$, and that we can assume $1 \leq k \leq \lfloor (n-1)/2 \rfloor + 1$. If $k=1$, then in the transformation given by the change of variables (8) the variable x_{n-2} is the only one which is always multiplied by g (in fact, in this case we could deal with $x_2 \cdot g$ as a unique variable) and this proves that

$$L_{n,1}(x_1,\dots,x_{n-3},-x_{n-2},-g) = L_{n,1}(x_1,\dots,x_{n-3},x_{n-2},g).$$

For $2 \leq k \leq \lfloor (n-1)/2 \rfloor + 1$, the variable g is always multiplied with either x_{n-k} or x_{n-k-1}; moreover, a product containing $x_{n-k-1}x_{n-k}g$ never appears. Hence, we have

$$L_{n,k}(x_1,\dots,-x_{n-k-1},-x_{n-k},\dots,-g) = L_{n,k}(x_1,\dots,x_{n-k-1},x_{n-k},\dots,g).$$

Both these symmetries show that we can reduce to the case $g \geq 0$ for the research of the maximum. ◻

Starting from this assumption, we now set the construction of graphical schemes as for the totally real case: we choose a vector of signs $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{n-2})$ and instead of studying the function $L_{n,k}$ we study the analogous function with $\prod_{k=i}^{j} x_k$ replaced by $\prod_{k=i}^{j} \varepsilon_k \prod_{k=i}^{j} x_k$, so that we have to study 2^{n-2} functions defined over $[0,1]^{n-1}$. Just like for the real case, we denote these functions as *configurations*. The main difference with the totally real case is that now we have to take into account also the terms of the ordering in which the variable g appears, together with the square root term: the factors containing g will be labeled with a mark " ′ " after the sign + or −, and the square root will be denoted as $R^{1/2}$. Thus, the graphical schemes of the configurations of $L_{6,1}$, $L_{6,2}$ and $L_{6,3}$ with the vector of signs $(+,-,-,+)$ are

$$
\begin{array}{|c|c|c|c|l}
\hline
+ & - & + & +' \\
\hline
\multicolumn{1}{c|}{} & - & + & +' \\
\cline{2-4}
\multicolumn{2}{c|}{} & - & -' \\
\cline{3-4}
\multicolumn{3}{c|}{} & +' & \; 2R^{1/2} \\
\cline{4-4}
\end{array}
\qquad
\begin{array}{|c|c|c|c|l}
\hline
+ & - & +' & + \\
\hline
\multicolumn{1}{c|}{} & - & +' & + \\
\cline{2-4}
\multicolumn{2}{c|}{} & -' & - \\
\cline{3-4}
\multicolumn{3}{c|}{} & +' & \; 2R^{1/2} \\
\cline{4-4}
\end{array}
\qquad
\begin{array}{|c|c|c|c|l}
\hline
+ & -' & + & + \\
\hline
\multicolumn{1}{c|}{} & -' & + & + \\
\cline{2-4}
\multicolumn{2}{c|}{} & -' & -' \\
\cline{3-4}
\multicolumn{3}{c|}{} & + & \; 2R^{1/2} \\
\cline{4-4}
\end{array}
$$

The three schemes above represent respectively the functions

$$
\begin{array}{llll}
(1-x_1) & (1+x_1x_2) & (1-x_1x_2x_3) & (1-2x_1x_2x_3x_4g+(x_1x_2x_3x_4)^2) \\
 & (1+x_2) & (1-x_2x_3) & (1-2x_2x_3x_4g+(x_2x_3x_4)^2) \\
 & & (1+x_3) & (1+2x_3x_4g+(x_3x_4)^2) \\
 & & & (1-2x_4g+x_4^2)\cdot 2\sqrt{1-g^2},
\end{array}
$$

$$
\begin{array}{llll}
(1-x_1) & (1+x_1x_2) & (1-2x_1x_2x_3g+(x_1x_2x_3)^2) & (1-x_1x_2x_3x_4) \\
 & (1+x_2) & (1-2x_2x_3g+(x_2x_3)^2) & (1-x_2x_3x_4) \\
 & & (1+2x_3g+x_3^2) & (1+x_3x_4) \\
 & & & (1-2x_4g+x_4^2)\cdot 2\sqrt{1-g^2},
\end{array}
$$

$$
\begin{array}{llll}
(1-x_1) & (1+2x_1x_2g+(x_1x_2)^2) & (1-x_1x_2x_3) & (1-x_1x_2x_3x_4) \\
 & (1+2x_2g+x_2^2) & (1-x_2x_3) & (1-x_2x_3x_4) \\
 & & (1+2x_3g+x_3^2) & (1+2x_3x_4g+(x_3x_4)^2) \\
 & & & (1-x_4)\cdot 2\sqrt{1-g^2}.
\end{array}
$$

A possible way to estimate this new kind of schemes could be using an algorithm employing dynamical estimates just like the one for totally real fields, so that all configurations are reduced to a standard one for which static estimates can be applied. Unfortunately, this algorithm does not seem to be available due to the new terms with the variable g. As an example, we no longer have an estimate of the form $\begin{array}{|c|c|}\hline + & -' \\ \hline\end{array} \le \begin{array}{|c|c|}\hline - & +' \\ \hline\end{array}$. In fact, this would be equivalent to the estimate $(1-x)(1+2xyg+(xy)^2) \le (1+x)(1-2xyg+(xy)^2)$ for $x, y, g \in [0,1]$, but it easy to verify that, while this is true whenever $g = 0$, there are instead values in the admissible range for which it is false. Similarly, the estimate $\begin{array}{|c|c|}\hline - & +' \\ \hline + & -' \\ \hline\end{array} \le \begin{array}{|c|c|}\hline + & -' \\ \hline - & +' \\ \hline\end{array}$ is not true. However, some dynamical estimates in this new setting are still possible, as proved in the following lemma.

Lemma 6. *We have* $\begin{array}{|c|c|}\hline +' & -' \\ \hline\end{array} \le \begin{array}{|c|c|}\hline -' & +' \\ \hline\end{array}$ *and* $\begin{array}{|c|}\hline - \\ \hline + \\ \hline\end{array} \le \begin{array}{|c|}\hline + \\ \hline - \\ \hline\end{array}$.

Proof. Both these estimates correspond to

$$(1-2xg+x^2)(1+2xyg+(xy)^2) \le (1+2xg+x^2)(1-2xyg+(xy)^2).$$

To prove this, subtract the left hand side from the right hand side: the result is factorized as $4xg(1-y)(1-yx^2) \ge 0$ (remember that we are assuming all variables to be in $[0,1]$). $\square$

Together with this dynamical estimate, it is also possible to detect new patterns which involve the terms with g, so that the scheme can be covered with patterns and estimated by multiplying the upper bounds of every pattern. An example of this static inequality is given by the following.

Lemma 7. *We have* $\begin{array}{|c|c|}\hline + & -' \\ \hline\end{array} \leq 32/27$ *and* $\begin{array}{|c|}\hline -' \\ \hline + \\ \hline\end{array} \leq 32/27$.

Proof. The claim is equivalent to proving that $(1-x)(1+2xyg+(xy)^2) \leq 32/27$. Now, under the assumption that the variables are all in $[0,1]$, it is immediate to see that the left hand side is maximized for $y = g = 1$, and it becomes $(1-x)(1+x)^2 = (1-x^2)(1+x)$: this quantity is maximized at $x = 1/3$, giving the value $32/27$ as maximum. □

The combination of dynamical and static estimates is the tool which allows to obtain estimates for the configurations in all orderings and degree between 6 and 9 as reported in Theorem 2. Differently from the totally real case, the main difficulty is now the static part of the approach: our method relies much more on this kind of estimates, since the presence of the terms with g forbids several dynamical estimates. As a consequence, the number of patterns that have to be recognized is much larger, around 150, and for several of them the optimization can be quite troublesome. We gathered all the patterns we recognized, together with the corresponding estimates and their proofs, in the dataset [5]: in the rest of this section we do not show the proof of every such inequality, but we discuss some of the most meaningful ones. Additional discussion on the dataset [5] and how we used it for the proof of Theorem 2 can be found in the next section.

6.1. **Dynamical estimates.** In some cases, an approach completely similar to the one of Lemma 1 can be useful: some configurations for our new graphical schemes can be proved to be less than other ones by using only dynamical estimates, which can be either the old ones described in Lemma 1 or the new ones given in Lemma 6. An example is the configuration of $P_{6,1}$, first ordering, defined by the vector of signs $(+,-,+,+)$:

$$\begin{array}{|c|c|c|c|l}
\hline
+ & - & - & -' & \\
\hline
\multicolumn{1}{c|}{} & - & - & -' & \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & + & +' & \\
\cline{3-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & +' & 2R^{1/2} \\
\cline{4-4}
\end{array}$$

In fact, we can apply the estimate H of Lemma 1 to the couple $\{(1,1),(1,2)\}$, the estimate V of the same lemma to $\{(2,3),(3,3)\}$ and the vertical estimate of Lemma 6 to $\{(1,4),(4,4)\}$. This proves that the considered configuration is estimated by the one (with degree 6 and with first ordering) defined by the vector of signs $(-,-,-,-)$. This new configuration requires an ad hoc estimate with static inequalities (which show that the upper bound is 32, see [5]), but we have proved that several configurations in this ordering reduce to this case. Similar reductions occur in other degrees and orderings, though being a small fraction of all the possible configurations.

6.2. **Easy static estimates.** Even if the presence of the terms with the g forbids to reduce the discussion to a mostly dynamical approach, several factors of the new graphical schemes are simple enough to be estimated with inequalities that do not require complicated optimization, a first example being the inequalities presented in Lemma 7. Other instances of this phenomenon is the inequality described in the following lemma.

Lemma 8. *We have* $\begin{array}{|c|c|c|}\hline + & + & -' \\ \hline\end{array} \leq 1$.

Proof. The claim is equivalent to proving that $(1-x)(1-xy)(1+2xyzg+(xyz)^2) \leq 1$ with all the variables in $[0,1]$. It is clear that the polynomial is maximized at $z = g = 1$, providing

$$(1-x)(1-xy)\cdot(1+xy)^2 \leq (1-x)(1-xy)\cdot(1+x)(1+xy) = (1-x^2)(1-(xy)^2) \leq 1.$$

□

An application of this can be seen for the estimate of the configuration of $P_{8,1}$, second ordering, defined by the signs $(+,-,-,+,-,-)$, whose graphical scheme is

$$\begin{array}{|c|c|c|c|c|c|l}
\hline
+ & - & + & + & -' & + & \\
\hline
 & - & + & + & -' & + & \\
\cline{2-6}
 & & - & - & +' & - & \\
\cline{3-6}
 & & & + & -' & + & \\
\cline{4-6}
 & & & & -' & + & \\
\cline{5-6}
 & & & & & -' & 2R^{1/2} \\
\cline{6-6}
\end{array} \tag{10}$$

Assuming an estimate for the configuration of $P_{6,1}$, second ordering, signs $(-,+,-,-)$ has been found ([5] reports that it is 12.33), one can find a bound for this configuration by simply estimating the first two lines. In fact, the signs at positions $\{(1,1),(1,2)\}$, $\{(1,6)\}$ and $\{(2,6)\}$ are patterns described in Lemma 2 and such that they are bounded by 1. Similarly, the positions $\{(1,3),(1,4),(1,5)\}$ and $\{(2,3),(2,4),(2,5)\}$ form the pattern described in Lemma 8 and so they are bounded by 1 too. Finally, the remaining position $\{(2,2)\}$ is trivially bounded by 2: this means that the first two lines of this scheme are bounded by 2 and thus the examined configuration is bounded by 24.66. This argument shows that the optimization of this configuration relies critically on the result for the configuration in degree 6, for which more accurate estimates are needed.

6.3. **Less easy static estimates.** As we said, the estimate of the previous configuration relies heavily on the upper bound for the graphical scheme

$$\begin{array}{|c|c|c|c|l}
\hline
- & - & +' & - & \\
\hline
 & + & -' & + & \\
\cline{2-4}
 & & -' & + & \\
\cline{3-4}
 & & & -' & 2R^{1/2} \\
\cline{4-4}
\end{array}$$

This can be decomposed in three blocks, which are $A := \{(1,1),(1,4),(2,4)\}$, $B := \{(1,2),(1,3),(2,2),(2,3)\}$ and $C := \{(3,3),(3,4),(4,4)\}$ (this one multiplied with $2R^{1/2}$). While the block A is immediately seen to be bounded by 2 thanks to Lemma 2, the remaining two blocks are more complicated and for them we cannot give a quick proof just like for the inequality of Lemma 8. Nonetheless, the optimization of these patterns can be obtained following the very same resultant tree strategy which we employed for the proof of Theorem 1.

Lemma 9. *We have* $\begin{array}{|c|c|}\hline - & +' \\ \hline + & -' \\ \hline\end{array} \leq 32/27$ *and* $\begin{array}{c|c|c|l} & j & j' & \\ \cline{2-3} i & - & + & \\ \cline{2-3} i' & & - & 2R^{1/2} \\ \cline{3-3}\end{array} \leq 5.2$ *for* $j' = j+1$ *or* $i' = i+1$.

Proof. Let us begin with the first pattern. We want to prove that the function

$$F = (1+xy)(1-2xyzg+(xyz)^2)(1-y)(1+2yzg+(yz)^2)$$

is at most 32/27 for $(x,y,z,g) \in [0,1]^4$. Just like for the proof of $L_{5,1}$, we begin the optimization by studying the behaviour of this function on the boundary of the hypercube, and then we shall look for eventual stationary points in $(0,1)^4$ (all the factorizations computed in the next lines are done via MAGMA, while all the computations of real roots of rational polynomials are done in PARI). We begin by studying what happens whenever one of the variables is 0 (notice that this is indeed a boundary computation, since differently from Theorem 1 we are assuming all the variables to be non-negative); later we shall study the case of the variables being equal to 1.

$x = 0$: the function becomes $(1-y)(1+2yzg+(yz)^2)$, which is the one described in Lemma 7: hence we already know its maximum is 32/27.

$y = 0$: the function trivially becomes identical to 1.

$z = 0$: the function becomes $(1+xy)(1-y) \leq (1-y^2) \leq 1$.

$g = 0$: the function is now $(1+xy)(1+(xyz)^2)(1-y)(1+(yz)^2)$: this is maximized at $x = z = 1$ giving

$$(1+y)(1+y^2)(1-y)(1+y^2) = (1-y^4)(1+y^2) \leq 32/27$$

(this follows again by Lemma 7 considering y^2 as the variable).

$x = 1$: the function has now the form

$$(1+y)(1-2yzg+(yz)^2)(1-y)(1+2yzg+(yz)^2) = (1-y^2)[(1+(yz)^2)^2 - 4(yzg)^2].$$

Notice that, under our hypotheses, this expression is maximized at $g = 0$: thus we reduce to the previous boundary computation, so that we already know that the function is at most $32/27$ in this case too.

$y = 1$: the function is trivially equal to 0.

$z = 1$: the function becomes $L = (1+xy)(1-2xyg+(xy)^2)(1-y)(1+2yg+y^2)$: this does not have a direct estimate and so we proceed with the resultant tree, i.e. we look for stationary points of L in $(0,1)^3$, i.e. we want to solve

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial g} = 0.$$

We factor these derivatives and denote by L_x, L_y and L_g the only factors of each derivative which are not trivial (in the sense that they have not only roots on the boundary or they are not always positive over the domain). Then we compute $\mathrm{Res}(L_x, L_g; g)$ and $\mathrm{Res}(L_y, L_g; g)$, and we call $\mathrm{res}1g$ and $\mathrm{res}2g$ their unique non-trivial factors. We are then considering the system $L_x = \mathrm{res}1g = \mathrm{res}2g = 0$ and thus we compute the resultant $\mathrm{Res}(\mathrm{res}1g, \mathrm{res}2g; x)$: however, the polynomial we obtain has only factors which are powers of y or $(y-1)$, hence trivial: this means that it is not possible to find a stationary point for L.

The discussion on L concludes by considering the only boundary condition we can add to $z = 1$, i.e. $g = 1$: in this subcase the function is now $(1+xy)(1-xy)^2(1-y)(1+y)^2$, and since its derivative in x has no zeros in $(0,1)$, this case reduces to previous ones and so is not of interest.

$g = 1$: this case is very similar to the one above, since again we have a function in three variables for which a resultant tree is required. Fortunately, this is even easier since, denoting by L_y and L_z the unique non-trivial factors of $\partial L/\partial y$ and $\partial L/\partial z$ are such that $\mathrm{Res}(L_y, L_z; z)$ has only trivial factors, and since we do not have additional boundary conditions to impose, we can skip this case.

Finally, we consider the study over the interior $(0,1)^4$. This requires a resultant tree starting from four derivatives, so a priori it appears to be more complicated: however, if F_z and F_g are the unique non-trivial factors of the derivatives of F with respect to z and g, one verifies that $\mathrm{Res}(F_z, F_g; g)$ has only trivial factors, so that the two derivatives have no common roots in the interior and thus stationary points for F do not exist. This concludes the optimization of the first pattern.

The optimization of the second pattern is completely similar, since it is equivalent to proving that

$$S = \underbrace{(1+2xg+x^2)(1-xy)(1+2yg+y^2)\cdot 2}_{F}\cdot\sqrt{1-g^2}$$

is bounded by 5.2 and this is done via a simpler resultant tree that only involves three variables. The only thing we remark is that the derivatives are made for the function F and that instead of the one with respect to g we consider the quantity $\left(\frac{\partial F}{\partial g}(1-g^2) - Fg\right)$. In this way, the resultant tree procedure shows that the function S has no stationary points in $(0,1)^3$ and its maximum is assumed at the boundary values $(x,y,z) = (0,1,1/2)$ and is equal to $3\sqrt{3} < 5.2$. $\square$

The results of Lemma 9 allow to estimate the blocks B and C of the scheme we are considering: we multiply then the upper bounds of every block and we end with the upper bound $2 \cdot 32/27 \cdot 5.2 = 12.3259\ldots < 12.33$ for this configuration.

Many other patterns we found are estimated with a resultant tree procedure in three or four variables like the one described above. However, not all the necessary patterns involve such a small number of variables or an easy procedure.

6.4. **Complicated static estimates.** The 2^4 different configurations of $L_{6,1}$ (i.e. the number of transformations of $P_{6,1}$ in the first ordering we get with the different choice of the signs of their variables) can be estimated using static or dynamic inequalities which are in similar shape to the ones we have briefly described before, or in the worst case not so much more complicated. This is no more the case whenever we change the ordering and/or the degree: though many of the configurations can still be bounded in similar fashion, some other ones do need of a more detailed investigation, especially in order to obtain an upper bound which is suitable for our original number-theoretic goal. We illustrate this with the following example, which is the configuration of $L_{8,4}$ (i.e. fourth ordering in degree 8) with the vector of signs $(+,-,-,-,-,-)$.

$$\begin{array}{cccccc l}
+ & - & +' & - & + & - & \\
  & - & +' & - & + & - & \\
  &   & -' & + & - & + & \\
  &   &   & -' & +' & -' & \\
  &   &   &   & - & + & \\
  &   &   &   &   & - & 2R^{1/2}
\end{array}$$

This is a complicated scheme to estimate, due to the presence of multiple factors to consider and the symmetric disposition of the g-terms. In fact, none of the estimates of the previous kind is able to give a satisfying estimate for this configuration (the concept of "satisfying" will be further explained in the next sections). It was thus necessary to provide more complicated decompositions of this scheme, each one requiring a much longer optimization process (similar to the one employed for the estimate of $L_{5,2}$) and the one we gave is made of the following two blocks: the first one is the gray one which also includes the factor $2R^{1/2}$, while the second one is formed by the remaining of the scheme.

$$\begin{array}{cccccc l}
+ & - & +' & - & + & - & \\
  & - & +' & - & + & - & \\
  &   & -' & + & - & + & \\
  &   &   & -' & +' & -' & \\
  &   &   &   & - & + & \\
  &   &   &   &   & - & 2R^{1/2}
\end{array}$$

Lemma 10. *The gray block multiplied with* $2R^{1/2}$ *is bounded by* 9.482*.*

We do not give here the complete proof of this lemma and we refer instead our database [5] which contains the complete proof written in MAGMA and PARI files; nonetheless, we show one of the main difficulties occurred in the estimate of this block. In fact, the function we need to estimate has the form

$$\begin{aligned} S = (1+xyzt)(1-xyzta)(1-2yzg+(yz)^2)(1+yztab)(1+2zg+z^2)(1-zt)\cdot \\ (1+2tg+t^2)(1-2tag+(ta)^2)\cdot 2\sqrt{1-g^2} \end{aligned}$$

We now have 6 variables to consider, and this results in considering a system of 6 polynomial equations to solve. Again, we start by the derivatives of the function with respect to every variable (with the caveat that instead of the derivative in g we take the more complicated quantity $\partial L/\partial g\cdot(1-g^2)-L\cdot g$, where $L=S/(2\sqrt{1-g^2})$). Factorization gives some factors that we gather into a system

$$L_x = L_y = L_z = L_t = L_a = L_g = 0$$

and the usual resultant tree and factorization process (done with resultants of these factors with each other with respect to the variable a, g and then x) gives

$$\mathrm{res}1a = \mathrm{res}2a = \mathrm{res}3a = \mathrm{res}4a = \mathrm{res}5a = 0,$$
$$\mathrm{res}1g = \mathrm{res}2g = \mathrm{res}3g = \mathrm{res}4g = 0,$$
$$\mathrm{res}1x = \mathrm{res}2x = \mathrm{res}3x = 0.$$

The next step consists in looking for the common zeros of the last 3 resultants, and so we compute and factor the resultants $\mathrm{Res}(\mathrm{res}2x,\mathrm{res}1x;y)$ and $\mathrm{Res}(\mathrm{res}3x,\mathrm{res}1x;y)$. However, now there is an abundance of non-trivial factors, and most of them are common between the two resultants: in fact, $\mathrm{Res}(\mathrm{res}2x, \mathrm{res}1x; y)$ and $\mathrm{Res}(\mathrm{res}3x, \mathrm{res}1x; y)$ share five common non-trivial factors $\mathrm{res}C1$, $\mathrm{res}C2$, $\mathrm{res}C3$, $\mathrm{res}C4$, $\mathrm{res}C5$ and they both posses a specific and different non-trivial factor, which are respectively $\mathrm{res}1y$ and $\mathrm{res}2y$. This forces to consider six subcases, even five of them are very similar.

CASES 1-5: we consider the common factor $\mathrm{res}Ci$ (with $i \in \{1, \dots, 5\}$) and we look for stationary points of R over the locus $\mathrm{res}Ci = 0$. These common factors are polynomial expressions which are simpler than the ones describing the factors of the derivatives, and thus one can consider the system

$$L_x = L_y = L_z = L_t = L_a = \mathrm{res}Ci = 0$$

and restarts the resultant tree process, which is now easier and with simpler resultants since a complicated condition has been replaced with a simpler one. Iteration of this process eventually leads to the desired polynomial in one variable from which we can extract roots and move backwards to search for stationary points.

In these specific cases, the factors $\mathrm{res}Ci$ are always linear in some variable, and thus the relation $\mathrm{res}Ci = 0$ allows to substitute one variable in a chosen previously computed polynomial: in particular, if we substitute in either $\mathrm{res}1x$ or $\mathrm{res}1g$, we obtain polynomials which have no roots in the open interval $(0, 1)$.

CASE 6: we proceed with the resultant tree computing $\mathrm{Res}(\mathrm{res}1y, \mathrm{res}2y; z)$: we multiply the non-trivial factors of this resultant obtaining a polynomial $\mathrm{res}1z$ and finally we solve the system

$$L_y = \mathrm{res}1a = \mathrm{res}1g = \mathrm{res}1x = \mathrm{res}1y = \mathrm{res}1z = 0$$

where the first polynomial has 6 variables, the second 5, the third 4 and so on. The research of the roots and the evaluation give some stationary points, which however give small local maximums (around $1.41\dots$) that are easily dominated by values on the boundary.

The detailed optimization of the interior and the boundary is reported in [5] and similar difficulties (sometimes with worse expressions) are encountered in the estimate of the second block and in the optimization of some other configurations for which the old procedures do not work suitably. They all share, however, the same “resultant tree/common factors” behaviour that we described with this example.

A similar optimization procedure shows that the non-coloured part of the scheme, if considered as a unique block, is estimated by 8.641. Thus the scheme is bounded by the product of the upper bounds of the two blocks, which is 82.

7. Technical and computational remarks

In this section we gather several remarks about our computations and the estimates we needed in order to deal with all the graphical schemes we considered.

1) As we mentioned, the optimization in the degree 5 case, i.e. $P_{5,1}$, was made considering each variable lying in $[-1, 1]$: the function was simple enough in both orderings to provide a rigorous optimization. For $P_{n,1}$ with $n \in \{6, 7, 8, 9\}$, only variables between 0 and 1 were considered.

2) We have 3 (respectively 3, 4, 4) orderings for $P_{6,1}$ (respectively $P_{7,1}$, $P_{8,1}$, $P_{9,1}$). Each ordering unfolds into 16 configurations (respectively 32, 64, 128), each one defined by a vector of signs. Every configuration is estimated by recognizing patterns into it and either bounding them via static estimates or replacing them with other patterns via dynamic estimates. In several cases this is achieved just estimating the first line or the last column of the scheme and then multiplying the obtained upper bound with the one previously found for the remainder of the scheme, which results to be a configuration in smaller dimension which has been studied before. The scheme shown in (10) that we have already discusse earlier is an example is this procedure.

3) The dataset [5] contains the collection of all static and dynamic estimates we employed for the optimization of the orderings: more in detail, they are gathered in the file “estimates-general-database.txt”. The file contains slightly more than 150 estimates, divided into 9 groups. Group A) contains the estimates first employed for the totally real case and described in Lemma 1 and Lemma 2. Groups from B) to G) collect estimates involving the terms containing the new variable g: their labelling follows no particular order, apart from chronological appearance and similarities in their resolution. Moreover, these estimates are used mostly for the cases in degree

6, 7 and 8. Group H) contains inequalities which turned useful for dealing with the degree 9 case: some of them were later used in lower degree cases replacing some estimates of the previous groups since they turned to be easier to apply. Group I) contains unemployed but proved estimates. Finally, the tiny group RMK contains 4 inequalities which either allow to reduce the number of variables in the optimization of a specific scheme or permit to estimate a large amount of factors in a similar way to the "triangle estimate" of Lemma 2.

4) Apart from the very easy ones, the estimates in this collection are all proved by employing the resultant tree procedure explained in the previous section: the factorization process was carried most of the times in MAGMA (which is very suitable for the factorization of rational polynomials in several variables) while the computation of real roots and evaluation of polynomials was done mostly in PARI/GP. The dataset [5] contains a MAGMA file and a GP file for every proved estimate: the GP part especially contains a program which allows to compute all the roots of the examined polynomials.

5) The proof of some estimates (especially the ones in group G)) turned out to be longer, since they involved polynomials with 6 or 7 variables for which many subcases had to be considered during the optimization. Many of these cases required intermediate considerations due to the presence of the common factors between the considered resultants: they were dealt with in the same way as done in the proof of the second ordering of Theorem 1.

6) Many configurations are estimated by dividing them in blocks and by providing an estimate for every block via resultant trees. Sometimes, choosing the blocks for the decomposition is immediate; however, there are cases where a convenient division in blocks was not evident (such was, for example, the configuration of the fourth ordering $L_{8,4}$ in degree 8 given by the vector of signs $(+,-,-,-,-,-)$). We looked for a good decomposition by writing the object function in MATLAB and by applying the Global Optimization Toolbox, which provided several groups of blocks and their (numerical) maximums whenever the variables are between 0 and 1. The MATLAB results, however, were considered uniquely as suggestions and not as proofs: whenever we found what seemed a convenient decomposition, we have proved for each block that the output maximum was indeed a true maximum (the proof was carried again via the resultant tree procedure in MAGMA/PARI).

7) Sometimes, in order to improve the results and obtain upper bounds suitable for our number-theoretic goals, a simple division in blocks was not sufficient. In some specific cases, like the configuration of the third ordering $L_{8,3}$ in degree 8 given by the vector of signs $(+,-,+,+,-,+)$, we not only divided the function in two blocks, but we also considered for every block two precise subcases: the first one is given by assuming a specific variable x_i in an interval $[0,\alpha]$ (where $\alpha \in (0,1)$ is a convenient number), the second one by assuming $x_i \in [\alpha,1]$. This was done in order to exploit a cancellation behaviour, since the two blocks do not assume their maximum values on the same points: in particular, one block will assume its maximum for $x_i \in [0,\alpha]$ and be instead quite small in $[\alpha,1]$, while the other block will present the opposite behaviour. The necessary inequalities, with their MAGMA/PARI files, are present in the dataset.

8. Final remarks on the results

The following table represents the upper bounds we detected for the considered signatures and for all the possible orderings.

ordering \ degree n and $r_2=1$	5	6	7	8	9
1st	16M	32	32M	64M	155.1
2nd	16M	32M	54M	79.42	190.2
3rd		34.89	65.81	83.49	201.4
4th				83.90	233.1

First of all, we compare these upper bounds with the basic bound $n^{n/2}$: the new bounds are considerable improvements being reduced by a factor of circa 3.3 (respectively 6.2, 13.8, 48.8, 84.4) in the case of degree 5 (respectively 6, 7, 8, 9).

We also know from Theorem 1 that the new upper bounds $16M$ in degree 5 are the best possible for each ordering, since these values are attained in both cases on the point $(x_1, x_2, x_3, g) = (1/\sqrt{7}, -1, 1, 1/(2\sqrt{7}))$. Also the upper bound for degree 6 first ordering and degree 7 first ordering are sharp, since

$$L_{6,1}((-1, 0, -1, 1, 0)) = 32$$

and

$$L_{7,1}((-1, 0, 1/\sqrt{7}, -1, 1, 1/(2\sqrt{7}))) = 32M.$$

Notice in particular that the correct bound for degree 6 first ordering is two times the bound for the function $P_{4,1}$, and the one in degree 7 first ordering is twice the bound for $P_{5,1}$. We conjecture the following:

- For a fixed degree n, the maximum of $L_{n,k}$ should be the same independently from the chosen k-th ordering, exactly like it happens for $n \leq 5$.
- the maximum of $P_{n,1}$ follow a recursive formula starting from degree 4 and 5:

$$\sup P_{n+2,1} = 2 \sup P_{n,1} \quad \text{when } n \geq 4, \text{ so that} \quad \sup P_{n,1} = \begin{cases} 2^{(n+4)/2} & n \geq 4 \text{ even} \\ 2^{(n+3)/2}\, M & n \geq 5 \text{ odd}. \end{cases}$$

We conclude by taking into exam the case in degree 9. At the moment the known list of ten fields with signature $(7, 1)$ in LMFDB is not proved to be complete, but one conjectures that this is the case. Assume that this is true. The GP computation of the regulators yields their true value since the conditions of Proposition 4 are satisfied, and the smallest value is circa 18.0874 and comes from the field with smallest discriminant in the list, i.e. the field defined by the polynomial

$$x^9 - x^8 - 5x^7 + 6x^6 + 5x^5 - 11x^4 + 5x^3 + 6x^2 - 6x + 1 \tag{11}$$

and absolute discriminant equal to 1904081383. Adjusting the level R_0 in Proposition 2 to 18.1 and using the upper bound 233.1 in Theorem 2 for $P_{9,1}$ we get the bound $\log |d_K| \leq 47.334357\ldots$. Thus, in order to explore all $(7, 1)$ fields having a regulator ≤ 18.1 we should explore all fields with absolute discriminant up to $\exp(47.334357\ldots)$. Unfortunately in this case Proposition 3 does not help to reduce the range, since $4g_{7,1}(\exp(-47.334357))$ is negative, and the task remains impossible. The only thing we can conclude (using the classification method with this upper bound) is that, imposing $R_0 = 5.3$, Proposition 1 gives the upper bound $|d_K| \leq \exp(41.471548\ldots)$: we have also $2g_{7,1}(\exp(-41.471548\ldots)) = 32.365375 > 5.3$ and $4g_{7,1}(1/1904081382) = 16.786962 > 5.3$, therefore any field of signature $(7, 1)$ with $|d_K| > 1904081382$ has regulator above 5.3, and if 1904081383 is indeed the minimal discriminant in this signature, this would mean that every field with signature $(7, 1)$ would have regulator bigger than 5.2.

The situation does not improve even if we assume for $P_{9,1}$ the conjectured optimal upper bound equal to $64\,M = 66.786126\ldots$. This is because looking for fields with $R_K \leq 18.1$ is still unfeasible by the classification method since Proposition 1 now gives $\log |d_K| \leq 44.834413\ldots$ and $4g_{7,1}(\exp(-44.834413))$ is again negative. The classification remains then not possible and the only thing we can conclude, by operating like above and assuming the conjectural bound $64\,M$, is that all fields in signature $(7, 1)$ with $|d_K| > 1904081382$ have regulator which is at least 9.2.

It appears then that, at least for the fields with one complex embedding, the method for the classification of number fields with small regulator has reached its limits and new theoretical improvements are needed for the resolution of the problem: in particular, it would be of interest to know if more efficient versions of the lower bound (4) for R_K can be obtained, either by improving the existent result using explicit formulae of Dedekind Zeta functions or by new tools.

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