Princípios da Computação

Numeral systems. Arithmetic.

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Numeral systems

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Representing quantities

- Throughout history, there have been various ways to represent quantities.
- We commonly use the **Indo-Arabic** numeral system, characterised by:
  - **Decimal base**
    - It uses ten symbols: { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 }
  - **Positional notation**
    - The value of a digit depends on its relative position within the number.

Positional notation

- The **value of a digit** depends on the **order** (i.e. the relative position) it occupies in the composition of the number.
  - The first position to the left of the decimal point is order **zero**.
  - Orders **increase to the left** and **decrease to the right**.

243.87

ORDER	*2*	*1*	*0*	*-1*	*-2*
DIGIT	2	4	3	8	7

Positional notation

- The value of a digit is given by its **intrinsic value** multiplied by the **weight of the order**.
- The weight of the order is given by the base raised to the order.

243.87

ORDER	*2*	*1*	*0*	*-1*	*-2*
WEIGHT	10^2	10^1	10^0	10^{-1}	10^{-2}
DIGIT	**2**	**4**	**3**	**8**	**7**
VALUE	200	40	3	0.8	0.07

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Positional notation

- The **value of the number** is the sum of the values of the digits that compose it.
- It is represented in the form of a polynomial of powers of the base.
  - 1724.3
    - $= 1 \times 10^3 + 7 \times 10^2 + 2 \times 10^1 + 4 \times 10^0 + 3 \times 10^{-1}$
    - = 1 thousand, 7 hundreds, 2 tens, 4 units and 3 tenths.

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Other numeral bases?

- Besides the decimal, we can use any other base with at least 2 symbols to represent quantities:
  - **Binary.** 2 symbols: { 0, 1 }
  - **Hexadecimal.** 16 symbols: { 0, ..., 9, A, B, C, D, E, F }
  - **Octal.** 8 symbols: { 0, 1, 2, 3, 4, 5, 6, 7 }
  - or any other base with 2 or more symbols
    - One symbol for **nothing**
    - One symbol for the **unit**

Binary numbering system

- Today's digital computers use binary electronic circuits:
- They only operate in **TWO** valid states: **ON** / **OFF**.
- The two states can be represented by two binary symbols: the **binary digits** aka **bits**.
- Circuits operate simultaneously on groups of binary symbols: the **word**.

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Binary numbering system

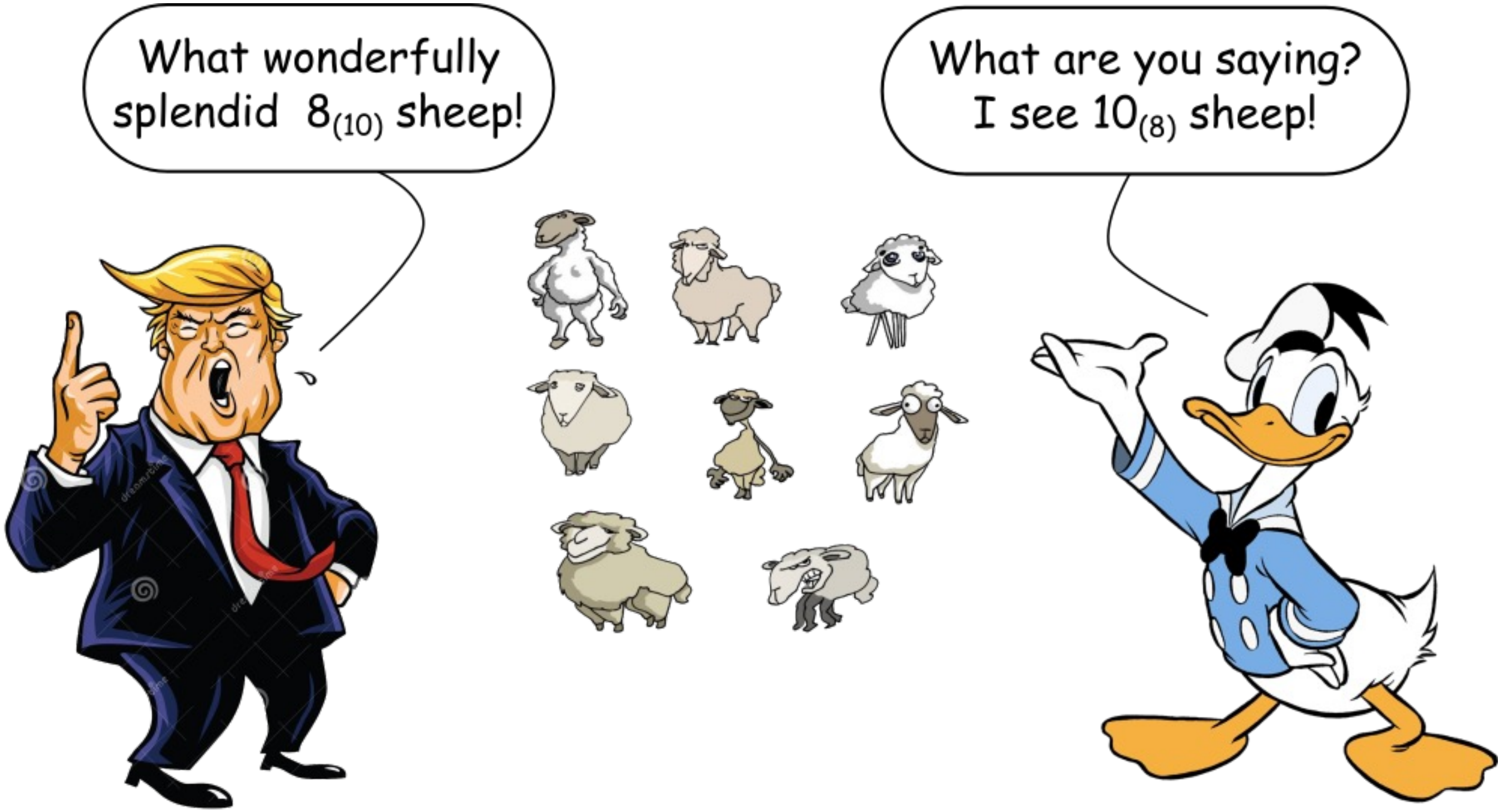
- Any information on a computer is represented by a sequence of bits: numbers, text, sound, video, etc...
- In other words, current digital computers **operate numbers** that are expressed in the binary number system.
- **Natural for computers, strange for humans!**

Conversion between bases

Quantities vs. numbers

- Numbers are graphical abstractions that represent quantities.
- The representation of a quantity depends on the number of symbols available in the numeral base.
- The same quantity is represented in different ways, in different numeral bases.
- It is possible to convert numbers from one numeral base to another.

Quantities vs. numbers



Ten-fingered Donald vs. Eight-fingered Donald

Binary-to-decimal conversion

- **Evaluate the polynomial of powers of the base.**
- *Problem: represent $1110.01_{(2)}$ in decimal form.*
- *Solution:*

$$1 \times 2^{3} + 1 \times 2^{2} + 1 \times 2^{1} + 0 \times 2^{0} + 0 \times 2^{-1} + 1 \times 2^{-2} =$$

$$= 8 + 4 + 2 + 0 + 0 + 0.25 =$$

$$= 14.25_{(10)}$$

- Remember that $1110.01_{(2)}$ and $14.25_{(10)}$ represent the same quantity: only the numeral bases are different.

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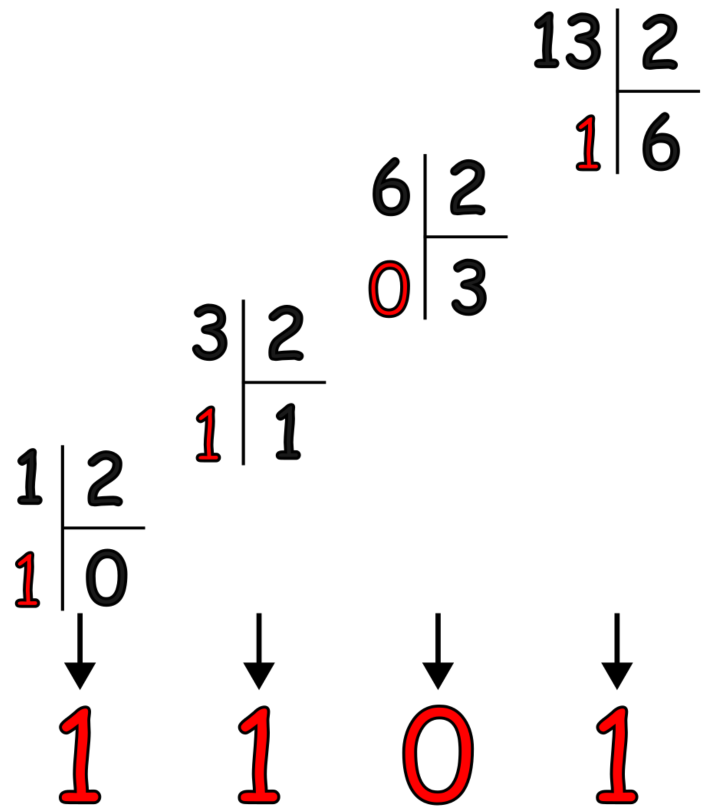
Decimal-to-binary conversion

- The conversion from decimal to binary consists of two distinct processes:
  - One process for the **integer part**: successive divisions by 2.
  - Another process for the **fractional part**: successive multiplications by 2.

Decimal-to-binary conversion (integer part)

- Perform **successive divisions by 2**.
- Record the remainders at each step.
- The binary representation is obtained by arranging the remainders in reverse order.



$$13_{(10)} = 1101_{(2)}$$

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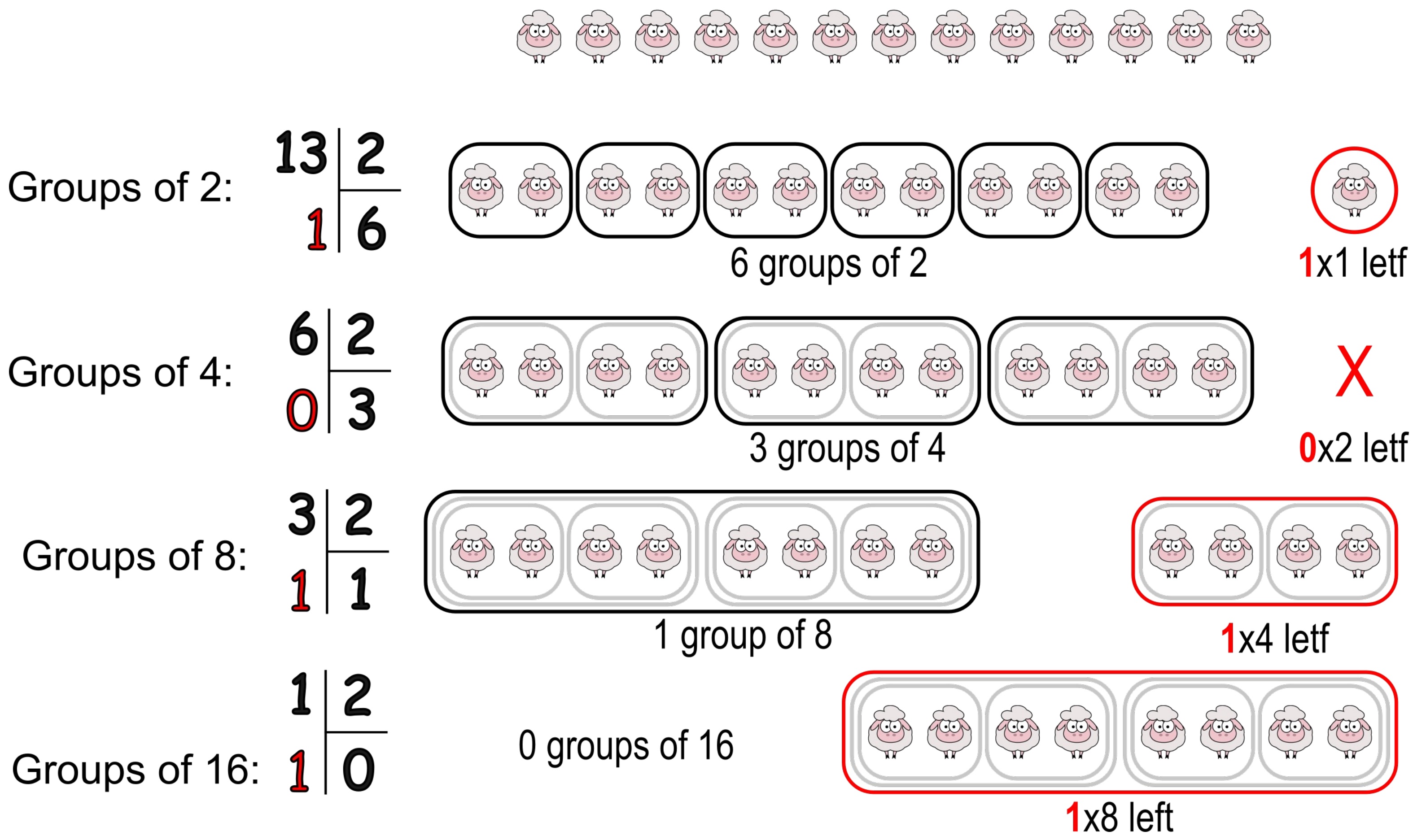
Visualisation of the process



We conclude that $13_{(10)} = 01101_{(2)}$

Decimal-to-binary conversion (fractional part)

- Perform **successive multiplications of the fractional part by 2**.
- Record the integer part (0 or 1) after each multiplication.
- The binary representation is obtained by reading these values in order.

$$0{,}75_{(10)} = \ldots$$

$$\begin{array}{r} 0{,}75 \\ \times 2 \\ \hline 1{,}50 \end{array} \qquad \begin{array}{r} 0{,}50 \\ \times 2 \\ \hline 1{,}00 \end{array} \qquad \begin{array}{r} 0{,}00 \\ \times 2 \\ \hline 0{,}00 \end{array}$$

$$\ldots = 0{,}110_{(2)}$$

Octal and hexadecimal systems

- Since the binary system only has two symbols, the representation of values is not very compact.
- This makes the binary system less easily understood by humans.
- Octal and hexadecimal systems represent values in a way that looks more alike the decimal system.
- They have the advantage of offering **direct conversions to and from binary**.

Binary-Hexadecimal conversion

- The hexadecimal system has 16 symbols:
  - { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F }
- A hexadecimal symbol directly represents 4 bits.
- Conversion is always carried out from the decimal point.

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Binary-Hexadecimal conversion

Bin	Hex	Dec
0000	0	*0*
0001	1	*1*
0010	2	*2*
0011	3	*3*
0100	4	*4*
0101	5	*5*
0110	6	*6*
0111	7	*7*

Bin	Hex	Dec
1000	8	*8*
1001	9	*9*
1010	A	*10*
1011	B	*11*
1100	C	*12*
1101	D	*13*
1110	E	*14*
1111	F	*15*

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Bin-to-Hex example

- To convert the binary number $1011011.101_{(2)}$ to hexadecimal:
  - Fill with zeros to make 4-bit groups
    - Binary number: 0101 1011 . 1010
  - Convert each 4-bit group to hexadecimal:
    - 0101 (binary) = 5 (hexadecimal)
    - 1011 (binary) = B (hexadecimal)
    - 1010 (binary) = A (hexadecimal)
- So, the hexadecimal representation is $5B.A_{(16)}$.

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Binary-Octal conversion

- The octal system has 8 symbols:
  - { 0, 1, 2, 3, 4, 5, 6, 7 }
- An octal symbol directly represents 3 bits.
- Conversion is always carried out from the decimal point.

Binary-Octal conversion

Bin	Oct	Dec
000	0	*0*
001	1	*1*
010	2	*2*
011	3	*3*
100	4	*4*
101	5	*5*
110	6	*6*
111	7	*7*

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Bin-to-Oct example

- To convert the binary number 1011011.101 to octal:
  - Fill with zeros to make 3-bit groups:
    - Binary number: 001 011 011 . 101
  - Convert each 3-bit group to octal:
    - 001 (binary) = 1 (octal)
    - 011 (binary) = 3 (octal)
    - 011 (binary) = 3 (octal)
    - 101 (binary) = 5 (octal)
- So, the octal representation is $133.5_{(8)}$.

Conversions (summary)

- Decimal to Base N
  - Integer part: Successive divisions by N
  - Fractional part: Successive multiplications by N
- Base N to Decimal
  - Evaluate the polynomial using the powers of N

Conversions (summary)

- Binary to Hexadecimal/Octal
  - Group 3 (for octal) or 4 (for hexadecimal) bits and convert each group directly to a symbol.
- Hexadecimal/Octal to Binary
  - Convert each symbol directly to 3 (for octal) or 4 (for hexadecimal) bits.

Arithmetic

Arithmetic vs. numeral systems

- The rules of arithmetic are universal, i.e., they do not depend on the numerical base used.
- Therefore, arithmetic operations in binary are carried out in the same way as in decimal.

Arithmetic operations in binary

- Although the rules are the same, let's review (with examples) the following operations:
  - Addition
  - Subtraction
  - Multiplication
  - Division

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Addition

- The terms are aligned by the decimal point.
- The sums are performed from the rightmost column towards the leftmost column.
- When the result of adding a column has more than one digit, the rightmost digit stays, and the remaining digits are carried over to the next column.

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Addition

$$
\begin{array}{rccc}
 & 1 & 1 & 1 \\
+ & 0 & 1 & 1 \\
\hline
 & & & ?
\end{array}
$$

Addition

$$\begin{array}{rccc} & & +1 & \\ & 1 & 1 & 1 \\ + & 0 & 1 & 1 \\ \hline & & & 0 \end{array}$$

Carry

Result of the column

$1 + 1 = 10$ (i.e. 2)

Addition

$$\begin{array}{rccc} & +1 & +1 & \\ & 1 & 1 & 1 \\ + & 0 & 1 & 1 \\ \hline & & 1 & 0 \end{array}$$

$1 + 1 + 1 = 11$ (i.e. 3)

Addition

$$\begin{array}{rcccc} & & +1 & & \\ & & 1 & 1 & 1 \\ + & & 0 & 1 & 1 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

1 + 0 + 1 = 10 (i.e. 3)

Subtraction

- The subtraction algorithm functions correctly only when the minuend (top number) has a greater absolute value than the subtrahend (bottom number).
- Numbers are aligned at the decimal point.
- Subtraction is performed from the rightmost column to the leftmost column.
- If the minuend digit is smaller than the subtrahend digit, a unit is borrowed from the next column.

Subtraction

$$
\begin{array}{r}
1\ 1\ 0\ 1 \\
-\ 1\ 0\ 1\ 0 \\
\hline
?
\end{array}
$$

Subtraction

$$\begin{array}{r} 1\ 1\ 0\ 1 \\ -\ 1\ 0\ 1\ 0 \\ \hline 1 \end{array}$$

$1 - 0 = 1$

Subtraction

$$\begin{array}{rcccc} & 1 & 1 & 0 & 1 \\ - & 1 & 0^{+1} & 1 & 0 \\ \hline & & & 1 & 1 \end{array}$$

Borrow

$\mathbf{1}0 - 1 = 1$

Subtraction

$$
\begin{array}{rcccc}
 & 1 & 1 & 0 & 1 \\
- & 1 & 0^{+1} & 1 & 0 \\
\hline
 & & 0 & 1 & 1
\end{array}
$$

$$1 - (0 + 1) = 0$$

Subtraction

$$
\begin{array}{r}
1\ 1\ 0\ 1 \\
-\ 1\ 0\ 1\ 0 \\
\hline
0\ 0\ 1\ 1
\end{array}
$$

$1 - 1 = 0$

Multiplication

- Multiplication requires knowing the times table for the base being used.
  - In binary, the times table is extremely simple!
- Numbers are aligned to the right, not by the decimal point.
- First, multiply the top factor by each digit of the bottom factor.
- Then, sum the partial results to get the final product.

Multiplication

$$
\begin{array}{r}
1\ 1\ 0\ 1 \\
\times\ \ 1\ 1\ 0 \\
\hline
?
\end{array}
$$

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Multiplication

First step: multiply...

$$
\begin{array}{r}
1\ 1\ 0\ 1 \\
\times\ \ 1\ 1\ 0 \\
\hline
0\ 0\ 0\ 0 \\
1\ 1\ 0\ 1\ 0 \\
+\ 1\ 1\ 0\ 1\ 0\ 0 \\
\hline
\end{array}
$$

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Multiplication

Second step: sum.

			1	1	0	1
			x	1	1	0
			0	0	0	0
		1	1	0	1	0
+	1	1	0	1	0	0
1	0	0	1	1	1	0

Division

- Division also requires knowing the times table for the base being used.
  - In binary, the times table is incredibly simple!
- At each step, you determine how many times the divisor fits into the current part of the dividend.
- In binary, the only options are
  - **0** (it doesn't fit), or
  - **1** (it fits).

Division

$$
\begin{array}{r|l}
1\ 0\ 0\ 0\ 1 & 1\ 1 \\
\hline
 & \quad ?
\end{array}
$$

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Division

$$
\begin{array}{rrrrr|rr}
1 & 0 & 0 & 0 & 1 & 1 & 1 \\ \cline{6-7}
- & 1 & 1 & & & 1 & \\ \cline{1-3}
0 & 0 & 1 & & & &
\end{array}
$$

Division

1	0	0	0	1		1	1
-	1	1				1	0
0	0	1	0				

Division

$$\begin{array}{ccccc|ccc} 1&0&0&0&1&1&1& \\ \cline{6-8} -&1&1&&\downarrow&1&0&1\\ \cline{1-3} 0&0&1&0&1&&&\\ &&-&1&1&&&\\ \cline{3-5} &&0&1&0&&&\end{array}$$