isep Instituto Superior de Engenharia do Porto

ALGAN	Exam 1th part	
Curso. LEI	Data: 2024 – 01 – 16	75 minutes

Name: ______________________ Number: __________ Class: ______

Repeatable component of the assessment. You must indicate your option: maintains the evaluation obtained in the test or intends to be evaluated in exam.

I declare that ______________________________________.

Part A (15 points)

M 1. **(2+2 val.)** Consider the following matrices

$$\mathbf{A} = \begin{bmatrix} -2 & -2 & 0 \\ -1 & 1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \quad \text{e } \mathbf{J} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

a) Use Gauss-Jordan's algorithm to compute $\mathbf{A}^{-1}$. ✓

b) Find, if possible, a matrix $\mathbf{B}$ that satisfies the relation $\mathbf{ABA^{-1} = J}$. How? / Tip 2

✱

m 2. **(3 val.)** Consider the matrice $\mathbf{A} = \begin{bmatrix} a & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & -1 & 2 \\ 0 & 0 & a & 0 & 0 \\ 1 & a-2 & 1 & 1 & -2 \\ 1 & 1 & 1 & -1 & -2 \end{bmatrix}$, where a is a real parameter.

If det ≠ 0 → det

For which values of the parameter a the matrix $\mathbf{A}$ is invertible?

✓ SLE 3. **(2,5+1,5 val.)** Consider the following system of linear equations (SLE), $\begin{cases} x+y+z=1 \\ -x+ay+az=b \\ x-ay+bz=a \end{cases}$ Where a e b are real parameters. Slide 12/30

2 a) Classify the SLE in function of the parameters a e b. → How? / Tip

b) Let $a=-1$ e $b=1$. Find the solution set of the SLE. → echelon form How? / Tip

✓ SLE 4. **(1+2+1 val.)** Consider the following system of linear equations, written on matricial form,

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 3 & 1 & 1 & 3 \\ 4 & -4 & 10 & 2 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ 10 \\ 28 \end{bmatrix}.$$

2 changing columns

a) Find an reduced echelon form of the completed matrix (without permute of columns).

b) Find the solution set of the SLE. →

c) Find, if possible, one solution $(s_1, s_2, s_3, s_4, s_5, s_6)$ such that: $s_2 = 1$, $s_6 = 1$ and $s_1+s_2+s_3+s_4+s_5+s_6 = 8$.

2

Part B (5 points)

To each question corresponds a unique right answer. The answers must be indicated on the following table with one option (A,B,C or D). Each correct answer worth **1 point** and each wrong resposta worth **-1/3 points.**

Questão	5	6	7	8	9	SR	E	C	Total
Respostas									

M 5. consider a matrix $\mathbf{A} \in \mathcal{M}_n$, such satisfies $\mathbf{A}^6 = \mathbf{I}$. Then we can conclude: 2

A. $\mathbf{A}^{-1} = \mathbf{A}^3$; B. $\mathbf{A}^{-1} = \mathbf{A}^2$; C. $\mathbf{A}^{-1} = \mathbf{A}^5$; D. We cannot conclude any of the above.

6. Consider the following matrices,

2

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \frac{1}{2}b & \frac{1}{2}e & \frac{1}{4}h \\ 2c & 2f & i \\ 2a & 2d & g \end{bmatrix}$$

Properties

Then we can conclude:

A. $\det(\mathbf{A}) = -\det(\mathbf{B})$; B. $\det(\mathbf{A}) = \det(\mathbf{B})$; C. $\det(\mathbf{A}) = -2\det(\mathbf{B})$; D. None of the above options are correct.

M 7. Consider a matrix $\mathbf{A} \in \mathcal{M}_3$. Suposing that $\operatorname{adj}(\mathbf{A}) = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -2 \\ -3 & 4 & 1 \end{bmatrix}$ we have:

2

A. $\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ -\frac{3}{5} & \frac{4}{5} & \frac{1}{5} \end{bmatrix}$ B. $\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{25} & -\frac{1}{25} & \frac{1}{25} \\ \frac{1}{25} & \frac{2}{25} & -\frac{2}{25} \\ -\frac{3}{25} & \frac{4}{25} & \frac{1}{25} \end{bmatrix}$ C. $\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{25} & \frac{1}{25} & -\frac{3}{25} \\ -\frac{1}{25} & \frac{2}{25} & \frac{4}{25} \\ \frac{1}{25} & -\frac{2}{25} & \frac{1}{25} \end{bmatrix}$.

D. None of the above options are correct.

SLE 8. Let $\mathbf{Ax} = \mathbf{b}$, be the matricial form of a non-homogeneous system of linear equations . If $\mathbf{s}$ e $\mathbf{s}'$ are solutions of $\mathbf{Ax} = \mathbf{b}$, $\mathbf{s} \neq \mathbf{s}'$ we can conclude:

2

A. $\mathbf{s} + \alpha\mathbf{s}'$ is a solution of the associated homogeneous system for all $\alpha \in \mathbb{R}$;

B. $\mathbf{Ax} = \mathbf{b}$ is a Cramer's system;

C. $\mathbf{s} + \alpha(\mathbf{s}' - \mathbf{s})$ is solution of $\mathbf{Ax} = \mathbf{b}$ for all $\alpha \in \mathbb{R}$;

D. We cannot conclude any of the above.

M 9. Consider the following matricial equation $[\mathbf{A}(\mathbf{B} + \mathbf{X})]^T = \mathbf{I}$, where all matrices are square matrices, with same size and invertibles. Than we have:

2

A. $\mathbf{X} = \mathbf{B}^T - (\mathbf{BA})^T$;

B. $\mathbf{X} = \mathbf{B}^T - (\mathbf{AB})^T$;

C. $\mathbf{X} = \mathbf{A}^{-1} - \mathbf{B}$;

D. None of the above options are correct.

isep Instituto Superior de Engenharia do Porto

ALGAN	Exam 2nd part	
Curso. LEI	Date: 2024 − 01 − 16	75 minutos

Name: ______________________ Number: __________ Class: ______

Part A (15 points)

1. (1,5+1+1,5 val.)

   a) Check if the set $S = \{(x, y, z, t) \in \mathbb{R}^4 \; : \; xyzt = 0\}$ is a linear subspace of $\mathbb{R}^4$ (with usual operations $\oplus$ e $\odot$).

   b) Consider the set $\mathbb{R}^4$ with the standard addition $\oplus$ and with the non standard operation $\odot$ defined by,

   $$\alpha \odot (x, y, z, t) = (\alpha x, y, z, \alpha t), \quad \forall \alpha \in \mathbb{R}, \forall (x, y, z, t) \in \mathbb{R}^4.$$

   i) Check if the axiom M_1 : is satisfied (see list of axioms at the end).

   ii) Indicate, if any, an axiom that is not satisfied (justify your answer).

2. (1,5 + 1,5 val.) Consider the linear space, $\mathbb{P}_3[x]$, of the polynomials with real coefficients, with degree up to 3 with standard operations.

   a) Find a basis for $\mathcal{F}$, where $\mathcal{F} = \{ax^3 + bx^2 + cx + d \in \mathbb{P}_3[x] \; : \; a + c = 0 \; \wedge \; a + b - c - 2d = 0\}$.

   b) Consider the linear transformation$\mathbf{T} : \mathbb{P}_3[x] \longrightarrow \mathbb{P}_3[x]$, where

   $$\mathbf{T}\left(ax^3 + bx^2 + cx + d\right) = (a + c + d)x^3 + (d - c)x^2 + (a + b)x + (2a + b + 2d).$$

   Check if $\mathbf{T}$ is injective (justify your answer).

3. (1+3 val.) Consider the linear transformation $\mathbf{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by,

   $$\mathbf{T}(x, y, z) = (x - 2y - z \; , \; -3x + 6y - 3z \; , \; -10x + 20y - 8z)$$

   a) Find the matrix $\mathbf{M}_{B_c,B_c}(\mathbf{T})$, where B_c is the standard basis of $\mathbb{R}^3$.

   b) Compute the eigenvalues of $\mathbf{T}$.

4. (2+2 val.)

   a) Consider the following basis of $\mathbb{R}^2$: $B_1 = (\,(1, -1), (1, 1)\,)$ e $B_2 = (\,(1, -2), (-1, 1)\,)$. Compute the matrix, $M_{B1,B2}$.

   b) Find the expression of $\mathbf{T}(x, y)$ where $\mathbf{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation that fullfill the conditions:

   (i) $\mathbf{T}$ has eigenvalues $\lambda_1 = 1$ e $\lambda_2 = -1$;

   (ii) $E_{\lambda_1} = \{(x, y) \; : \; x + 2y = 0\}$ e $E_{\lambda_2} = \{(x, y) \; : \; x - y = 0\}$

Part B (5 points)

To each question corresponds a unique right answer. The answers must be indicated on the following table with one option (A,B,C or D). Each correct answer worth **1 point** and each wrong resposta worth **-1/3 points.**

Question	5	6	7	8	9	SR	E	C	Total
Answer									

5. Let $\mathcal{E}$, be a vector space, $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ one basis of $\mathcal{E}$. Then we can conclude:

   A. For all $\mathbf{v} \in \mathcal{E}$ the list $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v})$ is linearly dependent.; B. Existe $\mathbf{v} \in \mathcal{E}$ tal que a lista $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v})$ é linearly independent.; C. $\mathcal{B}_1 = (\mathbf{u}_1, \mathbf{u}_2)$ is linearly dependent.; D. We cannot conclude any of the above.

6. Let $\mathcal{E}$ a vetor space with finite dimension and $\mathbf{T} : \mathcal{E} \longrightarrow \mathcal{E}$ a linear transformation. Then we can conclude:

   A. If $\mathbf{T}$ é bijective then $\lambda = 0$ is not an eigenvalue of $\mathbf{T}$; B. If $\dim(\mathrm{Im}(\mathbf{T})) + \dim(\mathrm{Ker}(\mathbf{T})) = \dim(\mathcal{E})$ then $\mathbf{T}$ é bijective; C. If $\lambda = 0$ is an eigenvalue of $\mathbf{T}$ then $\dim(\mathrm{Ker}(\mathbf{T})) = 0$; D. None of the above options is correct.

7. Let $\mathbf{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $\mathbf{T}((x, y, z)) = (z, x, y)$ and let B_c the standard baiss of $\mathbb{R}^3$. Than we have:

   A. $M_{B_c,B_c}(\mathbf{T} \circ \mathbf{T} \circ \mathbf{T}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; B. $M_{B_c,B_c}(\mathbf{T} \circ \mathbf{T} \circ \mathbf{T}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$; C. $M_{B_c,B_c}(\mathbf{T} \circ \mathbf{T} \circ \mathbf{T}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;
   D. None of the above options is correct.

8. Let $\mathbf{A} = \begin{bmatrix} a & -1 & 0 \\ -1 & a & 0 \\ 0 & -1 & a \end{bmatrix}$ the matrix that represents a linear transformation relatively to the standard basis B_c of $\mathbb{R}^3$ ($\mathbf{A} = \mathbf{M}_{B_c,B_c}(\mathbf{T})$). Then we have:

   A. (1,-1,1) is an eigenvector associated to the eigenvalue a;

   B. (1,-1,1) is an eigenvector associated to the eigenvalue $a - 1$;

   C. (1,-1,1) is an eigenvector associated to the eigenvalue $a + 1$;

   D. None of the above options is correct.

9. Consider $\mathcal{P} = \{\mathbf{X} \in \mathcal{M}_2 \ : \ \mathbf{XA} = \mathbf{AX}\}$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then we have:

   A. $\mathcal{P} \not\prec \mathcal{M}_2$;

   B. $\mathcal{P} \prec \mathcal{M}_2$ e $\dim(\mathrm{P}) = 1$;

   C. $\mathcal{P} \prec \mathcal{M}_2$ e $\dim(\mathrm{P}) = 3$;

   D. None of the above options is correct.

Axioms (of a vetor space)

A_1 : $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{E}$, $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$.
A_2 : $\exists \mathbf{o}_\oplus \in \mathcal{E}$, $\forall \mathbf{u} \in \mathcal{E}$, $\mathbf{u} \oplus \mathbf{o}_\oplus = \mathbf{o}_\oplus \oplus \mathbf{u} = \mathbf{u}$.
A_3 : $\forall \mathbf{u} \in \mathcal{E}$, $\exists \mathbf{u}' \in \mathcal{E}$, $\mathbf{u} \oplus \mathbf{u}' = \mathbf{u}' \oplus \mathbf{u} = \mathbf{o}_\oplus$.
A_4 : $\forall \mathbf{u}, \mathbf{v} \in \mathcal{E}$, $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$.

M_1 : $\forall \alpha, \beta \in \mathbb{R}$, $\forall \mathbf{u} \in \mathcal{E}$, $(\alpha + \beta) \odot \mathbf{u} = (\alpha \odot \mathbf{u}) \oplus (\beta \odot \mathbf{u})$.
M_2 : $\forall \alpha \in \mathbb{R}$, $\forall \mathbf{u}, \mathbf{v} \in \mathcal{E}$, $\alpha \odot (\mathbf{u} \oplus \mathbf{v}) = (\alpha \odot \mathbf{u}) \oplus (\alpha \odot \mathbf{v})$.
M_3 : $\forall \alpha, \beta \in \mathbb{R}$, $\forall \mathbf{u} \in \mathcal{E}$, $(\alpha \bullet \beta) \odot \mathbf{u} = \alpha \odot (\beta \odot \mathbf{u})$.
M_4 : $\forall \mathbf{u} \in \mathcal{E}$, $1 \odot \mathbf{u} = \mathbf{u}$.