MECHANICAL VIBRATIONS

MATLAB TEAMWORK PROJECT

A.A. 2020/21

COMFORT WITH ROLL MOTION

Group D

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INTRODUCTION

In this project, we will analyze the vibrations that will be induced in a vehicle during a test run on a straight track fitted out with two rows of triangular-ish-shaped bumps (in our case trapezoidal) with a phase displacement between them.

To study this problem we will use a multiple-degrees-of-freedom model to take into account the response of the suspensions and also of the cabin (sprung and unsprung masses).

The test will be conducted twice at constant acceleration starting from a predetermined initial velocity. The first time with an anti-roll bar fitted in the front axle and the second time without it.

The main objectives of the project are:

- Derive the equation of motion of the system and determine the natural frequencies and the mode shapes.
- Determine and plot the full transient response of the masses by which the system is composed.
- Compare the results obtained using the anti-roll bar with the ones obtained without using it.

The data used for the masses, the moments of inertia, the stiffnesses, and the damping coefficients are based on the BMW 530i F11 (see Table 3).

THEORY

Model development

For this project, we will use a 7 DOF tridimensional model to simplify the real vehicle's behavior. Despite its simplicity, this model is quite used in the literature [1].

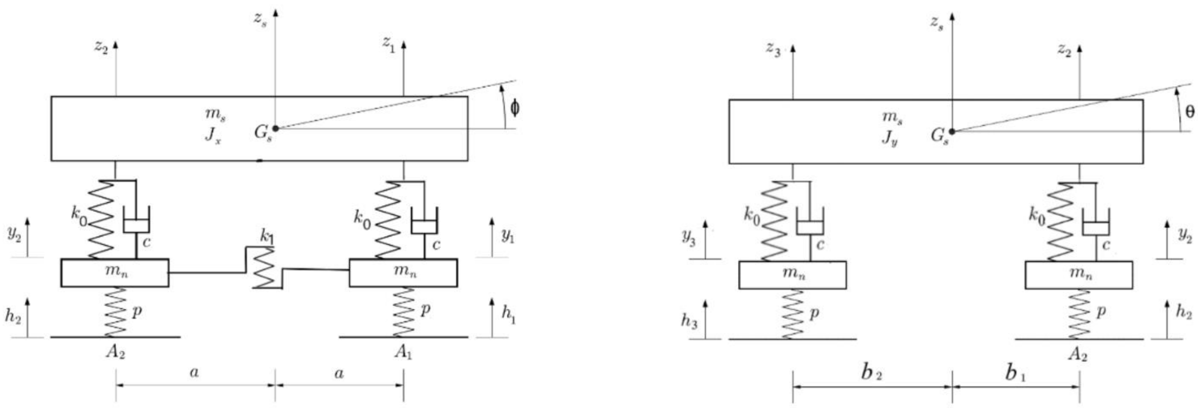

FIGURE 1: FRONT AND RIGHT-SIDE VIEW OF THE SUSPENSION MODEL

The hypothesis on which this model is based are the following:

- Wheels-road contact: modelled as a perfectly elastic vertical spring.
- Wheels: rigid body with only 1 DOF (vertical displacement).
- Springs and dampers: independent for each wheel, vertical orientation.
- Car body: rigid body with 3 DOF (vertical displacement, roll, pitch).
- Front anti-roll bar: simplified model, acts like a vertical spring that connects the two wheels of the front axle. No connection to the car body.

- Rear anti-roll bar: absent.
- Small displacements: rotation angles can be linearized.

The parameters and the variables of the systems are in Tab. 1:

TABLE 1: SYSTEM PARAMETERS AND VARIABLES

$z_s(t)$	Car body's COG vertical position	k_0	Springs stiffness
$\phi(t)$	Roll angle	k_1	Front anti-roll bar stiffness
$\theta(t)$	Pitch angle	c	Damping coefficient
m_s	Sprung mass	m_n	Unsprung mass per wheel
J_x	Inertia along longitudinal axis	a	Wheels-COG transversal distance
J_y	Inertia along transversal axis	b_1	Front wheels-COG distance
$z_i(t)$	Car body's suspension joints	b_2	Rear wheels-COG distance
$h_i(t)$	Road vertical position	p	Tyres stiffness
$y_i(t)$	Wheel vertical position		

To find the equation of motion the forces and the torques acting on the bodies have been calculated. The system was analyzed in function of the independent variables, keeping in mind that:

$$\begin{cases} z_1 = z_s + a\,sin\phi + b_1\,sin\theta \approx z_s + a\phi + b_1\theta \\ z_2 = z_s - a\,sin\phi + b_1\,sin\theta \approx z_s - a\phi + b_1\theta \\ z_3 = z_s - a\,sin\phi - b_2\,sin\theta \approx z_s - a\phi - b_2\theta \\ z_4 = z_s + a\,sin\phi - b_2\,sin\theta \approx z_s + a\phi - b_2\theta \end{cases} \tag{1}$$

The equations are 7, one for each DOF, and by putting them on a system we obtain the following matrices (for the case without anti-roll bar it is enough to put k_1=0 to obtain the corresponding EOM):

$$\boldsymbol{M} = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & J_x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_n & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_n \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 4c & 0 & 2c(b_1 - b_2) & -c & -c & -c & -c \\ 0 & 4ca^2 & 0 & -ca & ca & ca & -ca \\ 2c(b_1 - b_2) & 0 & 2c(b_1^2 + b_2^2) & -cb_1 & -cb_1 & cb_2 & cb_2 \\ -c & -ca & -cb_1 & c & 0 & 0 & 0 \\ -c & ca & -cb_1 & 0 & c & 0 & 0 \\ -c & ca & cb_2 & 0 & 0 & c & 0 \\ -c & -ca & cb_2 & 0 & 0 & 0 & c \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 4k_0 & 0 & 2k_0(b_1 - b_2) & -k_0 & -k_0 & -k_0 & -k_0 \\ 0 & 4k_0a^2 & 0 & -k_0a & k_0a & k_0a & -k_0a \\ 2k_0(b_1 - b_2) & 0 & 2k_0(b_1^2 + b_2^2) & -k_0b_1 & -k_0b_1 & k_0b_2 & k_0b_2 \\ -k_0 & -k_0a & -k_0b_1 & k_0 + k_1 + p & -k_1 & 0 & 0 \\ -k_0 & k_0a & -k_0b_1 & -k_1 & k_0 + k_1 + p & 0 & 0 \\ -k_0 & k_0a & k_0b_2 & 0 & 0 & k_0 + p & 0 \\ -k_0 & -k_0a & k_0b_2 & 0 & 0 & 0 & k_0 + p \end{bmatrix}$$

$$with \quad \boldsymbol{q} = \begin{vmatrix} z_s \\ \phi \\ \theta \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{vmatrix} \qquad \boldsymbol{f} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ ph_1 \\ ph_2 \\ ph_3 \\ ph_4 \end{vmatrix}$$

All these matrices create a 7-dimensional second order ODE:

$$\boldsymbol{M\ddot{q}} + \boldsymbol{C\dot{q}} + \boldsymbol{Kq} = \boldsymbol{f} \tag{2}$$

Modal analysis and resolution

The simplest modal analysis that can be done is the one without damping:

$$\boldsymbol{M\ddot{q}} + \boldsymbol{Kq} = \boldsymbol{0} \tag{3}$$

So the 7 eigenvalues λ_i, and the 7 corresponding eigenvectors $\boldsymbol{u}_i$ of the system can be found:

$$(-\lambda_i \boldsymbol{M} + \boldsymbol{K})\boldsymbol{u_i} = \boldsymbol{0} \tag{4}$$

The modal solution of each coordinate can be written as:

$$\boldsymbol{q_i}(t) = \boldsymbol{u_i} \cos{(\omega_i t)} \tag{5}$$

With this expression the natural frequencies and the relative displacements of each mass are known for all the modes of vibration.

With damping we can write:

$$\begin{cases} \boldsymbol{M\dot{q}} - \boldsymbol{M\dot{q}} = \boldsymbol{0} \\ \boldsymbol{M\ddot{q}} + \boldsymbol{C\dot{q}} + \boldsymbol{Kq} = \boldsymbol{f} \end{cases} \quad hence: \tag{6}$$

$$\boldsymbol{A\dot{y}} + \boldsymbol{By} = \boldsymbol{g} \quad where \quad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{M} \\ \boldsymbol{M} & \boldsymbol{C} \end{bmatrix}; \boldsymbol{B} = \begin{bmatrix} -\boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{K} \end{bmatrix}; \boldsymbol{y} = \begin{pmatrix} \boldsymbol{\dot{q}} \\ \boldsymbol{q} \end{pmatrix}; \boldsymbol{g} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{f} \end{pmatrix} \tag{7}$$

So calculating complex eigenvalues λ_i and correspondent eigenvectors $\boldsymbol{v}_i$ both of them are 7 couples of complex conjugate numbers:

$$(\lambda_i \boldsymbol{A} + \boldsymbol{B})\boldsymbol{v_i} = \boldsymbol{0} \tag{8}$$

Filling the matrix $\widetilde{\boldsymbol{V}} = [\widetilde{\boldsymbol{v_1}} \quad \dots \quad \widetilde{\boldsymbol{v_{2N}}}]$, we can diagonalize both the matrixes A, B:

$$\widetilde{\boldsymbol{V}}^T \boldsymbol{A} \widetilde{\boldsymbol{V}} = diag(a_i) \tag{9}$$

$$\widetilde{\boldsymbol{V}}^T \boldsymbol{B} \widetilde{\boldsymbol{V}} = diag(-\lambda_i a_i) \tag{10}$$

Consider now that the matrix $\widetilde{\boldsymbol{V}}$ is normalized obtaining $\boldsymbol{V}$ the modal matrix containing the modes of vibration, simply through $v_i = \frac{\tilde{v}_i}{a_i}$, it follows:

$$\boldsymbol{V}^T\boldsymbol{A}\boldsymbol{V} = \boldsymbol{I} \quad \boldsymbol{V}^T\boldsymbol{B}\boldsymbol{V} = diag(-\lambda_i) = \boldsymbol{\Lambda} \tag{11}$$

Projecting the solution **y** in the modal space:

$$\mathbf{y} = \mathbf{V}\boldsymbol{\eta} \tag{12}$$

Hence we can obtain a decoupled system of equations:

$$\boldsymbol{V}^T\boldsymbol{A}\boldsymbol{V}\dot{\boldsymbol{\eta}} + \boldsymbol{V}^T\boldsymbol{B}\boldsymbol{V}\boldsymbol{\eta} = \boldsymbol{V}^T\boldsymbol{g} = \boldsymbol{p} \tag{13}$$

Where, component by component:

$$\dot{\eta}_i = \lambda_i\eta_i + p_i \tag{14}$$

Solving, through convolution integral, we obtain:

$$\eta_i = \int_0^t e^{\lambda_i(t-\tau)}\, p_i(\tau)d\tau \tag{15}$$

From this we can calculate $\boldsymbol{y}$, thus $\boldsymbol{q}$.

The application of the modal analysis to an unforced system to evaluate the free response, once the eigenvalues and eigenvectors have been calculated (8), the equation (12) becomes:

$$\boldsymbol{y}_i = \boldsymbol{v}_i e^{\lambda_i t} + \boldsymbol{v}_i{}^*\, e^{\lambda_i t} = 2Re\left(\boldsymbol{v}_i e^{\lambda_i t}\right) = 2e^{\lambda_i^R t}(\boldsymbol{v}_i^R cos\ \lambda_i^I t - \boldsymbol{v}_i^I sin\ \lambda_i^I t) \tag{16}$$

NUMERICAL IMPLEMENTATION

Road surface modelling

The road surface is composed of a series of alternated triangular bumps, with a phase shift between right and left ones. These are the chosen parameters to define the track:

TABLE 2 –ROAD SURFACE CHARACTERISTICS

Height	150 mm	**Number of bumps**	10
Slope	45°	**Distance of the first bump**	2968 mm
Top flat surface length	50 mm	**Length of test track**	88 m
Pitch between two bumps	3318 mm		

The creation of each h_i is trivial: an appropriate phase-shift is added to the road profile. The vehicle is assumed as accelerating from zero with a constant acceleration of 1 m/s^2. The surface road, defined as a function of the space, has been shifted to a time function through a variable change:

$$t = \sqrt{\frac{2\,x}{a}} \tag{16}$$

Natural frequencies and modes of vibration

The undamped system has been considered to obtain the natural frequencies and modes of vibration. The eigenvalue problem has been solved simply using the MATLAB command “eig”. To find the damped frequencies and the behaviour of the system under damping the Duncan approach has been used to find the complex eigenvalues and eigenvectors.

Calculating the response

To obtain the system response to the excitation two methods have been implemented: the former based on the Duncan approach and the convolution integral, the latter on the ode45 direct simulation. The results have been compared for validation purposes.

In the case of ode45 code to prevent error propagation and a long computational time due to the discontinuity of the road surface as it is defined, every bump has been approximated by a sine wave that has the same length and height as the original bump through a “for” loop that analyses the surface vector. This solution increases by far the efficiency of the code despite a little approximation error.

RESULTS

TABLE 3 –VEHICLE CHARACTERISTICS

BMW 530i (F11)

Car total mass	1547 kg [5]	**Longitudinal axis inertia**	486 kg m^2 [4]
Sprung mass	1359 kg [3]	**Unsprung mass (per wheel)**	47 kg [3]
Wheelbase	2968 mm [5]	**Spring stiffness**	30 000 N/m [3]
Vehicle track	1632 mm [5]	**Damping coefficient**	1450 N s/m [3]
Weight distribution (front-rear)	50,9-49,1 % [3]	**Front anti-roll bar stiffness**	44 000 N/m [6]
Transversal axis inertia	2366 kg m^2 [4]	**Tyre stiffness**	310 000 N/m [3]

b1 = 1457 mm, b2 =1511 mm (calculated from wheelbase and weight distribution)

Modes of vibration

TABLE 4 – NATURAL FREQUENCIES IN ASCENDING ORDER

Anti-roll bar	1	2	3	4	5	6	7
With [Hz]	1.4262	1.6063	1.9566	13.5435	13.5440	13.5460	15.1926
Without [Hz]	1.4262	1.6063	1.9466	13.5366	13.5440	13.5460	13.5505

The natural frequencies obtained show that the anti-roll bar produces a variation in the system’s behaviour, and this is evident also in the modes of vibration.

The normalized modal matrices obtained, respectively with and without anti-roll bar are:

$$\boldsymbol{U_{with}} = \begin{bmatrix} -0.0270 & -0.0021 & 7.76\cdot10^{-5} & 8.96\cdot10^{-4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0012 & 0.0453 & -0.0015 \\ -0.0016 & 0.0205 & -7.67\cdot10^{-4} & 5.21\cdot10^{-5} & 0 & 0 & 0 \\ -0.0026 & 0.0025 & 0.0663 & -0.0789 & -0.1031 & 0.0026 & -4.08\cdot10^{-4} \\ -0.0026 & 0.0025 & 0.0663 & -0.0789 & 0.1031 & -0.0026 & 4.08\cdot10^{-4} \\ -0.0022 & -0.0030 & -0.0789 & -0.0664 & 3.23\cdot10^{-4} & -0.0033 & -0.1031 \\ -0.0022 & -0.0030 & -0.0789 & -0.0664 & -3.23\cdot10^{-4} & 0.0033 & 0.1031 \end{bmatrix}$$

$$\boldsymbol{U_{without}} = \begin{bmatrix} -0.0270 & 0.0021 & 7.76\cdot10^{-5} & -8.96\cdot10^{-4} & -7.76\cdot10^{-5} & 0 & 0 \\ 0 & 0 & -0.0453 & 0 & 0 & 0.0021 & 0 \\ -0.0016 & -0.0205 & 0 & -5.21\cdot10^{-5} & 7.67\cdot10^{-4} & 0 & 0 \\ -0.0026 & -0.0025 & -0.0033 & 0.0789 & -0.0663 & -0.0729 & -0.0729 \\ -0.0026 & -0.0025 & 0.0033 & 0.0789 & -0.0663 & 0.0729 & 0.0729 \\ -0.0022 & 0.0030 & -0.0033 & 0.0664 & 0.0789 & 0.0729 & 0.0729 \\ -0.0022 & 0.0030 & -0.0033 & 0.0664 & 0.0789 & -0.0729 & -0.0729 \end{bmatrix}$$

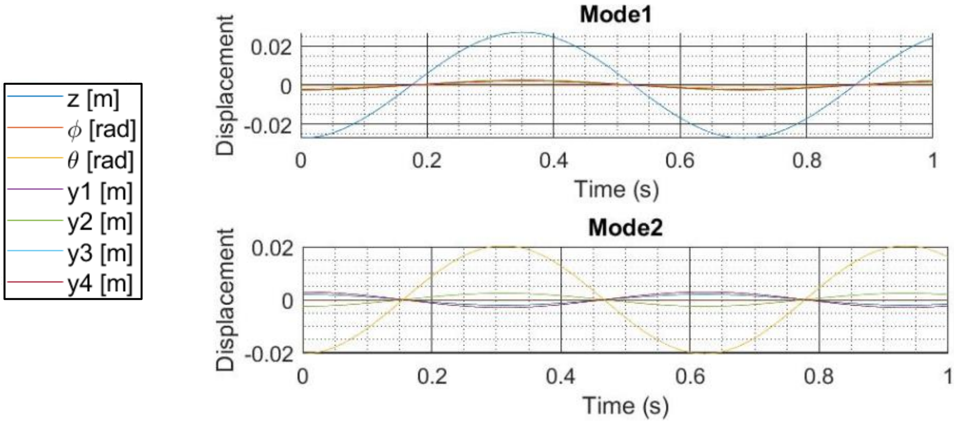

FIGURE 2: EXAMPLES OF MODE SHAPES

The detailed plot of all modes of vibration can be found in the MATLAB code *T1_modes*.

Vehicle response and effect of the anti-roll bar

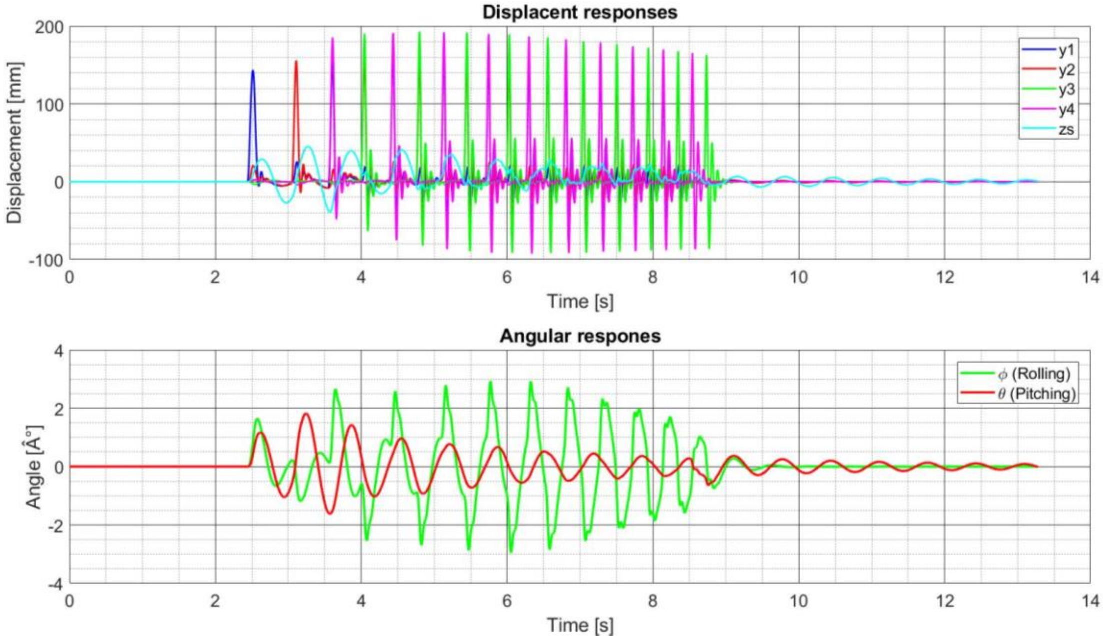

FIGURE 3: FULL TRANSIENT RESPONSE

In the figures below are shown the variables affected by the presence of the anti-roll bar, namely the response of the front wheels and the roll of the cabin. In particular the latter is reduced as expected, while we can see that with the anti-roll bar the amplitude of the displacement is slightly decreased, but the connection between the two front suspensions causes the wheel not engaged in the bump to oscillate as well.

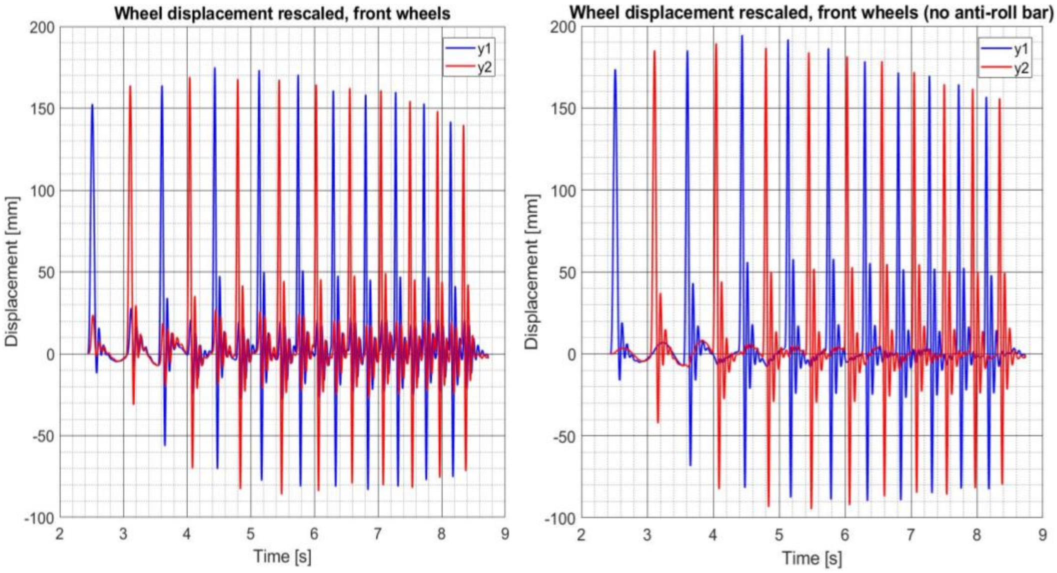

FIGURE 4: WHEEL DISPLACEMENT IN TIME AND EFFECT OF ANTI-ROLL BAR

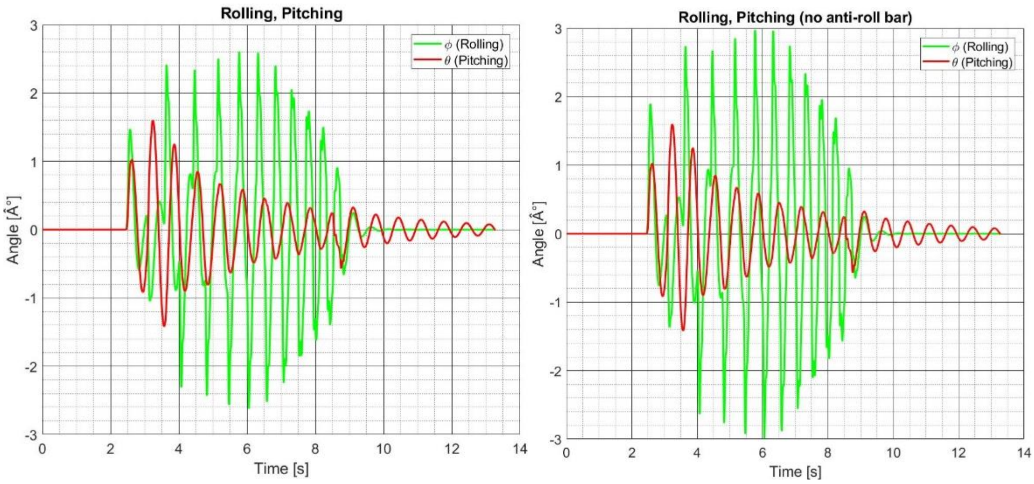

FIGURE 5: ROLLING AND PITCHING IN TIME AND EFFECT OF ANTI-ROLL BAR

It is important to note that the tyres are modelled as simple springs; this is a limitation when the road surface variation is too fast and the wheel, due to its inertia, doesn't follow perfectly the shape of the bumps. In reality, this implicates the detachment of the wheel from the ground, while in this model a traction force between wheel and road is present. The error that is generated is however little and doesn't affect the physical meaning of the model.

CONCLUSION AND FURTHER DEVELOPMENTS

With this work, we have simulated the dynamic response of a vehicle. The modal analysis has been evaluated both for the damped and undamped system, with and without the anti-roll bar. The response of the car allows also to show the effect of the anti-roll bar: its function of reducing the roll is achieved, producing a different behaviour of the front axle. This simulation has been validated through the use of the Duncan approach and direct integration in time.

Despite its simplicity, this model is very used in the literature [1] and offers a good reference in estimating vehicle behavior. Of course, it is possible to make it even more similar to reality, giving a more accurate response. First, usually in vehicles the front and rear springs and dampers have different stiffnesses and coefficients (and even different damping coefficients for compression and extension). The same is for the unsprung masses due to the different brake discs, suspension systems, and sometimes even different wheel sizes. Moreover, the center of gravity of the car body is never perfectly in the middle and changes modifying the car payload (fuel, liquids, passengers, luggage). It is also possible to improve the anti-roll bar modelling: instead of a simple vertical spring that connects the wheels, a torsion bar that connects the axle to the car body is more realistic. Another advanced improvement is the tyre modelling: instead of considering them simply as springs, it is possible to add a damping coefficient and eventually consider that the wheels can detach themselves from the ground and this can change the global response.

REFERENCES

[1] B. Cebolla Bono, "Modelado y Caracterización de Sistemas de Suspensión en Vehículos Automóviles", Universitat Politècnica de València, 2017

[2] Francesco Pellicano, "Mechanical Vibrations", 2020

[3] T.P.J. van der Sandea, B.L.J. Gysenb, I.J.M. Besselinka, J.J.H. Paulidesb, E.A. Lomonovab, H. Nijmeijer, "Robust control of an electromagnetic active suspension system: Simulations and measurements", Eindhoven University of Technology, 2013

[4] Dongchan Lee, "Summary of Typical Vehicle Parameters Measurements", 2018

[5] BMW 5 Series (F10) - Wikipedia

[6] Zhong-Xing Li, Rong-Zhou Xu, Hong Jiang, "Roll Stiffness Optimization for Anti-roll Bar in Interconnected Air Suspension", Jiangsu University, 2016