Coupled Stokes-Darcy Problem

Francesco Egidio Faggion and Martina Silvestri

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Contents

1 Introduction

The "Coupled Stokes-Darcy problem" represents a fundamental mathematical challenge to describe fluid motion across two domains with different physical characteristics. In our case, we consider a free-flow domain (Ω_1 in Fig. 1), where the flow is governed by the Stokes equations, and a porous domain (Ω_2 in Fig. 1), where the flow is described by Darcy's model. These conditions typically occur in environmental and engineering applications, such as surface-water and ground-water interactions.

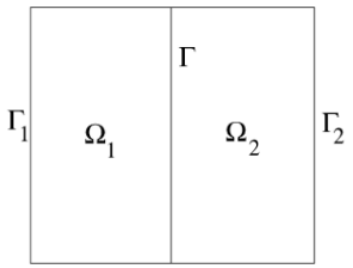

Figure 1. *Domain for the Stokes-Darcy coupled problem. Image taken from J. Urquiza, D. N'dri, A. Garon, and M. Delfour. Coupling Stokes and Darcy equations. Applied Numerical Mathematics, 58(5):525–538, 2008.*

We address the numerical solution of the coupled Stokes-Darcy problem (1) on a subdivided domain, as shown in Fig.1. The interface Γ between the two subdomains requires specific coupling conditions: continuity of normal flux and pressure equilibrium (or normal force balance).

$$\begin{cases} -\mu\Delta u_1 + \nabla p_1 = f_1 & \text{in } \Omega_1, \\ \nabla \cdot u_1 = 0 & \text{in } \Omega_1, \\ \mu u_2 + K\nabla p_2 = 0 & \text{in } \Omega_2, \\ \nabla \cdot u_2 = f_2 & \text{in } \Omega_2, \\ u_1 \cdot n = u_2 \cdot n & \text{on } \Gamma, \\ \mu\nabla u_1 n - p_1 n = -p_2 n & \text{on } \Gamma. \end{cases} \tag{1}$$

The goal of this project is to implement and test a partitioned iterative solver for the coupled Stokes-Darcy problem, focusing on its performance and convergence properties. This approach follows the methodology proposed by Urquiza, N'dri, Garon, and Delfour in *Coupling Stokes and Darcy Equations* (*Applied Numerical Mathematics*, 58(5):525–538, 2008), where they introduce a partitioned iterative solver that treats the two subdomains independently and facilitates the exchange of information at the interface between them.

We will also address computational aspects related to numerical implementation, boundary conditions, and interface management.

2 Stokes Problem

In the following paragraph we'll proceed to unpack Stokes problem singularly, focusing on how the problem is formulated, our approach to obtain the weak formulation of the problem and a brief analysis of the resultant matrix.

2.1 Introduction to Stokes Problem

Stokes problem models the slow, viscous flow of an incompressible fluid where inertial effects are negligible. This is usually exploited to model small duct flow, or biological flows. In the context of the coupled Stokes-Darcy problem, Stokes model is implemented in the free domain Ω_1 to represent the fluid's motion outside of the porous media.

We worked on the classical differential form of the **stationary Stokes** problem in Ω_1 (2), where: u_1 represents the velocity field, p_1 is the pressure, μ is the viscosity of the fluid and f_1 is the external force function.

$$\begin{cases} -\mu\Delta u_1 + \nabla p_1 = f_1 & \text{in } \Omega_1, \\ \nabla \cdot u_1 = 0 & \text{in } \Omega_1. \end{cases} \tag{2}$$

The first equation (2) represents the momentum equation, and the second is the incompressibility condition.

2.2 Mathematical Approach

In order to solve the problem we need to take a variational approach and reduce the problem to its weak formulation:

find $u_1 \in V = \{v \in [H_1(\Omega_1)]^2 : v$ meets Derichlet conditions with datum $= 0\}$ and $p_1 \in Q = L^2(\Omega_1)$, such that, for all test function $v \in V$ and $q \in Q$, we have

$$\begin{cases} a(u_1, v) + b(v, p_1) = F_u(v) \\ b(u_1, q) = 0 \end{cases} \tag{3}$$

where:

- $a(u_1, v) = \mu \int_{\Omega_1} \nabla u_1 \cdot \nabla v \, dx$ is the bilinear form of the viscosity,
- $b(v, p_1) = -\int_{\Omega_1} (\nabla \cdot v) p_1 \, dx$ is the coupling term between velocity and pressure,
- $F_u(v) = \int_{\Omega_1} f_1 \cdot v \, dx - R$ is the source term.

The first equation (3) represents the momentum equation, and the second is the incompressibility condition. R is the so called residual that allows us to impose

the pressure equilibrium boundary condition on Γ, we will provide a more solid explanation later on.

This formulation is coherent with the finite element method and it allows us to use mixed functional spaces where we can use the inf-sup condition to demonstrate the problem's stability.

2.3 Matrix Structure

We want to analyze the structure of the algebraic system associated to the Galerkin approximation of the Stokes problem.
We define:

$$V_h = \Big\{ \mathbf{v}_h \in [C^0(\overline{\Omega}_1)]^2 : \ \mathbf{v}_h|_T \in [P_r(T)]^2 \ \forall T \in \mathcal{T}_h, \ \mathbf{v}_h = \mathbf{0} \text{ su } \Gamma_D \Big\},$$

$$Q_h = \Big\{ q_h \in C^0(\overline{\Omega}_1) : \ q_h|_T \in P_{r-1}(T) \ \forall T \in \mathcal{T}_h \Big\}.$$

Where: $\mathcal{T}_h$ is a triangulation of Ω_1, T is a generic element of $\mathcal{T}_h$ and $P_r(T)$ is the space of polynomials of degree r defined on T.

Denote with $\varphi_j \in V_h$, $\phi_k \in Q_h$, the Lagrangian basis functions of the spaces V_h and Q_h respectively and expand the discrete solutions u_1 and p_1 with respect to such bases:

$$u_h(x) = \sum_{j=1}^{N} u_{1j}\varphi_j(x), \quad p_h(x) = \sum_{k=1}^{M} p_{1k}\phi_k(x). \tag{4}$$

Where N=dim$V$ and M=dim$Q$.

By using basis functions as test functions we obtain the following block system:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} F_u \\ 0 \end{bmatrix} \tag{5}$$

where $A \in R^{N\times N}$ and $B \in R^{M\times N}$ are the matrices related respectively to the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, whose elements are given by $A = [a_{ij}] = [a(\varphi_j, \varphi_i)]$, $B = [b_{km}] = [b(\varphi_m, \phi_k)]$, while U and P are the vectors of the unknowns $U = [u_{1j}]$ and $P = [p_{1j}]$.

The $(N + M) \times (N + M)$ block-symmetric matrix

$$S = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \tag{6}$$

is non-singular if and only if no eigenvalue is null, a property that follows from the inf-sup condition.

3 Darcy's Problem

3.1 Introduction to Darcy's Model

Darcy's model describes fluid flow through a porous media, where the fluid is characterized by high viscosity forces and low velocity. Under these conditions, the fluid macroscopic behavior is described by a linear relation between velocity and pressure gradient. In differential form, on a porous domain Ω_2 (1), Darcy's law is:

$$\mu u_2 = -K\,\nabla p_2 \tag{7}$$

where: μ is the viscosity coefficient, u_2 is the seepage velocity, p_2 is the fluid pressure, K is the permeability constant of the media and ∇p_2 is the pressure gradient.
Combining Darcy's law (7) with the law of conservation of mass for an incompressible fluid ($\nabla \cdot u_2 = f_2$) we obtain stationary Darcy's model:

$$\begin{cases} \mu u_2 + K\nabla p_2 = 0 & \text{in } \Omega_2, \\ \nabla \cdot u_2 = f_2 & \text{in } \Omega_2. \end{cases} \tag{8}$$

where f_2 is the source term, for example a well in the porous media.

3.2 Mathematical Approach

Darcy's problem is dealt with a mixed type variational formulation, where velocity u_2 and pressure p_2 are both treated as unknowns.

We have the equations in their variational formulation:

$$\begin{cases} c(u_2, v) + b(p_2, v) = 0 & \forall v \in V, \\ (\nabla \cdot q, u_2) = (F_p, q) & \forall q \in Q. \end{cases} \tag{9}$$

where:

- V and Q are similarly defined as the ones in the Stokes case,
- $c(u_2, v) = \int_{\Omega_2} \mu K^{-1} u_2 \cdot v\, dx$ represents the inverse permeability term (with K permeability matrix of the media),
- $b(p_2, v) = -\int_{\Omega_2} p_2 \nabla \cdot v\, dx$ is the coupling therm between pressure and velocity,
- $b(q, u_2) = \int_{\Omega_2} q \nabla \cdot u_2\, dx$ impose mass conservation,
- $F_p = -\int_{\Omega_2} f_2 q\ dx$ is the known term,
- Please notice that, since we will not be using Neumann B.C.s on the Darcy Problem, the first equation keeps the rhs=0.

3.3 Matrix Structure

We need to analyze the structure of the algebraic system associated to the Galerkin approximation of the Darcy model.

We define:

$$V_h = \Big\{ \mathbf{v}_h \in H_1(\Omega_2) \ : \ \mathbf{v}_h|_T \in [P_r(T)]^2 \quad \forall T \in \mathcal{T}_h \Big\},$$

$$Q_h = \Big\{ q_h \in L^2(\Omega_2) \ : \ q_h|_T \in P_{r-1}(T) \quad \forall T \in \mathcal{T}_h \Big\}.$$

Where: $\mathcal{T}_h$ is a triangulation (or mesh) of the domain Ω_2, T is a generic element of the mesh, $P_r(T)$ is the space of polynomials of degree r defined on T.

Denote with $\varphi_j \in V_h$, $\phi_k \in Q_h$, the Lagrangian basis funtions of the spaces V_h and Q_h respectively and expand the discrete solutions u_2 and p_2 with respect to such bases:

$$u_h(x) = \sum_{j=1}^{N} u_{1j}\varphi_j(x), \quad p_h(x) = \sum_{k=1}^{M} p_{1k}\phi_k(x). \tag{10}$$

Where N=dim$V$ and M=dim$Q$.

By choosing as test function in the same basis functions we obtain the following block system:

$$\begin{bmatrix} C & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ F_p \end{bmatrix} \tag{11}$$

where $C \in R^{N\times N}$ and $B \in R^{M\times N}$ are the matrices related respectively to the bilinear forms $c(\cdot,\cdot)$ and $b(\cdot,\cdot)$, whose elements are given by $C = [c_{ij}] = [c(\varphi_j, \varphi_i)]$ and $B = [b_{kj}] = [b(\varphi_j, p_k)]$, while U and P are the vectors of the unknowns $U = [u_{2j}]$ and $P = [p_{2k}]$.

We can also scale the system by its permeability constant K. In this case we consider $C = \frac{1}{K}A$, where A is a modified mass matrix, we obtain:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} 0 \\ KF_p \end{bmatrix} \tag{12}$$

This resulting system has a block structure similar to Stokes problem. However, in this case, matrix A is positive definite; the system in its entirety corresponds to a symmetric but not positive definite matrix. The corresponding linear system can be solved with iterative methods.

4 Coupled Problem

In order to numerically solve the coupled Stokes-Darcy problem, we need to analyze the variational formulation of the model and the conditions required to have existence, uniqueness, and stability of the solution.

In the cited article, we can find a rigorous approach to the coupled weak formulation of the problem and an iterative numerical pattern to solve it.

In this section we will analyze the problem's well-posedness, the convergence of its numerical method and the system's global algebraic structure obtained from the Galerkin approximation.

4.1 Variational Formulation of the Coupled Problem

As mentioned in the previous sections, the coupled problem is composed of two subproblems:

- **Stokes problem** on region Ω_1 for $(u_1, p_1) \in V_1 \times Q_1$,
- **Darcy's problem** on region Ω_2 for $(u_2, p_2) \in V_2 \times Q_2$,

coupled with the boundary conditions on the interface Γ. The variational problem is: find $(u_1, p_1, u_2, p_2) \in V_1 \times Q_1 \times V_2 \times Q_2$ such that:

$$\begin{aligned} a_1(u_1, v_1) + b_1(v_1, p_1) + c(v_1, u_1) &= f_1, \quad \forall v_1 \in V_1 \\ b_1(u_1, q_1) &= 0, \quad \forall q_1 \in Q_1 \\ a_2(u_2, v_2) + b_2(v_2, p_2) &= 0, \quad \forall v_2 \in V_2 \\ b_2(u_2, q_2) &= f_2(q_2), \quad \forall q_2 \in Q_2. \end{aligned} \tag{13}$$

where

- $a_1(u_1, v_1) = \int_{\Omega_1} 2\mu\epsilon(u_1) : \epsilon(v_1)\, dx$, where ϵ is the strain rate tensor (symmetric part of the velocity gradient);
- $b_1(v_1, p_1) = -\int_{\Omega_1} p_1 \nabla \cdot v_1\, dx$,
- $c(v_1, u_1) = \alpha \int_\Gamma (u_1 \cdot \tau)(v_1 \cdot \tau)\, ds$ represents the tangential slip constraint on Γ, α is the friction on the interface Γ and τ is its verser;
- $a_2(u_2, v_2) = \int_{\Omega_2} K^{-1} u_2 \cdot v_2\, dx$,
- $b_2(v_2, p_2) = -\int_{\Omega_2} p_2 \nabla \cdot v_2\, dx$.

4.2 Well-Posedness and Saddle-Point Problem

This system could be traced back to a saddle-point problem. In order to guarantee that the problem has a stable unique solution, the following conditions must be satisfied:

1. **Continuity** of every bilinear form: $a_i(\cdot\ ,\cdot), b_i(\cdot,\cdot), c(\cdot,\cdot)$ have to be continuous.
2. **Coercivity** on subspaces: $a_1 + c$ on V_1 and a_2 on V_2 have to be coercive on $ker(b_1)$ and $ker(b_2)$.
3. **inf-sup condition**:

$$\inf_{q\in Q_i}\sup_{v\in V_i}\frac{b_i(v,q)}{\|v\|_{V_i}\|q\|_{Q_i}} \geq \beta > 0, \qquad i = 1,2 \tag{14}$$

It can be demonstrated that, if V_i and Q_i satisfy the inf-sup condition and $\alpha > 0$, then the coupled problem is well-posed, so it exists a unique solution which is continuous with respect to the data.

4.3 Global Matrix Form

We can rewrite the whole coupled problem in matrix form through Galerkin approximation:

$$\underbrace{\begin{bmatrix} A & B_1^T & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & 0 & D & B_2^T \\ 0 & 0 & B_2 & 0 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} U_1 \\ P_1 \\ U_2 \\ P_2 \end{bmatrix}}_{\mathcal{U}} = \underbrace{\begin{bmatrix} F_{u_1} \\ 0 \\ 0 \\ F_{p_2} \end{bmatrix}}_{\mathcal{F}} \tag{15}$$

where A represents the Stokes matrix (symmetric and positive definite part), B_1 and B_2 are the coupling matrices due to the divergence conditions in the two regions, D is the Darcy's matrix (symmetric and positive definite), and F_{u_1} and F_{p_2} are the known terms of the data f_1 and f_2.The obtained global system S is symmetric and positive definite, as is usual in saddle-point problems.

The matrix structure perfectly reflects the decoupling between the two domains, which can be computationally exploited with iterative methods. The convergence of these methods is demonstrated with the spectral analysis of the system.

4.4 Coupled Iterative Method

In the paper, a coupling iterative method is proposed. The entire domain is divided into two subdomains, where the two subproblems (Stokes and Darcy) are alternately solved. This method is based on a fixed point algorithm defined on the interface Γ:

1. An initial condition is chosen on the interface (i.e. normal velocity).
2. The Darcy problem is solved with this initial condition.
3. The Darcy solution is used to compute the residual, which will be used to apply the Neumann condition on the Stokes problem.
4. The Stokes problem is solved.
5. The interface condition (the normal velocity) is updated.
6. These steps are iterated until convergence is reached.

Motivation for the residual

Since $p \in Q = L^2(\Omega_2)$, it is a function in a Lebesgue space, and hence it is defined almost everywhere. This means that we cannot directly evaluate p on the interface Γ, since Γ has null measure with respect to L^2.
To overcome this difficulty, we proceed by integrating by parts the Darcy equation. This yields:

$$-\int_\Gamma p\,\mathbf{v}\cdot\mathbf{n}\,ds = \frac{1}{k}\left(\mu\int_{\Omega_2}\mathbf{u}\cdot\mathbf{v}\,dx - K\int_{\Omega_2} p\,\nabla\cdot\mathbf{v}\,dx\right),$$

which allows us to reinterpret the boundary term in a variational sense.

5 Implementation

The numerical implementation of the coupled Stokes-Darcy problem follows an object-orientated approach that allows us to write an easily readable, reusable, and extendable code.

The implementation is composed of two principal classes: `Stokes` and `Darcy`; both in charge of the formulation, resolution, and management of their individual subproblems. Each class includes the data and the functionalities required to solve its own system, such as: the known terms (boundary function, forcing terms, etc), the boundary conditions, and the key methods `setup`, `assemble`, `solve`, and `output`.

In the following, we describe the class specific methods that handle the communication between the two subproblems.

In `Darcy` class:

1. a method `apply_gamma_boundary_conditions` that takes the current interface normal velocity and applies it as a B.C. to the Darcy problem.
2. a method `compute_residual` that based on the current solution computes and returns the residual vector.

3. a method `get_boundary_data` to extract the boundary data for every iteration, just to do the convergence check for the stopping criteria.

In `Stokes` class:

1. a method `apply_gamma_boundary_conditions` that applies the pressure equilibrium boundary condition on Γ by summing the residual vector to the velocity part of the rhs of the linear system.

2. a method `get_boundary_data` to extract the boundary data at every iteration, to update the interface normal velocity and to perform the convergence check.

These methods allow for information exchange between Stokes and Darcy along the Γ interface, generating in-code communication that mirrors the physical coupling model.

Inside the `main` program, the two classes are instantiated and coupled with a ping-pong iterative solver: starting from an initial condition on the boundary (initial guess), the Stokes and Darcy problems are alternately solved, updating the interface conditions during each iteration. The loop terminates when a stopping criterion is satisfied or a maximum number of iterations is reached. The stopping criterion is based on the comparison of two sequential solutions on Gamma: if the norm of the difference between the two solutions is smaller than a chosen tolerance, the algorithm stops.

5.1 Solvers and Preconditioners

To solve the discretized linear systems, we use the GMRES iterative method. Both matrices are characterized by zero-blocks on the diagonal, which makes the systems badly conditioned. To improve the convergence of GMRES, we employ block preconditioners specifically tailored to each problem:

- **Darcy Problem**: we use a block triangular preconditioner, constructed from the velocity block A, the coupling block B^T, and the pressure mass matrix.
- **Stokes Problem**: similarly, we use a block triangular preconditioner built from the viscous block A, the divergence block B^T, and the pressure mass matrix.

These block preconditioners allow GMRES to converge efficiently in both subproblems of the coupled system.

6 Results

Numerical simulations were carried out on a coupled Stokes–Darcy domain, with the interface located at the geometric center. Spatial discretization was

performed using the finite element method.

Starting from some known analytical solutions, we tested our solver building costumized forcing terms:

1. **Polinomial Function Test**
   - $u_1(x, y) = (2x^2y, -2xy^2)$
   - $p_1(x, y) = x - 2y$
   - $p_2(x, y) = (x - 1)^2 - y^2$
   - $u_2(x, y) = (-2K(x - 1)/\mu, 2Ky/\mu)$
   - $f_1(x, y) = (1 - 4\mu y, -2 + 4\mu x)$
   - $g(x, y) = \nabla \cdot u_2 = 0$
2. **Trigonometrical Function Test**
   - $u_1(x, y) = (\pi sin(\pi x)cos(\pi y), -\pi cos(\pi x)sin(\pi y))$
   - $p_1(x, y) = cos(\pi x) - sin(\pi y)$
   - $p_2(x, y) = y$
   - $f_1(x, y) = (2\mu\pi^3 sin(\pi x)cos(piy), -2\mu\pi^3 cos(\pi x)sin(\pi y) - \pi cos(\pi y))$
   - $g(x, y) = \nabla \cdot u_2 = 0$

From this tests we can observe that the normal velocity at the interface, $u \cdot n$, converges rapidly in approximately ten iterations for all test cases considered, regardless of the physical parameters employed, confirming the correct implementation of the coupling.

In contrast to velocity, the pressure does not converge fully: the values at the two sides of the interface do not match exactly. Although the Stokes and Darcy subproblems each reach numerical convergence, it cannot be guaranteed that the overall solution converges to the exact solution of the coupled problem. This behavior highlights a limitation of the current iterative strategy, while still preserving numerical stability and qualitative consistency of the flow. To overcome this obstacle, we could improve the solver by adding relaxation and refining the mesh.

The sub-solvers for the Stokes and Darcy problems were tested independently and shown to perform correctly, consistently and producing stable results across all simulations. This confirms that the observed discrepancies in pressure convergence arise from the coupling procedure itself, rather than from the accuracy of the individual solvers.

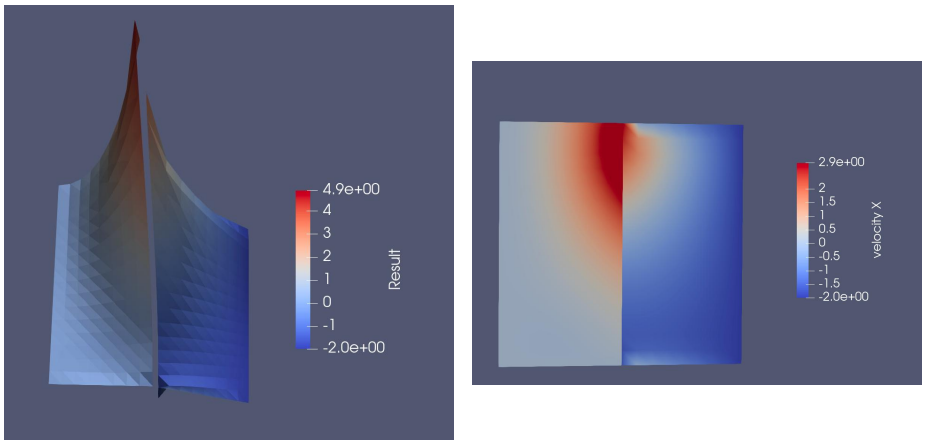

Figure 2. *Representation of the x component of velocity in the example with polynomial function at the last iteration*

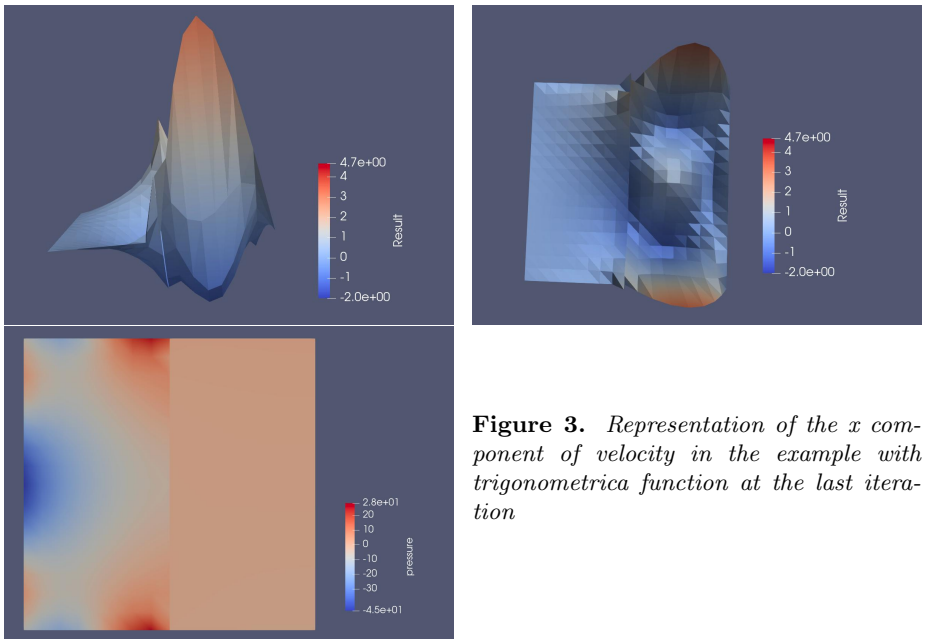

Figure 3. *Representation of the x component of velocity in the example with trigonometrica function at the last iteration*

7 Conclusion

In this work, we have implemented and analyzed a partitioned iterative solver for the coupled Stokes–Darcy problem. The adopted strategy, based on the independent resolution of the two subproblems and the exchange of interface data, has proven effective in capturing the main physical features of the coupled system. In particular, the normal velocity across the interface converges rapidly

within approximately ten iterations for all tested cases.

Nevertheless, the numerical experiments revealed limitations in the treatment of pressure continuity at the interface. While both the Stokes and Darcy subproblems reach convergence individually, the interface pressures do not perfectly match, and it remains unclear whether the global solution converges to the exact coupled solution. This issue is intrinsic to the iterative coupling procedure rather than to the sub-solvers themselves, which were separately validated and shown to be accurate and stable.

Overall, the results highlight both the potential and the limitations of the adopted approach. On the one hand, the solver successfully reproduces the expected flow structures and provides a robust framework for modular implementation. On the other hand, further refinement is required to guarantee full convergence, particularly in terms of pressure matching. Future developments may involve the introduction of relaxation strategies, enhanced interface conditions, or multilevel iterative schemes, as well as mesh refinement and spectral analysis of the coupling operator. Such improvements would extend the applicability of the method to more complex geometries and parameter regimes of practical interest in environmental and engineering contexts.