THE LINEAR AND TORSIONAL SPRING-MASS-DAMPER SYSTEM

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MOTIVATIONS & GOALS : As we need to measure motion of objects (translations and rotations) induced by external actions, it is worthwhile to give a complete description of the reference equations describing the system.

KINEMATICS OF RELATIVE MOTION

Reference systems (inertial, non-inertial)
Equations for translational motion

INERTIAL REFERENCE FRAME (SYSTEM) = a frame where the 1st Newton's law applies: an object moves at a constant velocity, if not perturbed by external forces

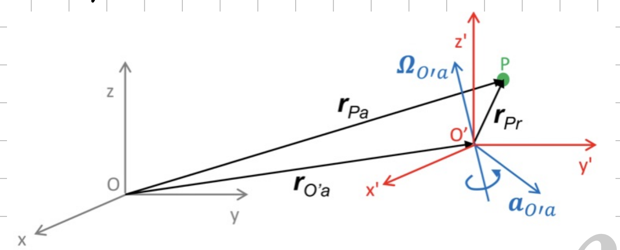
- all inertial reference frames are in a state of constant, rectilinear motion one another (they are not accelerating);
- Physical laws take exactly the same form in all inertial frames;

NON-INERTIAL REFERENCE FRAME = the laws of physics depend upon the particular frame of reference

→ the usual physical forces must be supplemented by "fictitious" or "apparent" forces

NOTE: the measurement of this fictitious forces will provide the information we need on the quantity to be measured.

Oxyz inertial (absolute) reference
O'x'y'z' non-inertial (relative) reference



$\boxed{\vec{r}_{O'a}}$ = position of the center of the relative reference w/rt the absolute one

$\boxed{\vec{a}_{O'a}}$ = acceleration of the relative reference w/rt the absolute one

$\boxed{\vec{\Omega}_{O'a}}$ = angular velocity of the relative reference w/rt the absolute one

$\boxed{\vec{r}_{Pa} = \vec{r}_{Pr} + \vec{r}_{O'a}}$: position of the mass w/rt the absolute reference

where $\boxed{\vec{r}_{Pr}}$ = position of the mass w/rt the relative reference

$\boxed{\vec{v}_{Pa} = \vec{v}_{Pr} + \vec{v}_{O'a} + (\vec{\Omega}_{O'a} \times \vec{r}_{Pr})}$: velocity of the mass w/rt the absolute reference

where $\boxed{\vec{v}_{Pr}}$ = velocity of the mass w/rt the relative reference

$\boxed{\vec{v}_{O'a}}$ = velocity of the relative reference w/rt the absolute one

$\boxed{(\vec{\Omega}_{O'a} \times \vec{r}_{Pr})}$ = tangential velocity

$$\boxed{\vec{a}_{Pa} = \vec{a}_{Pr} + \vec{a}_{O'a} + \cancel{\vec{\Omega}_{O'a} \times (\vec{\Omega}_{O'a} \times \vec{r}_{Pr})} + \cancel{(\dot{\vec{\Omega}}_{O'a} \times \vec{r}_{Pr})} + 2(\vec{\Omega}_{O'a} \times \vec{v}_{Pr})}$$ TRUE NEWTON ACCELERATION

where $\boxed{\vec{\Omega}_{O'a} \times (\vec{\Omega}_{O'a} \times \vec{r}_{Pr})}$ = centripetal acceleration
$\boxed{(\dot{\vec{\Omega}}_{O'a} \times \vec{r}_{Pr})}$ = angular acceleration
} negligible w/rt $\vec{a}_{cor}$

$\boxed{2(\vec{\Omega}_{O'a} \times \vec{v}_{Pr}) = \vec{a}_{cor}}$ CORIOLIS ACCELERATION

Earth = "good" inertial reference ; MEMS package = non-inertial reference frame

suspended mass = point-like mass

GOAL: • to measure the motion of our non-inertial system (O'x'y'z') w/rt the inertial one (Oxyz)

⇒ to do it we exploit the motion $\vec{r}_{Pr}$ of the MEMS mass P relative to O'x'y'z' described through ficticious forces

In reality this point mass P is not free to move, but it is constrained to the package of our MEMS through a system of springs and it is damped by the gaseous environment that encircles the suspended mass itself.
This gives us additional true forces to consider for the point-like-mass P:

ELASTIC FORCE $\boxed{F_{elastic} = -k \cdot r_{Pr}}$

DAMPING FORCE $\boxed{F_{damping} = -b \cdot v_{Pr}}$

} RESTORING FORCES: They tend to bring the mass P back to the rest position

By multiplying the former expression of the true Newton acceleration for the mass m, we obtain:

$$m \cdot \vec{a}_{Pa} = m \cdot \vec{a}_{Pr} + m \cdot \vec{a}_{O'a} + 2m \cdot (\vec{\Omega}_{O'a} \times \vec{v}_{Pr})$$

Hp: for the sake of semplicity we assume to be in a 1 dimension situation (along the x-axis)

$$\boxed{m \cdot \ddot{x}_{Pa} = m\ddot{x}_{Pr} + m a_{O'a} + 2m(\Omega_{O'a} \cdot \dot{y}_{Pr})}^{*_1}$$ TRUE NEWTONIAN FORCE

*₁ NOTE: The velocity of the point-mass P that generates the Coriolis acceleration should be orthogonal to the direction we are moving along and so either in the y or in the z direction

→ Hp: assume the velocity in the y-direction

The true Newton force is essentially given by the sum of the two restoring forces that we have seen before ($F_{elastic}$ and $F_{damping}$):

$$-b\dot{x}_{Pr} - kx_{Pr} = m\ddot{x}_{Pr} + m a_{O'a} + 2m(\Omega_{O'a} \cdot \dot{y}_{Pr})$$

NOTE: except for $\ddot{x}_{O'a}$ and $\Omega_{O'a}$, all the other terms refer to the motion of the mass P w/rt the relative system, we can therefore neglect the sub indexes

We can finally rewrite the expression in this way,

$$m\ddot{x} + b\dot{x} + kx = \underbrace{-m a_{O'a} - 2m(\Omega_{O'a} \cdot \dot{y})}$$

The first member represents the position of the mass P in the relative reference system so essentially the distance of the suspended mass from the fixed capacitive electrodes

GOAL: $a_{O'a}$ and $\Omega_{O'a}$ are the two values that we want to measure through our MEMS devices *₂

RESULT: This means that if we measure the capacitance values somewhat we precisely know the position of our suspended mass inside the MEMS and viceversa.

$$\boxed{-m a_{O'a} - 2m(\Omega_{O'a} \cdot \dot{y}) = F_{inertial}}$$ INERTIAL FORCE

$$\Rightarrow \boxed{m\ddot{x} + b\dot{x} + kx = F_{inertial}}$$ SPRING-MASS-DAMPER SYSTEM EQUATION

*₂ NOTE: • when we want to measure $a_{O'a}$ we have to design a device (ACCELEROMETER) in such a way the term related to the Coriolis force (the one related to the rotation, $\Omega_{O'a}$) is negligible.

• when we want to measure $\Omega_{O'a}$ we have to design a device (GYROSCOPE) in such a way the term related to the linear acceleration ($a_{O'a}$) is negligible.

THE LINEAR SPRING-MASS-DAMPER SYSTEM

Time description
Frequency description
MEMS operating regions

GOAL: to evaluate the transfer function between a generic external force and the displacement, in terms of both modulus and phase, as a function of the frequency of the force

LINEAR SPRING-MASS-DAMPER SYSTEM EQUATION

$$m\ddot{x} + b\dot{x} + kx = F_{ext} \xrightarrow{\mathcal{L}[\,]} ms^2X(s) + bsX(s) + kX(s) = F_{ext}(s)$$

- $X(s)$ = Laplace transform of the relative position between the pin-like mass P and the non-inertial frame
- $F_{ext}(s)$ = Laplace transform of the external force applied to the non-inertial frame
  - → This force is in general a combination of:
    - A contribution generated by the quantity to sense (e.g. the inertial force)
    - A contribution generated by the electrostatic sensing/driving of the MEMS

$$X(s)\left[ms^2 + bs + k\right] = F_{ext}(s)$$

$$T_{XF}(s) = \frac{X}{F_{ext}}(s) = \frac{1}{ms^2 + bs + k} = \frac{1/m}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

RESONANCE FREQUENCY: $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$ → it is the eigenvalue solution if have no applied force ($F_{ext} = 0$) and a null damping coefficient ($b = 0$)

QUALITY FACTOR: $\boxed{Q = \frac{\omega_0 \cdot m}{b} = \frac{1}{b}\sqrt{km}}$ → it represents the amplification factor (in terms of displacement) at resonance in the T_{XF} response

[in other words: the amplification in terms of displacement that we would have in terms of displacement in the response if we applied a force at the resonance frequency.]

→ $\boxed{T_{XF}(s) = \frac{1}{m} \cdot \frac{1}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}}$ TRANSFER FUNCTION DISPLACEMENT-FORCE

⇒ The system has a low-pass 2nd order transfer function w/ two singularities

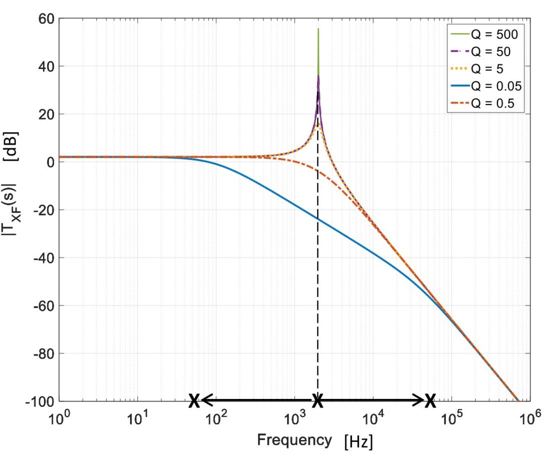
$$\Delta = \left(\frac{\omega_0}{Q}\right)^2 - 4\omega_0^2 = \left(\frac{1}{Q^2} - 4\right)\omega_0^2$$

- if $\Delta > 0 \equiv \boxed{Q < 0.5}$ → SPLIT REAL POLES → OVERDAMPED SYSTEM
- if $\Delta = 0 \equiv \boxed{Q = 0.5}$ → COINCIDENT REAL POLES
- if $\Delta < 0 \equiv \boxed{Q > 0.5}$ → COMPLEX CONJUGATE POLES → UNDERDAMPED SYSTEM

So we expect that the shape of T_{XF} is different as a function of the Q factor

TRANSFER FUNCTION MODULUS:

$$|T_{XF}(j\omega)| = \left|\frac{X(j\omega)}{F_{ext}(j\omega)}\right| = \left|\frac{1/m}{\omega_0^2 - \omega^2 + j\omega\frac{\omega_0}{Q}}\right| = \frac{1/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \left(\frac{\omega\omega_0}{Q}\right)^2}}$$



- @ DC: $\boxed{\omega \ll \omega_0} \rightarrow \boxed{|T_{XF}(j\omega)|_{@DC} \simeq \frac{1/m}{\omega_0^2} = \frac{1}{k}}$
- @ $\boxed{\omega = \omega_0} \rightarrow \boxed{|T_{XF}(j\omega)|_{\omega=\omega_0} \simeq \frac{1/m}{\sqrt{\left(\frac{\omega_0^2}{Q}\right)^2}} = \frac{Q/m}{\omega_0^2} = \frac{Q}{k}}$
- @ HF: $\boxed{\omega \gg \omega_0} \rightarrow \boxed{|T_{XF}(j\omega)|_{@HF} \simeq \frac{1/m}{\omega^2}}$

NOTE: if you change Q:

- nothing changes @ DC
- nothing changes @ HF
- @ $\omega = \omega_0$ →
  - if $Q < 0.5$ → 2 split real poles → under peaking
  - if $Q = 0.5$ → 2 coincident real poles → almost asymptotic
  - if $Q > 0.5$ → 2 complex conjugate poles → over peaking

TRANSFER FUNCTION PHASE:

As we have 2 poles we expect the phase decreases of $-180°$
The phase shift at the resonance frequency is exactly $90°$

$$\boxed{\varphi\left[T_{XF}(j\omega)\right] = \tan^{-1}\left[\frac{Im(T_{XF})}{Re(T_{XF})}\right]}$$

NOTE: Once again, the way the phase shift occurs depends on the Q factor

- Q ↑ ⇒ steeper the phase transition
- if the force is applied @ $\omega = \omega_0$ → the displacement will not be in phase w/ the force, but it will be in QUADRATURE w/ it.

MEMS devices operate in different regions of the transfer function:

- ACCELEROMETERS, MICROPHONES and PRESSURE SENSORS typical operate under forces that occur well before ω_0. They usually have low Q factors (< 10)
- RESONATORS operate at resonance and require high Q factor ($Q \sim 100 - 10k$)
- GYROSCOPES, MAGNETOMETERS and other sensors operate slightly before the resonance frequency
- No devices operate beyond the resonance frequency

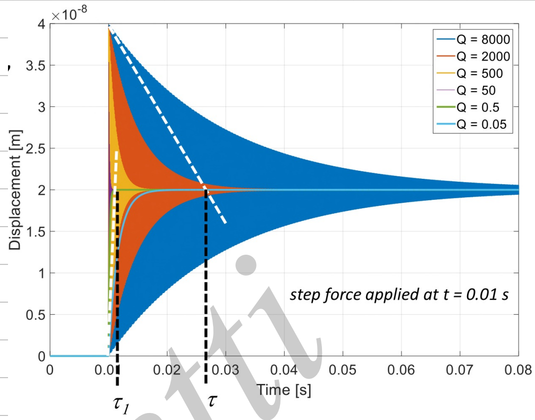
Att: The fact that w/ some devices we operate at the resonance frequency doesn't mean that the physical quantities are varying at that frequency but only that we will be able someway to modulate the effect of the physical quantities in frequency in such a way that the corresponding force will occur around the resonance frequency

What happens if we have a sudden shock or pulse that acts on the device?

A different quality factor implies not only a different amplification at resonance, but also a different response to pulses or steps

The response to a pulse essentially depends on what type of poles we have :

- SPLIT REAL POLES ($Q < 0.5$)
  → OVERDAMPED SYSTEM : The time constant is dominated by the 1st pole of Txf



- COINCIDENT REAL POLES ($Q = 0.5$)
  → The response will see no overshoot and so it will be the fastest response possible

- COMPLEX CONJUGATE POLES ($Q > 0.5$)
  → UNDERDAMPED SYSTEM: the time constant increases linearly w/ the Q factor and it is given by :

  TIME CONSTANT : $\boxed{\tau = \frac{Q}{\pi f_0}}$

THE TORSIONAL SPRING-MASS-DAMPER SYSTEM

Some MEMS operate through torsions of the structural elements, rather than through displacements. For this reason we cannot use anymore the equation found before : $\cancel{m\ddot{x} + b\dot{x} + kx = F_{ext}}$

We need to switch to the torsional counterpart which means that our quantity of interest is no more the displacement, but now it is the ROTATION ANGLE θ :

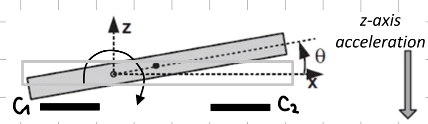
$\boxed{I\ddot{\theta} + b_\theta \dot{\theta} + k_\theta \theta = M_{ext}}$ TORSIONAL SPRING-MASS-DAMPER SYSTEM EQUATION

I = MOMENT OF INERTIA $[kg \cdot m^2]$ $\qquad \ddot{\theta}$ = ANGULAR ACCELERATION $[rad/s^2]$

b_θ = TORSIONAL DAMPING COEFFICIENT $[kg/s \cdot m^2]$ $\qquad \dot{\theta}$ = ANGULAR VELOCITY $[rad/s]$

k_θ = TORSIONAL STIFFNESS $[N \cdot m]$ $\qquad \theta$ = ROTATION ANGLE $[rad]$

M_{ext} = OVERALL EXTERNAL TORQUES



$\sum F = m \cdot \ddot{x} \longrightarrow \boxed{\sum M = I \cdot \ddot{\theta}}$ 2nd NEWTON'S LAW FOR ROTATIONAL MOTION

↳ The MOMENT OF INERTIA of a rigid body determines the torque needed for a desired angular acceleration about a rotational axis.
→ it depends on the body mass distribution around the chosen rotation axis

→ $M = F \cdot R$: obtained as the force applied in the gravity center times the arm between the rotation hinge and the gravity center.

$\boxed{\omega_0 = \sqrt{\frac{k_\theta}{m}}}$ $\qquad \boxed{Q = \frac{\omega_0 I}{b_\theta}}$ $\qquad$ indeed I is also known as ANGULAR MASS

ELECTROSTATIC FORCES

Parallel plates
Comb-fingers

GOAL: What we want to do in our inertial sensors is essentially to readout a capacitance in order to recover the information on the motion of our object and so on the force that we want to measure

→ To do it, we need to apply an electrical signal to the capacitors inside our MEMS (so to the suspended mass and to the fixed electrodes)

Att: As we are going to see, whenever we apply an electrical signal, an electrostatic forces arises inside our system and this force must be taken into account.

$m\ddot{x} + b\dot{x} + kx = F_{ext}$

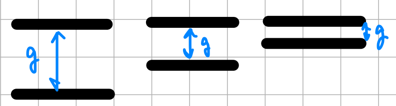
In the most general and common situation there are electrostatic forces that take part in the force balance:

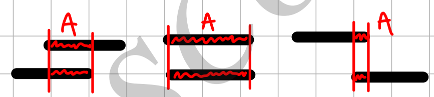
- MEMS AS CAPACITIVE ACTUATOR: a voltage is applied to generate a force (resonators)
- MEMS AS CAPACITIVE SENSOR: a voltage is applied to readout the value of a capacitance (almost all sensors)

VARIABLE CAPACITANCE $\boxed{C = \frac{\varepsilon_0 A N}{g}}$ can be of two types:
- **GAP VARYING** (PPs) $\boxed{C = \frac{\varepsilon_0 A N}{g \pm x}}$ $*_1$
- **AREA VARYING** (CFs) $\boxed{C = \frac{\varepsilon_0 h (x_0 \pm x) N}{g}}$ $*_2$

ε_0 = VACUUM PERMITTIVITY

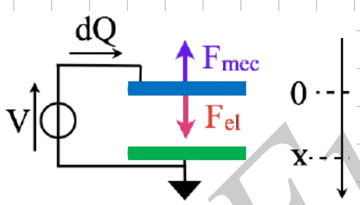
N = NUMBER OF SENSING CELLS

$*_1$ 

; $*_2$ 

ROTOR = moving part of the microsystem ; **STATOR** = fixed electrodes

- **SINGLE-ENDED PARALLEL PLATE CONFIGURATION**



Hp: assume to apply a DC voltage to its plates (QUASI-STATIONARY CONDITION)

PRINCIPLE OF VIRTUAL WORK: in equilibrium conditions, electrostatic forces balance the mechanical force

When we apply a voltage to a capacitor, essentially, what we are doing is to pull some charge in one plate and charge of the opposite will arise in the other plate.
↳ bcz $\boxed{Q = CV}$

Charges of opposite signs on the plates generate an electric field and thus an electrostatic forces that tends to attract the moving plate towards the fixed one

The variation of energy is given by 2 contributions: $\boxed{dE_c = dW_{mech} + dW_{elec}}$

Since we assumed to apply a DC voltage (quasi-stationary condition)

ELECTROSTATIC ENERGY STORED IN A CAPACITOR $\boxed{dE_c = d\left(\frac{1}{2}CV^2\right) = \frac{V^2}{2}dC}$

WORK DONE FOR A DISPLACEMENT dx: $\boxed{dW_{mech} = -F_{mech} \cdot dx}$

ELECTROSTATIC WORK DONE TO CHANGE THE VOLTAGE OVER A CAPACITOR : $\boxed{dW_{elec} = V dQ = V d(CV) = V(C dV + V dC) = V^2 dC}$

$$\rightarrow \frac{V^2}{2} dC = -F_{mech}\, dx + V^2 dC \longrightarrow F_{mech} \cdot dx = \frac{V^2}{2} dC$$

$$C = \frac{\varepsilon_0 A N}{g - x} \rightarrow \text{DERIVATIVE OF THE CAPACITANCE: } \boxed{dC = \frac{\varepsilon_0 A_{PP} N_{PP}}{(g-x)^2} dx}$$

virtual work principle

$$\rightarrow F_{mech} = \frac{V^2}{2} \frac{\varepsilon_0 A N}{(g-x)^2} \longrightarrow \boxed{F_{elec} \doteq F_{mech} = \frac{\Delta V^2}{2} \frac{dC}{dx} = \frac{V^2}{2} \frac{\varepsilon_0 A_{PP} N_{PP}}{(g-x)^2}}$$

ELECTROSTATIC FORCE IN A SINGLE-ENDED PARALLEL-PLATE CONFIGURATION

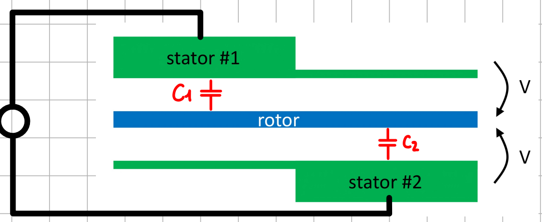
NOTE (IMP!): The electrostatic force is a non-linear function of the displacement

NOTE: $\vec{F}_{elec} = -\vec{F}_{mech}$

$$\rightarrow \boxed{m\ddot{x} + b\dot{x} + kx = F_{ext} + F_{elec}}$$

LINEAR S-M-D SYSTEM EQUATION CONSIDERING ELECTROSTATIC FORCES

- DIFFERENTIAL CONFIGURATION



Hp: The two stators are biased w/ the same voltage difference w/r't the rotor

In principle we would expect to have the a force that is equal and opposite from each of the two stators acting on the rotor.

This is true if and only if the device is in the rest position bcz only in this condition the two gaps are identical

In the most general situation (device not in rest position) : $F_{elec} = F_{elec,1} + F_{elec,2}$

$$\boxed{F_{elec} = \frac{\Delta V^2}{2} \frac{d(\Delta C)}{dx} = \frac{V^2}{2} \frac{\varepsilon_0 A_{PP} N_{PP}}{(g-x)^2} - \frac{V^2}{2} \frac{\varepsilon_0 A_{PP} N_{PP}}{(g+x)^2}}$$

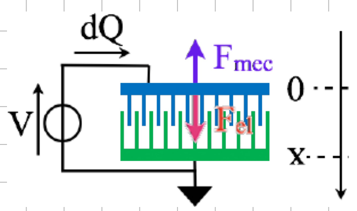
ELECTROSTATIC FORCE IN A DIFFERENTIAL CONFIGURATION

$$\longrightarrow \Delta C = C_1 + C_2 = \frac{\varepsilon_0 A N}{g - x} + \frac{\varepsilon_0 A N}{g + x}$$

Why is it better to use the differential configuration?

1. To minimize the effects of electrostatic forces;
2. To improve the linearity in the response of our sensor;
3. It allows to remove all undesired common mode disturbances: every time we have a common mode disturbance (a disturbance that affects evenly the entire system), if we have a differential readout, so a readout where the overall capacitance variation is calculated as the difference b/w C_1 and C_2, by taking this difference we will save the differential contributions by removing all the common mode terms and so also all the undesired disturbances related to them.

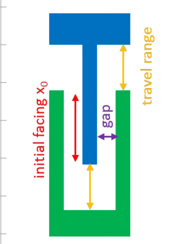
- COMB-FINGERS CONFIGURATION



Hp: assume to apply a DC voltage to its plates (QUASI-STATIONARY CONDITION)

NOTE: Again, according to the principle of the virtual work: $F_{elec} \doteq -F_{mech}$

→ Comb-finger capacitors are of area-varying type



NOTE: now, to evaluate the capacitance value, a factor
account bcz each finger gives origin to 2 needs
to be taken into two capacitances

$$\boxed{C = \frac{2\varepsilon_0 A N_{cf}}{g} = \frac{2\varepsilon_0 h (x_0 \pm x) N_{cf}}{g}}$$

COMB-FINGER CAPACITANCE

→ $$dC = \frac{2\varepsilon_0 h N}{g} dx$$

→ $$\boxed{F_{elec} = \frac{\Delta V^2}{2}\frac{dC}{dx} = \frac{V^2}{2}\frac{2\varepsilon_0 h N_{cf}}{g} = \frac{V^2 \varepsilon_0 h N_{cf}}{g}}$$

ELECTROSTATIC FORCE IN A COMB-FINGER CONFIGURATION

ADVANTAGE: The electrostatic force is no more a function of the displacement x