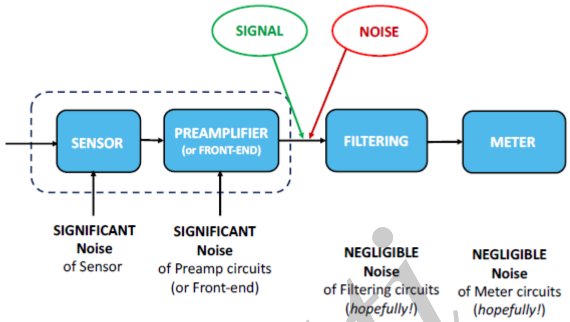
1. SIGNAL DESCRIPTION

1.1 SIGNALS

1.1.1 SET-UP FOR SENSOR MEASUREMENTS

First of all, we need:

- Sensor: to translate a physical quantity (e.g. temperature, light, etc.) into an electrical quantity.



Then, we have some block for signal conditioning and recording. In particular, we usually exploit:

- Amplifier: to acquire and amplify the signal:
- Filter: stage to select the signal of interest (or part of it)
- Converter: to translate the signal in the digital domain

One of the main issues in signal acquisition is the presence of noise.

The noise is a statistical signal that can be inherently associated with the signal of interest.

Its presence can be justified by observing that, if we consider an arbitrary physical process, there will always be some fluctuations related to it (e.g. electrons' thermal motion into a resistor causes fluctuations of the voltage across it).

This statistical phenomenon doesn't carry useful information, otherwise it corrupts in some way the information contained into the signal, so we need to use a particular class of block, called filters, in order to reduce as much as possible noise and enhance the signal of interest.

During this course we always speak about signal to noise ratio: it doesn't have any sense to speak about signals, without speaking about noise.

1.1.2 MATHEMATICAL DESCRIPTION OF SIGNALS

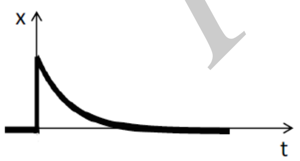
The point is that in order to introduce filters we have to find a way to distinguish signals and noise. To do it, we have to explore all the features about signals in order to be able to distinguish them from noise.

Time and frequency domain analysis: sometimes it's easier to analyze the signal in one domain, other times is easier in the other one, so we must be able to easily jump from one to the other.

1.2 TIME DOMAIN AND FREQUENCY DOMAIN ANALYSIS OF SIGNALS

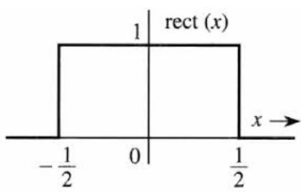
A signal is a function x that carries information about a phenomenon.

A deterministic signal is a signal whose value is defined for every instant of time: $x = x(t)$.



Two important functions for many applications are:

- **Exponential-decade-time signal**: $x(t) = 1(t)e^{-t/\tau}$



- **Rectangular signal**: $x(t) = 1(t) = \begin{cases} 1 & for\ |t|\le\frac{1}{2} \\ 0 & otherwise \end{cases}$

FrancescoGavetti

1.2.1 FOURIER TRANSFORM

The Fourier Transform is a very important mathematical tool that gives us the opportunity to analyze signals both in the time and in the frequency domain. The basic idea behind the Fourier-transform analysis is that any arbitrary signal can be seen as a superposition of sinusoids.

If we consider an arbitrary signal $x(t)$, it may be written as:

Anti-Fourier Transform: $x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}df$

In which $X(f)$ is the Fourier Transform of $x(t)$. In general, the quantity $X(f)$ is a complex number and it describes the frequency content of the initial signal $x(t)$.
In particular $|X(f)|$ denotes the amplitude of the complex sinusoid at each frequency in the superposition, and $\phi\big(X(f)\big)$ describes the phase of that complex sinusoid.

Fourier Transform: $X(f) = F[x(t)] = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}dt$

Att: we will never compute this integral, because from our point of view we have to make this computation in an intuitive way.

Considering at first the case $f = 0$, we can write:

$$X(0) = \int_{-\infty}^{+\infty} x(t)dt$$

It tells us that the frequency content at zero frequency, i.e. the DC value of the signal $x(t)$, is simply an average on time of the signal $x(t)$.

The main properties of the Fourier transform are:

P1. Value in zero and integrals:

$$for\ f = 0 \rightarrow\ X(0) = \int_{-\infty}^{+\infty} dt$$

$$for\ t = 0 \rightarrow\ x(0) = \int_{-\infty}^{+\infty} X(f)df$$

This means that the value in zero of the Fourier transform equals the integral of the function over all the time domain and that the value in zero of the time signal is equal to the integral of the Fourier transform over all the frequency domain.

P2. Duality:

If we look at the Fourier and the Anti-Fourier transforms, we can notice that they are almost equal, apart form a change in the sign of the exponent.

This means that not only the function $x(t)$ can be seen as the superposition of sinusoids, but also $X(f)$ can be seen as a superposition of complex exponentials whose weights are encoded in the function $x(t)$ itself.

The time and the frequency domain describe the same situation from two different perspectives.

P3. Fourier transform of a periodized function:

A periodization in time corresponds to sampling in frequency domain. Equally, sampling in time domain corresponds to a periodization in the frequency domain.

P4. Convolution and product:

The convolution of two signals in the time domain corresponds to the multiplication of their Fourier transforms in the frequency domain and vice versa.

$$x_1(t) * x_2(t)\ \leftrightarrow X_1(f)\, X_2(f)$$

FrancescoGavetti

In the same way, the multiplication of two signals in the time domain corresponds to the convolution of their Fourier transforms in the frequency domain and vice versa.

$$x_1(t)x_2(t) \leftrightarrow X_1(f) * X_2(f)$$

All properties and Fourier transform of the main signals:

FOURIER TRANSFORM PROPERTIES	
Linearity	$\alpha x_1(t) + \beta x_2(t) \leftrightarrow \alpha X_1(f) + \beta X_2(f)$
Symmetry	$x^*(t) \leftrightarrow X^*(-f), se\ x(t) \in \mathcal{R} \to x(-t) = X^*(f)$
Values in zero	$X(0) = \int_{-\infty}^{+\infty} x(t)dt\,, \qquad x(0) = \int_{-\infty}^{+\infty} X(f)df$
Duality	$x(t) \to X(f),\ X(t) \to x(-f)$
Scaling	$x(\alpha t) \leftrightarrow \frac{1}{\lvert\alpha\rvert} X\left(\frac{f}{\alpha}\right),\ \alpha \neq 0$
Translation in time	$x(t - t_0) \leftrightarrow X(f)e^{-j2\pi f t_0}$
Translation in frequency	$X(f - f_0) \leftrightarrow x(t)e^{j2\pi t f_0}$
Derivation in time	$\frac{dx(t)}{dt} \leftrightarrow j2\pi f\, X(f)$
Derivation in frequency	$\frac{dX(f)}{df} \leftrightarrow -j2\pi t\, x(t)$
Integration in time	$\int_{-\infty}^{t} x(\tau)d\tau = \frac{X(f)}{j2\pi f} + \frac{\delta(f)}{2} X(0)$
Integration in frequency	
Product theorem	$F[x(t) \cdot y(t)] = X(f) * Y(f)$
Convolution theorem	$F[x(t) * y(t)] = X(f) \cdot Y(f)$

MAIN FOURIER TRANSFORMS		
Signal		**Fourier Transform**
Rect	$x(t) = rect\left(\frac{t}{T}\right)$	$X(f) = Tsinc(fT) = T\frac{sin(fT)}{fT}$
Exponent	$x(t) = e^{j\omega_0 t}$	$X(f) = 2\pi\delta(\omega - \omega_0)$
Cosine	$x(t) = \cos(\omega_0 t)$	$X(f) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
Sine	$x(t) = \sin(\omega_0 t)$	$X(f) = \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
Sign function	$x(t) = sign(t) = \begin{cases} 1, & if\ t > 0 \\ -1, & if\ t > 0 \end{cases}$	$X(f) = \frac{1}{j\pi f}$
Unit step	$x(t) = u(t) = \begin{cases} 1, & if\ t \geq 0 \\ 0, & if\ t < 0 \end{cases}$	$X(f) = U(f) = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$

1.2.2 LINEAR SYSTEMS AND CONVOLUTION OPERATION

Consider a Constant-parameter linear filter = no switches, no time-variant components → everything is fixed.

A system is called linear if, when scaling the input signal by a certain constant, also the output is scaled by the same constant and when the sum of two signals $x_1 + x_2$ is applied at the input, the system produces an output equal to the sum of the individual responses to the input signals.

FrancescoGavetti

Linearity: $h[\alpha x_1(t) + \beta x_2(t)] = \alpha h[x_1(t)](t) + \beta h[x_2(t)](t)$

They are characterized by:

- $H(f) = F[h(t)]$: Fourier transform of the transfer function
- $h(t) = F^{-1}[H(f)]$: δ-response in time domain → function that gives the output for each possible input

Idea: if we can write a generic signal as a superposition of elementary signals and if we can calculate the response of the system to these elementary signals, then we can easily find the response to any arbitrary input. We can do it, for instance, using Dirac Delta functions.

The Dirac Delta centered at time τ needs to have an area equal to the value of the signal $x(t)$ at that point ($x(\tau)$).

There is an infinite number of Dirac Deltas, so the superposition has to be done by means of an integral operation:

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau$$

We can notice that our signal is finite at each time, so it is not completely correct to say that it can be written as superposition of Deltas, but it's more correct to say that it is given by the superposition of areas of Deltas.

<u>Remember</u>: a Dirac Delta $\delta(t-\tau)$ is a function centered in $t=\tau$, which has an infinite amplitude in that point, it has a finite and unitary area and is null for $t \neq \tau$.

Another interpretation could be given looking at the function $x(\tau)\delta(t-\tau)d\tau$ inside the integral. In general, by multiplying a signal $g(t)$ by a small element $d\tau$ we basically compute the local area of the function. If the function is finite, we obtain a null value, since as $d\tau$ goes to zero the area always decreases reaching the value 0. However, a Delta function is infinite for $t=\tau$ and with a finite local area, equal to 1, while its value is zero for $t \neq \tau$, so also its local area will be zero at any $t \neq \tau$.
Saying that, we can conclude: if we have functions that are 1 at a particular point and zero everywhere else, it is obvious that we can construct all possible signals by taking the superposition of these elementary functions.

The response of a linear system to an arbitrary input is given by:

$$y(t) = h[x(t)](t) = h\left[\int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)d\tau\right](t) = \int_{-\infty}^{+\infty} x(\tau)h[\delta(t-\tau)](t)d\tau$$

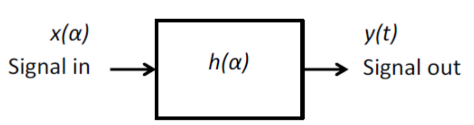
where we have used the linearity property.

Therefore, knowing the **impulse response** $h[\delta(t-\tau)](t)$, we can compute $y(t)$ for any $x(t)$. The impulse response depends on τ, this means that the system can change over time so, if we apply an impulse at $t=0$ we could obtain an output completely different from what we would get applying the impulse at a different time τ. Systems like that are called **time-variant systems**.

However, we will initially focus our attention on a particular class of systems called **Linear Time Invariant (LTI) systems**, where the response to an input signal does not depend on when the signal is applied and furthermore the output waveform is "synchronized" with the input signal, i.e. if we delay the input signal, we automatically delay the output one by the same quantity.

A time-invariant system is a system for which the **time invariancy** property is valid:

$$h[x(t-\tau)](t) = h[x(t)](t-\tau)$$



Again, a signal can be seen, in the frequency domain, as the superposition of sinusoids and, in the time domain, as the superposition of δ.

The input $x(\alpha)$ can be described as a linear superposition (sum) of elementary δ-pulses of amplitude $x(\alpha)\, d\alpha$.

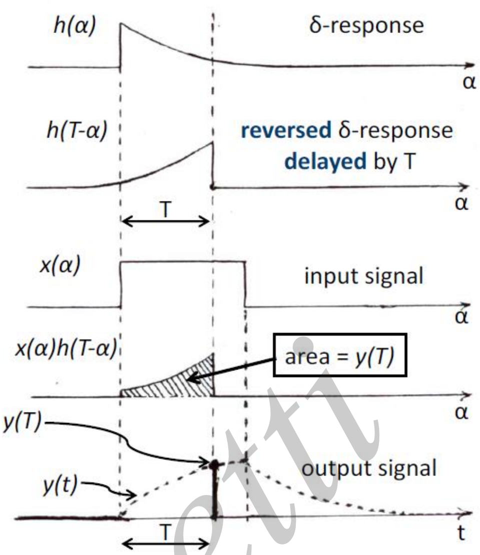
FrancescoGavetti

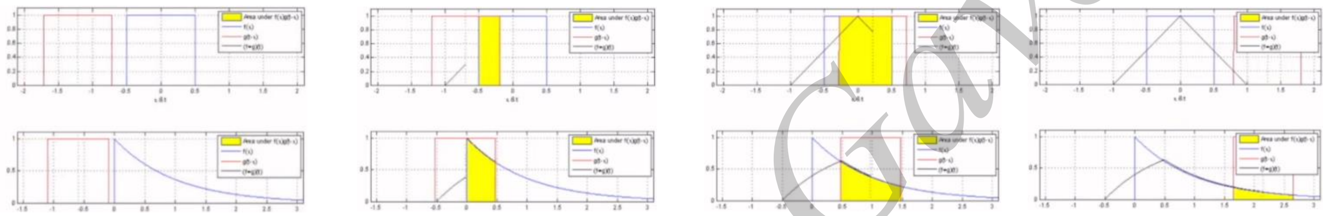
Therefore, the output $y(t)$ can be described as a linear (because the system is linear) superposition (sum) of elementary δ-pulses responses of amplitude $x(\alpha)\,d\alpha\,h(t-\alpha)$:

Convolution: $y(t) = x(\alpha) * h(\alpha) = \int_{-\infty}^{+\infty} x(\alpha)h(t-\alpha)d\alpha$

Convolution computation:

- Reverse the δ-response: $h(\alpha)$
- Shift the input signal by a quantity T rightwards, in order to obtain the desired function $h(T-\alpha)$
- Compute the integral (once fixed T)
- Repeat this procedure for any T



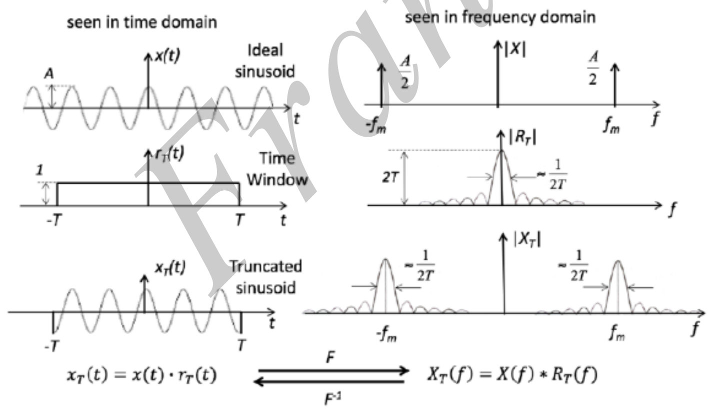


1.2.3 NOTE ON TRUNCATED SIGNALS

Let's consider a sinusoidal waveform. It is an infinite signal that goes from $-\infty$ to $+\infty$.

What's the problem? In real world the concept of infinite doesn't exist, so we have always to consider a finite part of signals, a piece of sinusoid in this case, which starts at a certain point and ends at another point after a certain interval.

What's the Fourier transform of this function? If we mathematically compute it, it would be a nightmare, so we should proceed with an intuitive approach, as we have done with the convolution before.



We know that the Fourier transform of a sinusoidal waveform is a couple of Deltas placed at $\pm f_0$ and with half of the sine's amplitude ($\frac{A}{2}$).

What's the Fourier transform of a truncated sinusoidal?

We can think about the sinusoidal as a multiplication between the entire sinusoidal and a $rect$ function whose transform is a $sinc$ function.

As we are making a multiplication in the time domain, it means that we'll have a convolution in the frequency domain. So, the Fourier transform of the truncated sinusoidal is given by the convolution between the Fourier transform of the sinusoidal and the Fourier transform of the $rect$ function.

We can now make some simple observations:

1. In reality, we always deal with truncated signals, since infinite signals do not exist;

FrancescoGavetti

2. Cropping in time corresponds to the convolution between the Fourier transform of the signal and the transform of a rectangle signal, which is a $sinc$ function.

3. The convolution is characterized by the fact that it spreads the signal in the frequency domain: it makes the signal wider and smoother.

4. The narrower is the time interval of the rectangle ($2T$), the wider is the $sinc$ and more significant is the signal spreading in frequency.

5. To properly apply the sampling theorem, we need to observe that the sampling frequency f_s to be employed for a truncated sinusoid of frequency f_m is NOT $f_s \approx 2f_m$, but it must be remarkably higher: $f_s \gg 2f_m$.

   We are increasing the maximum frequency of the signal, so to avoid aliasing we have to increase also the sampling frequency.

1.3 ENERGY SIGNALS AND CORRELATION FUNCTIONS

1.3.1 SIGNAL ENERGY

The real goal is to find a way to distinguish signals from noise and to do that we have to understand some characteristic properties.

The first information that can help us is energy which is one of the main properties of a signal.

We can define the energy of a signal $x(t)$ as:

Energy $$E = \lim_{T\to\infty} \int_{-T}^{+T} x^2(\alpha)d\alpha = \int_{-\infty}^{+\infty} x^2(\alpha)d\alpha$$

Signal for which this quantity exists (in a finite way) are called **energy signals**. Typical energy signals are pulse signals.

<u>Intuitive view of energy</u>: Let $x(t)$ be a voltage pulse on a unitary resistance ($R = 1\Omega$), $P = \frac{V^2}{R}$, then E is the energy dissipated in R by the pulse.

1.3.2 SINGAL AUTO-CORRELATION FUNCTION (ENERGY-TYPE)

The autocorrelation function of a signal $x(t)$ is defined as:

Energy autocorrelation function $$k_{xx,E}(\tau) = \lim_{T\to+\infty} \int_{-T}^{+T} x(\alpha)x(\alpha+\tau)d\alpha = \int_{-\infty}^{+\infty} x(\alpha)x(\alpha+\tau)d\alpha$$

As you can see is quite similar to convolution.

<u>Intuitive interpretation of autocorrelation</u>:
The autocorrelation function $k_{xx}(\tau)$ gives us the degree of similarity of $x(t)$ with itself shifted by τ ($x(t+\tau)$).

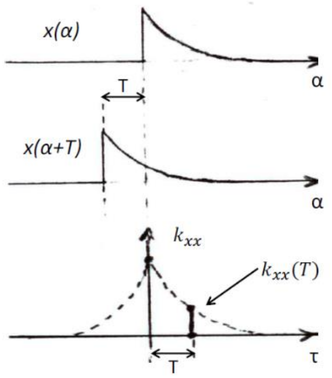
If we have an autocorrelation in time that is very concentrated at the origin this means that the signal is completely changing as we consider times larger than 0. Instead, if we have an autocorrelation very spread in time we expect our starting signal not to change so much for times larger than 0. Under this interpretation, looking at the autocorrelation waveform we can understand in some way where the information is localized in our signal of interest.

Why is it important? In a lot of applications, it is extremely useful, i.e. in biological applications it is used to understand the speed of molecules in diffusion processes.

We can easily notice that the signal energy is equal to the autocorrelation in zero ($\tau = 0$) of the signal itself:

$$E[x(t)] = \int_{-\infty}^{+\infty} x^2(\alpha)d\alpha = K_{xx,E}(0)$$

Francesco Gavetti

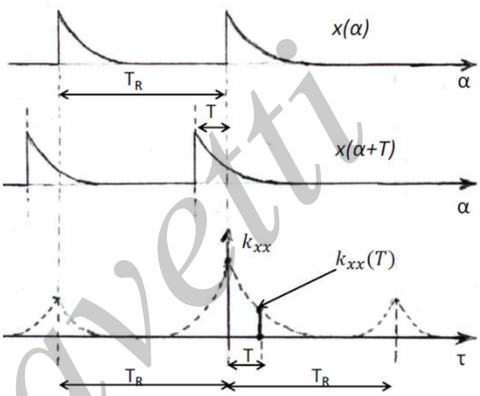


Example1: autocorrelation of an exponential decade time signal

$$x(\alpha) = 1(t)Ae^{-t/T_P}$$

1. We shift the signal by T ($x(\alpha + T)$)
2. We make the integral of the multiplication
3. We repeat the operation for any T

$$K_{xx,E}(\tau) = A^2 \frac{T_P}{2} e^{-|\tau|/T_P}$$



Example2: autocorrelation of two exponential decade time signals

$$x(t) = 1(t)Ae^{-t/T_P} + 1(t - T_R)Ae^{-(t-T_R)/T_P}$$

$$K_{xx}(\tau) = A^2 \frac{T_P}{2} e^{-|\tau|/T_P} + A^2 \frac{T_P}{2} e^{-|\tau - T_R|/T_P} + A^2 \frac{T_P}{2} e^{-|\tau + T_R|/T_P}$$

1.3.3 SIGNAL CROSS-CORRELATION FUNCTION (ENERGY-TYPE)

The cross-correlation function between $x(t)$ and $y(t)$ is defined as:

Energy cross-correlation function $K_{xy,E}(\tau) = \lim_{T\to\infty} \int_{-\infty}^{+\infty} x(\alpha)y(\alpha + \tau)d\alpha = \int_{-\infty}^{+\infty} x(\alpha)y(\alpha + \tau)d\alpha$

in which:

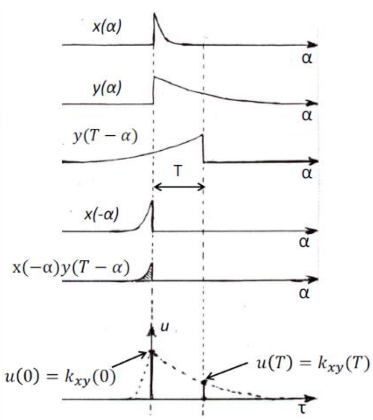
- $x(t)$ and $y(t)$ are two different energy signals
- $K_{xy}(\tau)$ gives the degree of similarity of (t) with $y(t)$ shifted by τ to left (towards earlier time)

Remember: the cross- and autocorrelation function are not defined for all the signals. In particular, it is possible that the integrals defining them do not converge. For this reason, these operations are defined just for a class of signals called energy signals, i.e. signals with finite energy, for which the integrals always converge. Later we will analyze how to generalize these concepts to signals with infinite energy.

Is it possible to connect the convolution to the cross-correlation from a graphical point of view?

The convolution is defined as $z(T) = x(\alpha) * y(\alpha) = \int_{-\infty}^{+\infty} x(\alpha)h(T - \alpha)d\alpha$ which is different from cross-correlation.

However, reversing the first signal in the convolution we'll have:



$x(-\alpha) * y(\alpha) = \int_{-\infty}^{+\infty} x(-\alpha)h(T - \alpha)d\alpha = K_{xy}(T)$

which is exactly equal to the cross-correlation.

Relation between convolution and cross-correlation:

$$k_{xy}(T) = x(-\alpha) * y(\alpha)$$

Now, we want to understand how energy is distributed in the spectrum (in frequency).

FrancescoGavetti

1.4 ENERGY SPECTRUM

1.4.1 FREQUENCY INTEPRETATION OF ENERGY

Applying the Parseval theorem we can obtain:

$$E[x(t)] = \int_{-\infty}^{+\infty} x^2(t)dt = \int_{-\infty}^{+\infty} |X(f)|^2 df = 2\int_{0}^{+\infty} |X(f)|^2 df$$

The term $|X(f)|^2 df$ in the previous formula may be thought as the contribution of energy from the component at frequency f of our signal.

Energy spectral density (energy spectrum) $S_{x,E}(f) = |X(f)|^2$

Intuitive interpretation of energy spectrum:

1. Let $x(t)$ be a voltage on a unitary resistance $R = 1\Omega$
2. $x(t)$ = sum of sinusoid components with frequency f and amplitude $|X(f)|df$
3. Sinusoids are orthogonal functions: no power from multiplication of different components (different f)

In this way we can compute the energy in a small amount of spectrum df, like:

$$dE = 2|X(f)|^2 df$$

which also represents the energy contribution given by every component at frequency f.

Spectrum gives us the distribution of energy, so we can cut parts of the spectrum where's no energy from the signal.

1.4.2 FOURIER TRANSFORM OF ENERGY AUTOCORRELATION, RELATION WITH ENERGY SPECTRUM

We have seen before that the energy of a signal corresponds to the value in zero of its autocorrelation function:

$$K_{xx,E}(0) = E[x(t)] = \int_{-\infty}^{+\infty} S_{x,E}(f)df$$

Recalling the "value in zero" property of the Fourier transform, according which the integral of the Fourier transform is equal to the value in zero of the original signal in time, we can guess that Fourier transform of the autocorrelation is the energy spectral density.

Demonstration:
To formally demonstrate it, let's use the expression of the autocorrelation in terms of convolution:

$$K_{xx,E}(\tau) = x(t) * x(-t)$$

Taking the Fourier transform of both sides and using the convolution property, we get:

$$F\left[K_{xx,E}(\tau)\right](f) = F[x(t) * x(-t)](f) = F[x(t)](f)\, F[x(-t)](f)$$

It is very easy to prove that $F[x(-t)](f) = F[x(t)](-f)$, i.e. flipping a function with respect the time axis also flips the Fourier transform with respect to the frequency axis, so:

$$F\left[K_{xx,E}(\tau)\right](f) = X(f)\overline{X(f)} = |X(f)|^2$$

But $|X(f)|^2$ is the energy spectral density as we have seen before. So it is true, the Fourier transform of the autocorrelation is the energy spectral density.

Energy spectral density (energy spectrum) $S_{x,E}(f) = F\left[K_{xx,E}(\tau)\right]$

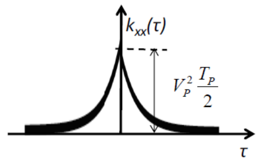
■

Francesco Gavetti

As we have already seen, the autocorrelation function gives us the intuition of how the information is distributed in time in our signal. This results shows that the concept of information is strictly related to the concept of energy.

<u>Example</u>:

Let's compute the energy spectral density of the signal $x(t) = V_p e^{-t/T_P}$



- Method 1:

$$K_{xx,E}(\tau) = V_p^2 \frac{T_P}{2} e^{-|\tau|/T_P}$$

We also know that the energy is the value in zero of the autocorrelation:

$$E = K_{xx,E}(0) = V_p^2 \frac{T_P}{2}$$

Provided that the spectrum $|X(f)|^2$ is the Fourier transform of the autocorrelation:

$$S_{x,E}(f) = |X(f)|^2 = F[K_{xx}(\tau)](f) = \int_{-\infty}^{+\infty} V_p^2 \frac{T_P}{2} e^{-|\tau|/T_P} e^{-j2\pi f\tau} d\tau = V_p^2 T_P^2 \frac{1}{1+(2\pi f T_P)^2}$$

- Method 2:

1.5 POWER SIGNALS, CORRELATION FUNCTIONS AND POWER SPECTRUM

1.5.1 POWER SIGNALS

So far, we considered only energy signals, that are signals with finite energy. Anyway, there are also signals with infinite energy, i.e. a constant (DC) signal, sinusoids, periodic signals, etc.

Let's take in consideration an infinite energy signal, if we look at it for a finite amount of time, obviously, it will have a finite energy, so the idea could be to understand which is the "flow" of energy for unit time. In order to do that, we can look at the signal for a finite amount of time T, compute its energy and then divide it by the time of observation itself.

Power $$P[x(t)] = \lim_{T\to\infty} \int_{-\frac{T}{2}}^{+\frac{T}{2}} \frac{x^2(\alpha)}{T} d\alpha = \lim_{T\to\infty} \int_{-T}^{+T} \frac{x^2(\alpha)}{2T} d\alpha = \lim_{T\to\infty} \frac{1}{T} E[x_T(t)]$$

where $x_T(t)$ is the truncated version of $x(t)$ and is defined as:

Truncated signal $$x_T(t) = \begin{cases} x(t), & |t| < \frac{T}{2} \\ 0, & |t| > \frac{T}{2} \end{cases}$$

Signal for which this quantity exists (in a finite way) are called **power signals**.

1.5.2 FREQUENCY DOMAIN INTERPRETATION OF POWER

Now, we want to give an intuitive interpretation of power in frequency domain, as we have already done for energy.

Truncated signals are finite energy signals, so we can apply all the theory developed so far to them.

If $x(t)$ is a non-periodic signal and $x_T(t)$ is the truncated signal as we have previously defined, we can write:

$$E[x_T(t)] = \int_{-\infty}^{+\infty} |X_T(f)|^2 df$$

FrancescoGavetti

in which $X_T(f)$ and $|X_T(f)|^2$ are respectively the Fourier transform and the energy spectral density of the truncated signal $x_T(t)$.

Recalling the power expression, we can thus write:

$$P[x(t)] = \lim_{T\to\infty} \frac{1}{T}\, E[x_T(t)] = \int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{1}{T}\, |X_T(f)|^2\, df$$

We can define:

Power spectral density (power spectrum) $S_{x,P}(f) = \lim_{T\to\infty} \frac{1}{T}\, |X_T(f)|^2$

In this way the signal power can be rewritten as:

$$P[x(t)] = \int_{-\infty}^{+\infty} S_{x,P}(f)\, df$$

Physical interpretation of power spectral density:
$S_x(f)$ is the density of power at the frequency f of our initial signal. In particular, multiplying $S_x(f)$ by a small frequency interval Δf we obtain the power content of our signal in the frequency interval $[f, f + \Delta f]$.

1.5.3 EXTENSION OF CORRELATION FUNCTIONS TO POWER SIGNALS

The auto-correlation function for power signals is defined as:

Power autocorrelation function $K_{xx,P}(\tau) = \lim_{T\to\infty} \int_{-T}^{+T} \frac{x(\alpha)x(\alpha+\tau)}{2T}\, d\alpha = \lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{x_T(\alpha)x_T(\alpha+\tau)}{2T}\, d\alpha = \lim_{T\to\infty} \frac{K_{xx,T}(\tau)}{2T}$

We can notice that, also this time, the signal power is equal to the autocorrelation in zero ($\tau = 0$) of the signal itself:

$$P[x(t)] = K_{xx,P}(0)$$

Also this time, we can definite the power spectrum as the Fourier transform of the autocorrelation function:

Power spectral density (power spectrum) $S_{x,P}(f) = F\left[K_{xx,P}(\tau)\right]$

Taking into account these last two equations, we can rewrite the signal power as:

Power $P[x(t)] = K_{xx,P}(0) = \int_{-\infty}^{+\infty} S_{x,P}(f) df$

The power cross-correlation functions for power signal, instead, is defined as:

Power cross-correlation function $K_{xy,P}(\tau) = \lim_{T\to\infty} \int_{-T}^{+T} \frac{x(\alpha)y(\alpha+\tau)}{2T}\, d\alpha = \lim_{T\to\infty} \int_{-\infty}^{+\infty} \frac{x_T(\alpha)y_T(\alpha+\tau)}{2T}\, d\alpha$

in which $x(t)$ and $y(t)$ are obviously two power signals.

Some considerations:

- The autocorrelation $K_{xx}(\tau)$ and the cross-correlation $K_{xy}(\tau)$ measures the degree of similarity of $x(t)$ with respect to itself and to $y(t)$, respectively

- If even only one of the two signals $x(t)$ and $y(t)$ is an energy-type signal, the energy-type cross-correlation must be employed.
  If one of the two signals is an energy signal and we apply the power cross-correlation, anyway, it turns that $K_{xy}(\tau) = 0$.

FrancescoGavetti