

# Fisher matrix for gravitational wave forecasting

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*When we build Einstein Telescope, how many compact binary signals will it be able to see? How well will it localize them in the sky? How well will it measure their parameters, such as the radii of neutron stars?*

# Gravitational wave data analysis

Gravitational wave data is modelled as signal plus additive noise:

$$d(t) = n(t) + h_{\theta}(t).$$

We can estimate the parameters  $\theta$  by exploring the posterior distribution

$$p(\theta|d) = \mathcal{L}(d|\theta)\pi(\theta) = \mathcal{N} \exp \left( (d|h_\theta) - \frac{1}{2}(h_\theta|h_\theta) \right) \pi(\theta) ,$$

where

$$(a|b) = 4\Re \int_0^\infty \frac{a(f)b^*(f)}{S_n(f)} \mathrm{d}f ,$$

while  $\pi(\theta)$  is our prior distribution on the parameters.

The posterior is explored stochastically, and we can compute summary statistics:

- ▶ mean  $\langle \theta_i \rangle$ ,
- ▶ variance  $\sigma_i^2 = \langle (\theta_i - \langle \theta_i \rangle)^2 \rangle$ ,
- ▶ covariance  $\mathcal{C}_{ij} = \langle (\theta_i - \langle \theta_i \rangle)(\theta_j - \langle \theta_j \rangle) \rangle$ .

At this stage, we are not making any approximation, and the covariance matrix is just a summary tool.

## Signal-to-noise ratio

The signal-to-noise ratio (SNR) statistic is  $\rho = (d|h)/\sqrt{(h|h)}$ .

With the expected noise realization,

$$\rho \approx \sqrt{(h|h)} = 2\sqrt{\int_0^\infty \frac{|h(f)|^2}{S_n(f)} df}.$$

## SNR thresholds

Without time shifts nor non-Gaussianities, the SNR would simply follow a  $\chi^2$  distribution with two degrees of freedom: “five  $\sigma$ ” significance with a threshold of  $\rho = 5.5$ .

In real data, typically:

$$\text{FAR} = \text{FAR}_8 \exp \left( -\frac{\rho - 8}{\alpha} \right) .$$

For BNS in O1:  $\alpha = 0.13$  and  $\text{FAR}_8 = 30000\text{yr}^{-1}$ .

## Parameter dependence of CBC signals

A discussion of the parameters a BNS signal depends on, with relative error ( $\sigma_x/x$ ) values computed from the parameter estimation of GW170817.



## Intrinsic parameters

- ▶ **masses**  $m_1$  and  $m_2$ :  $\sigma_x/x \sim 10\%$ ,
- ▶ **chirp mass**  $\mathcal{M}$ :  $\sigma_x/x \sim 0.1\%$ .

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

- ▶ **mass ratio**  $q = m_1/m_2$ :  $\sigma_x/x \sim 20\%$ .

We are measuring the *detector-frame* mass:

$$\mathcal{M} = \mathcal{M}_{\text{source}}(1 + z)$$

Alternative parametrization:

- ▶ **symmetric mass ratio**  $\nu = \mu/M = q/(1+q)^2$ :  $\sigma_x/x \approx 4\%$
- ▶ **total mass**  $M = m_1 + m_2$ :  $\sigma_x/x \approx 3\%$

- ▶ **aligned spin:**  $\chi_{1z}$  and  $\chi_{2z}$ :  $\sigma_x/x \sim 3$  and 10 respectively,
- ▶ **effective aligned spin**  $\chi_{\text{eff}} = (m_1\chi_{1z} + m_2\chi_{2z})/(m_1 + m_2)$ :  
 $\sigma_x/x \sim 1$  (compatible with zero)
- ▶ **precessing spin**  $\chi_p$ : compatible with zero,

- ▶ **tidal polarizability**  $\Lambda_1$  and  $\Lambda_2$ :  $\sigma_x/x \sim 1.5$ ,
- ▶ **effective tidal parameter**  $\tilde{\Lambda}$ :  $\sigma_x/x \sim 0.6$ .

$$\Lambda_i = \frac{2}{3} \kappa_2 \left( \frac{R_i c^2}{G m_i} \right)^5$$

## Extrinsic parameters

- ▶ **distance**  $d_L$   $\sigma_x/x \sim 20\%$ ,
- ▶ degeneracy with the **inclination** of the source,  $\iota$ :  
 $\sigma_x/x \sim 10\%$ ,
- ▶ **arrival time** at geocenter  $t_{\oplus}$ ,
- ▶ **phase**  $\phi$ ,
- ▶ **polarization** angle  $\psi$ :  $\sigma_x \sim 0.3\text{rad}$ ,
- ▶ **sky position** (ra, dec):  $\sigma_x \sim 2\text{deg}$  and  $9\text{deg}$ .

Sky area:

$$\Delta\Omega \approx -2\pi \log(1-0.9) |\cos(\text{dec})| \sqrt{\sigma_{\text{ra}}^2 \sigma_{\text{dec}}^2 - \text{cov}_{\text{ra, dec}}^2} \times \left( \frac{180 \text{ deg}}{\pi \text{ rad}} \right)^2$$

For GW170817, using the posterior covariance matrix, this approximation yields  $28 \text{ deg}^2$ .

## GW150914 comparison

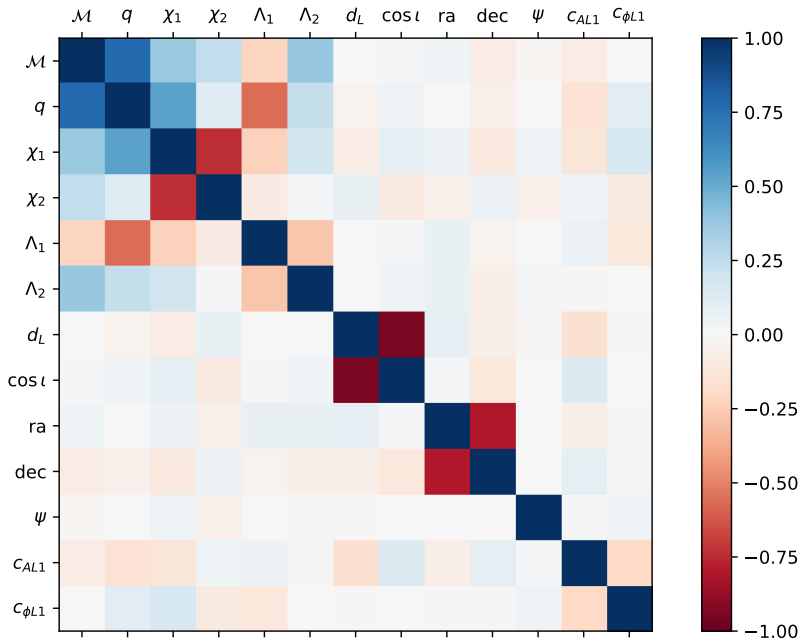
- ▶  $\sigma_{\mathcal{M}}/\mathcal{M} = \sigma_M/M \approx 3\%$ : not so many cycles
- ▶ two-detector event: sky area was  $600\text{deg}^2$ , but the Gaussian approximation gives  $1800\text{deg}^2$ .

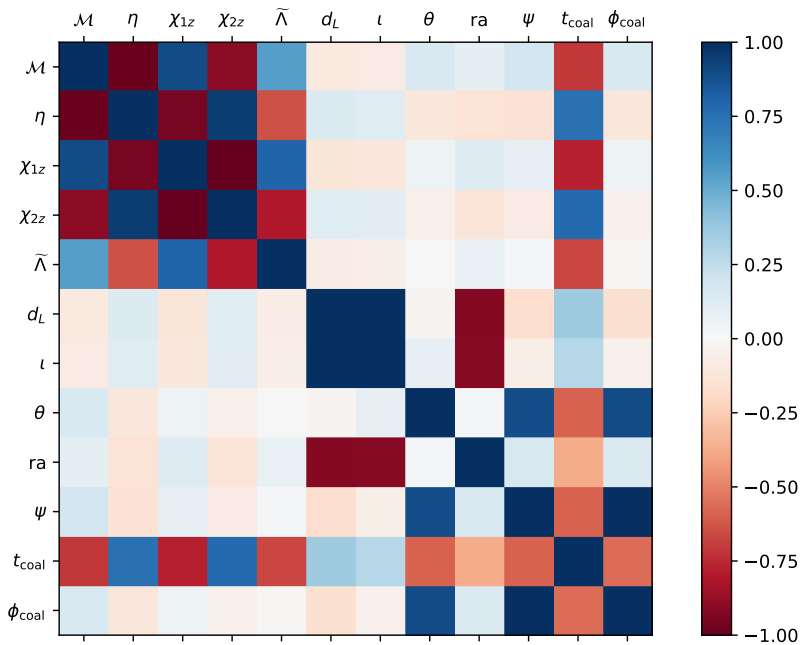
## Correlation structure

We can compute correlation coefficients:

$$\rho_{ij} = \frac{\text{cov}(\theta_i, \theta_j)}{\sigma_i \sigma_j}$$







## Antenna pattern

The strain at the detector depends on the antenna pattern:

$$h(t) = h_{ij}(t)D_{ij}(t) .$$

## Fisher matrix

In the Fisher matrix approximation, we are approximating the likelihood as

$$\mathcal{L}(d|\theta) \approx \mathcal{N} \exp \left( -\frac{1}{2} \Delta\theta^i \mathcal{F}_{ij} \Delta\theta^j \right)$$

where  $\Delta\theta^i = \theta^i - \langle \theta^i \rangle$ .

A **multivariate normal distribution**, with covariance matrix  $\mathcal{C}_{ij} = \mathcal{F}_{ij}^{-1}$ . This is a good approximation in the high-SNR limit.

The Fisher matrix  $\mathcal{F}_{ij}$  can be computed as the scalar product of the derivatives of waveforms:

$$\mathcal{F}_{ij} = \langle \partial_i \partial_j \mathcal{L} \rangle \big|_{n=0, \theta=\langle \theta \rangle} = (\partial_i h | \partial_j h) = 4\Re \int_0^\infty \frac{1}{S_n(f)} \frac{\partial h}{\partial \theta_i} \frac{\partial h^*}{\partial \theta_j} df.$$

The covariance matrix can be evaluated in  $\lesssim 1\text{s}$ , while full parameter estimation takes hours to weeks.

Tricky step computationally: inverting  $\mathcal{F}_{ij}$  to get  $\mathcal{C}_{ij}$ .

