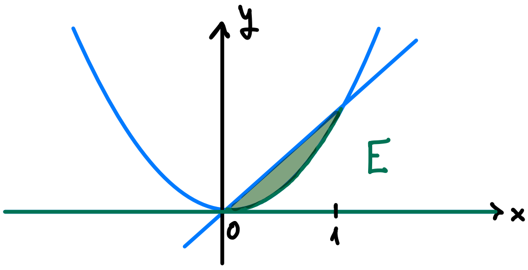
Foglio 4 : risoluzione

Posto $E = \{(x,y)^T : x^2 \le y \le x\} \cup \{(x,0)^T : x \in \mathbb{R}\}$,



si ha:

$$\mathrm{cl}\, E = \{(x,y)^T : x^2 \le y \le x\} \cup \{(x,y)^T : y = 0\}$$

$$\mathrm{int}\, E = \{(x,y)^T : x^2 < y < x\} = \{(x,y)^T : x^2 < y < x,\ 0 < x < 1\}$$

$$\mathrm{fr}\, E = \{(x,y)^T : y = x,\ 0 \le x \le 1\} \cup \{(x,y)^T : y = x^2,\ 0 \le x \le 1\} \cup \{(x,y)^T : y = 0\}$$

$$\mathrm{cl}(\mathrm{int}\, E) = \{(x,y)^T : x^2 \le y \le x\}$$

$$\mathrm{int}(\mathrm{fr}\, E) = \emptyset$$

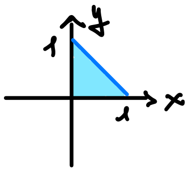
$$\mathrm{cl}(\mathrm{fr}\, E) = \mathrm{fr}\, E$$

$$\mathrm{fr}(\mathrm{fr}\, E) = \mathrm{fr}\, E$$

Descrivere geometricamente la sfera $B[\underline{0}, 1]$ in $(\mathbb{R}^2, d_p)$, con $1 \le p \le \infty$.

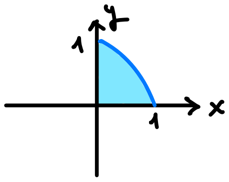
Per ragioni di simmetria basta descrivere l'intersezione di $B[\underline{0}, 1]$ con il I quadrante $[0,+\infty[\times [0,+\infty[$.

- $p = 1$:



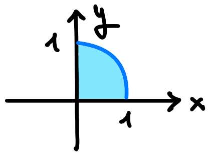
$$\{(x,y)^T : |x| + |y| \le 1\} \cap [0,+\infty[\times [0,+\infty[$$
$$= \{(x,y)^T : 0 \le y \le 1 - x,\ 0 \le x \le 1\}$$

- $1 < p < 2$:



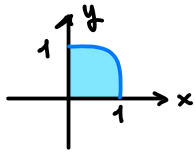
$$\{(x,y)^T : (|x|^p + |y|^p)^{\frac{1}{p}} \le 1\} \cap [0,+\infty[\times [0,+\infty[$$
$$= \{(x,y)^T : 0 \le y \le (1 - x^p)^{\frac{1}{p}},\ 0 \le x \le 1\}$$

- $p = 2$:



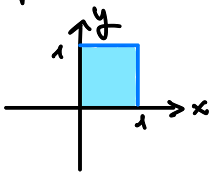
$$\{(x,y)^T : (x^2+y^2)^{\frac{1}{2}} \le 1\} \cap [0,+\infty[\times [0,+\infty[$$
$$= \{(x,y)^T : 0 \le y \le \sqrt{1-x^2}, 0 \le x \le 1\}$$

- $1 < p < \infty$:



$$\{(x,y)^T : (|x|^p + |y|^p)^{\frac{1}{p}} \le 1\} \cap [0,+\infty[\times [0,+\infty[$$
$$= \{(x,y)^T : 0 \le y \le (1-x^p)^{\frac{1}{p}}, \; 0 \le x \le 1\}$$

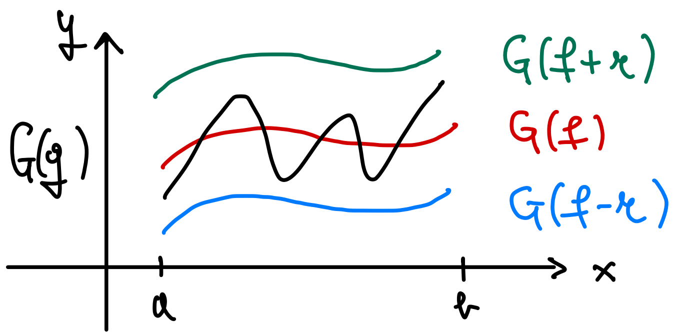
- $p = \infty$:



$$\{(x,y)^T : \max\{|x|, |y|\} \le 1\} \cap [0,+\infty[\times [0,+\infty[$$
$$= \{(x,y)^T : 0 \le x \le 1, \; 0 \le y \le 1\}$$

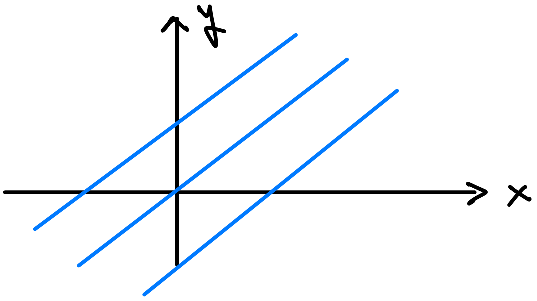
Descrivere le sfere $B(f,r)$ in $(\mathcal{C}^0([a,b]), d_\infty)$ per mezzo dei grafici.

$$B(f,r) = \{g \in \mathcal{C}^0([a,b]) : d_\infty(g,f) < r\}$$
$$= \{g \in \mathcal{C}^0([a,b]) : \max_{[a,b]} |g(x) - f(x)| < r\}$$
$$= \{g \in \mathcal{C}^0([a,b]) : |g(x) - f(x)| < r \text{ per ogni } x \in [a,b]\}$$
$$= \{g \in \mathcal{C}^0([a,b]) : f(x) - r < g(x) < f(x) + r \text{ per ogni } x \in [a,b]\}$$

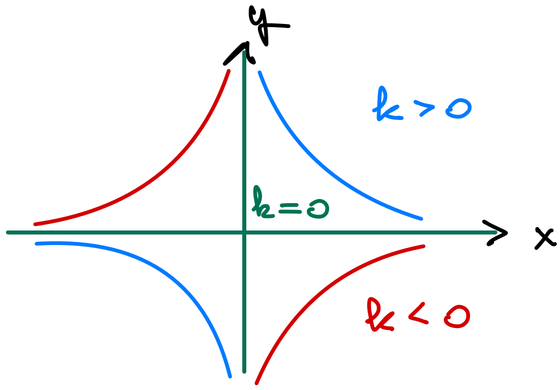


Tracciare gli insiemi di livello di :

$f(x,y) = x - y$: $\operatorname{dom} f = \mathbb{R}^2$, $L_k(f) = \{(x,y)^T \in \operatorname{dom} f : x - y = k\}$



$f(x,y) = xy$: $\operatorname{dom} f = \mathbb{R}^2$, $L_k(f) = \{(x,y)^T \in \operatorname{dom} f : xy = k\}$

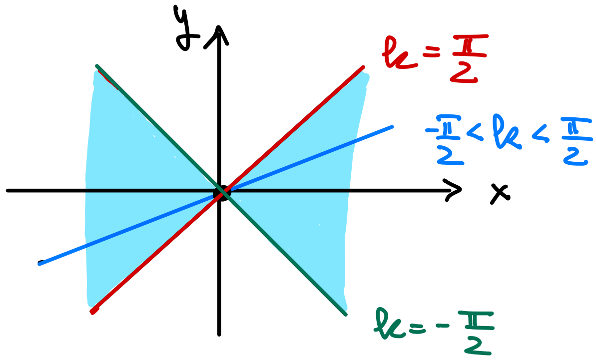


$f(x,y) = \arcsin\left(\frac{y}{x}\right)$: $\operatorname{dom} f = \{(x,y)^T : x \neq 0, \ \left|\frac{y}{x}\right| \leq 1\}$

$L_k(f) = \{(x,y)^T \in \operatorname{dom} f : \arcsin\left(\frac{y}{x}\right) = k\}$

$L_k(f) = \emptyset$ se $|k| > \frac{\pi}{2}$,

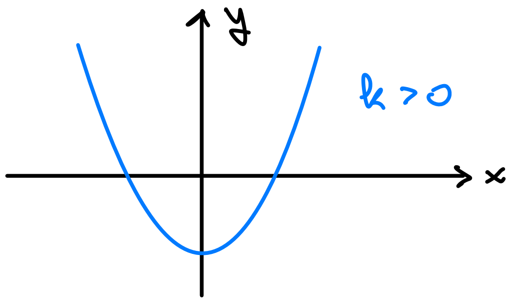
$L_k(f) = \{(x,y)^T : x \neq 0, \ y = (\sin k) x\}$ se $|k| \leq \frac{\pi}{2}$



$f(x,y) = e^{y-x^2}$: $\operatorname{dom} f = \mathbb{R}^2$, $L_k(f) = \{(x,y)^T \in \operatorname{dom} f : e^{y-x^2} = k\}$

$L_k(f) = \emptyset$ se $k \leq 0$,

$L_k(f) = \{(x,y)^T : y - x^2 = \log k\}$ se $k > 0$



$f(x,y,z) = x^2 - 2x + y^2 + z^2 + 1$: $\operatorname{dom} f = \mathbb{R}^3$

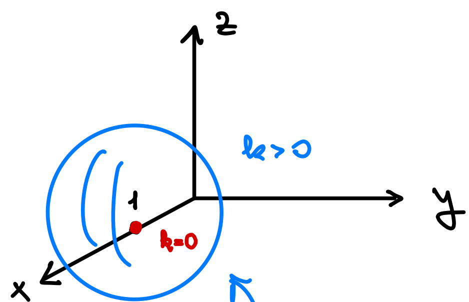
$L_k(f) = \{(x,y,z)^T \in \operatorname{dom} f : x^2 - 2x + y^2 + z^2 + 1 = k\}$

$= \{(x,y,z)^T : (x-1)^2 + y^2 + z^2 = k\}$

$L_k(f) = \emptyset$ se $k < 0$

$L_k(f) = \{(1,0,0)^T\}$ se $k = 0$

$L_k(f) = \{(x,y,z)^T : (x-1)^2 + y^2 + z^2 = (\sqrt{k})^2\}$ se $k > 0$



superficie sferica di centro $(1,0,0)^T$
e raggio $\sqrt{k}$

Provare che
$$\lim_{(x,y)^T \to (0,0)^T} \frac{x+1}{x^2+y^2+1} = 1,$$
cioè
$$(\forall \varepsilon>0)(\exists \delta>0)(\forall (x,y)^T)\left(0<\sqrt{x^2+y^2}<\delta \Rightarrow \left|\frac{x+1}{x^2+y^2+1}-1\right|<\varepsilon\right)$$
Fissato $\varepsilon>0$, si cerca $\delta>0$ tale che
$$\left|\frac{x+1}{x^2+y^2+1}-1\right| = \left|\frac{x+1-x^2-y^2-1}{x^2+y^2+1}\right| = \frac{|x^2+y^2-x|}{x^2+y^2+1} < \varepsilon,$$
se $0<\sqrt{x^2+y^2}<\delta$.

Poiché
$$|x^2+y^2-x| \le x^2+y^2+|x| \quad \text{e} \quad x^2+y^2+1 \ge 1$$
risulta
$$\frac{|x^2+y^2-x|}{x^2+y^2+1} \le x^2+y^2+|x| \le x^2+y^2+\sqrt{x^2+y^2}$$
$$= \left(\sqrt{x^2+y^2}\right)^2 + \sqrt{x^2+y^2} < \varepsilon,$$
se $0<\sqrt{x^2+y^2}<\delta$, con $\delta^2+\delta<\varepsilon \iff$
$$0<\delta<\frac{-1+\sqrt{1+4\varepsilon}}{2}$$

Provare che
$$\lim_{(x,y)^T \to (0,0)^T} \frac{1+x^2}{x^2+y^2} = +\infty,$$
cioè
$$(\forall K>0)(\exists \delta>0)(\forall (x,y)^T)\left(0<\sqrt{x^2+y^2}<\delta \Rightarrow \frac{1+x^2}{x^2+y^2}>K\right)$$

Fissato $K>0$, si cerca $\delta>0$ tale che

$$\frac{1+x^2}{x^2+y^2} > K, \qquad \text{se} \qquad 0 < \sqrt{x^2+y^2} < \delta.$$

Poiché

$$\frac{1+x^2}{x^2+y^2} \geq \frac{1}{x^2+y^2},$$

si ha

$$\frac{1+x^2}{x^2+y^2} \geq \frac{1}{x^2+y^2} > K \qquad \text{se } 0<x^2+y^2<\frac{1}{K},$$

cioè $0<\sqrt{x^2+y^2}<\delta$, con $\delta=\sqrt{\frac{1}{K}}>0$.

Provare che non esiste

$$\lim_{(x,y)^T\to(0,0)^T} \frac{xy}{x^2+y^2}.$$

Osserviamo che, posto $f(x,y)=\frac{xy}{x^2+y^2}$, si ha, per $x\neq 0$,

$$f(x,mx) = \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}, \qquad \text{cioè } f \text{ è costante}$$

sulle rette $y=mx$, con $m\in\mathbb{R}$.

Quindi risulta, in particolare,

$$\lim_{x\to 0} f(x,x) = \frac{1}{2} \qquad \text{e} \qquad \lim_{x\to 0} f(x,-x) = -\frac{1}{2},$$

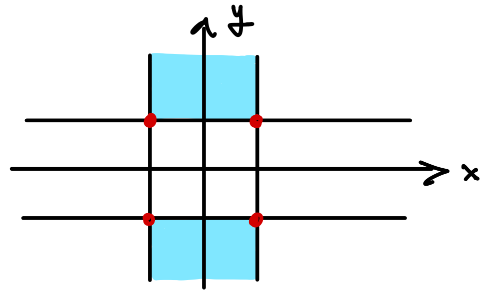
cioè le restrizioni di f a rette diverse hanno limiti diversi per $x\to 0$.

Se esistesse $\lim_{(x,y)^T\to(0,0)^T} f(x,y) = \ell$, allora

sarebbe $$\lim_{x\to 0} f(x,mx) = \ell \qquad \text{per ogni } m\in\mathbb{R}.$$

Determinare il dominio e stabilire i segni di:

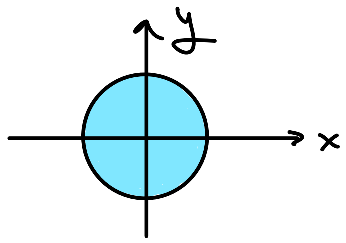
- $f(x,y) = \sqrt{1-x^2} + \sqrt{y^2-1}$:



$\mathrm{dom}\, f = \{(x,y)^T : \ |x| \leq 1,\ |y| \geq 1\}$

$f(x,y) \geq 0$ in $\mathrm{dom}\, f$, $f(x,y) = 0 \iff (x,y)^T = (\pm 1, \pm 1)^T$

- $f(x,y) = \log(1 - x^2 - y^2)$:

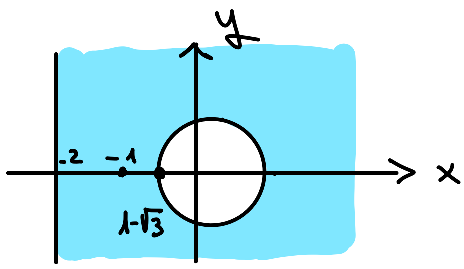


$\mathrm{dom}\, f = \{(x,y)^T : \ x^2 + y^2 < 1\}$

$f(x,y) \leq 0$ in $\mathrm{dom}\, f$, $f(x,y) = 0 \iff (x,y)^T = (0,0)^T$

- $f(x,y) = \log(x+2)\sqrt{x^2 - 2x + y^2 - 2}$

$\mathrm{dom}\, f = \{(x,y)^T : \ x > -2,\ (x-1)^2 + y^2 \geq 3\}$



$$f(x,y) > 0 \iff \begin{cases} x+2 > 1 \\ (x-1)^2 + y^2 < 3 \end{cases} \iff \begin{cases} x > -1 \\ (x-1)^2 + y^2 < 3 \end{cases}$$

$$f(x,y) = 0 \iff (x-1)^2 + y^2 = 3 \ \vee\ x = -1$$

- $f(x,y,z) = \sqrt{xy} + \sqrt{z-1}$

$\mathrm{dom}\, f = \{(x,y,z)^T : \ xy \geq 0,\ z \leq 1\}$

$f(x,y,z) \geq 0$ in $\mathrm{dom}\, f$

$$f(x,y,z) = 0 \iff \begin{cases} x = 0 \\ z = 1 \end{cases} \vee \begin{cases} y = 0 \\ z = 1 \end{cases}$$

- $f(x,y,z) = \dfrac{x-1}{\sqrt{x^2+y^2+z^2-1}}$

$\operatorname{dom} f = \{(x,y,z)^T : \ x^2+y^2+z^2>1\}$

$f(x,y,z)>0 \iff \begin{cases} x>1 \\ x^2+y^2+z^2>1 \end{cases}$

$f(x,y,z)=0 \iff \begin{cases} x=1 \\ x^2+y^2+z^2>1 \end{cases}$

- $f(x,y,z) = z\log(1-x^2-y^2)$

$\operatorname{dom} f = \{(x,y,z)^T : \ x^2+y^2<1\}$

$f(x,y,z)>0 \iff \begin{cases} z<0 \\ x^2+y^2<1 \end{cases}$

$f(x,y,z)=0 \iff \begin{cases} z=0 \\ x^2+y^2<1 \end{cases} \quad \vee \quad x^2+y^2=0$