$$\underset{n\times d}{\mathbb{X}} = \begin{bmatrix} | & | & & | \\ x^1 & \boxed{x^2} & \cdots & x^d \\ | & | & & | \end{bmatrix} = \begin{bmatrix} - & x_1^T & - \\ - & \boxed{x_2^T} & - \\ & \vdots & \\ - & x_n^T & - \end{bmatrix}$$

FEATURE $x^j \in \mathbb{R}^n$

OBSERVATION $x_i \in \mathbb{R}^n$

$$\text{cov}[\mathbb{X}] = \frac{1}{n}\mathbb{X}^T\mathbb{X} - \frac{1}{n}\mathbb{X}^T\mathbb{1}\,\frac{1}{n}\mathbb{1}^T\mathbb{X} = \frac{1}{n}\mathbb{X}^T\left(I_d - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right)\mathbb{X}$$

$$= \begin{bmatrix} \text{Var}[x^1] & \text{cov}[x^1,x^2] & \cdots & \text{cov}[x^1,x^d] \\ \text{cov}[x^2,x^1] & \text{Var}[x^2] & \cdots & \text{cov}[x^2,x^d] \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}[x^d,x^1] & \text{cov}[x^d,x^2] & \cdots & \text{Var}[x^d] \end{bmatrix}$$

$$= \frac{1}{n}\mathbb{X}^T\mathbb{X}$$ if the problem is CENTERED in 0_v.

$$\text{cov}[\mathbb{X}] = \begin{bmatrix} | & | & & | \\ v_1 & \boxed{v_2} & \cdots & v_d \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \phi \\ & \lambda_2 & \\ & & \ddots \\ \phi & & \lambda_d \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ & \vdots & \\ - & v_d^T & - \end{bmatrix}$$

PRINCIPAL DIRECTIONS $v_j \in \mathbb{R}^d$, $j = 1, \dots, d$

$$v_j \in \underset{\|u\|_2 = 1}{\text{argmax}}\; u^T \text{cov}[\mathbb{X}]\, u$$

PRINCIPAL COMPONENTS

$$\underset{n\times d}{y} = \begin{bmatrix} | & | & & | \\ y^1 & y^2 & \cdots & y^d \\ | & | & & | \end{bmatrix} = \mathbb{X}V = \begin{bmatrix} | & | & & | \\ x^1 & x^2 & \cdots & x^d \\ | & | & & | \end{bmatrix} \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_d \\ | & | & & | \end{bmatrix} \Rightarrow y^j \in \mathbb{R}^n$$

$$\begin{cases} y^1 = x_1 v_{11} + \dots + x_d v_{d1} \\ y^2 = x_1 v_{12} + \dots + x_d v_{d2} \\ \vdots \\ y^d = x_1 v_{1d} + \dots + x_d v_{dd} \end{cases}$$

$$y^1 = \begin{bmatrix} - & x_1^T & - \\ - & x_2^T & - \\ & \vdots & \\ - & x_n^T & - \end{bmatrix} \begin{bmatrix} | \\ v_1 \\ | \end{bmatrix} \longrightarrow$$ PROJECTION of OBSERVATIONS on the first PRINCIPAL DIRECTION

$$v_1 = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{d1} \end{bmatrix}$$

$$y = \mathbb{X}V \Rightarrow \mathbb{X} = yV^{-1} = yV^T = \begin{bmatrix} | & & | \\ y^1 & \cdots & y^d \\ | & & | \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ & \vdots & \\ - & v_d^T & - \end{bmatrix}$$

In the BIPLOT I have only 2 PRINCIPAL COMPONENTS, for example y^1 and y^2.
The coordinates of factors in the $\langle y^1, y^2 \rangle$ space are the components of the EIGENVECTORS $v_1, \dots, v_d$ associated to y^1 and y^2:

$$\underset{n\times d}{\mathbb{X}} = \underset{n\times 2}{\begin{bmatrix} | & | \\ y_1 & y_2 \\ | & | \end{bmatrix}} \underset{2\times d}{\begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \end{bmatrix}} = \underset{n\times 2}{\begin{bmatrix} | & | \\ y^1 & y^2 \\ | & | \end{bmatrix}} \underset{2\times d}{\begin{bmatrix} v_{11} & v_{21} & \cdots & v_{d1} \\ v_{12} & v_{22} & \cdots & v_{d2} \end{bmatrix}} = y^1 v_1 + y^2 v_2$$

So the generic feature $x^j = y^1 \cdot v_{j1} + y^2 \cdot v_{j2} \longrightarrow$ LINEAR COMBINATION of y^1 & y^2 using components of v_1 and v_2 as weights

component of v_1 associated to the original feature x^j

$$\begin{array}{cc} & \\ \begin{bmatrix} v_{11} & v_{12} \\ \boxed{v_{21} \quad v_{22}} \\ \vdots & \vdots \\ v_{d1} & v_{d2} \end{bmatrix} \\ \uparrow \; y^1 \quad \uparrow \; y^2 \end{array}$$

coordinates of feature x^2 in the space $\langle y^1, y^2 \rangle$

Supposing 2 features $x^A, x^B \in \mathbb{R}^n$, to draw the projection line corresponding to the first principal direction, we need only 2 point projected:

DOT PRODUCT

$$\begin{bmatrix} | & | \\ x^A & x^B \\ | & | \end{bmatrix} = \begin{bmatrix} x_1^A & x_1^B \\ x_2^A & x_2^B \end{bmatrix} \longrightarrow \begin{array}{l} P_1 = (x_1^A, x_1^B) \longrightarrow proj(P_1) = (v_1^T P_1) v \\ P_2 = (x_2^A, x_2^B) \longrightarrow proj(P_2) = (v_1^T P_2) v \\ v = \begin{bmatrix} v^A \\ v^B \end{bmatrix} \end{array}$$

$$proj(P_1) = \overbrace{(v^A x_1^A + v^B x_1^B)}^{\text{SCALAR}} \begin{bmatrix} v^A \\ v^B \end{bmatrix} = Q_1 \qquad \psi$$

$$proj(P_2) = (v^A x_2^A + v^B x_2^B) \begin{bmatrix} v^A \\ v^B \end{bmatrix} = Q_2 \qquad \varphi$$

I want the equation of the line $x^B = m \cdot x^A + q$ to draw on the space $\langle x^A, x^B \rangle$

Since the problem is centered $\Rightarrow q = 0$

$$\text{EQ.} \Rightarrow \frac{x^B - Q_1^B}{Q_2^B - Q_1^B} = \frac{x^A - Q_1^A}{Q_2^A - Q_1^A} \Rightarrow \frac{x^B - \psi v^B}{(\varphi - \psi) v^B} = \frac{x^A - \psi v^A}{(\varphi - \psi) v^A} \Rightarrow \boxed{x^B = \cancel{\psi v^B} + x^A \frac{v^B}{v^A} - \cancel{\psi v^B}}$$