Imperial College London
Department of Mathematics
Dr Igor Shevchenko

**Numerical Solution of Ordinary Differential Equations**



**I pledge that the work submitted for this coursework, both the report and the MATLAB code, is my own unassisted work unless stated otherwise.**

CID........................ 01730821

Are you a Year 4 student?.. yes

## Coursework 2

Fill in your CID and include the problem sheet in the coursework. Before you start working on the coursework, read the coursework guidelines. Any marks received for this coursework are only indicative and may be subject to moderation and scaling. *The mastery component is marked with a star.*

### Exercise 1 (Explicit LMM) — % of course mark: /5.0

Develop an explicit 3-step convergent method of your own design with the global error of order 3. Note that the methods from the Adams–Bashforth and Adams–Moulton families cannot be used in the development.

### Exercise 2 (Implicit LMM) — % of course mark: /5.0

Develop an implicit 3-step convergent method of your own design with the global error of order 3. Note that the methods from the Adams–Bashforth and Adams–Moulton families cannot be used in the development.

### Exercise 3 (Explicit and Implicit LMMs) — % of course mark: /5.0

Solve the initial value problem

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1000 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 2\sin(t) \\ 1000(\cos(t) - \sin(t)) \end{pmatrix}, \quad \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad t = [0, 30] \tag{1}$$

with the explicit and implicit LMM you developed.

### Exercise 4 (Global error and LMMs) — % of course mark: /4.0★

Theoretically estimate the global error of your LMMs and compare it with the global error computed numerically for system (1).

**Coursework mark:** **% of course mark**

## Coursework Guidelines

Below is a set of guidelines to help you understand what coursework is and how to improve it.

### Coursework

- The coursework requires more than just following what has been done in the lectures, some amount of individual work is expected.
- The coursework report should describe in a concise, clear, and coherent way of what you did, how you did it, and what results you have.
- The report should be understandable to the reader with the mathematical background, but unfamiliar with your current work.
- Do not bloat the report by paraphrasing or presenting the results in different forms.
- Use high-quality and carefully constructed figures with captions and annotated axis, put figures where they belong.
- All numerical solutions should be presented as graphs.
- Use tables only if they are more explanatory than figures. The maximum table length is a half page.
- All figures and tables should be embedded in the report. The report should contain all discussions and explanations of the methods and algorithms, and interpretations of your results and further conclusions.
- The report should be typeset in LaTeX or Word Editor and submitted as a single pdf-file.
- The maximum length of the report is ten A4-pages (additional 3 pages is allowed for Year 4 students); the problem sheet is not included in these ten pages.
- Do not include any codes in the report.
- Marks are not based solely on correctness. The results must be described and interpreted. The presentation and discussion is as important as the correctness of the results.

### Codes

- You cannot use third party numerical software in the coursework.
- The code you developed should be well-structured and organised, as well as properly commented to allow the reader to understand what the code does and how it works.
- All codes should run out of the box and require no modification to generate the results presented in the report.

### Submission

- The coursework submission must be made via Turnitin on your Blackboard page. You must complete and submit the coursework anonymously, **the deadline is 1pm on the date of submission** (unless stated otherwise). The coursework should be submitted via two separate Turnitin drop boxes as a pdf-file of the report and a zip-file containing MATLAB (m-files only) or Python (py-files only) code. The code should be in the directory named `CID_Coursework#`. The report and the zip-file should be named as `CID_Coursework#.pdf` and `CID_Coursework#.zip` , respectively. The executable MATLAB (or Python) scripts for the exercises should be named as follows: exercise1.m, exercise2.m, etc.

# Numerical odes - Coursework 2

CID: 01730921

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## 1 General LMM convergence order 3

The general form of a linear 3-step method is given by:

$$\sum_{m=0}^{3} \alpha_m x_{n+m} \; = \; h \sum_{m=0}^{3} \beta_m f(t_{n+m}, x_{n+m}) \tag{1}$$

where $\alpha_m$, $\beta_m \; = \; const \in \mathbb{R}$ and we take $\alpha_3 = 1$ as a normalizing constant. Furthermore, if the method is explicit then $\beta_3 = 0$.

**Theorem 1.** *[**Dahlquist equivalence theorem**] An LMM is convergent $\iff$ it is both consistent and zero-stable.*

**Theorem 2.** *The global error of a convergent LMM equals to its order of consistency.*

Therefore, in order to find a convergent method with global error of order 3, it is enough to show consistency of order 3 and zero-stability. Definition of consistency and zero-stability will be given in their respective section.

### 1.1 Consistency equation

For consistency to be satisfied we define the linear difference operator associated with the LMM to be

$$\mathcal{L}_h x(t) = \sum_{m=0}^{3} \alpha_m x(t + mh) \; - \; h \sum_{m=0}^{3} \beta_m x'(t + mh) \tag{2}$$

**Definition 1.** *The linear difference operator $\mathcal{L}_h$ is said to be consistent of order $p$ if*

$$\mathcal{L}_h x(t) = \mathcal{O}(h^{p+1}), \; p > 0 \tag{3}$$

In order to find $p$ such that (3) is true, we Taylor expand $x(t+mh)$ and $x'(t+mh)$ in (2) to get:

$$\mathcal{L}_h x(t_n) \;=\; \sum_{m=0}^{3} \alpha_m \sum_{k=0}^{\infty} \frac{(mh)^k}{k!} x^{(k)}(t_n) \;-\; h \sum_{m=0}^{3} \beta_m \sum_{k=0}^{\infty} \frac{(mh)^k}{k!} x^{(k+1)}(t_n) \tag{4}$$

recalling that $\alpha_3 = 1$ by definition.

Expanding (4), we consider terms with the same factor $h^p$. To get consistency of order $p$ this terms are set equal to 0 up to the one containing $h^p$ (included).

1. At p=0

$$x(t_n) \sum_{m=0}^{3} \alpha_m x(t_n) = 0 \implies 1 + \alpha_2 + \alpha_1 + \alpha_0 = 0$$

2. At p=1:

$$hx'(t_n) \sum_{m=0}^{3} (m\alpha_m - \beta_m) = 0 \implies 3 + 2\alpha_2 + \alpha_1 = \beta_3 + \beta_2 + \beta_1 + \beta_0$$

3. At p=2:

$$h^2 x''(t_n) \sum_{m=0}^{3} \left( \frac{\alpha_m m^2}{2} - m\beta_m \right) = 0 \implies 9 + 4\alpha_2 + \alpha_1 = 6\beta_3 + 4\beta_2 + 2\beta_1$$

4. At p=3:

$$h^3 x'''(t_n) \sum_{m=0}^{3} \left( \frac{\alpha_m m^3}{6} - \frac{\beta_m m^2}{2} \right) = 0 \implies 27 + 8\alpha_2 + \alpha_1 = 27\beta_3 + 12\beta_2 + 3\beta_1$$

Finally, we want to check that the method is not of order grater than 4, thus at p=4

$$h^4 x^{(4)}(t_n) \sum_{m=0}^{3} \left( \frac{\alpha_m m^4}{24} - \frac{\beta_m m^3}{6} \right) \neq 0 \implies 81 + 16\alpha_2 + \alpha_1 \neq 108\beta_3 + 32\beta_2 + 4\beta_1 \tag{5}$$

Rearranging in a system of equations we conclude that *necessary conditions* for consistency of order 3 are

$$\begin{cases} 1 + \alpha_2 + \alpha_1 + \alpha_0 = 0 \\ 3 + 2\alpha_2 + \alpha_1 = \beta_3 + \beta_2 + \beta_1 + \beta_0 \\ 9 + 4\alpha_2 + \alpha_1 = 6\beta_3 + 4\beta_2 + 2\beta_1 \\ 27 + 8\alpha_2 + \alpha_1 = 27\beta_3 + 12\beta_2 + 3\beta_1 \end{cases} \tag{6}$$

## 1.2 Zero-stability

**Definition 2.** *For a 3-step LMM, define its first characteristic polynomial as*

$$\rho(r) = \sum_{m=0}^{3} \alpha_m r^m$$

*and its second characteristic polynomial as*

$$\sigma(r) = \sum_{m=0}^{3} \beta_m r^m$$

**Definition 3** (**The root condition**)**.** *A polynomial of degree $n$ is said to satisfy the root condition iff all its roots $|r_i|_{i\in[1,n]} < 1$.*

**Definition 4** (**Zero stability**)**.** *An LMM is said to be zero-stable is its first characteristic polynomial, $\rho(r)$, satisfies the root condition.*

To enforce zero-stability we seek first characteristic polynomial that suits our needs. Since we want to keep numbers as simple as possible, and since we know that setting $\alpha_1 = \alpha_0 = 0$ would lead us to either Adam-Bashforth or Adam-Moulton methods, we choose the characteristic polynomial to be of the form

$$\rho(r) = r(r-1)(r-c) = r^3 - (1+c)r^2 + cr, \qquad c \in (-1, 1)$$

The root condition is then satisfied as the roots are 0, 1 and $c$. Moreover, $\alpha_3 = 1$, $\alpha_2 = -(1+c)$, $\alpha_1 = c$, and $\alpha_0 = 0$, hence $1 + \alpha_2 + \alpha_1 + \alpha_0 \ = \ 1 - (1+c) + c = 0$ for any $c \in (-1, 1)$, implying that the first of the consistency equations is also satisfied.

For both explicit and implicit method we then choose $c = 1/2$, implying

$$\alpha_2 \ = \ -3/2, \quad \alpha_1 \ = \ 1/2, \quad \alpha_0 \ = \ 0 \tag{7}$$

## 1.3 Question 1 - Explicit method

As we are looking for an explicit method we set $\beta_3 = 0$.

Substituting the values (7) and $\beta_3 = 0$ into the consistency equations (6), we get the following system of equations that is not under determined anymore (i.e. it has at most 1 solution):

$$\begin{cases} \beta_2 + \beta_1 + \beta_0 = 1/2 \\ 4\beta_2 + 2\beta_1 = 7/2 \\ 12\beta_2 + 3\beta_1 = 31/2 \end{cases} \tag{8}$$

The system above has a unique solution that is

$$\beta_2 = 41/24, \quad \beta_1 = -5/3, \quad \beta_0 = 11/24 \tag{9}$$

To check that the order is not greater than 3 we substitute this values into (5)

$$115/2 \;=\; 81 + 16(-3/2) + (1/2) \;\neq\; 108(0) + 32(41/24) + 4(-5/3) \;=\; 48$$

Since the above relation is true, we use Theorems 1 and 2 to affirm we have found an explicit 3-step LMM with global order 3.

Hence, given the IVP, $x' = f(t, x),\ x(t_0) = x_0$, the explicit method we have found is

$$x_{n+3} = \frac{3}{2}x_{n+2} - \frac{1}{2}x_{n+1} + \frac{h}{24}\left(41f_{n+2} - 40f_{n+1} + 11f_n\right) \tag{10}$$

or more precisely:

$$x(t_{n+3}) = \frac{3}{2}x(t_{n+2}) - \frac{1}{2}x(t_{n+1}) + \frac{h}{24}\left(41f(t_{n+2}, x(t_{n+2})) - 40f(t_{n+1}, x(t_{n+1})) + 11f(t_n, x(t_n))\right) \tag{11}$$

## 1.4 Question 1 - Implicit method

For the implicit case, $\beta_3 \neq 0$. Also, for simplicity we set $\beta_1 = 0$. Considering again the consistency equations (6) and the chosen values for the alphas (7), the system of equation for the implicit method is:

$$\begin{cases} \beta_3 + \beta_2 + \beta_0 = 1/2 \\ 6\beta_3 + 4\beta_2 = 7/2 \\ 27\beta_3 + 12\beta_2 = 31/2 \end{cases} \tag{12}$$

Again the only solution is given by

$$\beta_3 = 5/9, \quad \beta_2 = 1/24, \quad \beta_0 = -7/72 \tag{13}$$

To check that the order is not greater than 3 we substitute this values into (5)

$$115/2 \;=\; 81 + 16(-3/2) + (1/2) \;\neq\; 108(5/9) + 32(1/24) + 4(0) \;=\; 184/3$$

Since the above relation is true, we use Theorems 1 and 2 to affirm we have found an implicit 3-step LMM with global order 3.

Hence, given the IVP, $x = f(t, x),\ x(t_0) = x_0$, the explicit method is

$$x_{n+3} = \frac{3}{2}x_{n+2} - \frac{1}{2}x_{n+1} + \frac{h}{72}\left(40f_{n+3} + 3f_{n+2} - 7f_n\right) \tag{14}$$

or more precisely:

$$x(t_{n+3}) = \frac{3}{2}x(t_{n+2}) - \frac{1}{2}x(t_{n+1}) + \frac{h}{72}\left(40f(t_{n+3}, x(t_{n+3})) + 3f(t_{n+2}, x(t_{n+2})) - 7f(t_n, x(t_n))\right) \tag{15}$$

## 2 Numerical solution to IVP

The task here is to solve numerically and analytically

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1000 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2\sin(t) \\ 1000(\cos(t) - \sin(t)) \end{bmatrix}, \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{16}$$

and $t = [0, 30]$.

To begin with, we re-write the above system of ODEs in matrix form:

$$\mathbf{x}'(t) \;=\; \mathbf{A}\mathbf{x}(t) \;+\; \mathbf{b}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{17}$$

where:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} -1 & 1 \\ 1 & -1000 \end{bmatrix}, \qquad \mathbf{b}(t) = \begin{bmatrix} 2\sin(t) \\ 1000(\cos(t) - \sin(t)) \end{bmatrix} \qquad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Furthermore, accordingly to the definition of IVP, denote the function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^2$ as

$$f(t, \mathbf{x}(t)) \;=\; \mathbf{A}\mathbf{x}(t) + \mathbf{b}(t) \tag{18}$$

From now on we will denote the numerical solution at time $t_n$ as $\mathbf{x}_n$ and $\mathbf{b}(t_n)$ as $\mathbf{b}_n$.

Matrix $\mathbf{A}$ can be written in terms of the diagonal matrix $\mathbf{\Lambda}$ and the eigenvector matrix $\mathbf{V}$: $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$. In this case:

$$\mathbf{\Lambda} = \mathrm{diag}\,(\frac{-\sqrt{998005} - 1001}{2},\; \frac{-\sqrt{998005} + 1001}{2}) \tag{19}$$

Diagonalisation is needed to compute the analytical solution, as we know that the solution to the homogeneous problem is:

$$\mathbf{x}(t) = c_1 \exp(\lambda_1)\mathbf{v}_1 + c_2 \exp(\lambda_2)\mathbf{v}_2$$

where $c_1, c_2 \;=\; const.$, $\mathbf{v}_1$, $\mathbf{v}_2$ are the e.vectors and $\lambda_1$, $\lambda_2$ the e.values. We substitute initial conditions to find $c_1, c_2$. Finally we seek solution of the in-homogeneous part by Ansatz:

$$\mathbf{b}^*(t) = (K_1\cos(t) + M_1\sin(t),\; K_2\cos(t) + M_2\sin(t)) \qquad K_1, K_2, M_1, M_2 \;=\; const.$$

After solving the system of equation for $K_1, K_2, M_1, M_2$, we find the analytical solution to (16)

$$
\begin{aligned}
x_1(t) \;&=\; [(998005 - 1003\sqrt{998005})\exp(\lambda_1 t) + (998005 + 1003\sqrt{998005})\exp(\lambda_2 t)]/1996010 \\
&\quad + (-1004\cos(t) + 2001002\sin(t))/1998005 \\
x_2(t) \;&=\; [(998005 + 998\sqrt{998005})\exp(\lambda_1 t) + (998005 - 998\sqrt{998005})\exp(\lambda_2 t)]/998005 \\
&\quad + (1999998\cos(t) - 1994004\sin(t))/1998005
\end{aligned}
$$

As we are dealing with 3-step methods, to start the numerical scheme we need to compute $\mathbf{x}_1 = \mathbf{x}(t_0 + h)$ $\mathbf{x}_2 = \mathbf{x}(t_0 + 2h)$. We choose to apply forward Euler method twice.

Recall that Euler method of an IVP $\mathbf{x}' = f(t, \mathbf{x})$, is given by $\mathbf{x}_{n+1} = \mathbf{x}_n + hf(t_n, \mathbf{x}_n)$.

Therefore:

$$
\mathbf{x}_1 \;=\; \mathbf{x}_0 + h(\mathbf{A}\mathbf{x}_0 + \mathbf{b}_0), \qquad \mathbf{x}_2 \;=\; \mathbf{x}_1 + h(\mathbf{A}\mathbf{x}_1 + \mathbf{b}_1)
$$

**Explicit method**

The implementation of the python code has the following steps:

1. Initialisation of function $f$ that takes as input $(t, \mathbf{x})$ and outputs $\mathbf{A}\mathbf{x}(t) + \mathbf{b}(t)$.
2. Initialisation of number of steps given by $30/h$.
3. Initialisation of *solution_matrix* of dimension (2, steps + 1). The first column is given by the initial values of the IVP.
4. Computation of the second and third value of the solution using forward Euler method.
5. Implementation of a for loop where the values for the following steps are computed using (10).

**Implicit method**

The method we wish to implement on the given IVP is:

$$
\mathbf{x}_{n+3} = \frac{3}{2}\mathbf{x}_{n+2} - \frac{1}{2}\mathbf{x}_{n+1} + \frac{h}{72}\left(40f_{n+3} + 3f_{n+2} - 7f_n\right)
$$

where as usual: $\mathbf{x}_{n+m} = (x^1_{n+m}, x^2_{n+m})$ and $f_{n+m} = f(t, \mathbf{x}_{n+m})$ is defined in (18).

By definition we have that $f_{n+3} = \mathbf{A}\mathbf{x}_{n+3} + \mathbf{b}_{n+3}$, thus substituting into the implicit scheme we get

$$\mathbf{x}_{n+3} = \frac{3}{2}\mathbf{x}_{n+2} - \frac{1}{2}\mathbf{x}_{n+1} + \frac{h}{72}\left(40(\mathbf{A}\mathbf{x}_{n+3} + \mathbf{b}_{n+3}) + 3f_{n+2} - 7f_n\right)$$

rearranging we find

$$\mathbf{x}_{n+3} = \left(\mathrm{Id}_2 - \frac{40h}{72}\mathbf{A}\right)^{-1}\left(\frac{3}{2}\mathbf{x}_{n+2} - \frac{1}{2}\mathbf{x}_{n+1} + \frac{h}{72}\left(40\mathbf{b}_{n+3} + 3f_{n+2} - 7f_n\right)\right) \tag{20}$$

The Python code is the same of the explicit scheme and the only difference is at step 5, as the solution is computed using (20).

To ensure convergence, we seek a value of h small enough that is within the stability region. $h = 10^{-4}$ works for both our methods.

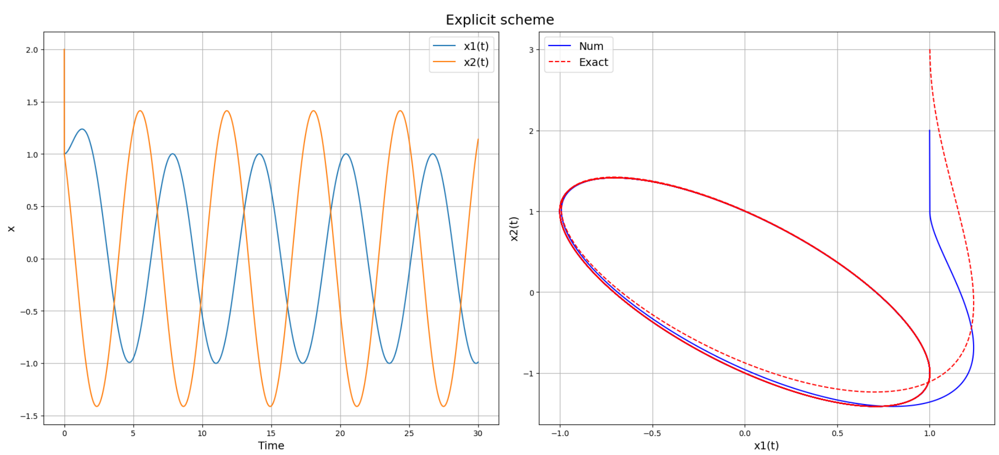


Figure 1: Numerical solution of the aforementioned explicit scheme for the given IVP. Here $h = 0.00001$. In the second subplot the exact solution is plotted as a dashed red line. We appreciate convergence even if it is not the fastest.

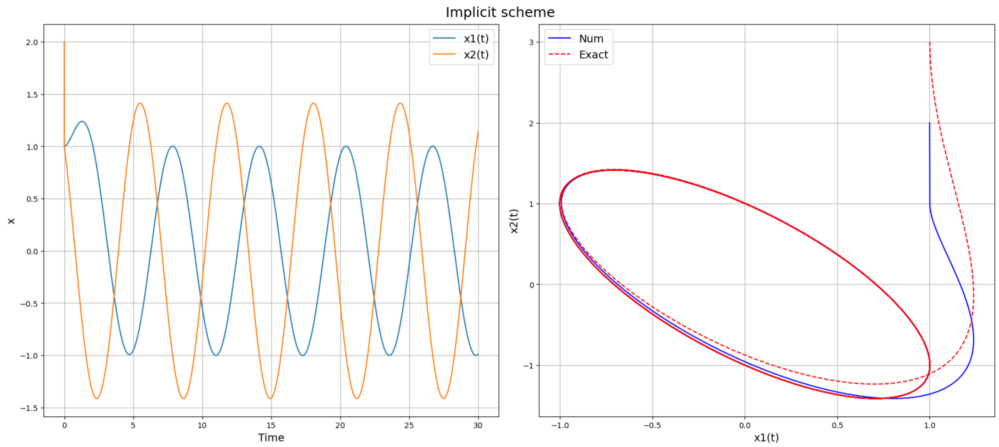


Figure 2: Numerical solution of the aforementioned implicit scheme for the given IVP. Here $h = 0.00001$. In the second subplot the exact solution is plotted as a dashed red line. We appreciate convergence even if it is not the fastest.

# 3 Global error estimation

**Theorem 3** (**LTE order**). *If the consistency order is $p$, the local truncation error order is $p+1$.*

**Lemma 1** (**Lipscitch continuity**). *$f$ Lipschitz continuous means that*

$|f(t_n, x(t_n)) - f_n| \leq L|x(t_n) - x_n|, \; L = const \geq 0$

As both of our methods are of consistency order 3 the local truncation error is of order 4.

In terms of notation in the section below $\mathbf{x}(t_n)$ represents the exact solution and $\mathbf{x}_n$ is the numerical one.

## 3.1 Explicit method

As it follows from the analysis of the local truncation error of explicit method, letting $R = \mathcal{O}(h^4)$, we get:

$$
\begin{aligned}
e_{n+1} &= x(t_n) - x_n \\
&= \frac{3}{2}(\mathbf{x}(t_n) - \mathbf{x}_n) - \frac{1}{2}(\mathbf{x}(t_{n-1}) - \mathbf{x}_{n-1}) + \frac{41h}{24}(f(t_n, \mathbf{x}(t_n)) - f_n) - \frac{40h}{24}(f(t_{n-1}, \mathbf{x}(t_{n-1})) - f_{n-1}) \\
&+ \frac{11h}{24}(f(t_{n-2}, \mathbf{x}(t_{n-2})) - f_{n-2})) + R \\
&= \frac{3}{2}e_n - \frac{1}{2}e_{n-1} + \frac{41h}{24}(f(t_n, \mathbf{x}(t_n)) - f_n) - \frac{40h}{24}(f(t_{n-1}, \mathbf{x}(t_{n-1})) - f_{n-1}) + \frac{11h}{24}(f(t_{n-2}, \mathbf{x}(t_{n-2})) - f_{n-2}) + R
\end{aligned}
$$

Taking absolute values and by triangle inequality:

$$\begin{aligned}|e_{n+1}| \;\leq\; & \frac{3}{2}|e_n| + \frac{1}{2}|e_{n-1}| + \frac{41h}{24}|f(t_n, \mathbf{x}(t_n)) - f_n| + \frac{40h}{24}|f(t_{n-1}, \mathbf{x}(t_{n-1})) - f_{n-1}| \\ & + \frac{11h}{24}|f(t_{n-2}, \mathbf{x}(t_{n-2})) - f_{n-2}| + |R|\end{aligned}$$

Using Lipschitz continuity and setting $\hat{h} := Lh$:

$$\begin{aligned}|e_{n+1}| \;\leq\; & \frac{3}{2}|e_n| + \frac{1}{2}|e_{n-1}| + \frac{41\hat{h}}{24}|x(t_n) - \mathbf{x}_n| + \frac{40\hat{h}}{24}|x(t_{n-1}) - \mathbf{x}_{n-1}| + \frac{11\hat{h}}{24}|x(t_{n-2}) - \mathbf{x}_{n-2}| + |R| \\ = \; & \frac{3}{2}|e_n| + \frac{1}{2}|e_{n-1}| + \frac{41\hat{h}}{24}|e_n| + \frac{40\hat{h}}{24}|e_{n-1}| + \frac{11\hat{h}}{24}|e_{n-2}| + |R| \\ = \; & \left(\frac{3}{2} + \frac{41\hat{h}}{24}\right)|e_n| + \left(\frac{1}{2} + \frac{40\hat{h}}{24}\right)|e_{n-1}| + \frac{11\hat{h}}{24}|e_{n-2}| + |R|\end{aligned}$$

Introducing the error bounding function $\delta_n = \max_{\substack{0\leq i\leq n \\ n\in[0,N]}} |e_n|$ and rewriting above equation in terms of $\delta_n$:

$$\delta_{n+1} \leq \left(\frac{3}{2} + \frac{41\hat{h}}{24}\right)\delta_n + \left(\frac{1}{2} + \frac{40\hat{h}}{24}\right)\delta_n + \frac{11\hat{h}}{24}\delta_n + |R| \;=\; \left(2 + \frac{23\hat{h}}{6}\right)\delta_n + |R|$$

For $n = 0, 1, 2, \ldots, N-1$ we find that:

$$\begin{aligned}\delta_0 \;&=\; x(t_0) - x_0 = 0, \\ \delta_1 \;&\leq\; |R_1|, \\ \delta_2 \;&\leq\; \left(2 + \frac{23\hat{h}}{6}\right)\delta_1 + |R_2| = \left(2 + \frac{23\hat{h}}{6}\right)|R_1| + |R_2|, \\ \delta_3 \;&\leq\; \left(2 + \frac{23\hat{h}}{6}\right)\delta_2 + |R_3| = \left(2 + \frac{23\hat{h}}{6}\right)^2|R_1| + \left(2 + \frac{23\hat{h}}{6}\right)|R_2| + |R_3|, \\ &\;\;\vdots \\ \delta_N \;&\leq\; \sum_{i=1}^{N}\left(2 + \frac{23\hat{h}}{6}\right)^{N-i}|R_i|\end{aligned}$$

In order to estimate $|\delta_N|$, we use the following inequality

$$2\left(1 + \frac{23\hat{h}}{12}\right) \leq 2\exp\left(\frac{23\hat{h}}{12}\right) \tag{21}$$

Exponentiating (21) to the power of $(N-i)$ and the fact that $2 < e$ gives

$$\left(2 + \frac{23\hat{h}}{6}\right)^{N-i} \leq 2^{N-i} \exp\left(\frac{23\hat{h}}{12}(N-i)\right) \leq \exp\left(\frac{23\hat{h}}{12}(N-i) + (N-i)\right) = \exp\left(\frac{35\hat{h}}{12}(N-i)\right) \tag{22}$$

Using (22) and the fact that $\hat{h}(N-i) < \hat{h}N = LhN = Lt_N$ the sum becomes:

$$\begin{aligned}
\delta_N &\leq \sum_{i=1}^{N} \exp\left(\frac{35}{12}Lt_N\right)|R_i| \\
&\leq N \exp\left(\frac{35}{12}Lt_N\right)Ch^4, \quad \text{since } |R_i| \leq Ch^4, \;\; C = const > 0 \\
&\leq \exp\left(\frac{35}{12}Lt_N\right)Ch^3 t_N \quad \text{since } t_N = Nh
\end{aligned}$$

Finally, we have

$$\max_{n \in [0,N]} |x(t_n) - x_n| \leq Bh^3, \quad B := \exp\left(\frac{35}{12}Lt_N\right)Ct_N \tag{23}$$

### 3.2 Implicit method

As it follows from the analysis of the local truncation error of implicit method, letting $R = \mathcal{O}(h^4)$, we get:

$$\begin{aligned}
e_{n+1} &= x(t_n) - x_n \\
&= \frac{3}{2}(\mathbf{x}(t_n) - \mathbf{x}_n) - \frac{1}{2}(\mathbf{x}(t_{n-1}) - \mathbf{x}_{n-1}) + \frac{5h}{9}(f(t_{n+1}, \mathbf{x}(t_{n+1})) - f_{n+1}) + \frac{h}{24}(f(t_n, \mathbf{x}(t_n)) - f_n) \\
&- \frac{7h}{72}(f(t_{n-2}, \mathbf{x}(t_{n-2})) - f_{n-2})) + R \\
&= \frac{3}{2}e_n - \frac{1}{2}e_{n-1} + \frac{5h}{9}(f(t_{n+1}, \mathbf{x}(t_{n+1})) - f_{n+1}) + \frac{h}{24}(f(t_n, \mathbf{x}(t_n)) - f_n) - \frac{7h}{72}(f(t_{n-2}, \mathbf{x}(t_{n-2})) - f_{n-2}) + R
\end{aligned}$$

Taking absolute values and by triangle inequality

$$\begin{aligned}
|e_{n+1}| \leq\; & \frac{3}{2}|e_n| + \frac{1}{2}|e_{n-1}| + \frac{5h}{9}|f(t_{n+1}, \mathbf{x}(t_{n+1})) - f_{n+1}| + \frac{h}{24}|f(t_n, \mathbf{x}(t_n)) - f_n| \\
&+ \frac{7h}{72}|f(t_{n-2}, \mathbf{x}(t_{n-2})) - f_{n-2}| + |R|
\end{aligned}$$

Using Lipschitz continuity and setting $\hat{h} := Lh$:

$$
\begin{aligned}
|e_{n+1}| &\leq \frac{3}{2}|e_n| + \frac{1}{2}|e_{n-1}| + \frac{5\hat{h}}{9}|x(t_{n+1}) - \mathbf{x}_{n+1}| + \frac{\hat{h}}{24}|x(t_n) - \mathbf{x}_n| + \frac{7\hat{h}}{72}|x(t_{n-2}) - \mathbf{x}_{n-2}| + |R| \\
&= \frac{3}{2}|e_n| + \frac{1}{2}|e_{n-1}| + \frac{5\hat{h}}{9}|e_{n+1}| + \frac{\hat{h}}{24}|e_n| + \frac{7\hat{h}}{72}|e_{n-2}| + |R| \\
&= \frac{5\hat{h}}{9}|e_{n+1}| + \left(\frac{3}{2} + \frac{\hat{h}}{24}\right)|e_n| + \frac{1}{2}|e_{n-1}| + \frac{7\hat{h}}{72}|e_{n-2}| + |R| \\
&= \left(\frac{3}{2} + \frac{\hat{h}}{24}\right)\tilde{Q}^{-1}|e_n| + \frac{1}{2}\tilde{Q}^{-1}|e_{n-1}| + \frac{7\hat{h}}{72}\tilde{Q}^{-1}|e_{n-2}| + \tilde{Q}^{-1}|R|, \qquad \tilde{Q} := 1 - \frac{5\hat{h}}{9}
\end{aligned}
$$

Introducing the error bounding function $\delta_n = \max_{0 \leq i \leq n} |e_n|$ and rewriting above equation in terms of $\delta_n$:

$$
\delta_{n+1} \leq Q\delta_n + \tilde{Q}^{-1}|R|, \qquad Q := \left(2 + \frac{5\hat{h}}{36}\right)\tilde{Q}^{-1}
$$

For $n = 0, 1, 2, \ldots, N-1$ we find that

$$
\begin{aligned}
\delta_0 &= x(t_0) - x_0 = 0, \\
\delta_1 &\leq Q\delta_0 + \tilde{Q}^{-1}|R_1|, \\
\delta_2 &\leq Q\delta_1 + \tilde{Q}^{-1}|R_2| = Q^2\delta_0 + (Q|R_1| + |R_2|)\tilde{Q}^{-1}, \\
\delta_3 &\leq Q\delta_2 + \tilde{Q}^{-1}|R_3| = Q^3\delta_0 + (Q^2|R_1| + Q^2|R_2| + |R_3|)\tilde{Q}^{-1}, \\
&\vdots \\
\delta_N &\leq \sum_{i=1}^{N} Q^{N-i}|R_i|\tilde{Q}^{-1}
\end{aligned}
$$

Since $|R_i| \leq Ch^4,\ C = \mathit{const} > 0$ and using geometric series properties, we can estimate the above sum as

$$
\delta_n \leq \frac{Q^N - 1}{Q - 1} Ch^4 \tilde{Q}^{-1} \tag{24}
$$

Then we rewrite $Q := 1 + 25\tilde{Q}^{-1}\hat{h}/36$ and substituting using similar arguments as before it leads to

$$
\delta_n \leq Bh^3, \quad B := 36C(\exp(t_N \frac{25L}{36}\tilde{Q}^{-1}) - 1)/(25L) \tag{25}
$$

### 3.3 Numerical error

```
-----------------------------------
Global error analysis
-----------------------------------
h = 10e-5
Exact values =  [-0.98959118  1.14045841]
Numerical values explicit =  [-0.98959118  1.14045841]
Error explicit =  4.832076163458377e-11
Error/h^3 explicit =  48.32076163458376
Numerical values implicit =  [-0.98959118  1.14045841]
Error implicit =  4.8311759022404794e-11
Error/h^3 implicit =  48.311759022404786
-----------------------------------
h =  0.0005
Exact values =  [-0.98959118  1.14045841]
Numerical values explicit =  [-0.98959118  1.14045841]
Error explicit =  5.579787337626384e-11
Error/h^3 explicit =  0.4463829870101107
Numerical values implicit =  [-0.98959118  1.14045841]
Error implicit =  3.441263570593456e-11
Error/h^3 implicit =  0.27530108564747646
-----------------------------------
h =  1e-05
Exact values =  [-0.98959118  1.14045841]
Numerical values explicit =  [-0.98959118  1.14045841]
Error explicit =  8.239831483992516e-10
Error/h^3 explicit =  823983.1483992514
Numerical values implicit =  [-0.98959118  1.14045841]
Error implicit =  8.220815763442669e-10
Error/h^3 implicit =  822081.5763442667
-----------------------------------
```

Figure 3: The error analysis is not very satisfactory as the precision of the machine compromise our results. For $h = 1e - 3$ the method does not converge, for smaller $h$ we instead appreciate a very high precision, with errors of order of magnitude smaller than $-10$. This unfortunately prevents us to assess the increasing convergence varying h. Finally, we assess that the implicit method very slightly outperforms the explicit one