Risk reserve model and its diffusion approximation

Francesco Piatti

https://imperial.cloud.panopto.eu/Panopto/Pages/Viewer.aspx?id=ff20e82d-ac01-4ba8-89ac-abd800991815

Imperial College London

Outline

By definition, risk is something hardly predictable and difficult to model. In this project risk is analysed form a stochastic point of view, throughout the risk reserve model (*or Cramér-Lundberg model*), typically used by insurance companies. In particular, a compound poisson process is used to represent claims insurance companies have to pay. This model has been introduced by Cramér and Lundberg (from whom it takes its name) in early 1900s. Finally, its behaviour and diffusion is studied in comparison with brownian motion.

Risk Reserve Model [1],[2]

A risk reserve process $\{R_t\}_{t\geq 0}$ is a simple model for the time evolution of the reserves of an insurance company and can be denoted as:

$$R_t = u + pt - \sum_{i=1}^{N_t} U_i \tag{1}$$

where:

1. $u = R_0$ represents the initial reserve.
2. U_n are non-negative and i.i.d. r.v. and denote the size of the nth claim.
3. Premiums flow in at rate p per unit time.
4. $\{N_t\}_{t\geq 0}$ is a poisson process. This implies there are only finite many claims in finite time intervals. More precisely, there are exactly $N(t)$ claims in $[0, t]$.
5. The process $S_t = u - R_t = \sum_{i=1}^{N_t} U_i - t$ is the *claim surplus process*.

In this project, we make the assumption that the sum $\sum_{i=1}^{N_t} U_i$ converges almost surely as $t \to \infty$ to a costant ρ, whose interpretation is the average amount of claim per unit time.

The probability $\psi(u)$ of ultimate ruin is defined as:

$$\psi(u) = \mathbb{P}\left(\inf_{t\geq 0} R_t < 0 \,|\, R_0 = u\right) \tag{2}$$

A further assumption it is made in this project is that we consider *light-tailed distributions*. In particular we will be dealing with exponential distributions.

Python code for risk reserve model

```python
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
from numpy import random as rnd

def risk_reserve_process(u, p, T, beta, delta):
    """ Generates a Cramer-Ludberg model with R_0 = u, premiums per unit time = p, parameter of poisson distribution = beta
    parameter of exponential distribution (claim sizes) = delta, prints out a dataframe """

    def compound_poisson_process(u, p, T, beta, delta):
        """Generates a compound poisson process of lenght N(t) with exponential jumps"""

        time_claims = [n for n in np.cumsum(rnd.poisson(1/beta, rnd.poisson(beta*T)))]
        time_claims_scaled = np.array(time_claims)/time_claims[-1]*T
        time_claims_list = [int(n) for n in time_claims_scaled]
        event_times = [n for n in range(time_claims_list[-1])] +  time_claims_list
        event_times_sorted = sorted(event_times, key=int)
        total = u
        increments = []

        for n in range(len(event_times_sorted)):
            if event_times_sorted[n] > event_times_sorted[n-1]:
                total += p
                increments.append(total)
            else:
                total += -rnd.exponential(1/delta)
                increments.append(total)
        return event_times_sorted, increments

    #let x_val be the number of the week and y_val the reserve amount
    x_val, y_val = compound_poisson_process(u, p, T, beta, delta)

    dataframe_dict = { 'x_val' : x_val, 'y_val' : y_val } #generating data frame
    reserve_df = pd.DataFrame(dataframe_dict)

    return reserve_df
```

Ruin probability [1],[3]

Consider a risk reserve model with $p = 1$, $N_t \sim \text{Poi}(\beta)$ and $U_i \sim \textbf{Exp}(\delta)$. Then:

$$\psi(u) = \rho e^{-(\delta-\beta)\,u} \tag{3}$$

Note that the formula is valid for $\rho < 1$ (i.e. $\delta - \beta > 0$), otherwise $\psi(u)$=1. **Proof:**

- Denote by $\kappa(\alpha) = \log \hat{U}[\alpha]$ the cumulant generating function ($\hat{U}[\alpha]$ is the m.g.f). For $U_i \sim \text{Exp}(\delta)$ the solution $\kappa(\gamma) = 0 \implies \gamma = \delta - \beta$.
- Consider the following *stopping times*: $\tau(u) = \inf\{t \geq 0 : R_t \leq 0\}$, $\tau_+(a) = \inf\{t \geq 0 : R_t \geq a\}$ and $\tau(u, a) = \min(\tau(u), \tau_+(a))$
- The process $M_n = e^{\alpha(-S_1-\dots-S_n)-n\kappa(\alpha)}$ is a **martingale** (easy to show). I.e. $\mathbb{E}[M_{n+1}|R_1, \dots, R_n] = M_n$
- We can thus exploit the *optional stopping theorem* which states that the expected value of a martingale at a stopping time is equal to its initial expected value.

Now, notice that $\mathbb{P}(U_i > x) = e^{-\delta x}$. Assume $R_{\tau(u)} = x > 0$, and let $Z = -R_{\tau(u)} + x$ be the size of the claim leading to ruin. But $Z \sim \text{Exp}(\delta)$ as a result of the *memoryless property* of the exponential distribution. Therefore $\mathbb{E}[e^{-\gamma R_{\tau(u,a)}}\,|R_{\tau(u,a)} \leq 0] = \delta/(\delta - \gamma)$ since it is the moment generating function of Z with $t = \gamma$.

$$\begin{aligned}
e^{-\gamma u} &= \mathbb{E}[e^{-\gamma R_0}] \\
&= \mathbb{E}[e^{-\gamma R_{\tau(u,a)}}\,|R_{\tau(u,a)} \leq 0]\,\mathbb{P}(R_{\tau(u,a)} \leq 0) + e^{-\gamma u}\mathbb{P}(R_{\tau(u,a)} = a) \\
&= \frac{\delta}{\delta - \gamma}\psi_a(u) + e^{-\gamma u}(1 - \psi_a(u)) \\
\implies \quad \psi_a(u) &= \frac{e^{-\gamma u} - e^{-\gamma a}}{\delta/\beta - e^{-\gamma a}}
\end{aligned}$$

Now, letting $a \to \infty$ yields to the classical expression (3).

Monte Carlo simulation for $\psi(u)$

In this paragraph we want to simulate $\psi(u)$ using Monte Carlo methods. The main idea is to run a large number of simulations, say n, for a large number of u-values. Each simulation is represented by an indicator function where $R_{u,k}$ is the kth-simulation of a risk reserve model with $R_0 = u$, and u given:

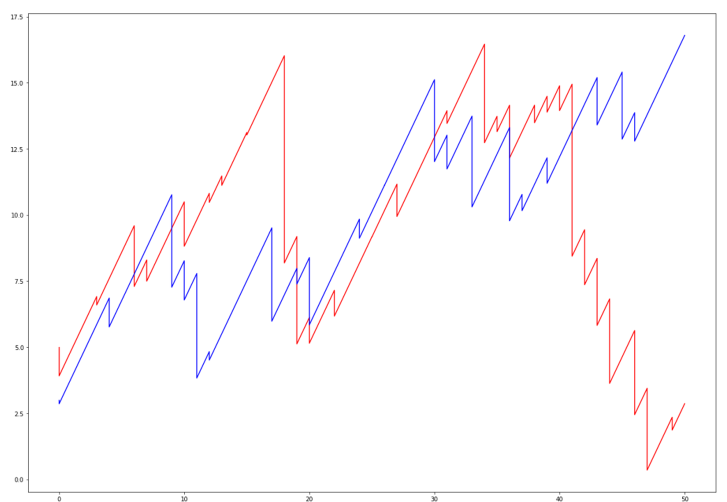
$$\mathbf{1}_{R_{u,k}} = \begin{cases} 1 & \text{if } \inf_{t\geq 0} R_{u,k} < 0 \text{ (ruin does happen)} \\ 0 & \text{if } \inf_{t\geq 0} R_{u,k} > 0 \text{ (ruin does not happen)} \end{cases}$$

Now, by the law of large numbers (i.e. $\mathbb{P}(\omega : |S_n(\omega)/n - \mu| > \epsilon) \to 0$) we deduce that:

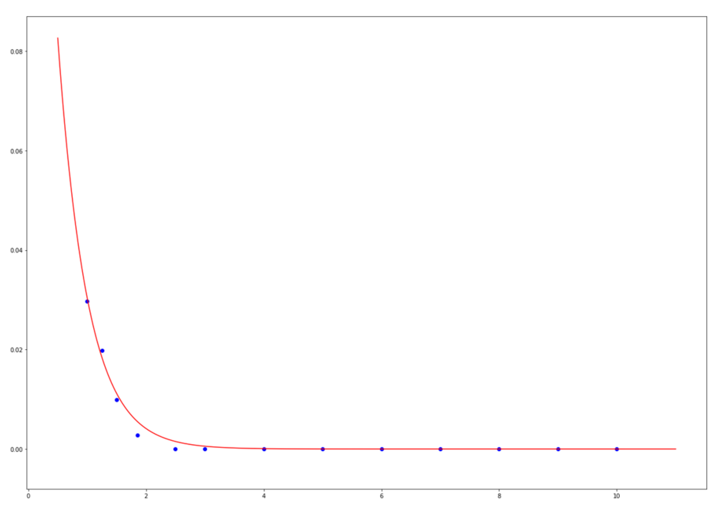
$$\frac{1}{n}\sum_{k=1}^{n} \mathbf{1}_{R_{u,k}} \to \mathbb{E}[\mathbf{1}_{R_u}] = \psi(u) \quad \text{where } u \text{ is given} \tag{4}$$

Iterating this process for an array of u-values we obtain a scatter plot that provides us with a good approximation of $\psi(u)$ as a function of u.

Graphs



RR with $u = 5$, $\beta = 0.5$, $\delta = 0.6$.



Monte Carlo simulation for $\psi(u)$.

Diffusion approximation [1],[4]

Mean and variance of the risk reserve model

$$\mathbb{E}[R_t] = u - \mathbb{E}[S_t] = u - t(\beta\mu_U - 1) = u - t(\rho - 1) \tag{5}$$

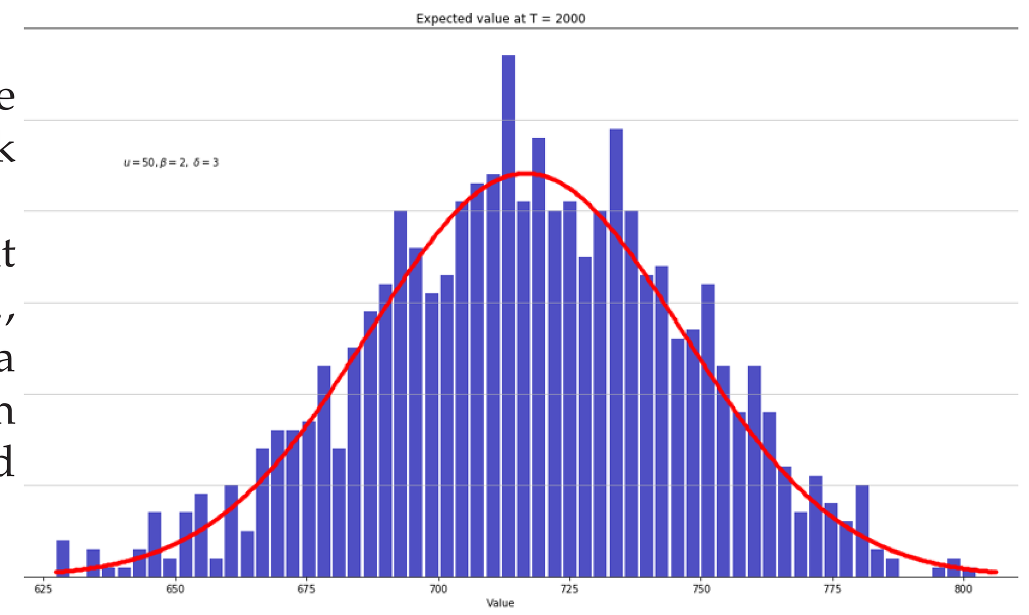
$$\text{Var}[R_t] = t\beta\mu_U^{(2)} \tag{6}$$

where $\mu_U^{(2)}$ is the second raw moment of U. Proof of the expectation of S_t:

$$\begin{aligned}
\mathbb{E}[\,S_t\,] &= \mathbb{E}\left[\sum_{i=1}^{N_t} U_i - t\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N_t} U_i \,\middle|\, N_t\right]\right] - t \\
&= \mathbb{E}[N_t\mu_U] - t = \beta t\mu_U - t = t(\rho - 1)
\end{aligned}$$

Central limit theorem

In the graph on the right, there have been simulated 1500 risk reserve models with $u = 50$, $\beta = 2$, $\delta = 3$ and as a result of the CLT, *since they all are i.i.d.*, the histograms approximates a normal distribution with mean $= u - t(\beta/\delta - 1)$ (see (5)), and variance $= 2t\beta/\delta^2$ (see (6)).



Functional central limit theorem or Donsker's invariance principle

The idea behind the diffusion approximations is to first approximate the claim surplus process by a Brownian motion with drift by matching the first two moments. The mathematical result behind this is the F.C.L.T: given a random walk $\{S_n\}_{n=0,1,\dots}$ in discrete time with $\mu = \mathbb{E}[S_1] = \rho - p$ (drift) and $\sigma^2 = \text{Var}[S_1] = \beta\mu_B^{(2)}$ then

$$\left\{\frac{1}{\sigma\sqrt{c}}(S_{\lfloor tc \rfloor} - tc\mu)\right\} \xrightarrow{\mathcal{D}} \{W_0(t)\}, \quad c \to \infty \tag{7}$$

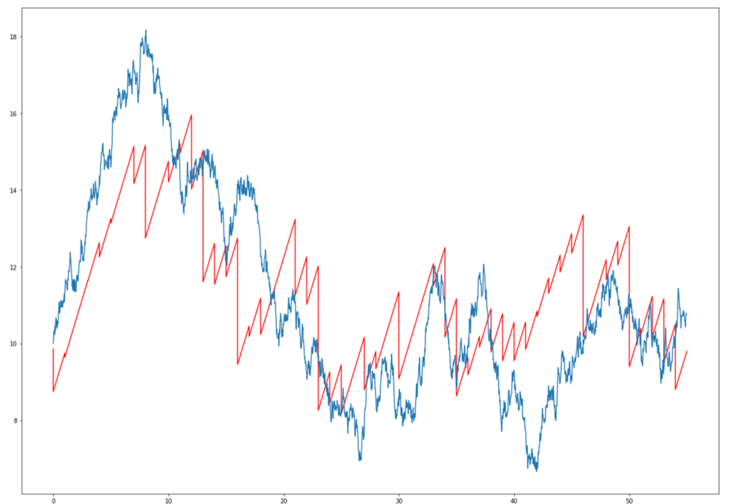
Where $W_\zeta(t)$ is a brownian motion with drift ζ and variance 1.

This approximation works well under the assumption that $p \downarrow \rho$ (and thus $\mu \to 0$). In fact, letting $c = \sigma^2/\mu_p^2$ we get that $c \to \infty$ and substituting into (7) we obtain:

$$\left\{\frac{|\mu|}{\sigma^2}S^{(p)}_{t\sigma^2/\mu^2} + t\right\} \xrightarrow{\mathcal{D}} \{W_0(t)\}$$

$$\left\{\frac{|\mu|}{\sigma^2}S^{(p)}_{t\sigma^2/\mu^2}\right\} \xrightarrow{\mathcal{D}} \{W_0(t) - t\} = \{W_{-1}(t)\}$$

using the properties of Wiener process.



Approximation of a risk reserve process with $p \downarrow \rho$ to a Wiener process with $\zeta = 1$.

Conclusions

We have successfully shown that under certain assumptions, risk reserve model can be well approximated by brownian motion, and that Montecarlo-simulation is a powerful method to approximates curves. Moreover, as an immediate application, this project shows that increasing linearly the amount of initial capital, the ruin-probability *exponentially* decreases.

References

[1] Asmussen, S., Albrecher, H. (2010) Ruin probabilities (Vol. 14). Singapore: World scientific.

[2] Feller, W.. (1968) An Introduction to Probability Theory and Its Applications, Vol. 1. John Wiley Sons Inc.

[3] Feller, W.. (1971) An Introduction to Probability Theory and Its Applications, Vol. 2, 3rd Edition. John Wiley Sons Inc.

[4] Billingsley, P.. (2013) Convergence of probability measures. John Wiley Sons Inc.