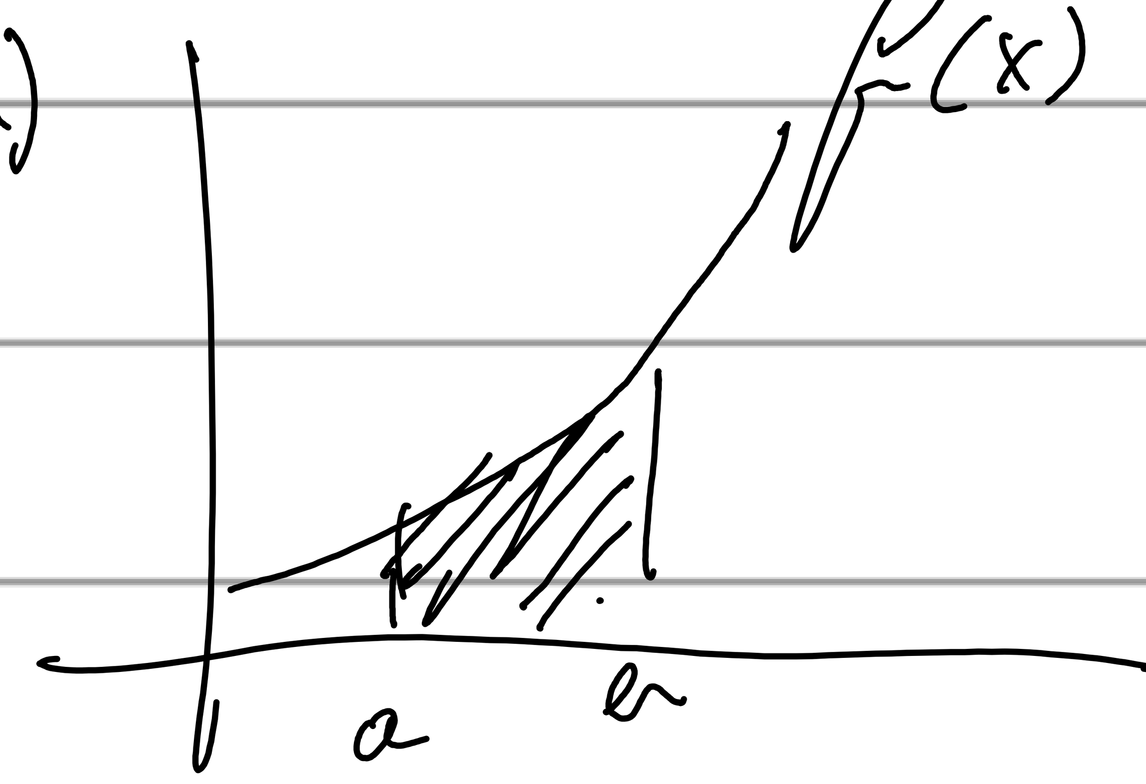
Hfst 5

Dubbel integraal

Wat?

normale $\int_a^b f(x)\,dx$ (analytisch)



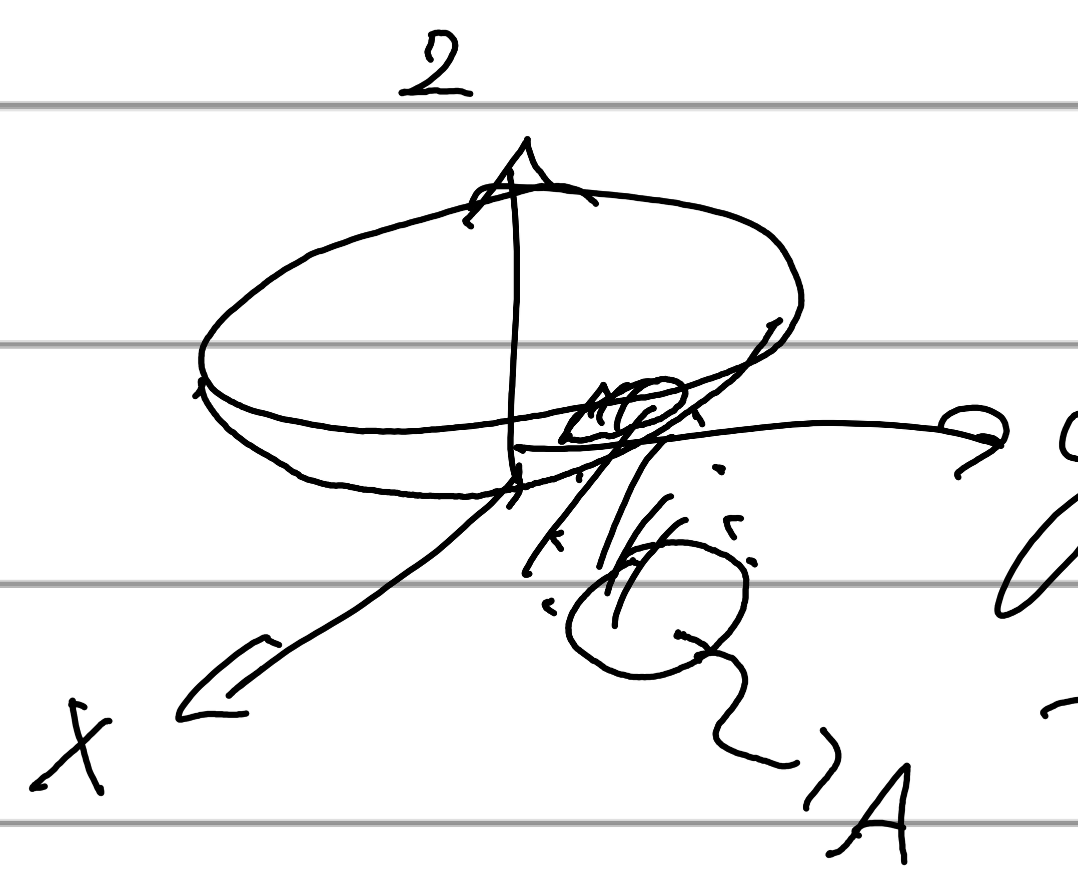
(meetkundig

Meetkundig is dat de oppervlakte

$\int_a^b dx$ (afstand tussen a en b) lengte dus

$z = f(x,y)$

$\iint_A f(x,y)\,dx\,dy$: in 3D



grenzen liggen in xy vlak

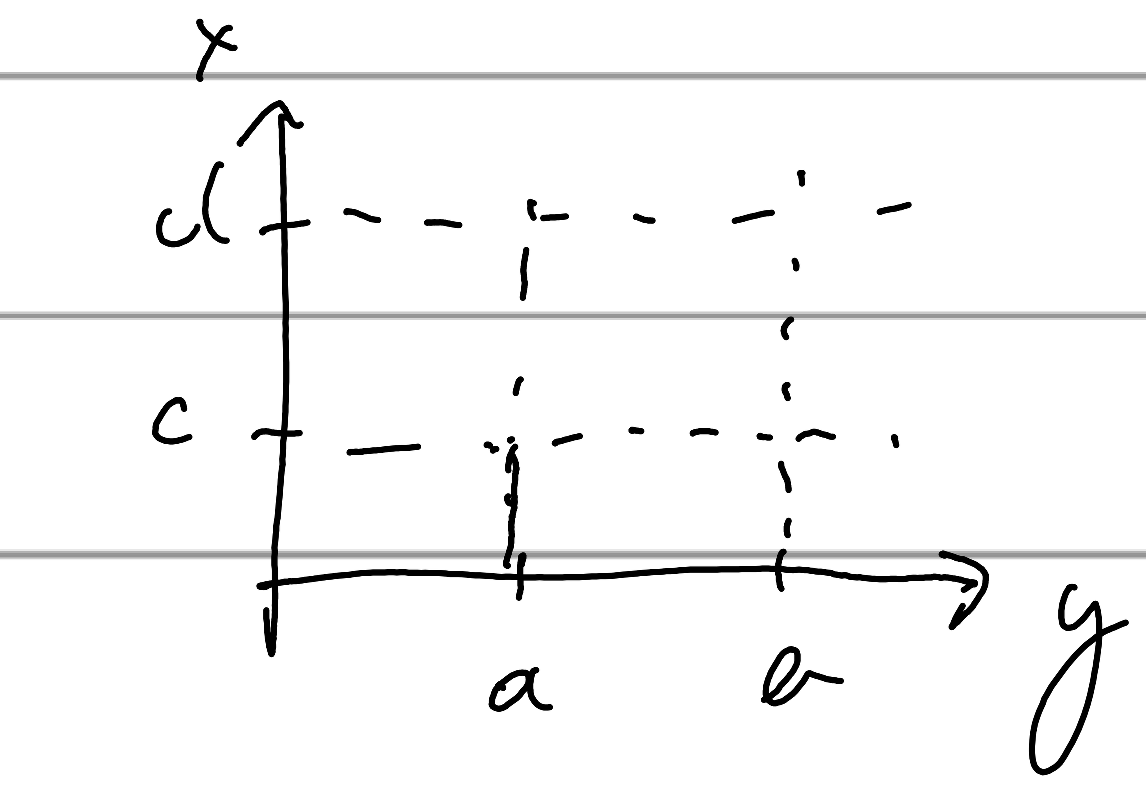
=> volume begrensd tot gebied A tot aan oppervlak.

$\iint_A dx\,dy$: oppervlakte van gebied A

Stelling Fubini

dus oppervlakte onder gebied $\times h$

dus een Volume



$$= \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b dx \int_c^d f(x,y)\,dy$$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d dy \int_a^b f(x,y)\,dx$$

Vb Opg $z = 100 - 6x^2y$

$A = 0 \le x \le 2 \,, -1 \le y \le 1$

Opl

$$\int_0^2 dx \int_{-1}^{1} (100 - 6x^2y)\,dy$$

$$= \int_0^2 \left[100y - \frac{6x^2y^2}{2}\right]_{-1}^{1} dx$$

dubbele integraal analytisch

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\, \Delta x_k$$ (gewone integraal)

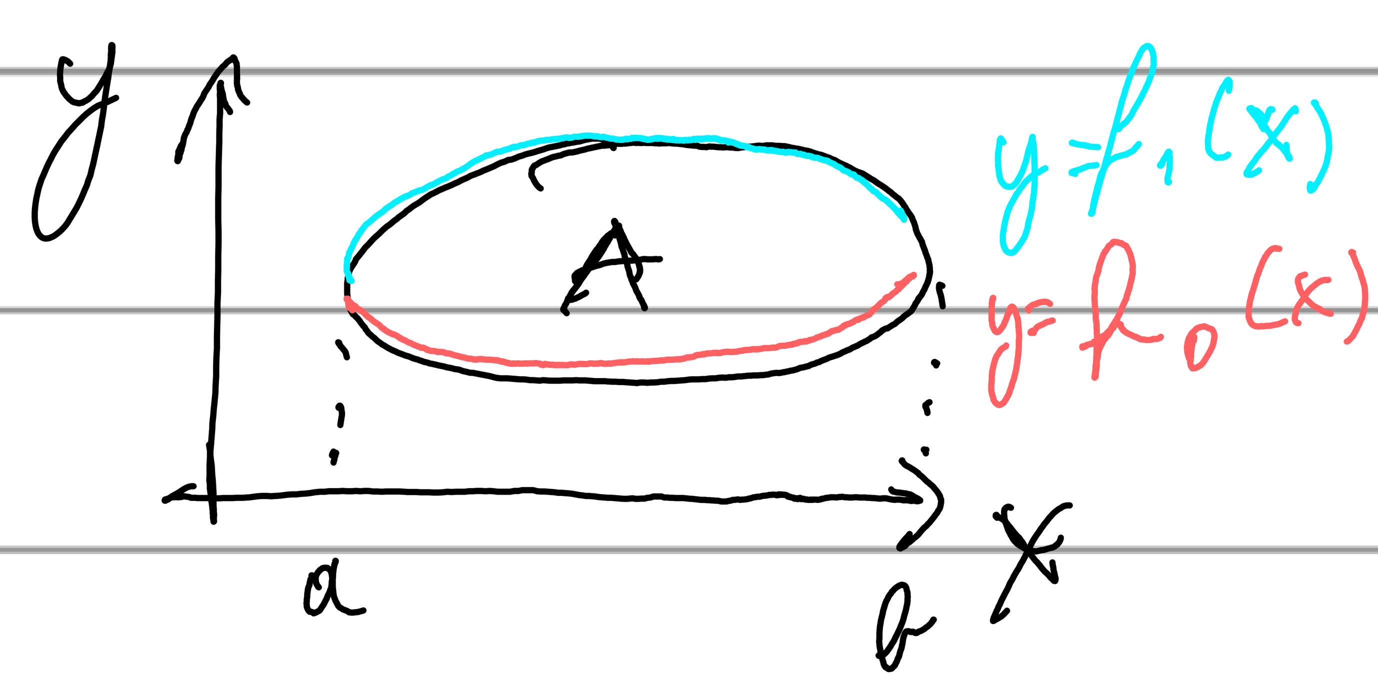
wordt $$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta x_k\, \Delta y_k$$

$$\int_0^2 \left[100 - \cancel{\frac{\cancel{6}^{3}x^2}{\cancel{2}}} + 100\cdot\cancel{\left(\frac{\cancel{6}^{3}x^2}{\cancel{2}}\right)}\right] dx$$

$$= \int_0^2 200x \; dx$$

$$= 400$$

<u>Willekeurig gebied</u>

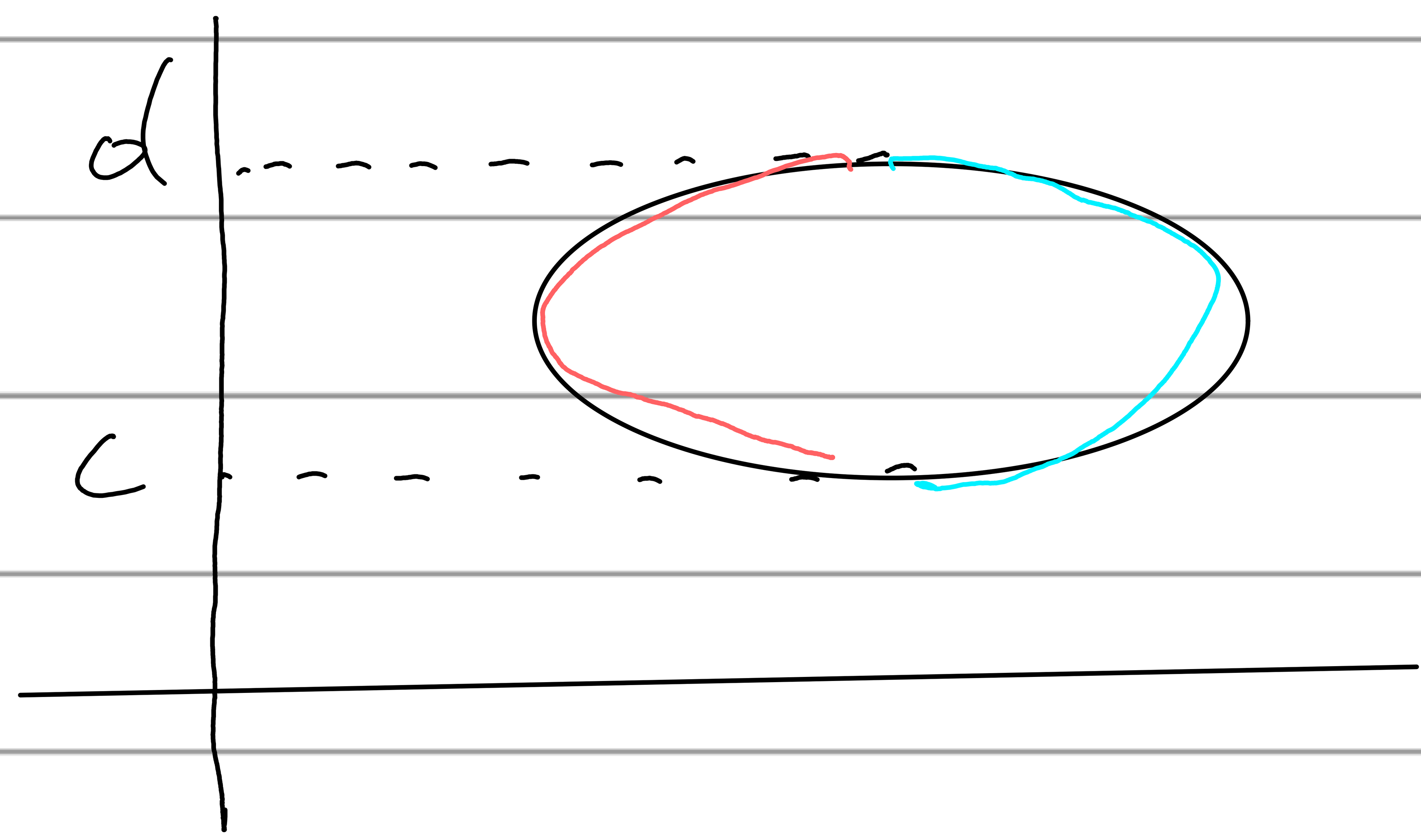


$$\int_a^b dx$$

Dan gaan we twee functies bepalen

$$\int_a^b dx \int_{f_0(x)}^{f_1(x)} f(x,y)\, dy$$

soms makkelijker eerst y dan x doen.



$x = f_0(y)$

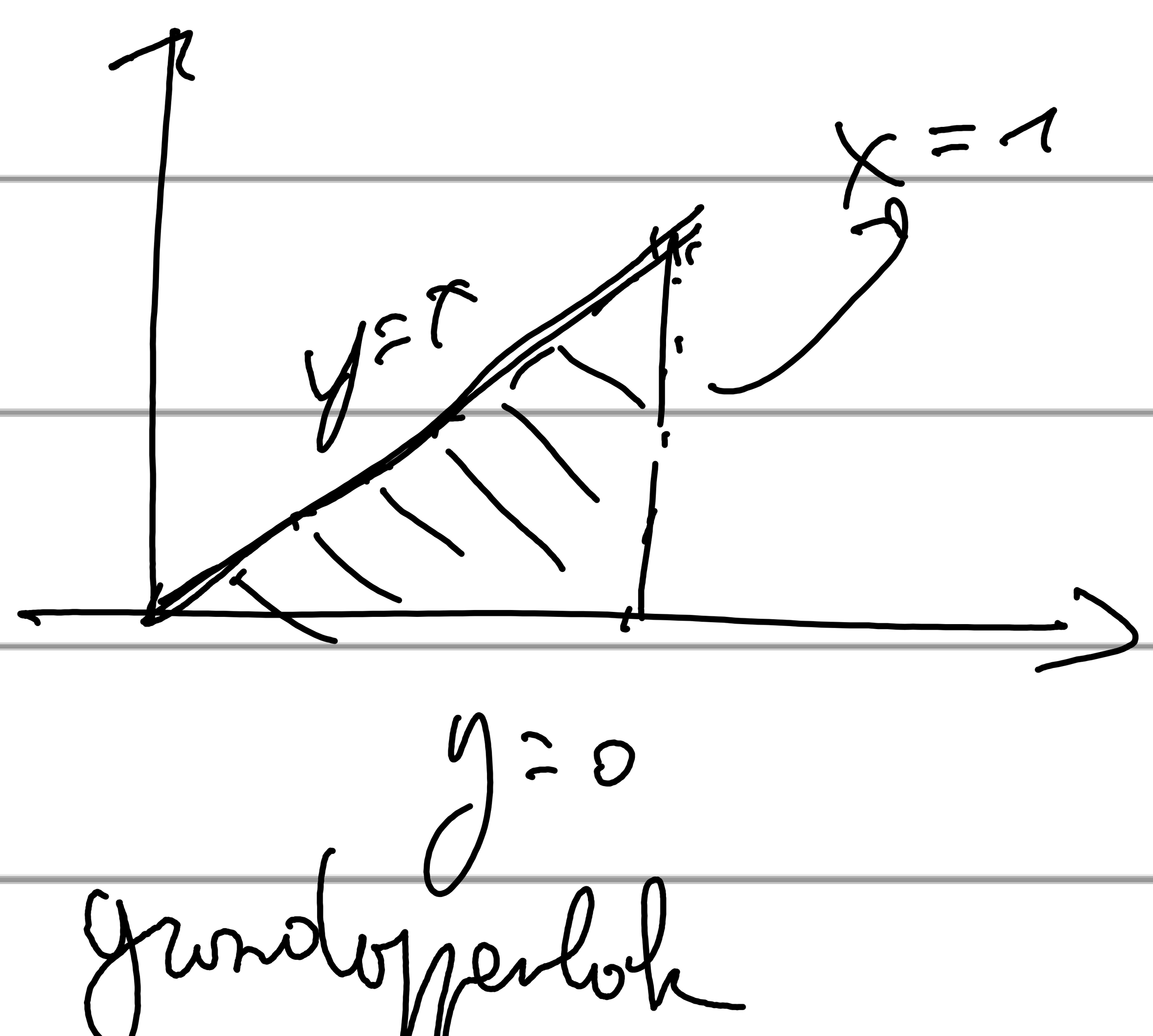
$x = f_1(y)$

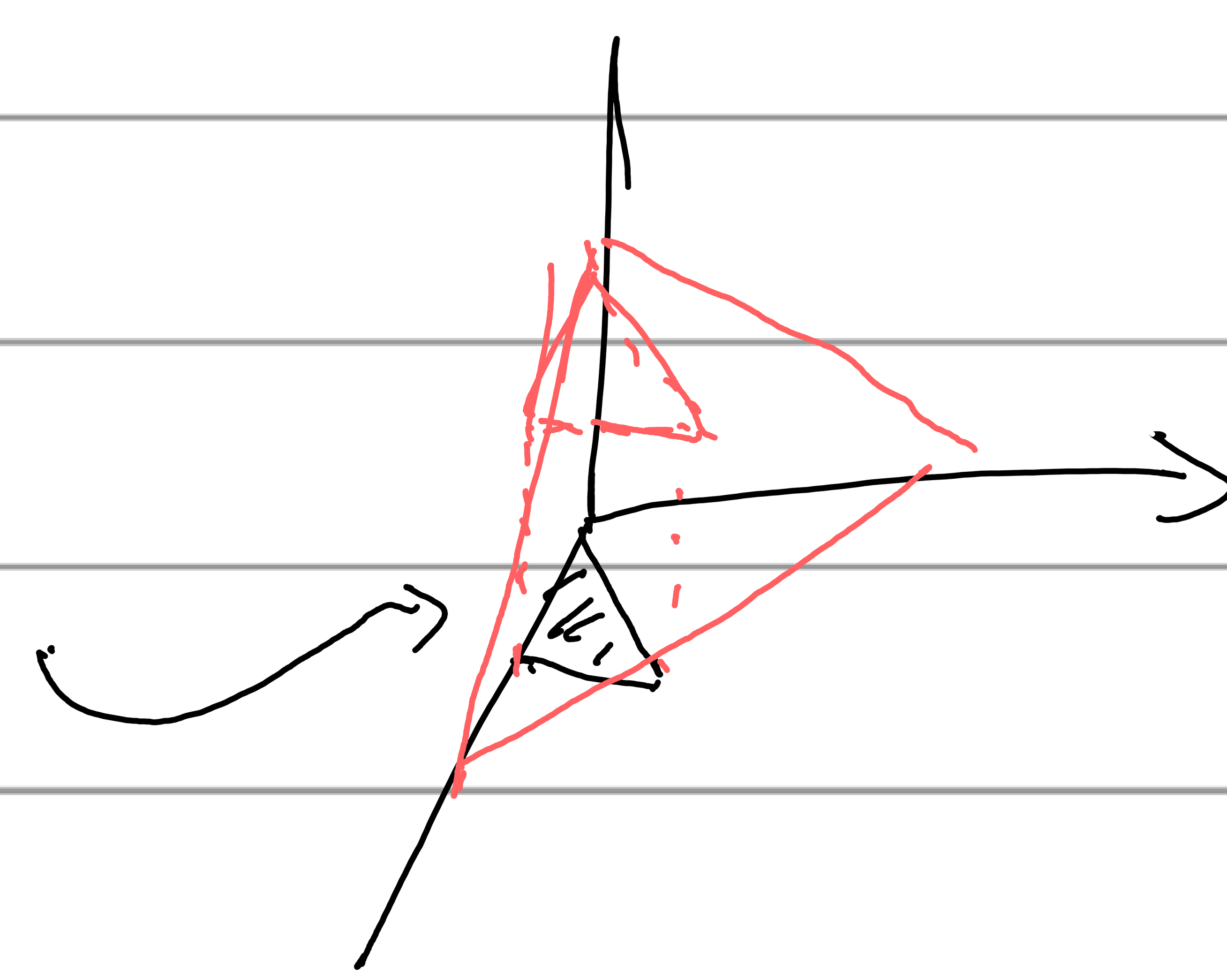
$$\int_c^d dy \int_{f_0(y)}^{f_1(y)} f(x,y)\,dx$$

16 Stel Bepaal volume van prisma

$$\begin{cases} x\text{-as} \ (\Rightarrow y = 0) \\ y = x \\ x = 1 \end{cases}$$

top van prisma ligt op vlak $Z = 3 - x - y$





$x = 0 \Rightarrow (0, 0, 3)$

$y = 0 \Rightarrow (0, 3, 0)$

$z = 0 \Rightarrow (3, 0, 0)$

↳ nulpunten bepalen van $Z = 3 - x - y$

$$\int_0^1 dx \int_{y=0}^{y=x} 3-x-y \, dy$$

↳ grafisch limieten bepalen

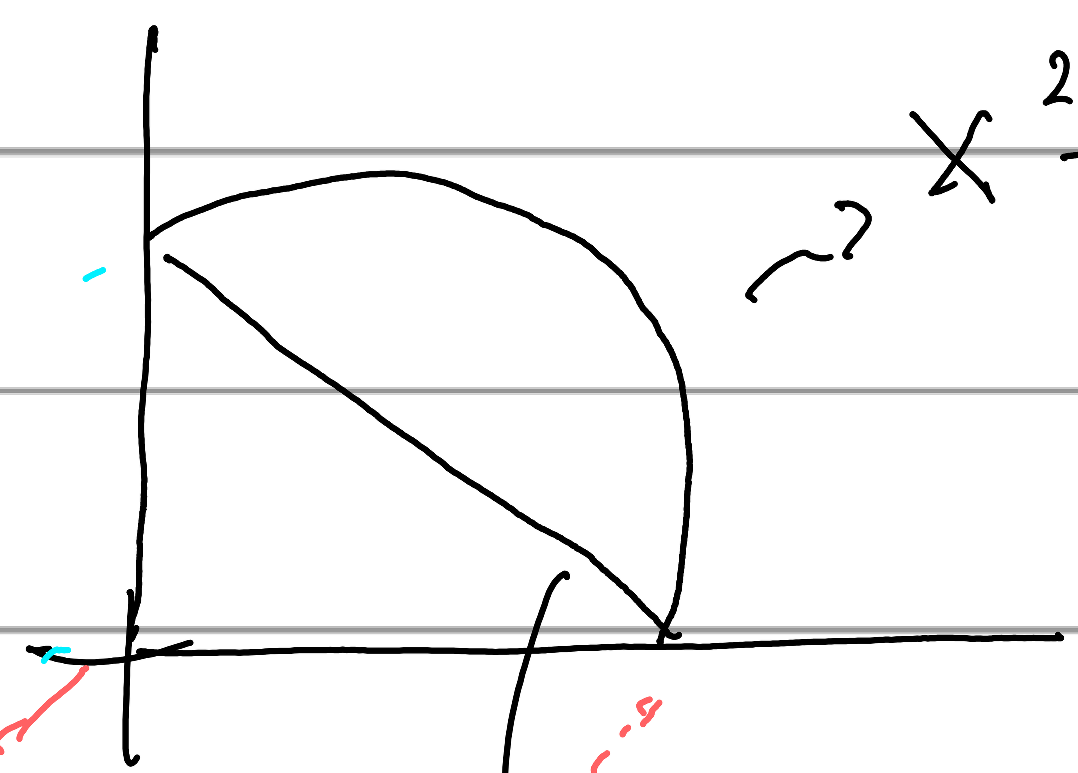
$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left[3x - x^2 - \frac{x^2}{2}\right] dx = \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx$$

$$= \left[\frac{3x^2}{2} - \frac{3x^3}{6}\right]_0^1$$

$$= 1$$

Ve



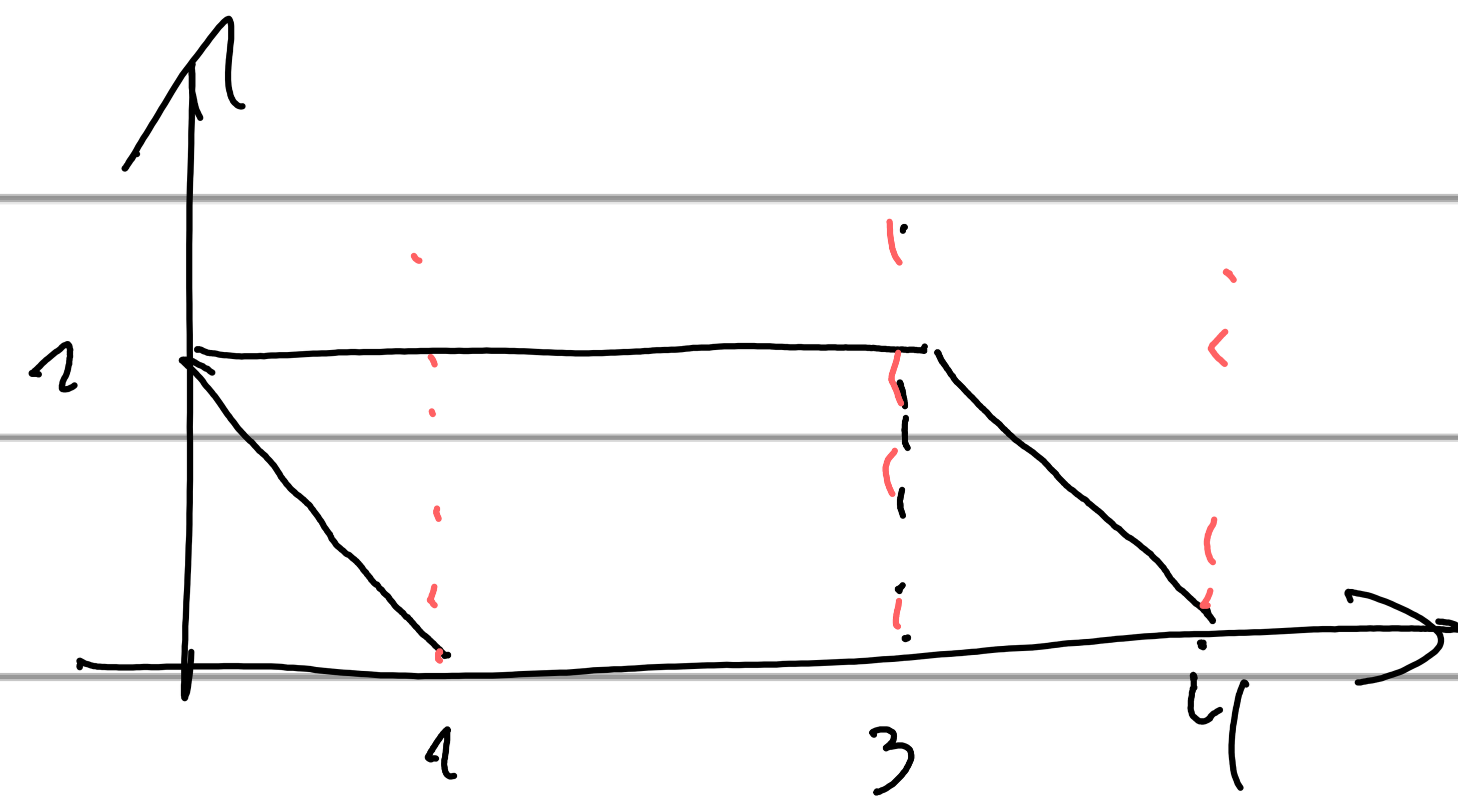
$$\int_0^1 dx \int_{1-x}^{\sqrt{1-x^2}} f \, dy \quad \text{of} \quad \int_0^1 dy \int_{x=1-y}^{\sqrt{1-y^2}} f \, dx$$

↳ minimum waarde van ⊙

$$\int_0^1 dy \int_{1-y}^{\sqrt{1-y^2}} f \, dx$$

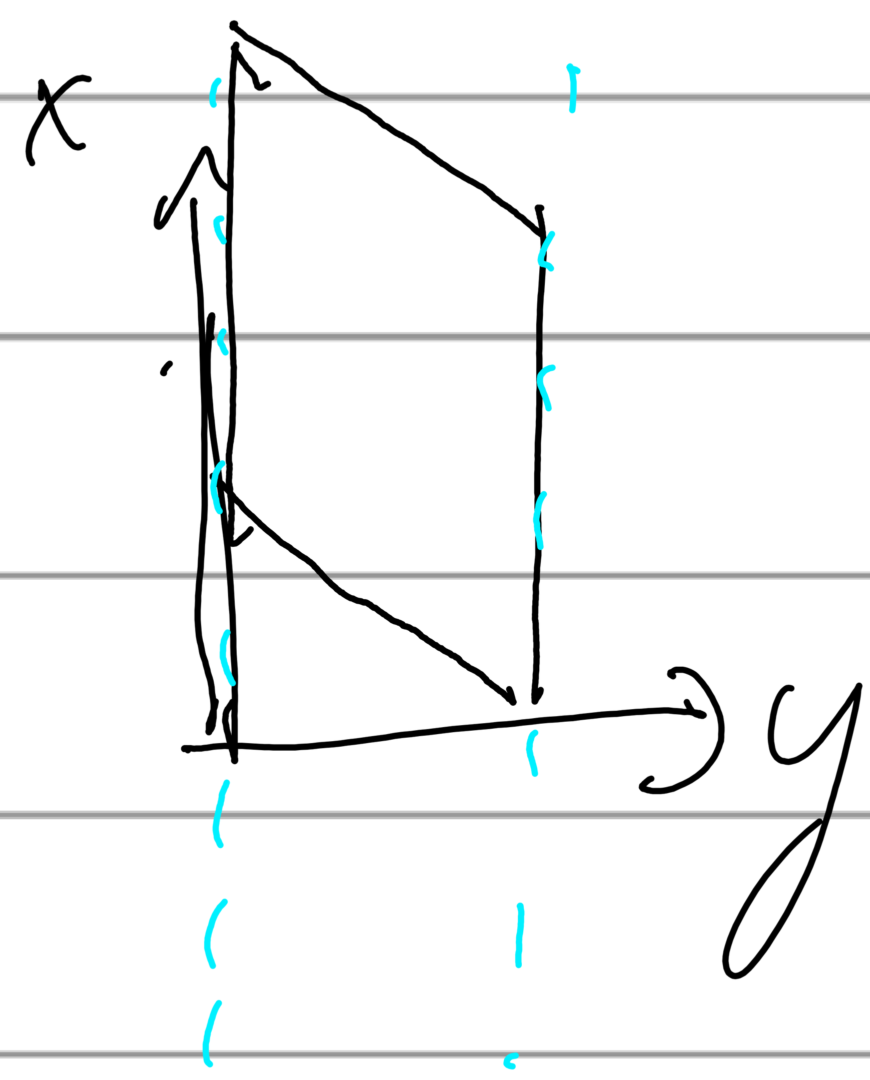
Stel de f staat er niet: dan wordt het de oppervlakte van het gebied!

9e



Stel we doen eerst x en dan y heb je 3 integralen

Maar als we vanuit y werken hebben we maar 1 boven en onder



$$\int_0^1 dy \int_{1-y}^{-2y+4} dx$$

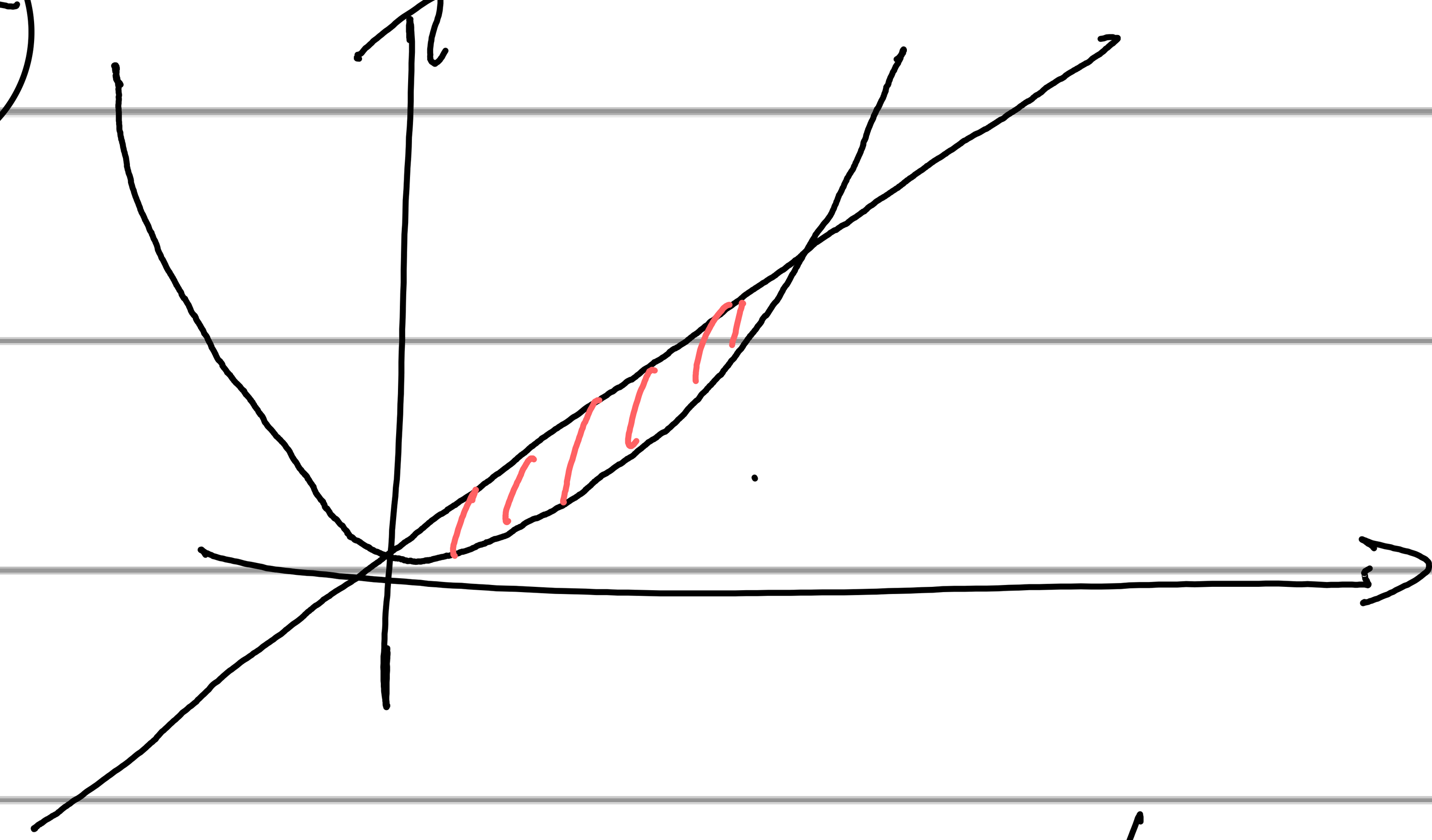
Rekenregels: gelijk normale integraal

$$\iint c f \, dx\, dy$$

$$\iint (f+g)\, dx\, dy = f \text{ en } g \text{ apart}$$

etc.

Vb



$y = 2x$

$y = x^2$

We willen zowel gebied kennen.

Eerst limiet kennen van x

$$2x = x^2 \qquad \text{dus } x = 0, \; x = 2$$

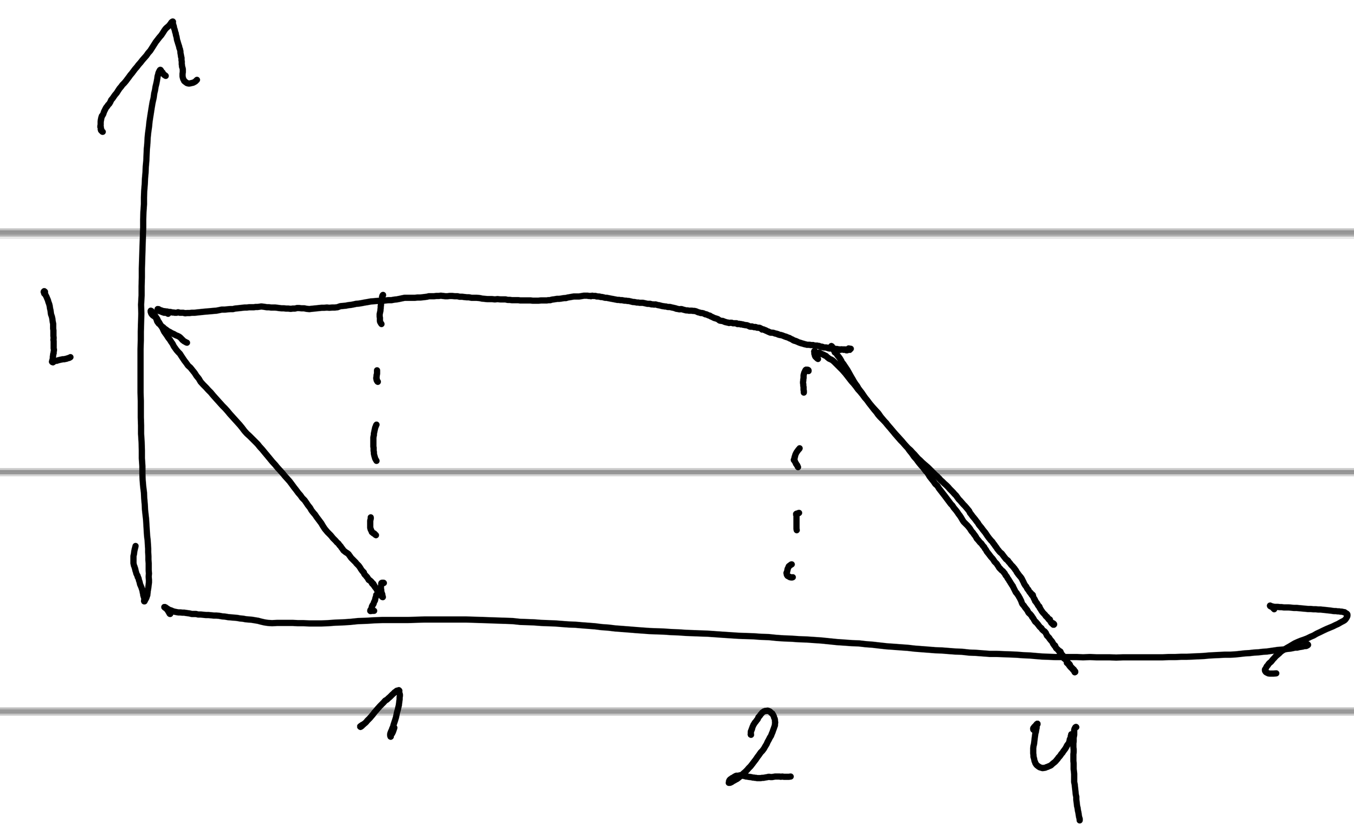
$$\int_0^2 dx \int_{x^2}^{2x} dy$$

Stel omgekeerd

$$\frac{y}{2} = x \text{ en } \sqrt{y} = x \qquad \frac{y}{2} = \sqrt{y} < \begin{matrix} 0 \\ 4 \end{matrix}$$

$$\int_0^4 dy \int_{2x}^{x^2} dx$$

Vb



$\int dx \int dy$ = probleem

x gaat door 3 gebieden

$\nabla + \square + \triangle$

$y = -x$

$$\int dy \int dx \Rightarrow \int_0^1 dy \int^{-2y+4} dx$$

$y = -(x-1)$

$= -x+1$

$(-y+1) = x$

$y = 2 - \frac{x}{2}$

$y - 2 = -\frac{x}{2}$

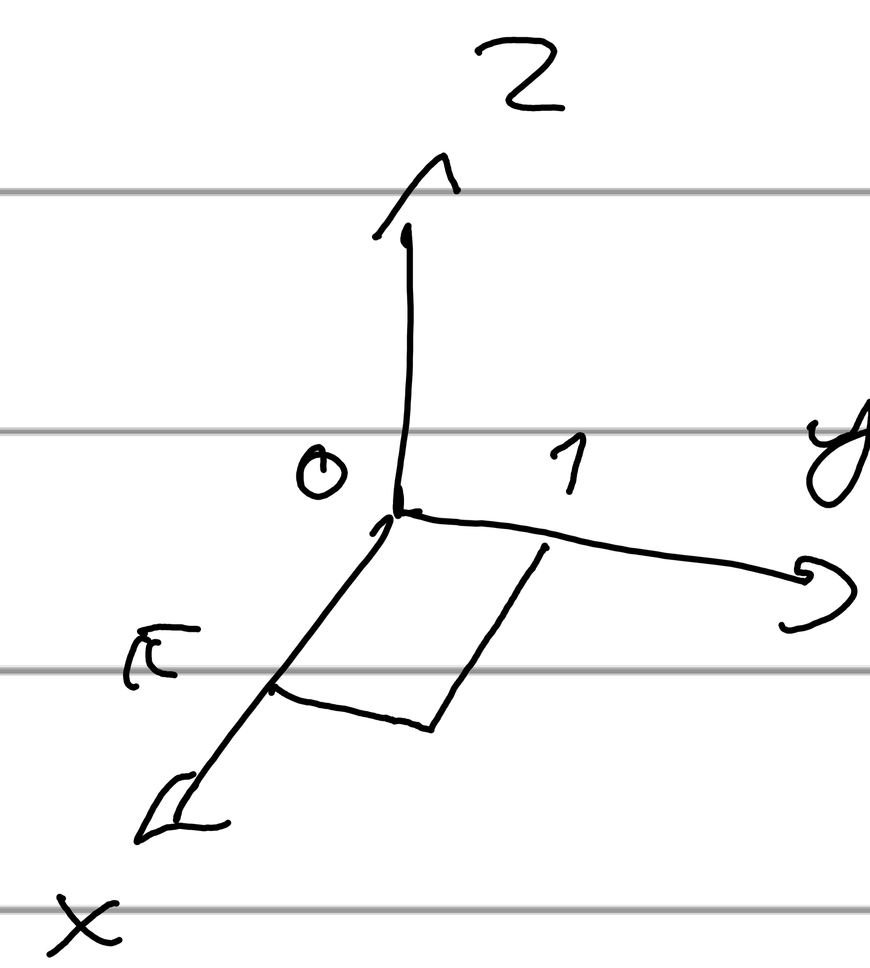
$-2(y-2) =$

$-2y+4 = x$

Tip Gem waarde van f over gebied A.

$$f(x,y) = x\cos(xy)$$

$$A = 0 \leq x \leq \pi$$
$$0 \leq y \leq 1$$



$$= \frac{\int_0^1 \int_0^\pi x\cos(xy)\,dx\,dy}{\int_0^1 \int_0^\pi dx\,dy}$$

$$\boxed{\text{Gem } \frac{\iint f\,dx\,dy}{\iint dx\,dy}}$$

= moeilijk op te lossen

= uitwerken en dan wordt het $\frac{2}{\pi}$

Toep Massamiddelpunt

$$M = \rho V$$

↳ volume

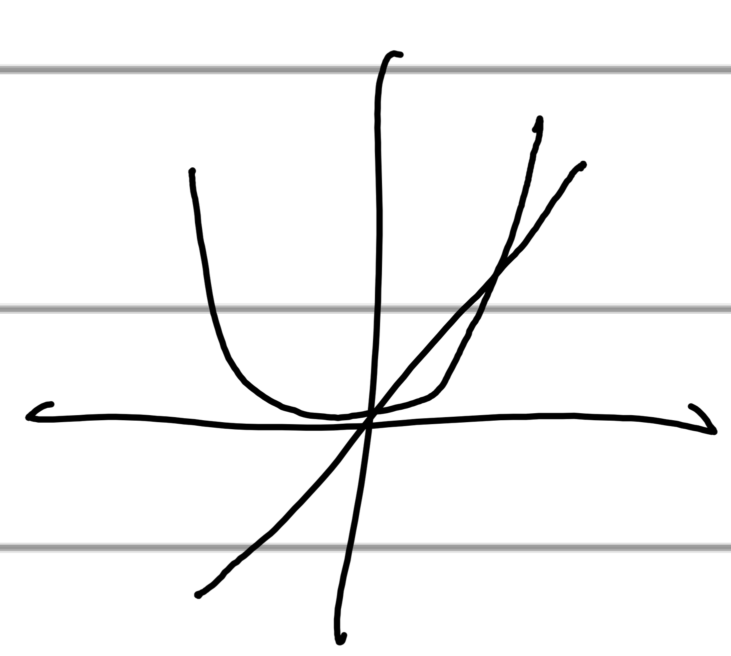
↳ massadichtheid

$$= \iint \rho \, dV$$

$$M_x = \frac{\iint \rho x \, dV}{M} \qquad M_y = \frac{\iint \rho y \, dV}{M}$$

ρ kan schrijven als constant

Vb



$y = x$

$y = x^2$

Massamiddelpunt binnen dit gebied

$$M = \iint \rho \, dV = \int_0^1 dx \int_{x^2}^{x} dy = \frac{1}{6}$$

$$(M_x)_{teller} = \int_0^1 dx \int_{x^2}^{x} x \, dy = \frac{1}{12}$$

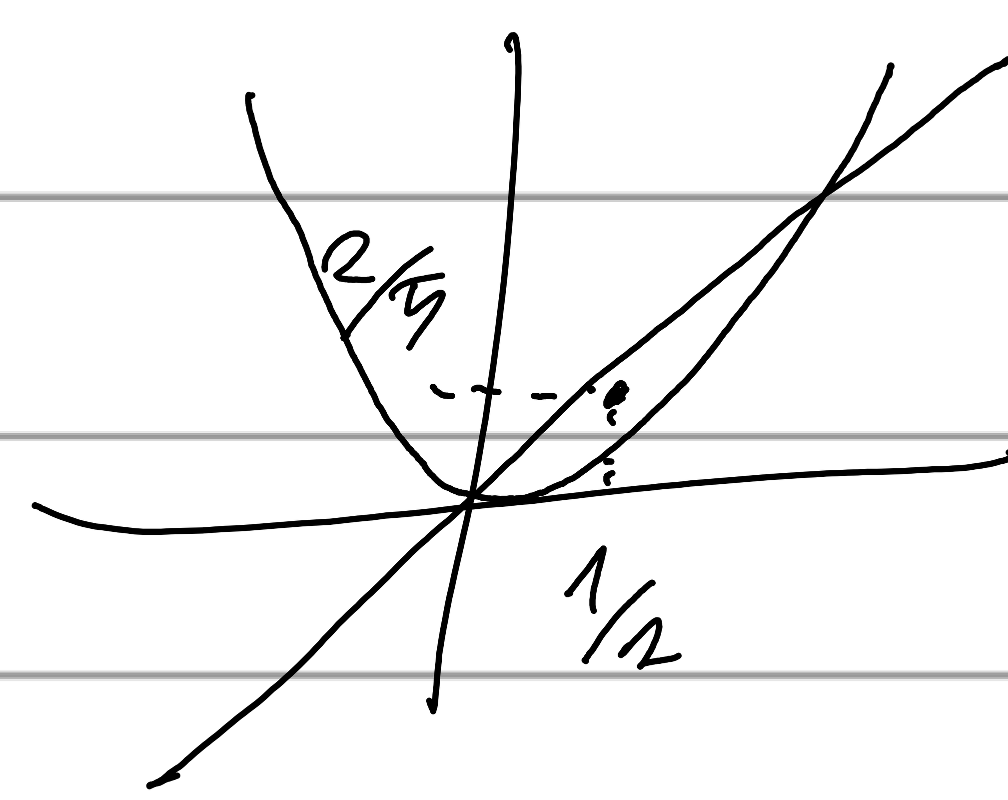
$$\rightsquigarrow \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}$$

$$M_{y_{\text{tele}}} \int_0^1 dx \int_{x^2}^{x} y\,dy = \frac{1}{15}$$

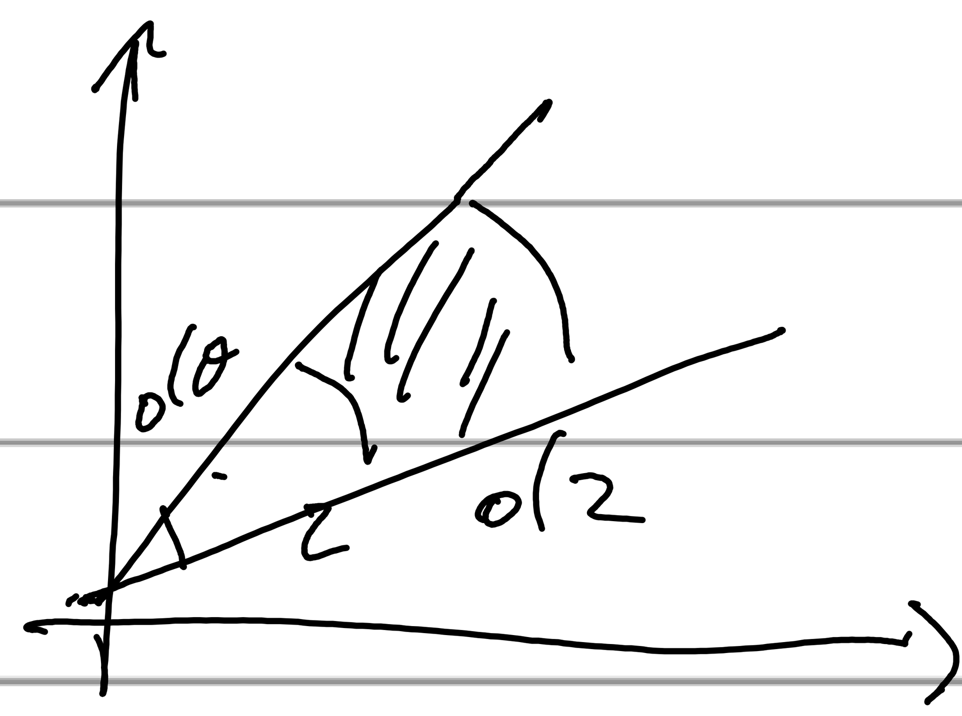
$$\frac{\frac{1}{15}}{\frac{1}{6}} = \frac{2}{5}$$

$M_x = \frac{1}{2}$

$M_y = \frac{2}{5}$

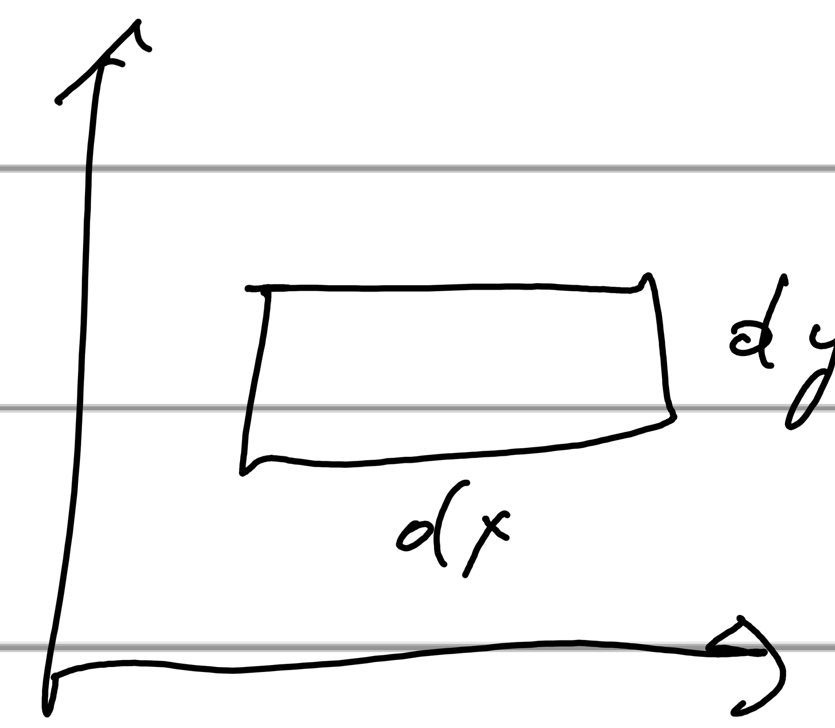


Dubbele integraal in poolcoördinaten



Pool

$r\,dr\,d\theta$



cartesisch

$$\iint_A f(x,y)\,dx\,dy \Rightarrow \iint_A f\left(r\cos\theta, r\sin\theta\right)\, r\,dr\,d\theta$$

Jacobiaan

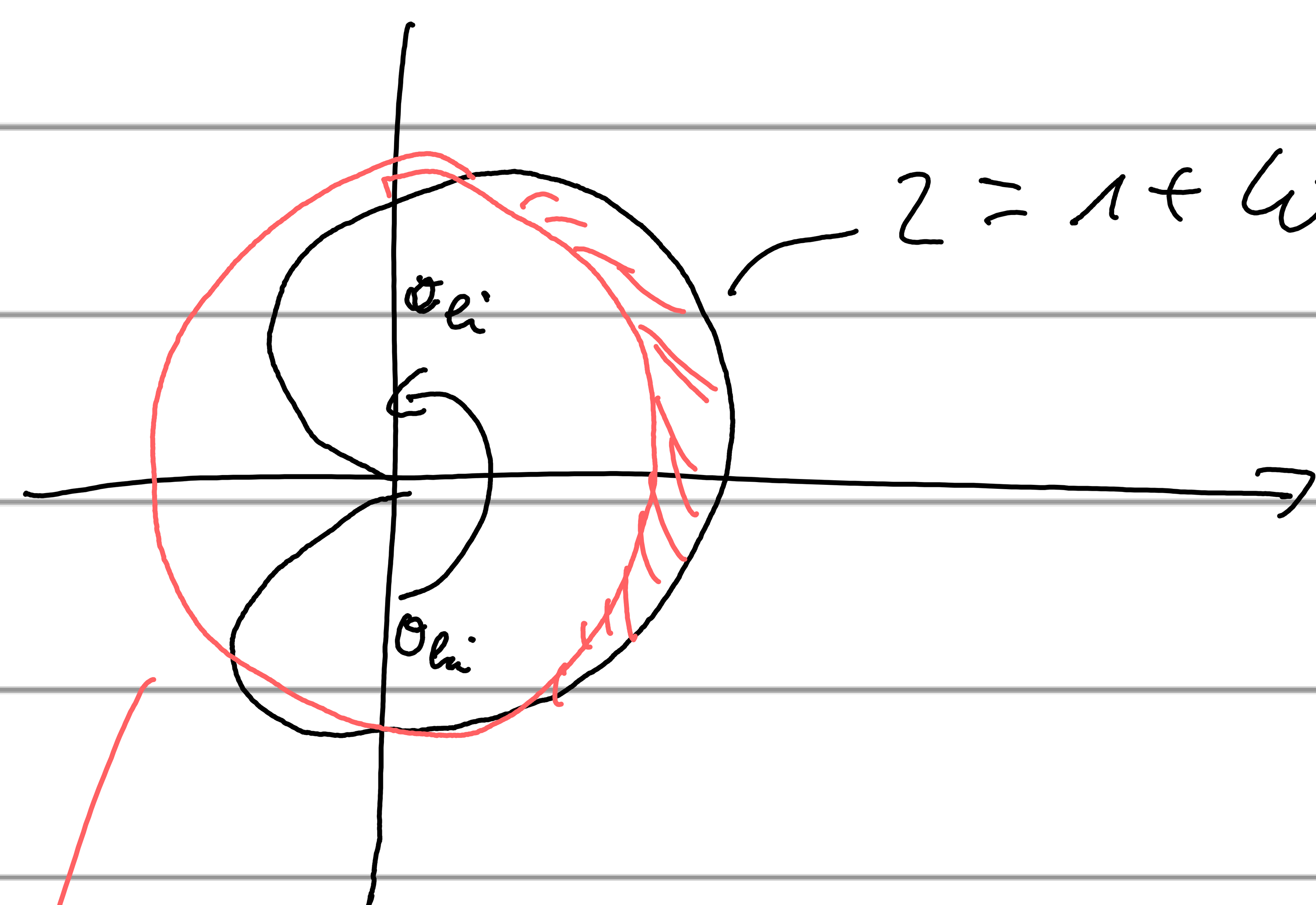
$$= \int_{\theta_1}^{\theta_2} d\theta \int_{r=f_{binnen}(\theta)}^{r=f_{buiten}(\theta)} f\, r\,dr$$

$dy\,dx = r\,dr\,d\theta$

Jacobiaan

θ als buitenste grenzen meestal

Ve) Bepaal oppervlakte van gebied dat in cardioide ligt $r = 1 + \cos\theta$ maar buiten cirkel $x^2 + y^2 = 1$ ligt



$x^2 + y^2 = 1 \rightarrow r = 1$

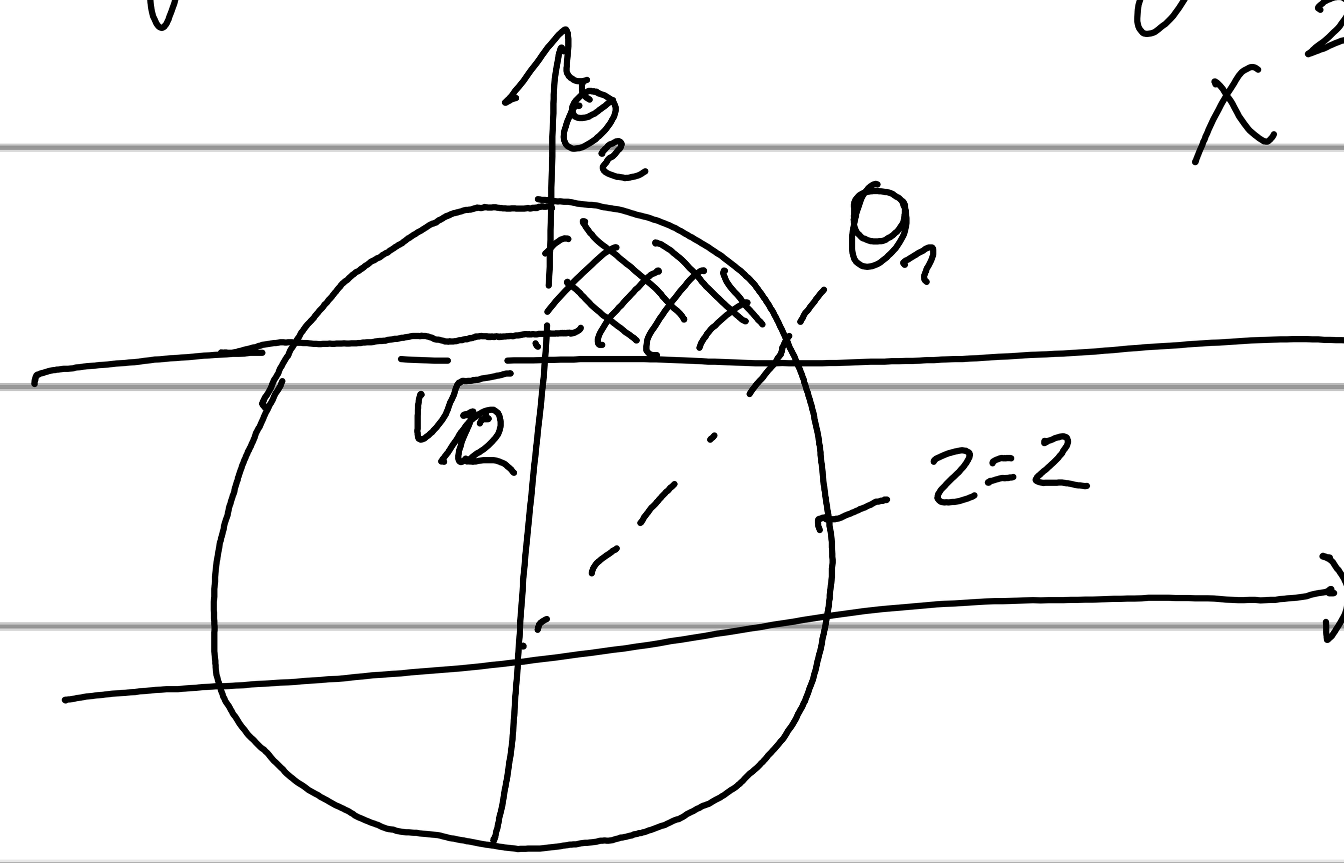
dus we stellen
$x^2 + y^2 = 1 = 1 + \cos\theta$
$1 = 1 + \cos\theta$
dan kan θ gelijk zijn aan $\frac{\pi}{2}$ of $-\frac{\pi}{2}$

$$\int_{-\pi/2}^{\pi/2} d\theta \int_{1}^{1+\cos\theta} r\,dr$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_1^{1+\cos\theta} d\theta$$

(Vl) Stel gelijk $y = \sqrt{2}$ 1,ko

$x^2 + y^2 = 4$



$\int d\theta \int z \, dz$

$\theta_2 = \frac{\pi}{2}$

$\theta_1 = ?$ $\rightsquigarrow$ $y = \sqrt{4 - x^2}$ $\sqrt{2}$

$y = \sqrt{2}$ $-\sqrt{2}$

$x = \sqrt{2}$

$y = \sqrt{2}$ $\rightsquigarrow$ $\tan \frac{\sqrt{2}}{\sqrt{2}} = \frac{\pi}{4} = \theta$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_{\frac{\sqrt{2}}{\sin\theta}}^{2} z \, dz$$

$z = 2$

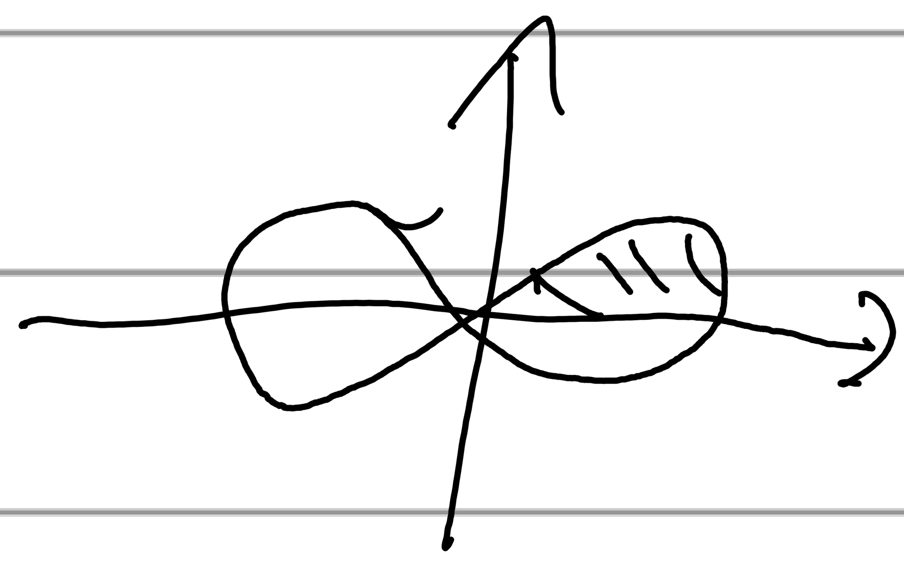
$y = \sqrt{2}$

$2 \sin \theta = \sqrt{2}$

$z = \frac{\sqrt{2}}{\sin \theta}$

VI

$r^2 = 4\cos(2\theta)$ $\quad r = 2\sqrt{\cos(2\theta)}$



$$4\int_0^{\pi/4} d\theta \int_0^{2\sqrt{\cos(2\theta)}} r\,dr = 4\cdot 2 = 8$$

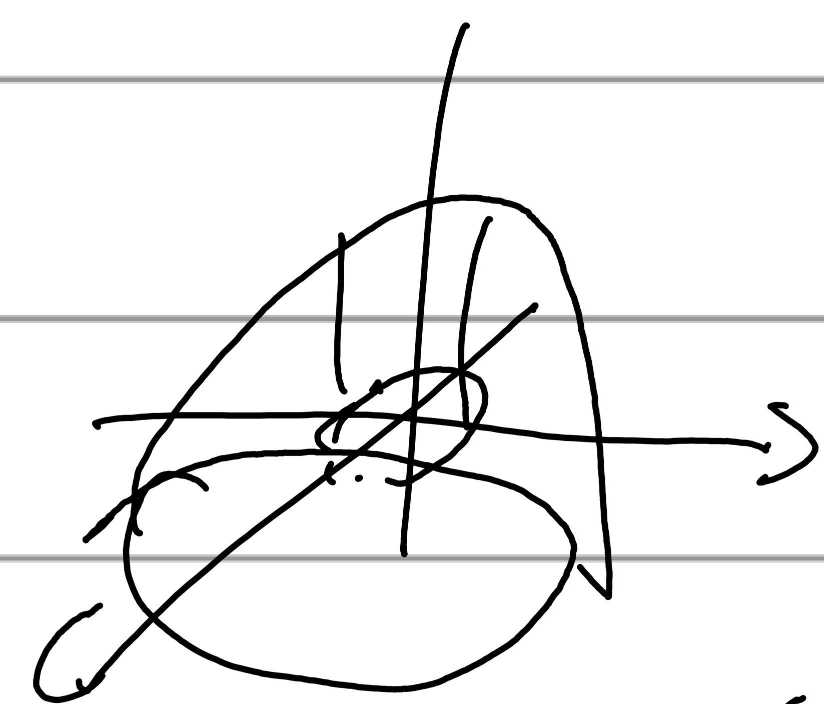
VI Bepaal oppervlakte cirkel met $r = 1$

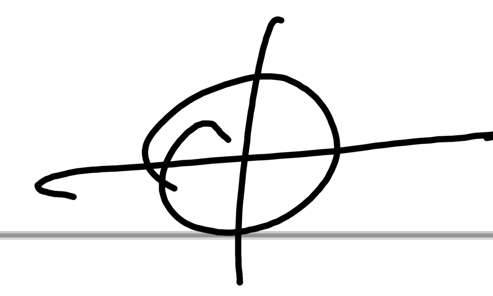
$$\int_0^{2\pi} d\theta \int_0^{1} r\,dr = \pi$$

Vb

vol van $z = 9 - x^2 - y^2$

Begrensd door $x^2 + y^2 = 1$





$$\int_0^{2\pi} d\theta \int_0^1 \rho\, z\, dz$$

z = niet in Polaircoörd

$$x^2 + y^2 = z^2$$

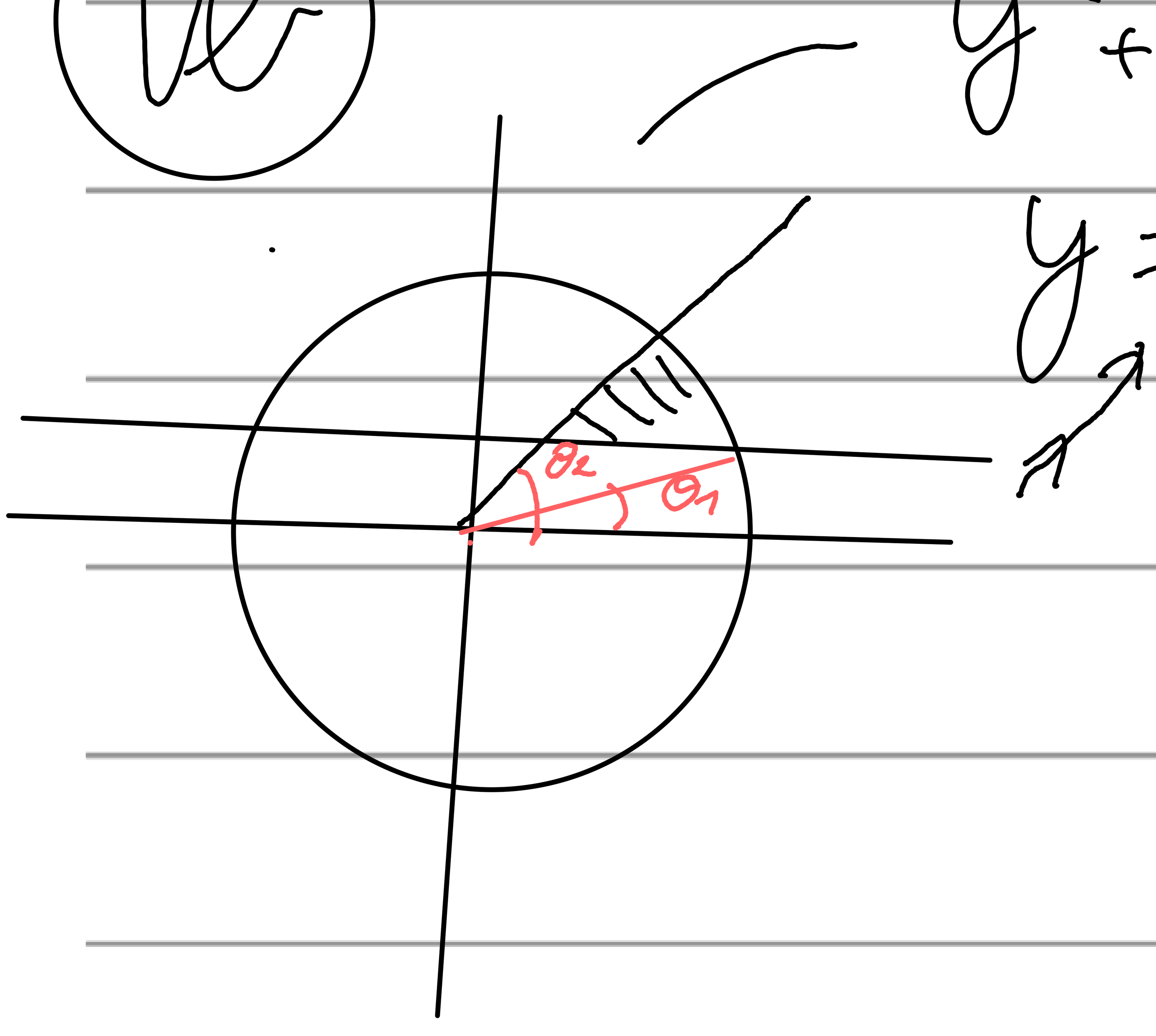
dus wordt $9 - z^2 = \rho$

Vb

$y = \sqrt{3}x$

$y^2 + x^2 = 4$

$y = 1$



$\theta_1 \Rightarrow tg\, \theta_1 = \frac{y}{\cancel{x}}$ 1

$y = \sqrt{\cancel{8}4 - x^2}$

$y = 1$ $\Rightarrow$ $x = \sqrt{3}$

$\theta_1 = \frac{\pi}{6}$

zelfde voor $\theta_2 =$ $x = 1$, $y = \sqrt{3}$ dus $\theta_2 = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

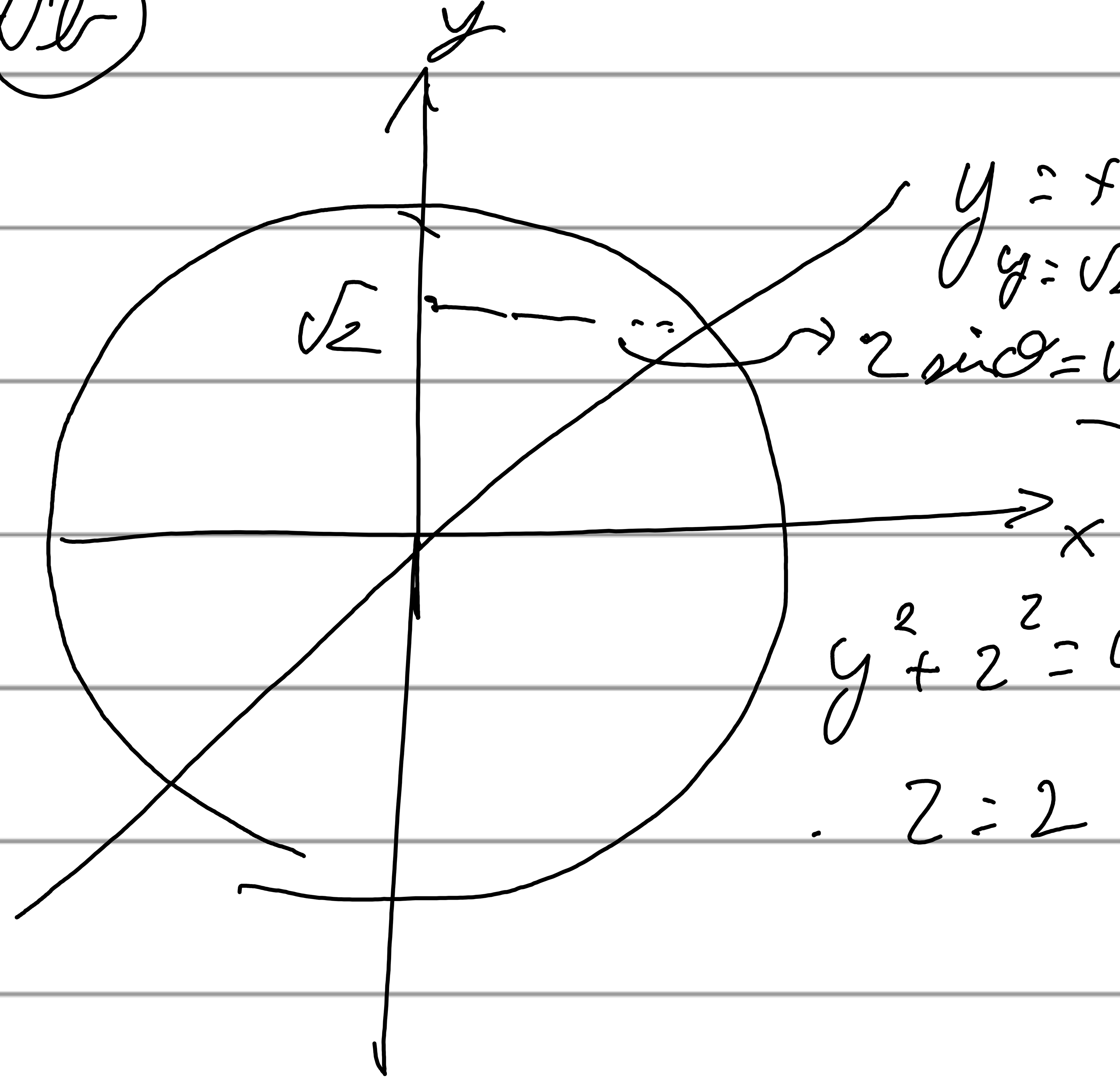
$$\int_{\pi/4}^{\pi/3} \cdot d\theta \int_{\frac{1}{\sin\theta}}^{2} r\, dr$$

$y = 1 \rightarrow \quad y = r \sin\theta = 1$

$r = \dfrac{1}{\sin\theta}$

de orde is genomen 2

1b



$y = x$

$y = \sqrt{2}$

$z \sin\theta = \sqrt{2}$

$y^2 + z^2 = 4$

$z = 2$

$$\int_{\pi/4}^{\pi/2} d\theta \int_{\frac{\sqrt{2}}{\sin\theta}}^{2} z\, dz$$

<u>Drievoud</u> $\iiint_V f(x,y,z)\, dx\, dy\, dz$

→ meetkundig geen betekenis

$\iiint_V dx\, dy\, dz \Rightarrow$ Volume van gebied

$\iiint_V f\, dx\, dy\, dz$

↳ onafhankelijk $= \int_a^b dx \int_{y=f_0(x)}^{y=f_1(x)} dy \int_{z=f_0(x,y)}^{z=f_1(x,y)} f\, dz$

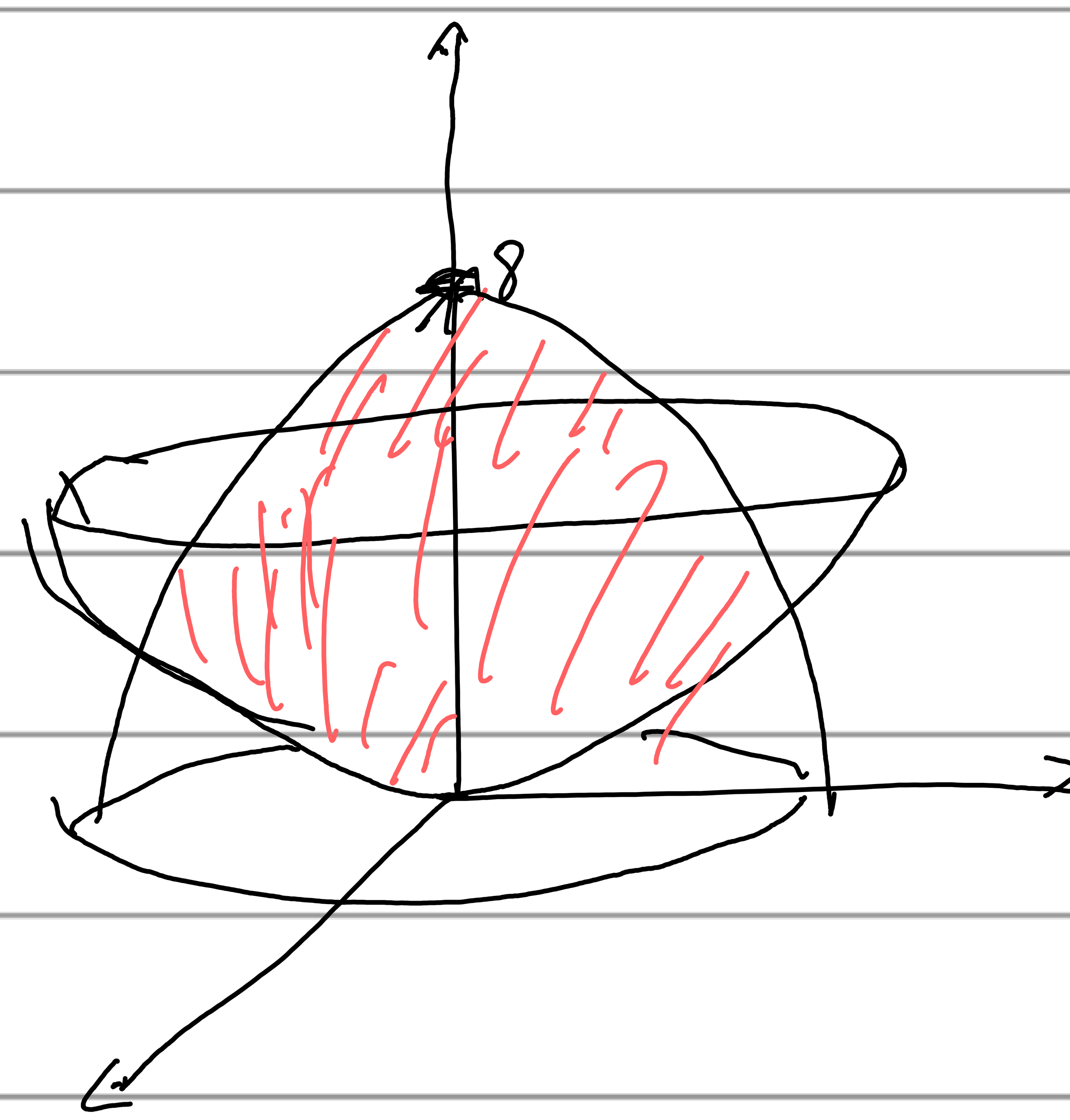
$z = f_1(x,y)$ bovenste vlak

$z = f_0(x,y)$ → onderste vlak

Vb Bepaal Volume begrensd door 2 oppervlakken

$z = 8 - x^2 - y^2$

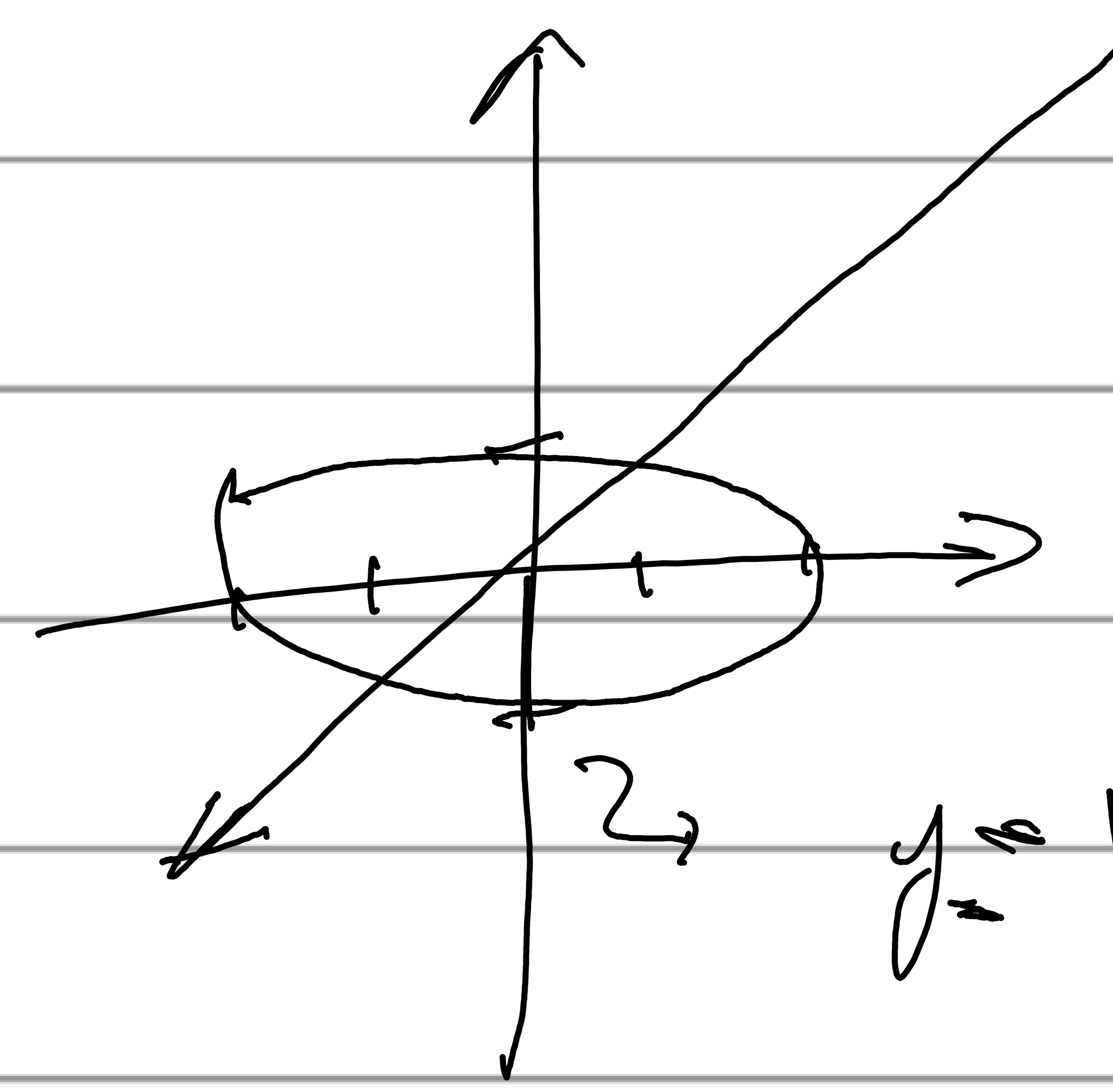
$z = x^2 + 3y^2$



$8 - x^2 - y^2 = x^2 + 3y^2$

$8 = 2x^2 + 4y^2$

$1 = \frac{x^2}{4} + \frac{y^2}{2} \longrightarrow$



$y = \sqrt{\frac{4 - x^2}{2}}$

D is elips dus voorbeeld

$$\int_{-2}^{2} dx \int_{-\sqrt{\frac{4-x^2}{4}}}^{+\sqrt{\frac{4-x^2}{4}}} dy \int_{x^2+3y^2}^{8-x^2-y^2} dz$$

Oef 2

Bepaal Volume

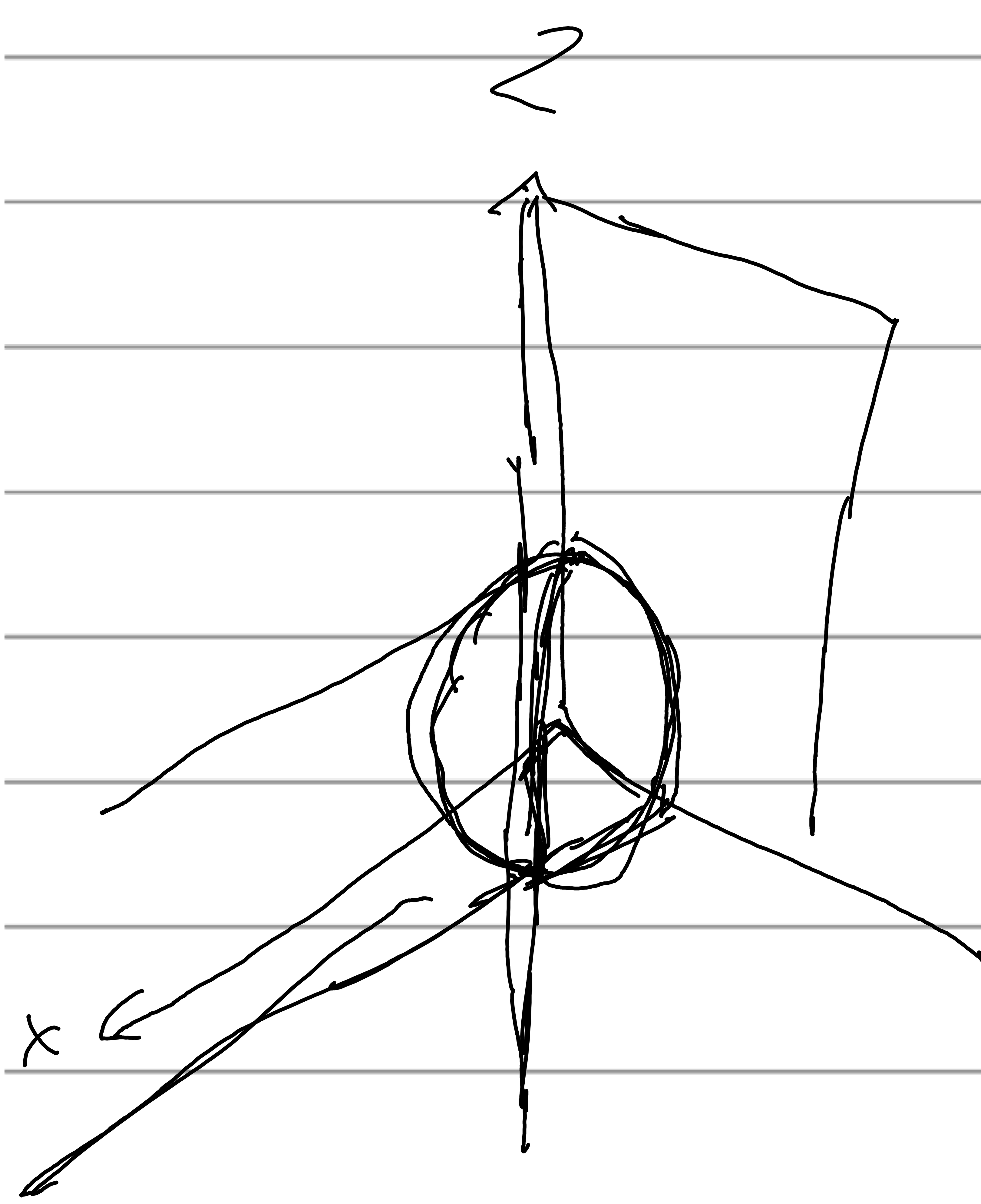
$$\begin{cases} y^2 + z^2 = 1 \\ y = x \\ z = 0 \\ x = 0 \end{cases}$$

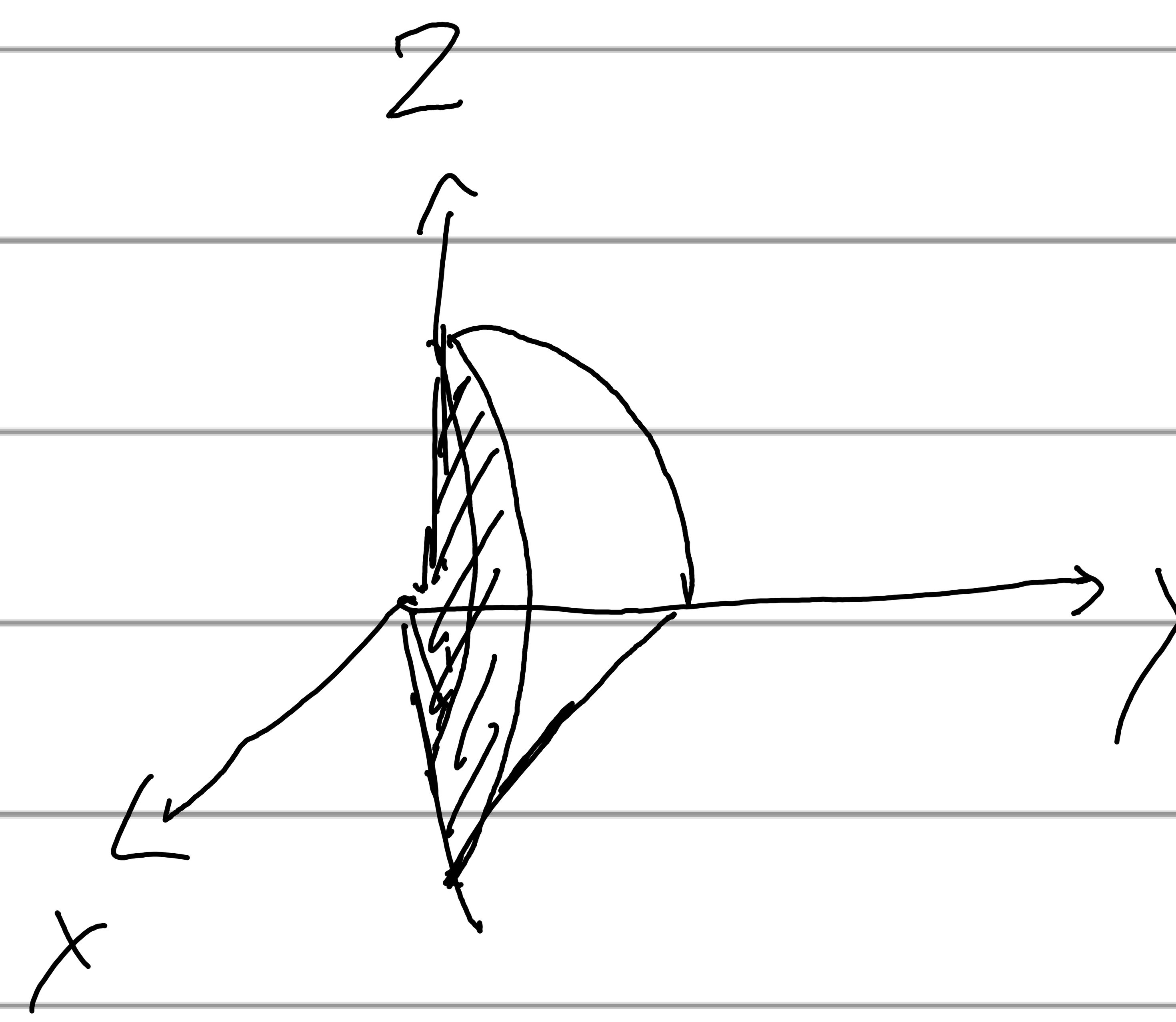
→ volgorde 3 manieren

$\int dx \int dy \int dz$

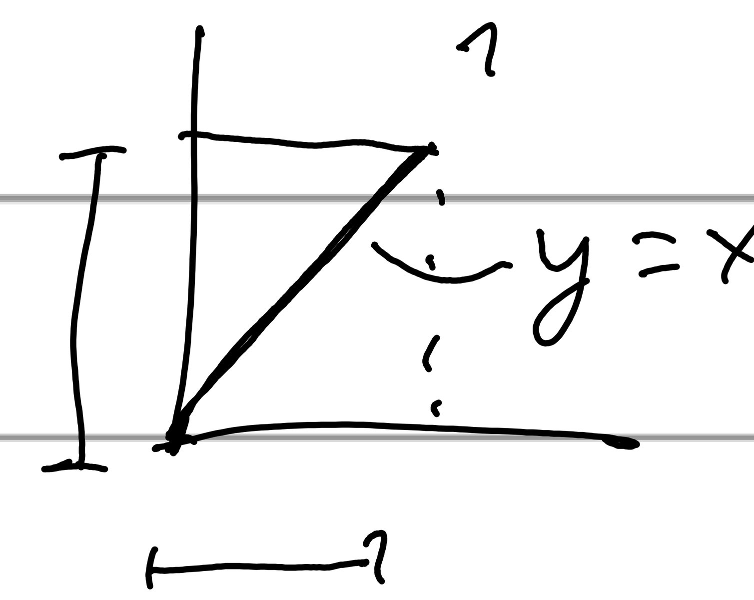
$\int dx \int dz \int dy$

$\int dy \int dz \int dx$



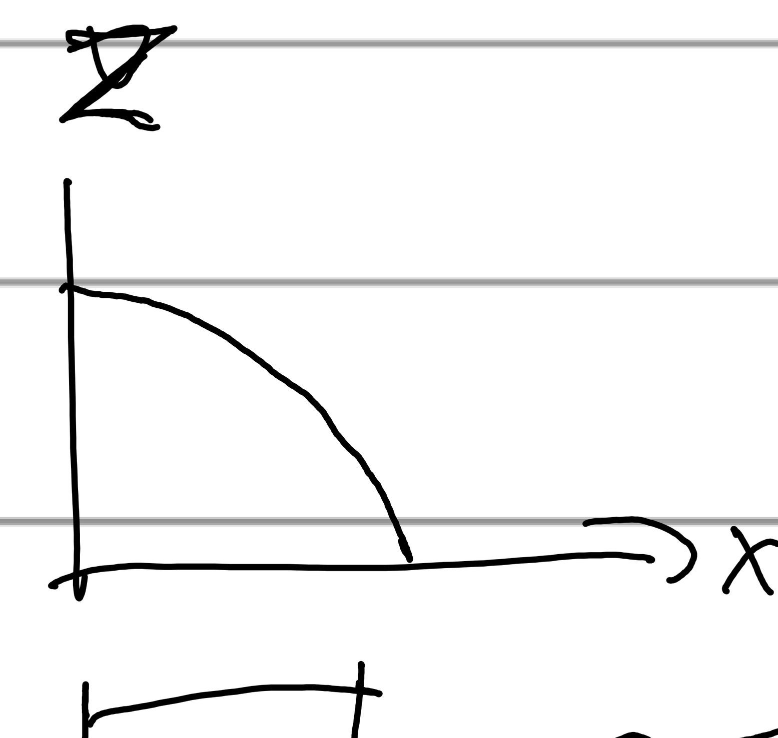


$\int dx \int dy \int dz \Rightarrow$ projectie op xy-vlak



$$\int_0^1 dx \int_x^1 dy \int_0^{\sqrt{1-z^2}} dz \rightarrow y^2+z^2=1$$

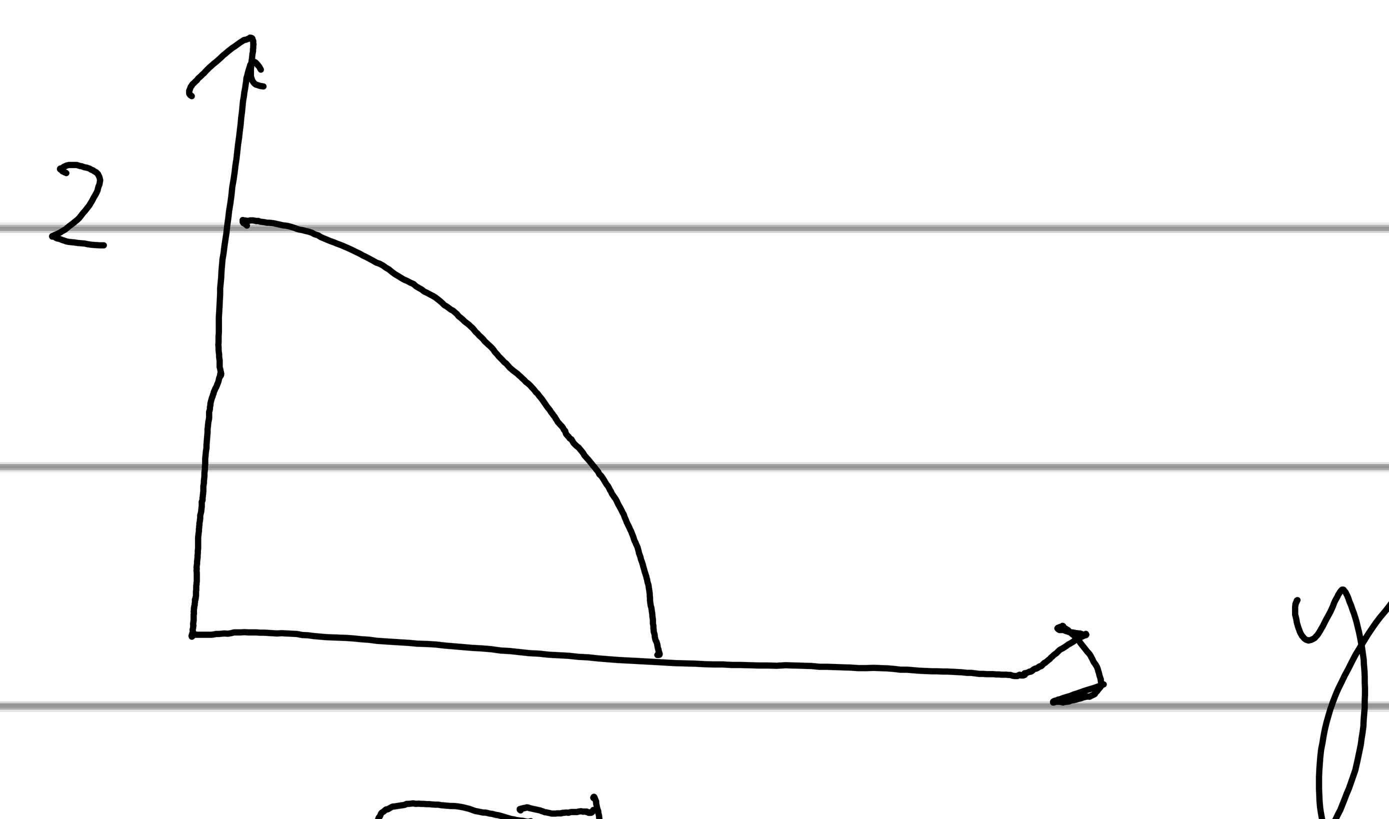
$\int dx \int dz \int dy \Rightarrow$



$y = x$

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dz \int_x^{\sqrt{1-z^2}} dy$$

$\int dy \int dz \int dx$



$$\int_0^1 dy \int_0^{\sqrt{1-y^2}} dz \int_0^y dx$$

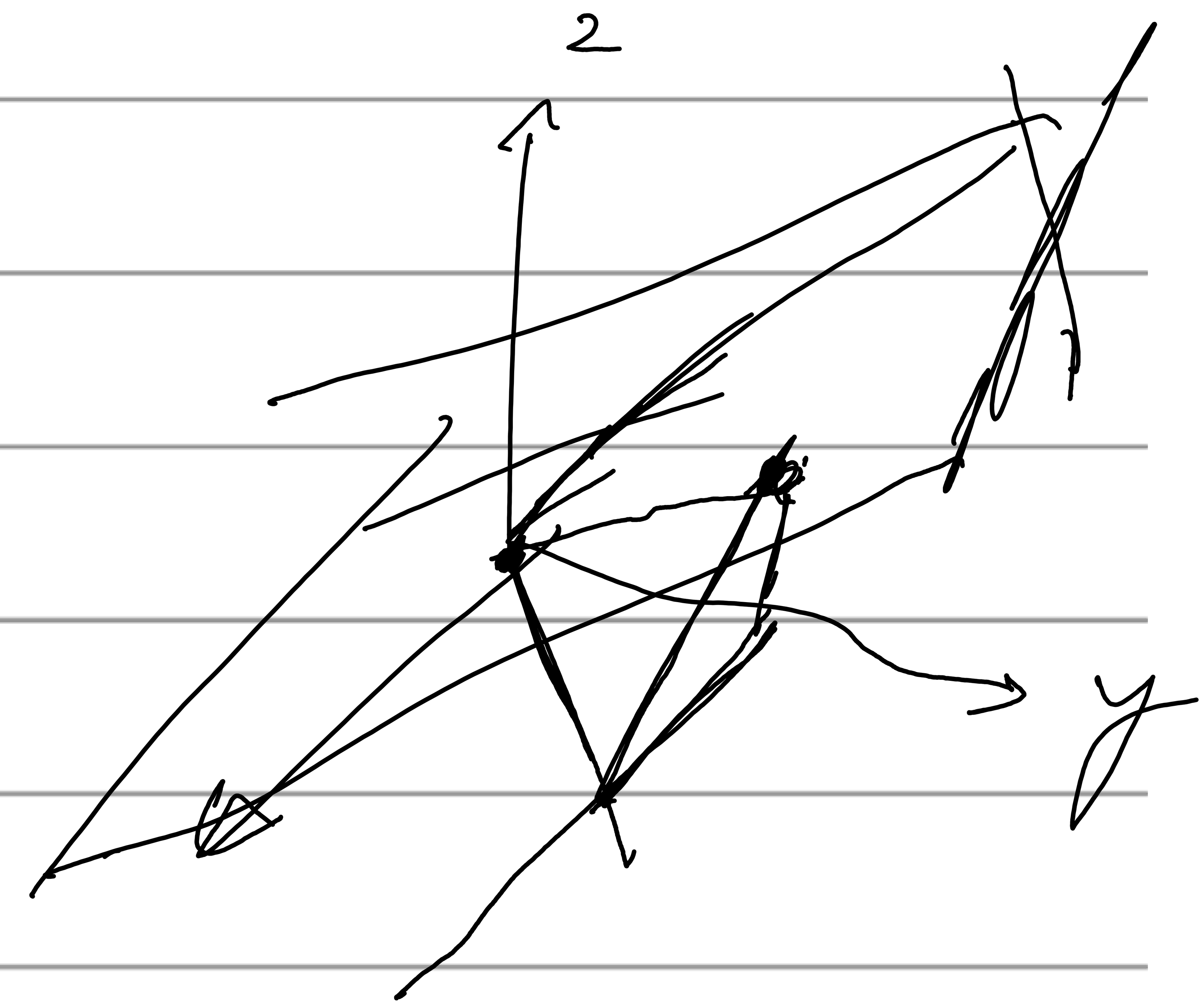
Vl-3 Beyond Volume der thetroede

$y = x$

$y = 1$

$x = 0$

$z = y - x$



$\iiint dz\,dy\,d$

$$\int_0^1 dx \int_0^1 dy \int_0^{y-x} dz = \frac{1}{6}$$

Toepassing massamiddelpunt

Massa bepalen van een lichaam in 3 dimensies

$$M = \iiint_V \rho \, dV$$

voor middelpunt

$$\bar{x} = \frac{\iiint_V \rho x \, dV}{M}$$

$$\bar{y} = \text{〃} \quad \text{〃} \quad \text{〃}$$

$$\bar{z} = \text{〃} \quad \text{〃} \quad \text{〃}$$

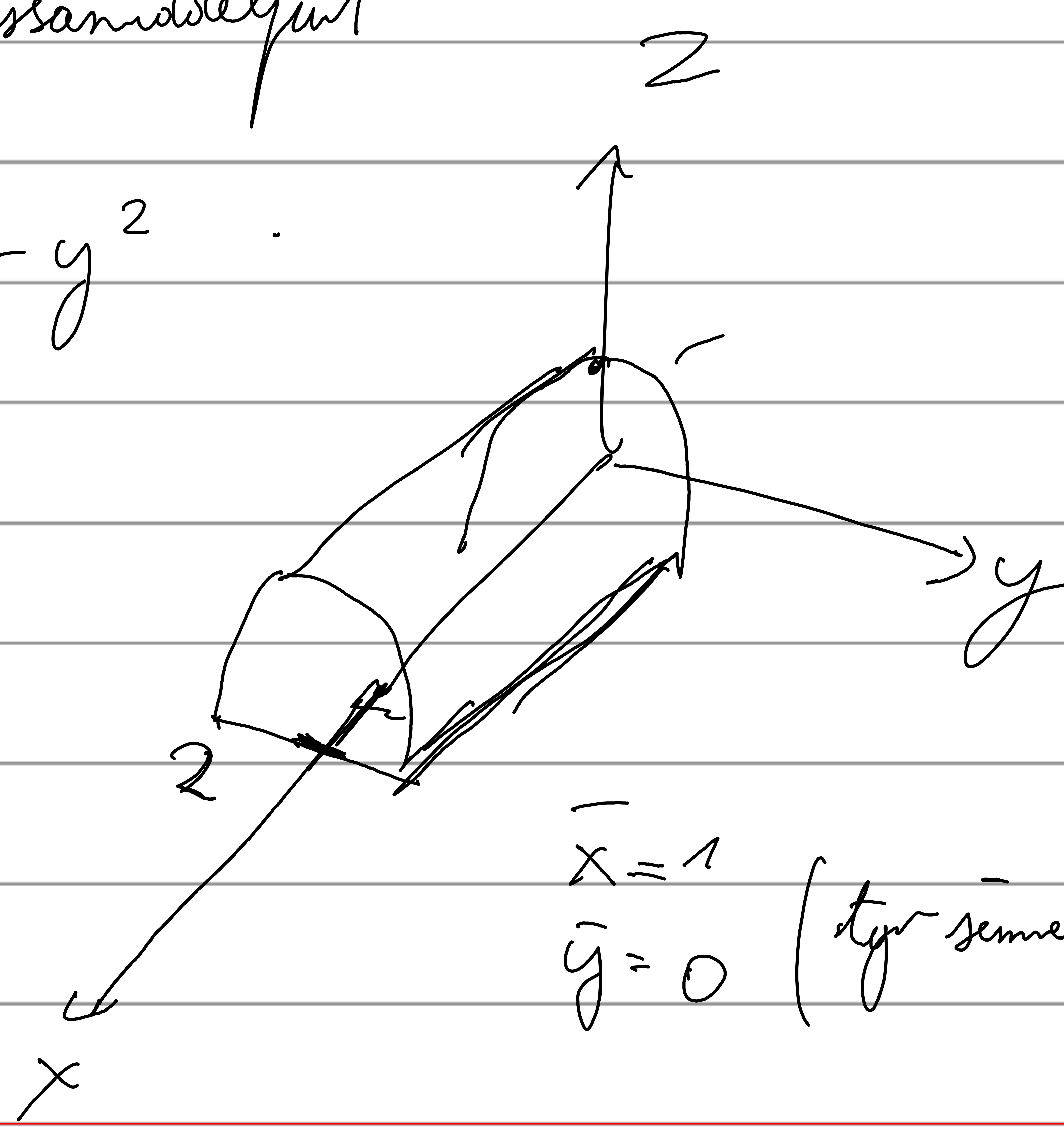
Stel bepaal massamiddelpunt

$z = 1 - y^2$

$x = 0$

$z = 0$

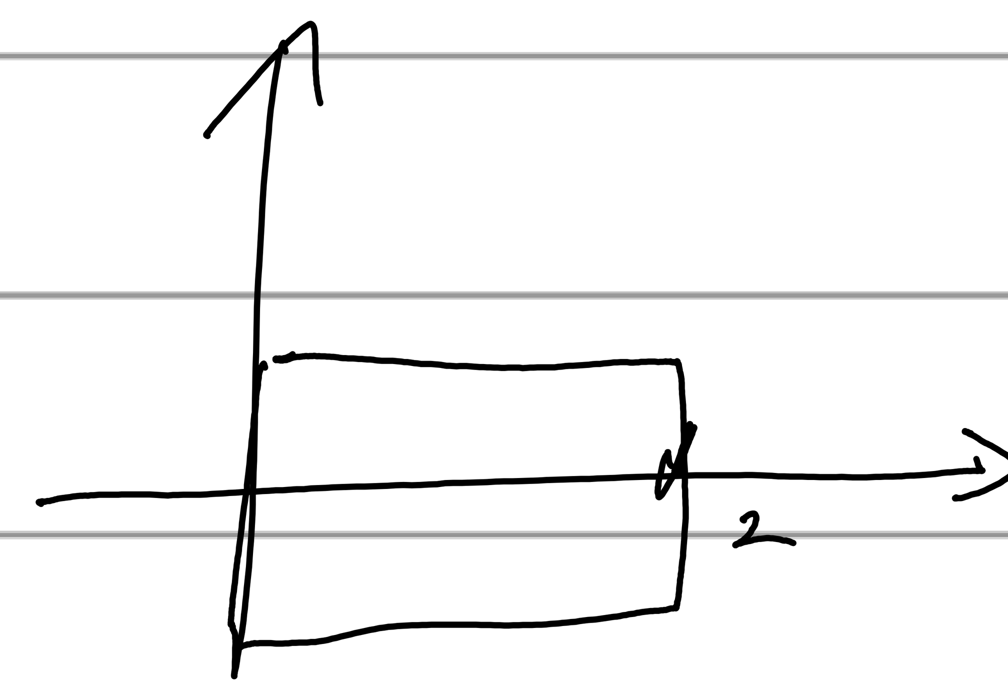
$x = 2$



$\bar{x} = 1$

$\bar{y} = 0$ (door symmetrie)

$$M = \iiint_V \rho \, dV$$



$$M = \int_0^2 dx \int_{-1}^{1} dy \int_0^{1-y^2} dz = \frac{8}{3}$$

$$M_z = \frac{2 \int_0^2 dx \int_0^1 dy \int_0^{1-y^2} z \, dz}{8/3} = \frac{2}{5}$$

Cilindro coordinate

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$r\,dr\,d\theta\,dz$

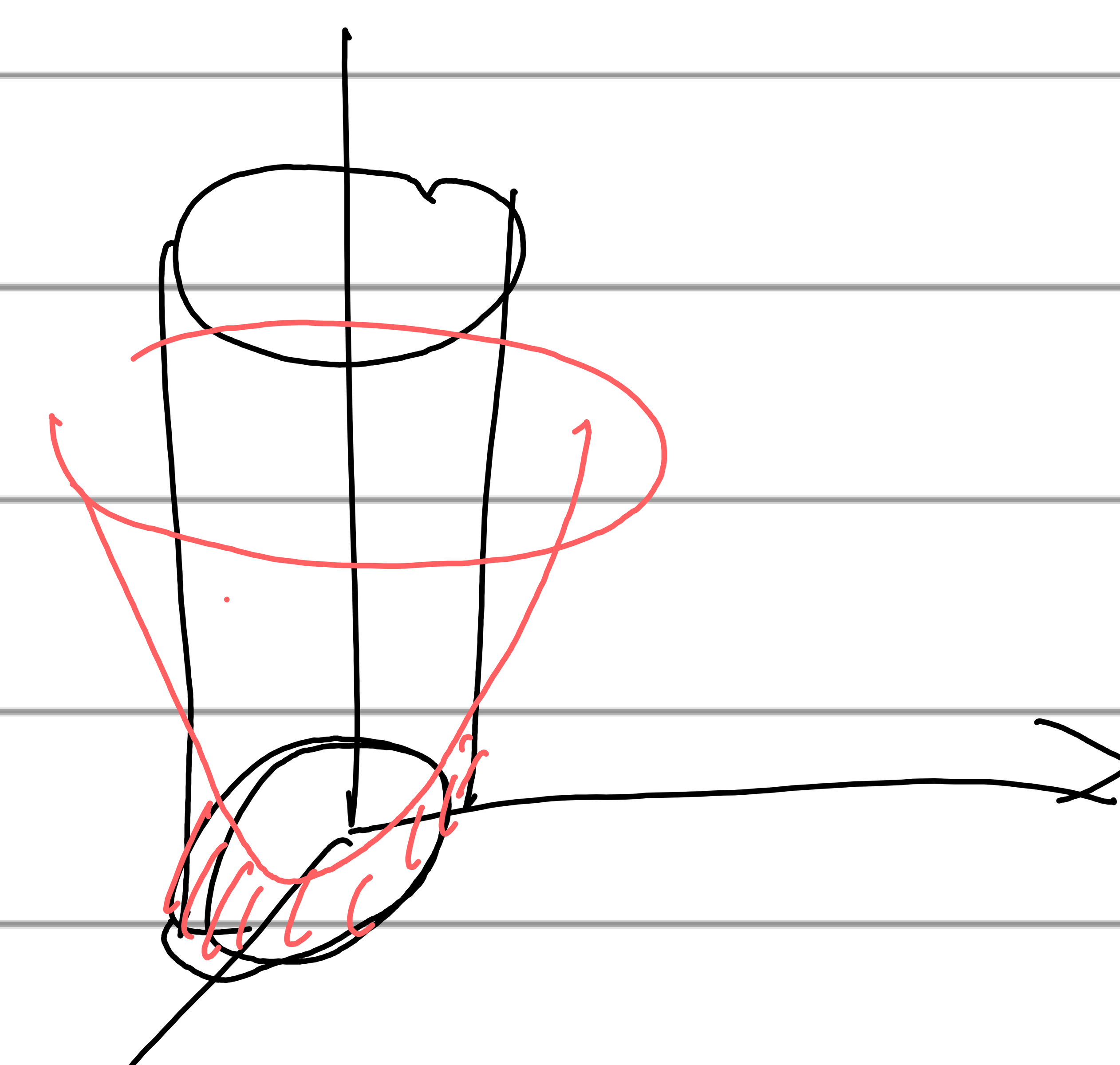
$$\int_{\theta_1}^{\theta_2} d\theta \int_{r=h_b(\theta)}^{r=h_a(\theta)} dr \int_{z=f_1(r,\theta)}^{z=f_2(r,\theta)} r\,dz$$

Es

$z = x^2 + y^2$

$x^2 + y^2 = 1$

$z = 0$



$$\int_0^{2\pi} d\theta \int_0^1 dr \int_0^{r^2} r\,dz$$

r^2 funzione

$z = x^2 + y^2$

$(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$

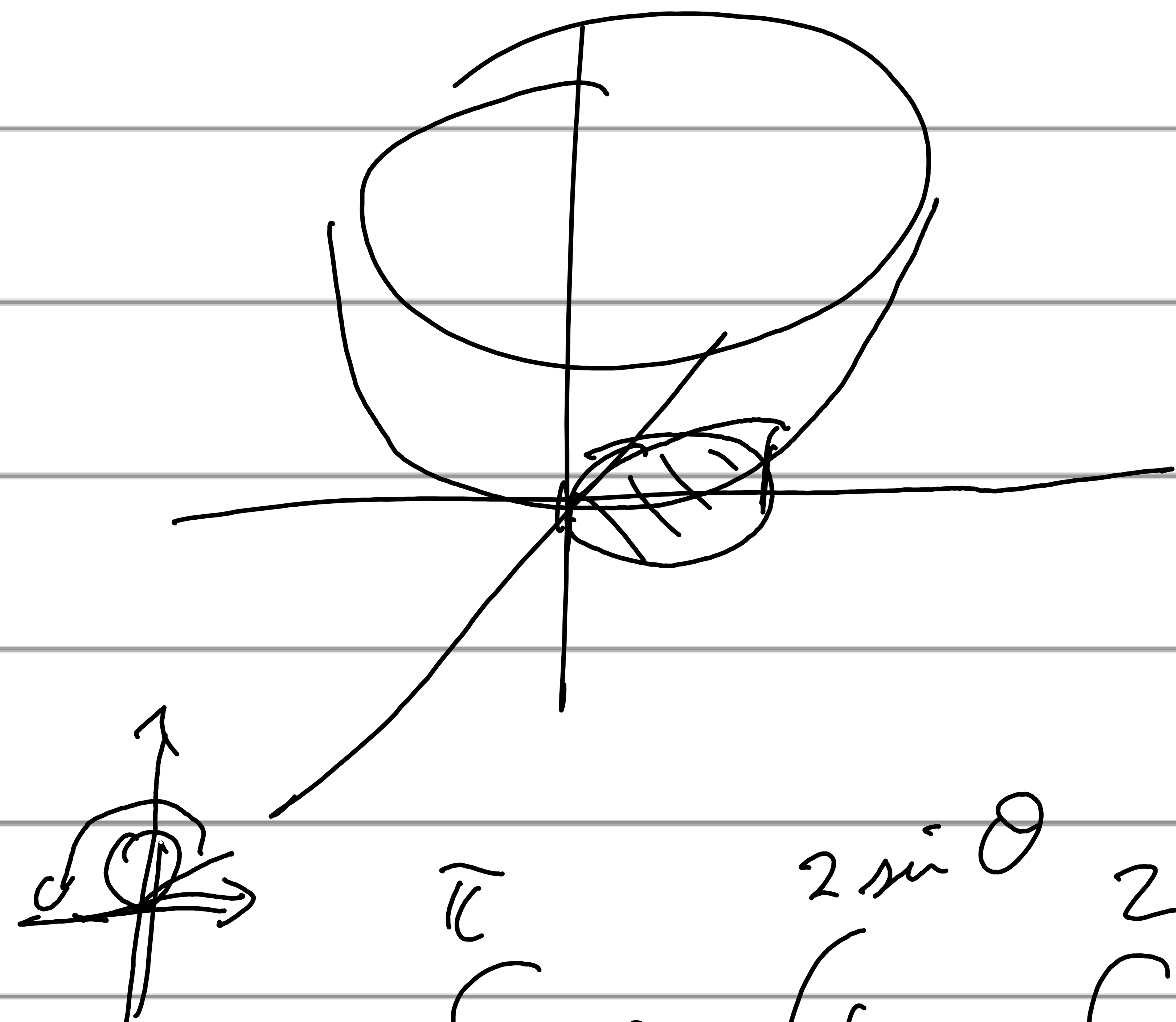
Ve

$$\left.\begin{aligned} & x^2+(y-1)^2=1 \\ & z=\underbrace{x^2+y^2}_{R^2} \end{aligned}\right\} \quad \underbrace{x^2+y^2}_{R^2}-2y+1=0$$

$$r^2-2r\sin\theta=0$$

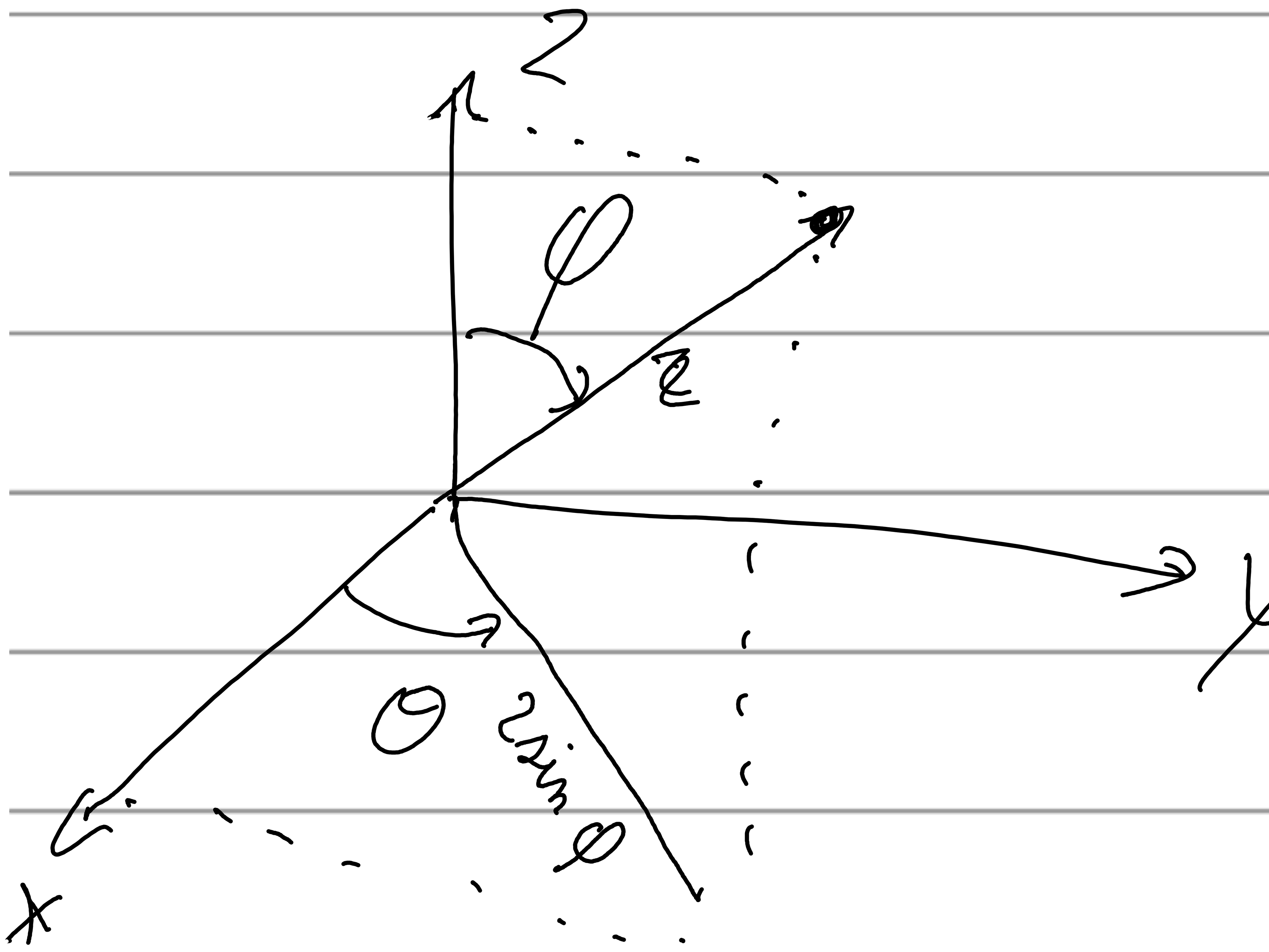
$$r^2=\cancel{2}\,2r\sin\theta$$

$$r=2\sin\theta$$



$$\int_0^{\pi} d\theta \int_0^{2\sin\theta} dr \int_0^{r^2} dz$$

Cirkelcoördinaten



$$\left.\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \end{aligned}\right\}$$

$$\int_{\theta_1}^{\theta_2} d\theta \int_{\varphi_1}^{\varphi_2} d\varphi \int_{\rho=\rho_e(\varphi,\theta)}^{\rho=\rho_e(\varphi,\theta)} \underbrace{r^2 \sin\theta}_{\text{Jacob}}\, dr$$

$\rightarrow x^2 + y^2 + z^2 = r^2$

$$(r \sin\varphi \cos\theta)^2 + (r \sin\varphi \sin\theta)^2 + (r \cos\varphi)^2$$

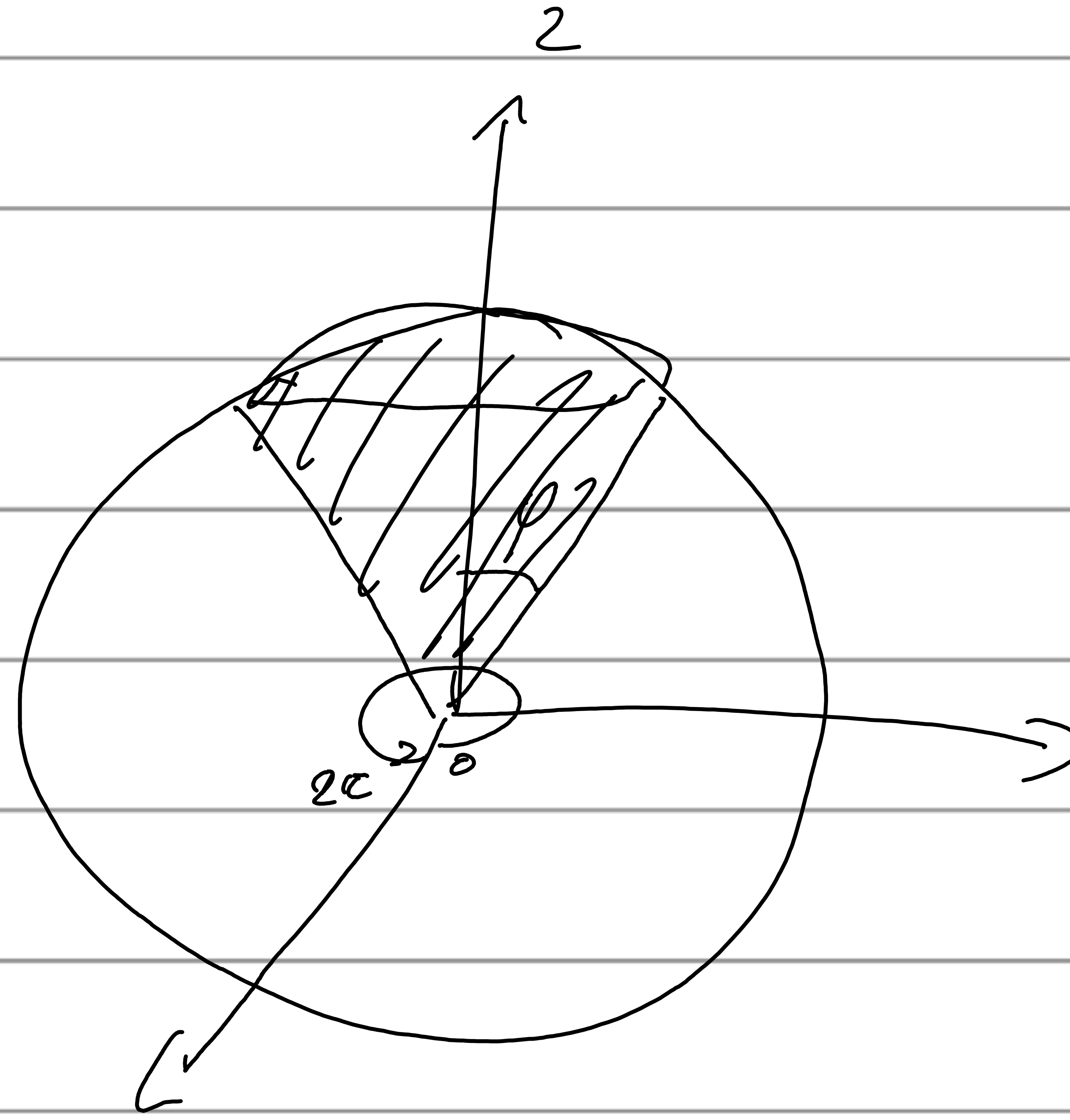
Je kunt invullen met cos en sin en bewijzen

$$r^2 \left(\sin(\cdot\varphi)^2 + 1 \right)$$

$$r^2 (1 - \cos(2\varphi)) + 1$$

$$r^2 \left(2 \cos 2(\varphi) \right) \text{ dofug?}$$

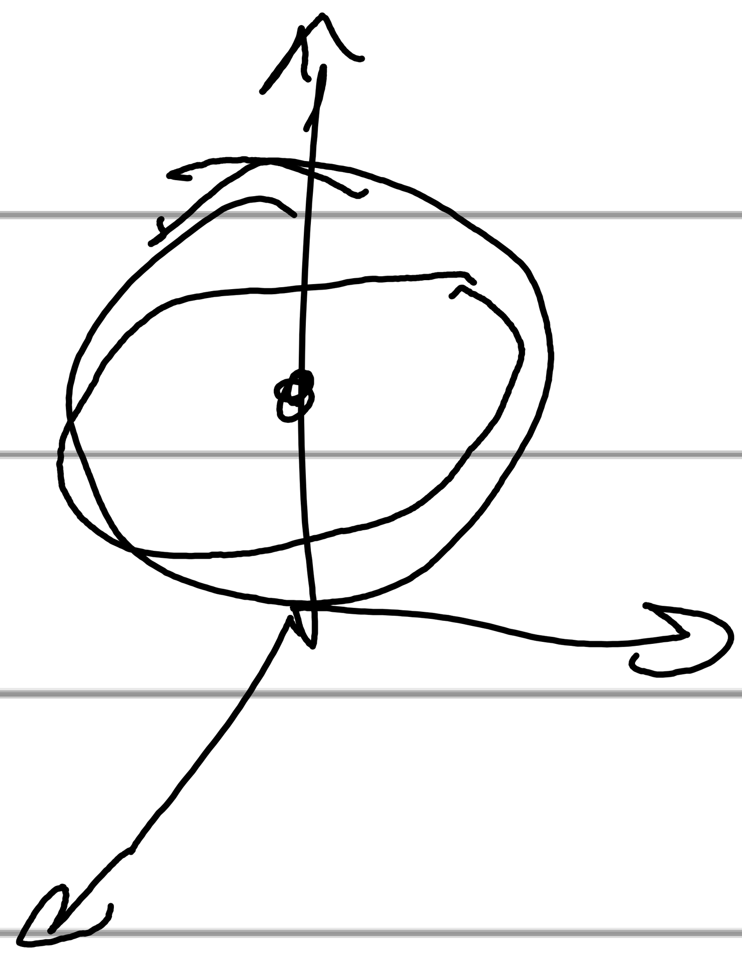
Vb Volume of cone



$$\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi$$

ϕ $\cancel{\mathbb{F}}$

Ve Vol $\underbrace{x^2+y^2+(z-1)^2=1}$



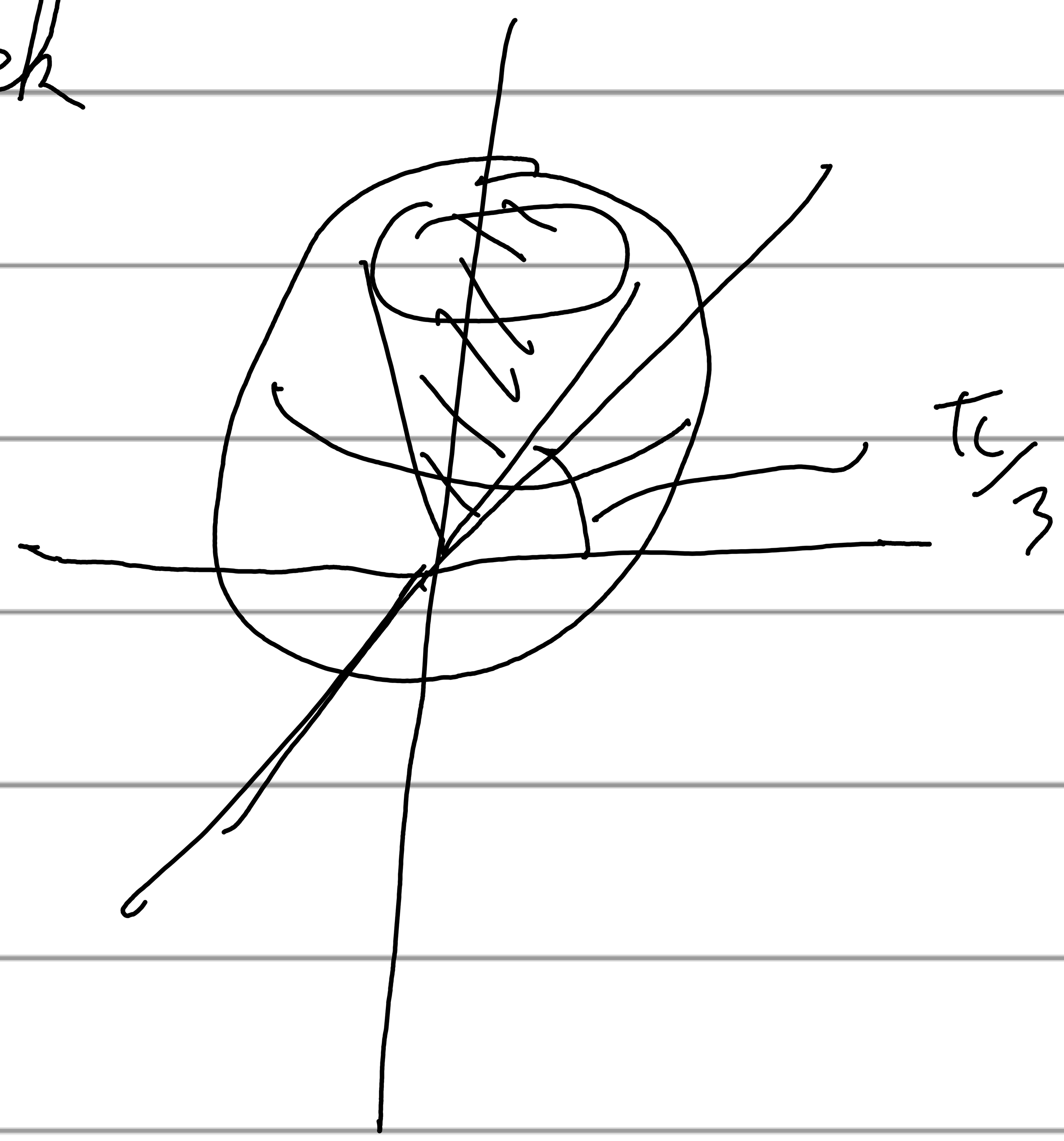
$$\underbrace{x^2+y^2+z^2}-2z+1=1$$

$$\rho^2-2\rho\cos\phi=0$$

$$\rho=2\cos\varphi$$

$$\int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^{2\cos\varphi} \rho^2 \sin\theta \, d\rho$$

ijskoek



$$\int_0^{2\pi} d\theta \int_0^{\pi/2 - \pi/3} d\varphi \int_0^1 \rho^2 \sin\theta \, d\rho$$

Enkelvoudig →	dubbel →	Drievoudig integraal
	||	||
$\int$	oppervlakte integraal	Volume integraal
	$\iint$	$\iiint$

Formule van de substitutie

$$\iint_A f(x,y)\,dx\,dy = \iint_{A'} f\big(x(u,v),\, y(u,v)\big)$$

$$x = x(u,v)$$
$$y = y(u,v)$$

Er komt iets extra bij: de Jacobiaan

Determinant

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} =$$

Bv $x = r\cos\theta \rightarrow x(r,\theta)$

$y = r\sin\theta \rightarrow y(r,\theta)$

$$J \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$= r =$ Jacobiaan

Theorie

$$\int \to \iint \to \iiint$$

Recording1
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$$\iiint_A f(x,y,z)\,dx\,dy\,dz$$

$$\to \iiint_{A(u,v,w)} f(x(u,v,w),\, y(u,v,w),\, z(u,v,w))\underset{|J|}{\wedge}\,du\,dv\,dw$$

$$x = 2\cos\theta$$
$$y = 2\sin\theta$$
$$z = z$$

$$\frac{\partial x}{\partial u} \quad \frac{\partial x}{\partial v} \quad \frac{\partial x}{\partial w}$$

$$\frac{\partial y}{\partial u} \quad \frac{\partial y}{\partial v} \quad \frac{\partial y}{\partial w}$$

$$\frac{\partial z}{\partial u} \quad \frac{\partial z}{\partial v} \quad \frac{\partial z}{\partial w}$$

$$|J| = \begin{vmatrix} \cos\theta & -2\sin\theta & 0 \\ \sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2$$