functies in meerdere variabelen

Def: $n = 2$ (2 variabelen) alles hoger dan 2 kan niet in de ruimte getekend worden.

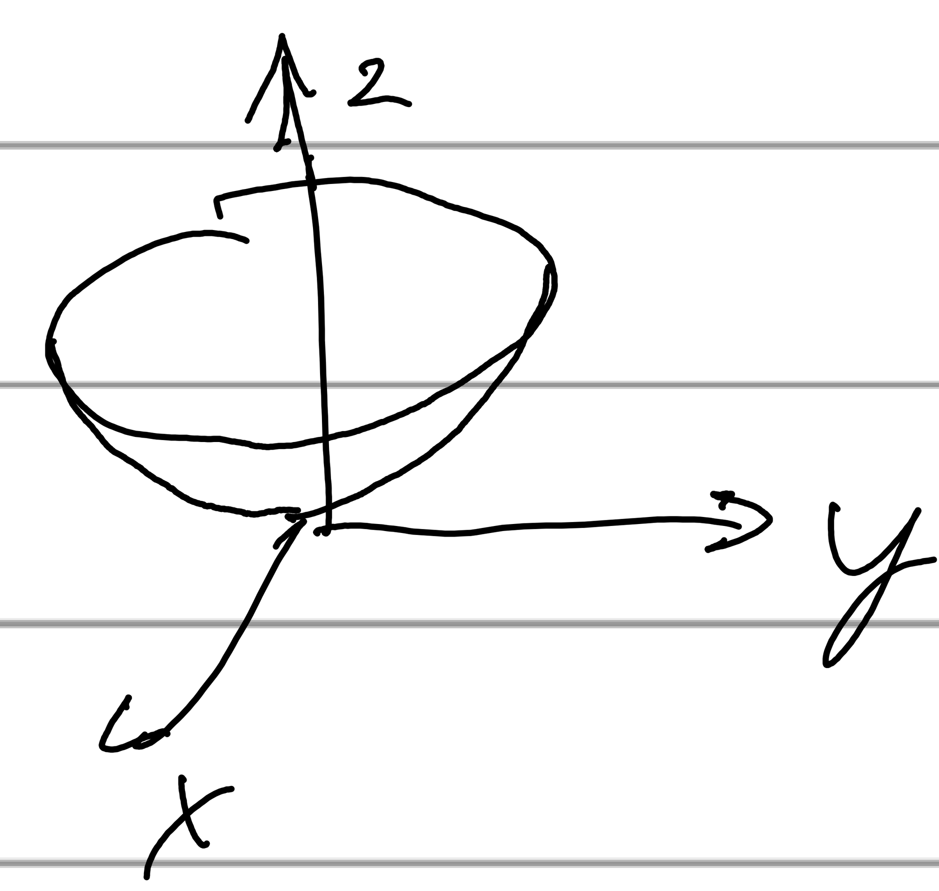
→ Explicit $z = f(x,y)$ of $z(x,y) : \mathbb{R}^2 \to \mathbb{R}$

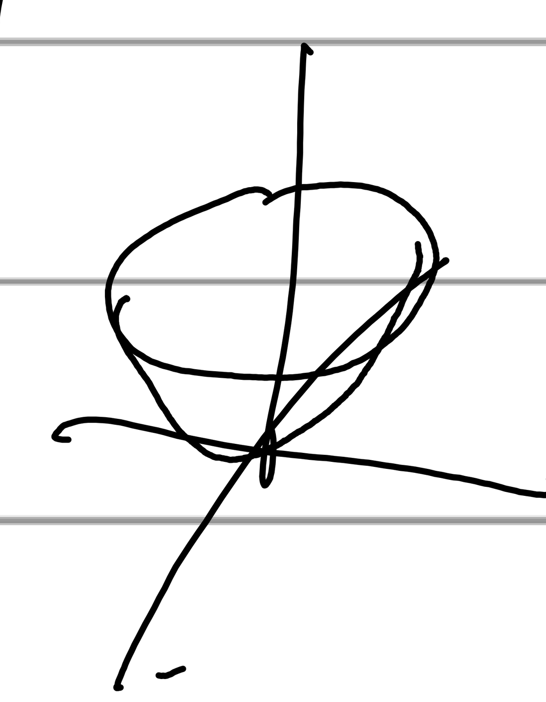
→ Implicit $F(x,y,z) = 0$

↓

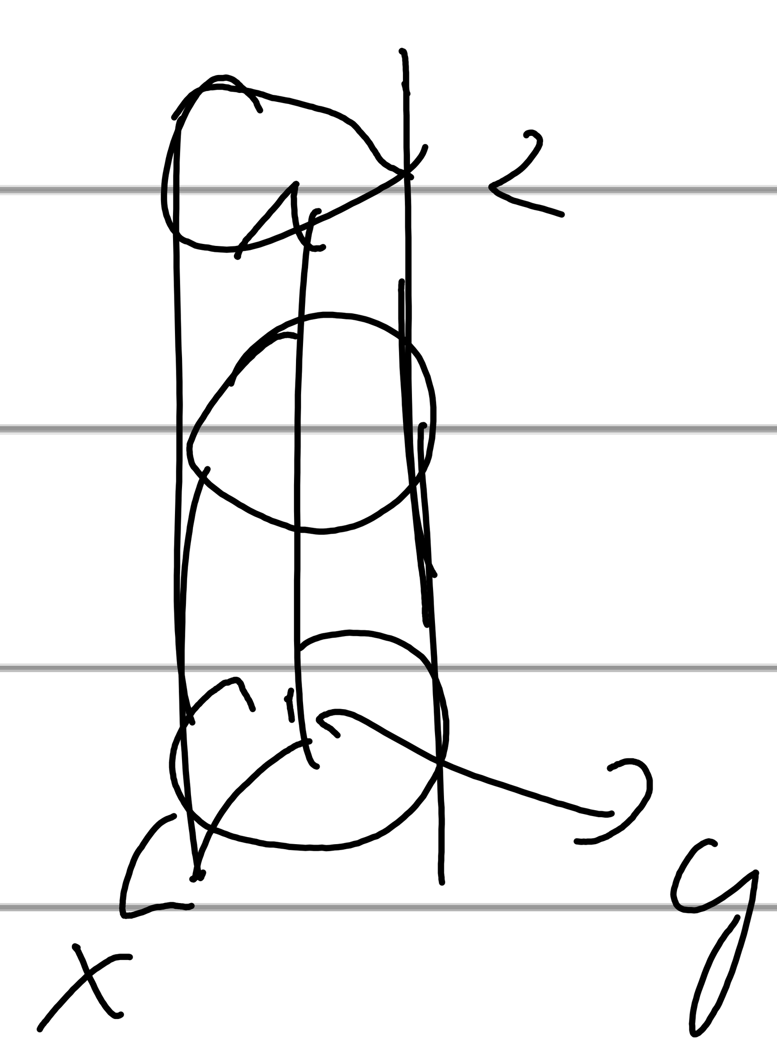
z is functie van x,y $z = f(x,y)$

$z^2 = x^2 + y^2$





$1 = x^2 + y^2$



$w = f(x,y,z)$
→ 4 dimensies

$w = f(x,y,z,t)$
→ waarde, tijd
plaats

$\mathbb{R}^n \to \mathbb{R}$

Vergl met vectorfunctie

$t \to (x(t), y(t), z(t)) \Rightarrow$ kromme

$\mathbb{R} \to \mathbb{R}^n$

t varieert en wordt afgebeeld

omgekeerd nu

met veel inputs naar 1 output

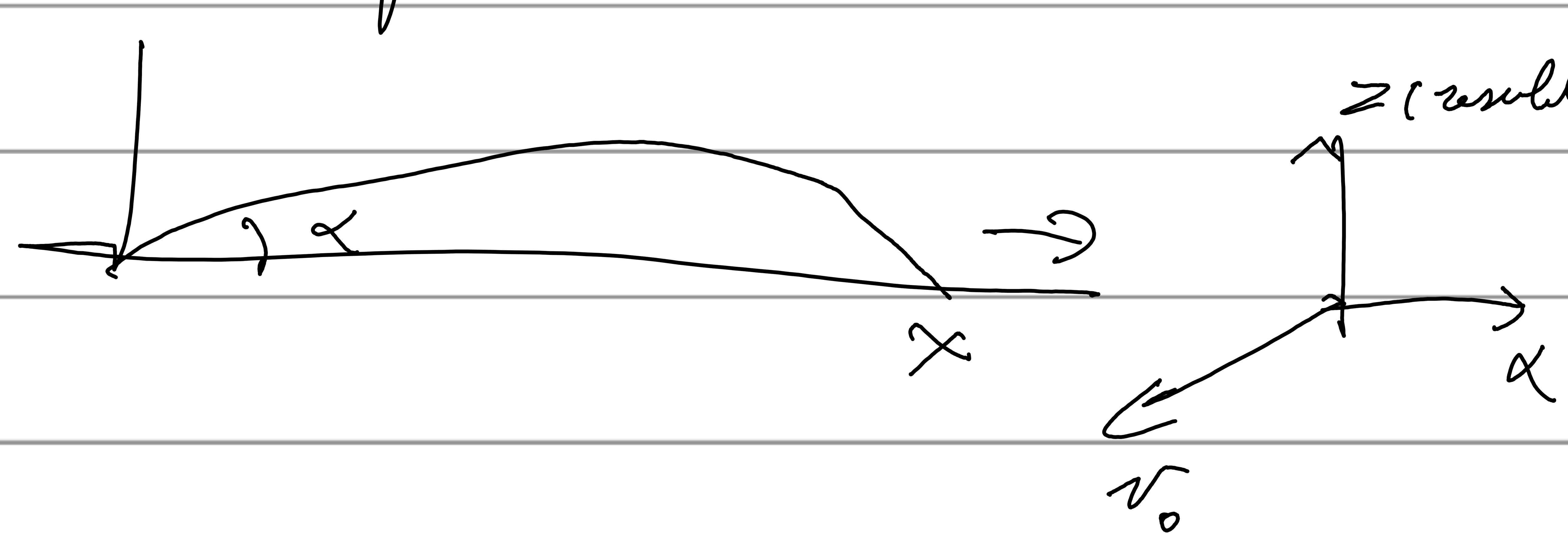
we werken nu niet met kromme maar met vlak

Def $\mathbb{R}^n \to \mathbb{R}$

n dimensionale vector $\Rightarrow$ Beeld

$$\underbrace{(x_1, x_2, \ldots, n)}_{\text{Domein}} \to \underbrace{f(x_1, x_2, n) \in \mathbb{R}}_{\text{Beeld}}$$

Vb $x = \dfrac{v_0^2 \sin 2\alpha}{g}$ $\to$ functie in functie van α en gravitatie



Beschrijf t° op een bepaalde locatie in t

$v = f(x, y, z)$

$w = f(x, y, z, t)$

$\mathbb{R}^2 \to \mathbb{R}$ maar kan naar hoger. $\mathbb{R}^n \to \mathbb{R}$

Voorstelling

$z = f(x, y)$

vlak $ax + by + cz + d = 0$

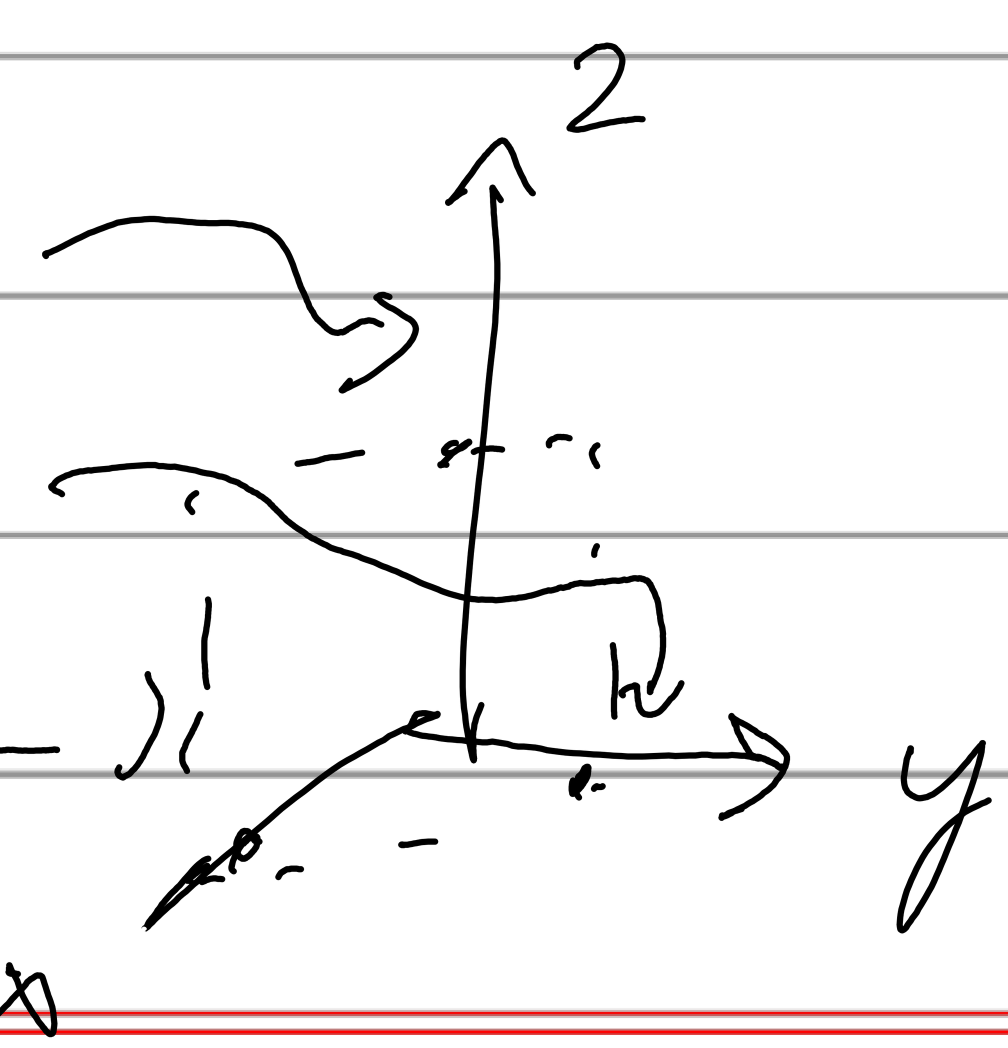
$x = 0 \iff yz$-vlak

vb $x + 2y + 4z - 8 = 0$

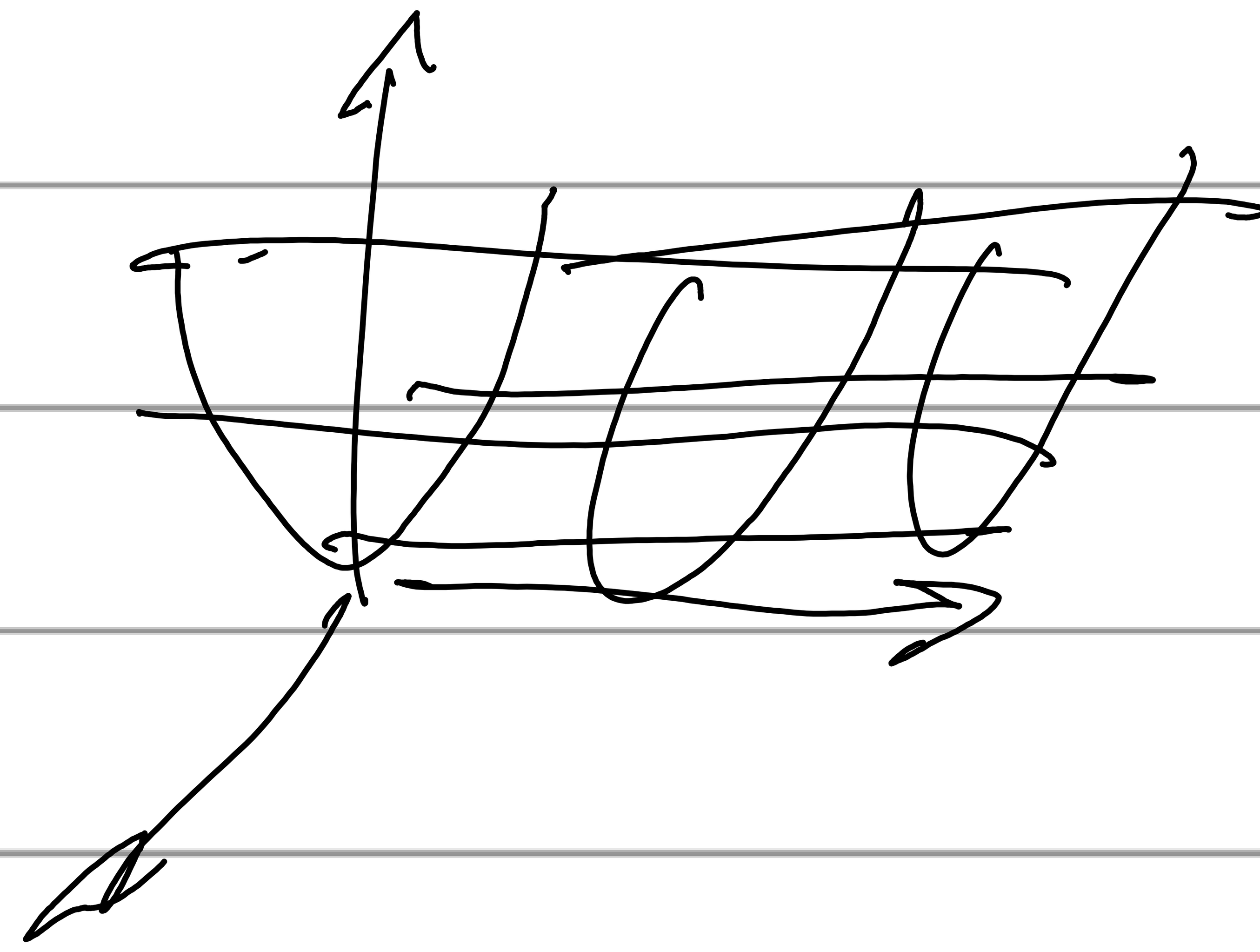
$(0, 0, 2)$

$(0, 4, 0)$

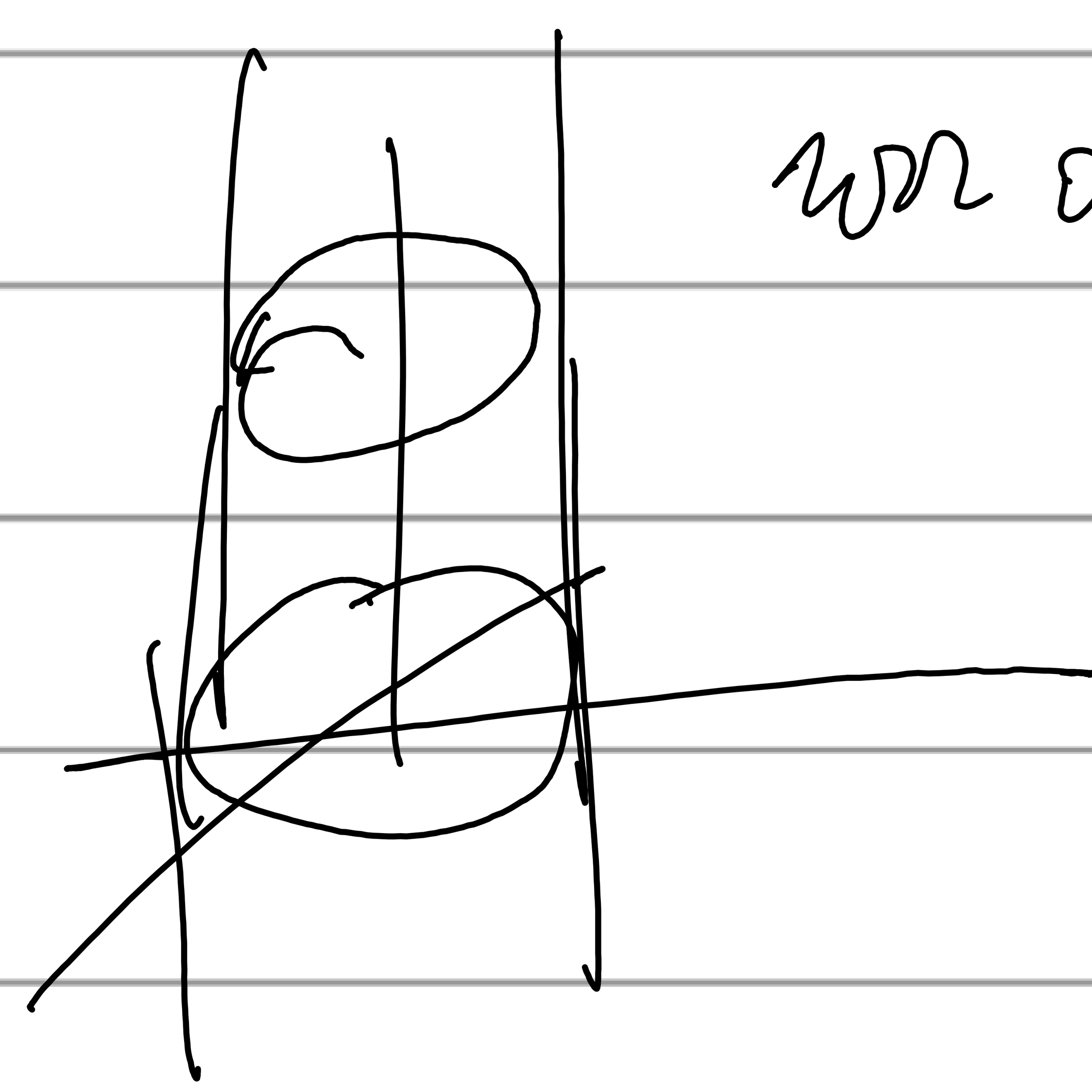
$(8, 0, 0)$



$z = f(x) = x^2$

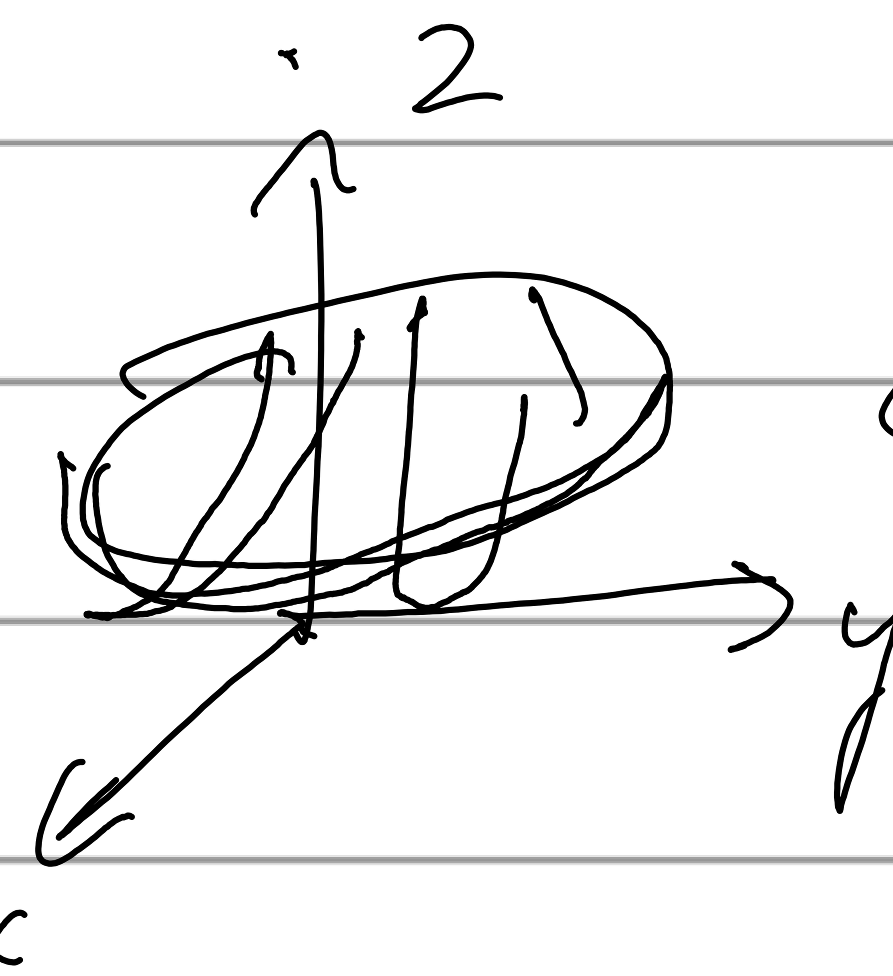


$x^2 + y^2 = 1$

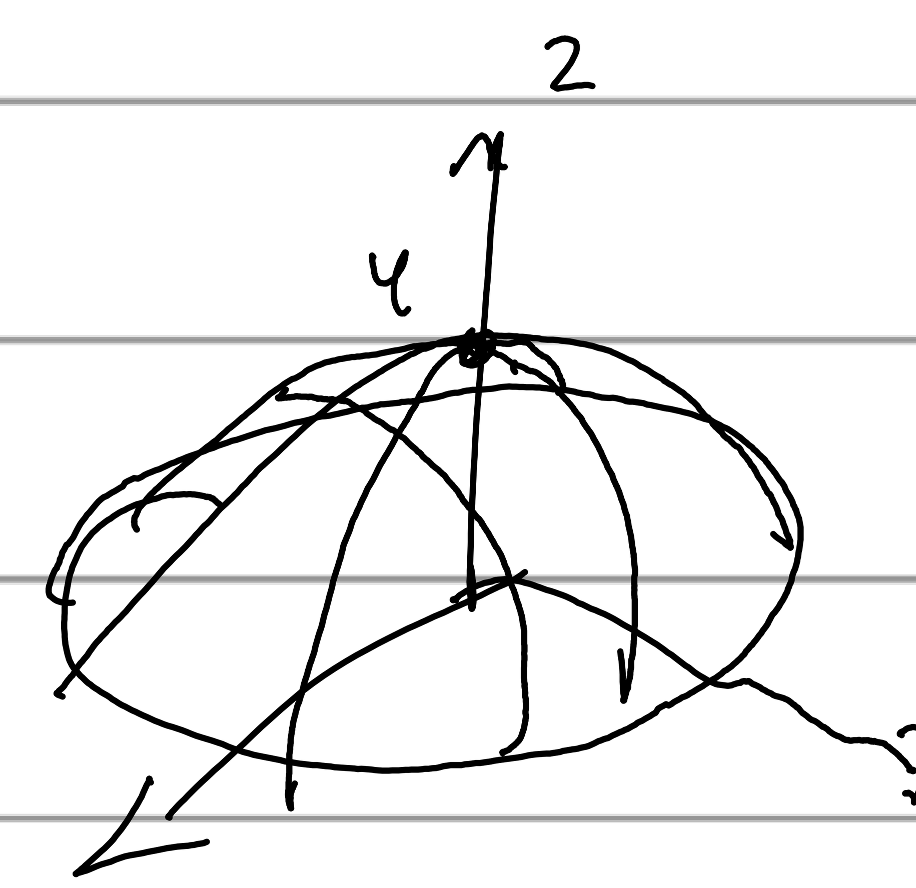


für alle z

$z = x^2 + y^2$



$\Leftrightarrow$ $z = 4 - x^2 - y^2$

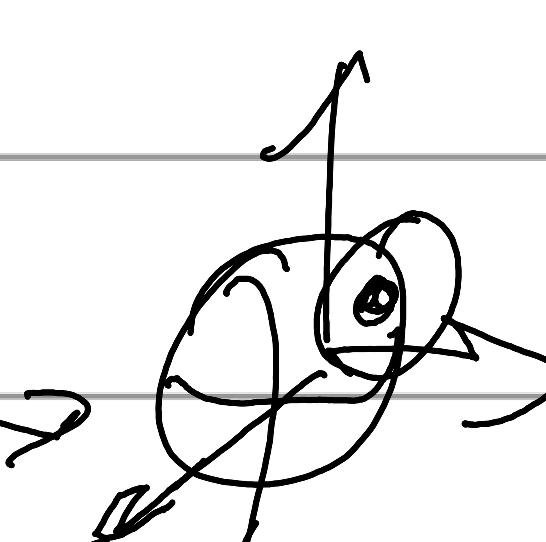


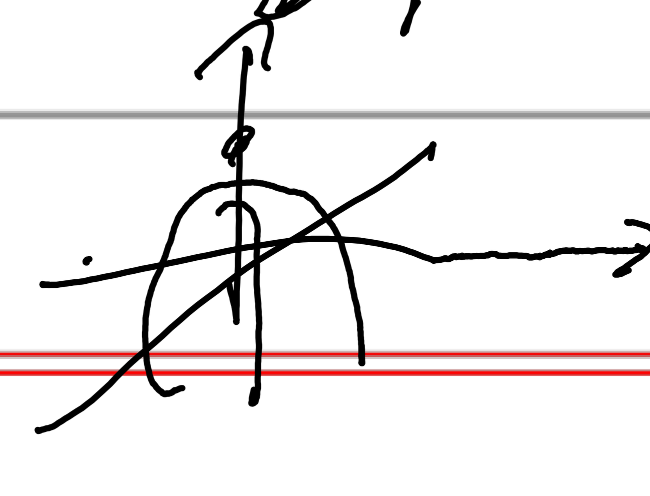
Domein en Beeld

functie	Domein	Beeld
$z=\sqrt{y-x^2}$	$y \geq x^2$ 	$[0, +\infty[$
$z=\dfrac{1}{xy}$	$xy \neq 0$ $\hookrightarrow \begin{cases} x \neq 0 \\ y \neq 0 \end{cases}$	$\mathbb{R}_0$
$z = \sin xy$	$\mathbb{R}^2$ alle x en y	$[-1, 1]$
$\omega=\sqrt{x^2+y^2+z^2}$	$\mathbb{R}^3$	$[0, +\infty[$
$\omega = xy \ln z$	$\underset{x,y}{\mathbb{R}^2} \times \underset{z}{\mathbb{R}^+_0}$	$\mathbb{R}$

Def:

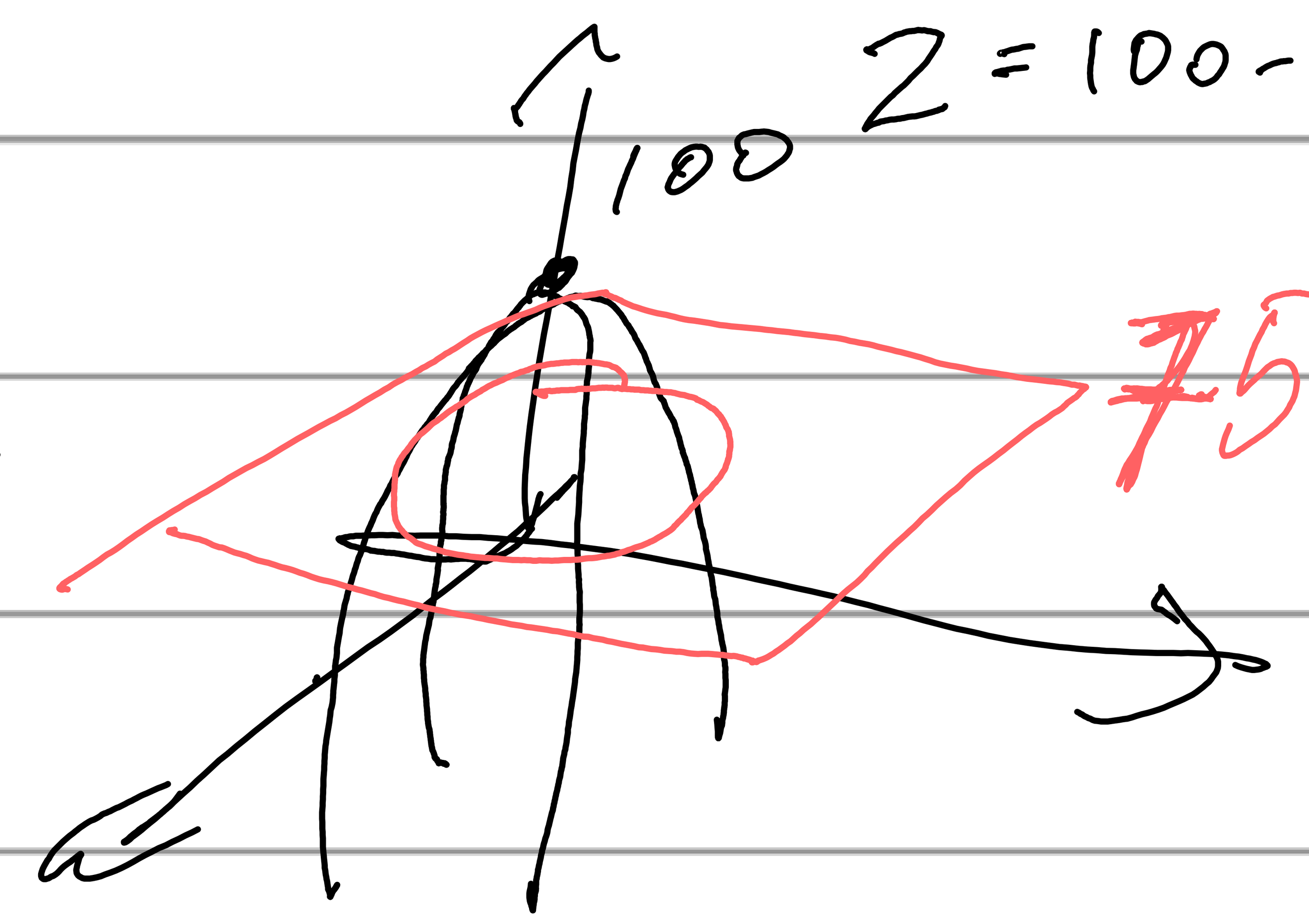
Inwendig punt: ligt op het oppervlak $\longrightarrow$ 

Grenspunt: 

niveaulijn hoogtelijn $z = 100 - x^2 - y^2$ 

Hoogtelijn :

niveaulijn = hoogtelijn op 't xy vlak



$z = 75$

$z = 75$

$75 = 100 - x^2 - y^2$

$5 = x^2 + y^2$

↳ cirkel in xy vlak z = g

Niveauoppervlak indien grotere orde

$\lim f(x,y) = f(x_0, y_0)$

continu: als bestaande geen gaten in oppervlak

(x en y vlak)

Vb $\lim_{(x,y)\to(0,1)} \dfrac{x - xy + 3}{x^2 y + 5xy - y^3} = \dfrac{3}{-1} = -3$

Vb $\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = 5$

Vb $\lim_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \dfrac{0}{0} \Rightarrow$ dus het wegwerken

$\dfrac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} = \dfrac{x\cancel{(x-y)}(\sqrt{x} + \sqrt{y})}{\cancel{x - y}} =$

$$\lim_{(x,y)\to(0,0)} \frac{-xy}{\sqrt{x^2+y^2}} = \lim_{(r,\theta)\to(0,0)} \frac{r^{\cancel{2}}\sin\theta\cos\theta}{\cancel{r}} = 0$$

Noor Prof

Parameter voorstelling

$$t \to (x(t), y(t), z(t)) \quad \mathbb{R} \to \mathbb{R}^3$$

Vectorvelden: afbeelding $\mathbb{R}^3 \to \mathbb{R}^3$

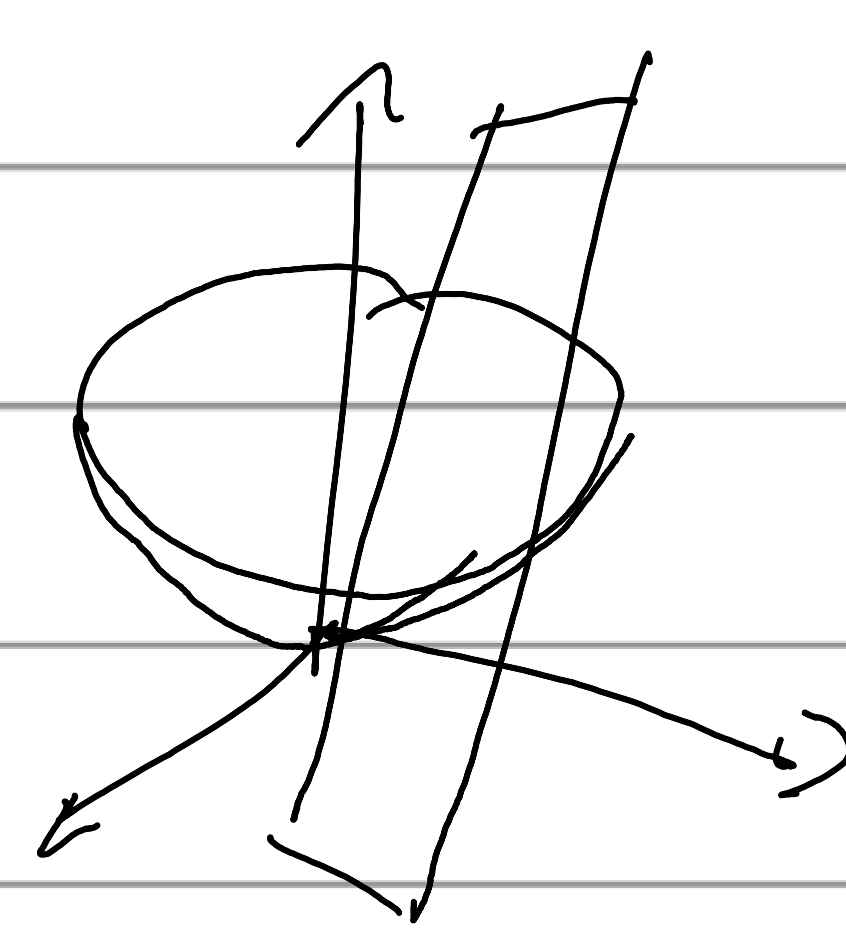
Continuïteit en limiet: lezen, niet belangrijk

↳ geen gaten in oppervlakte

Partiële afgeleide: meetkundig en analytisch uit.

Expliciete voorstelling

Meetkundig: $z^2 = x^2 + y^2$



Meetkundig

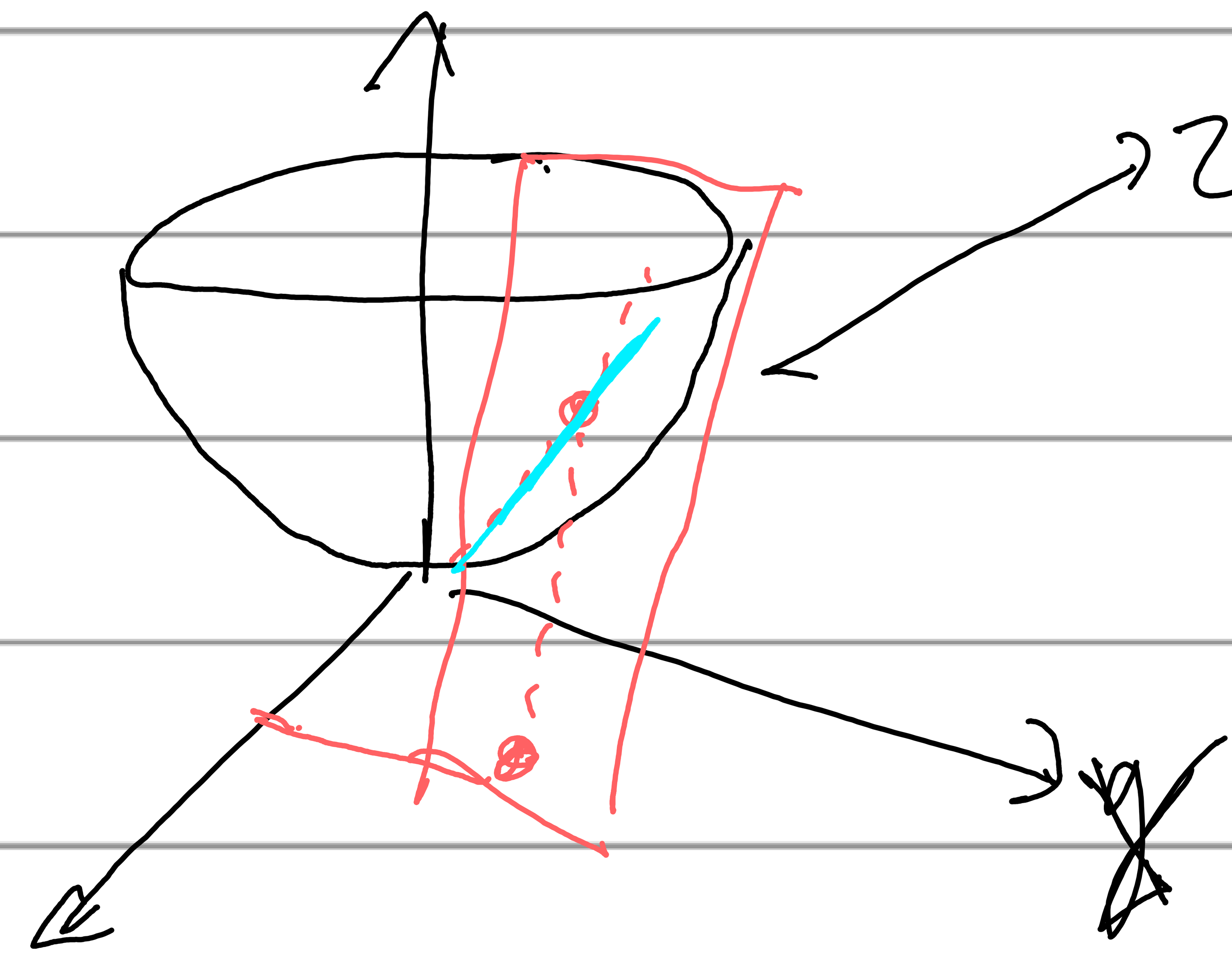
$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \text{ of } \frac{\partial f}{\partial y}$$

$$\boxed{f'_x = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}}$$

y = cst en varieer in x

(analytisch)

Meetkundig



→ rico van de raaklijn

Dus we nemen vlak waarvan y constant is, elk x varieert

Vlak y = constant

creëer een kromme in funct van x

grootte van rico is raaklijn

zelfde voor

f_y' $\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$

analytisch

Vb $f(x,y) = y \sin(xy)$

$\frac{\partial f}{\partial x} = y^2 \cos(xy)$ (y zien als constante)

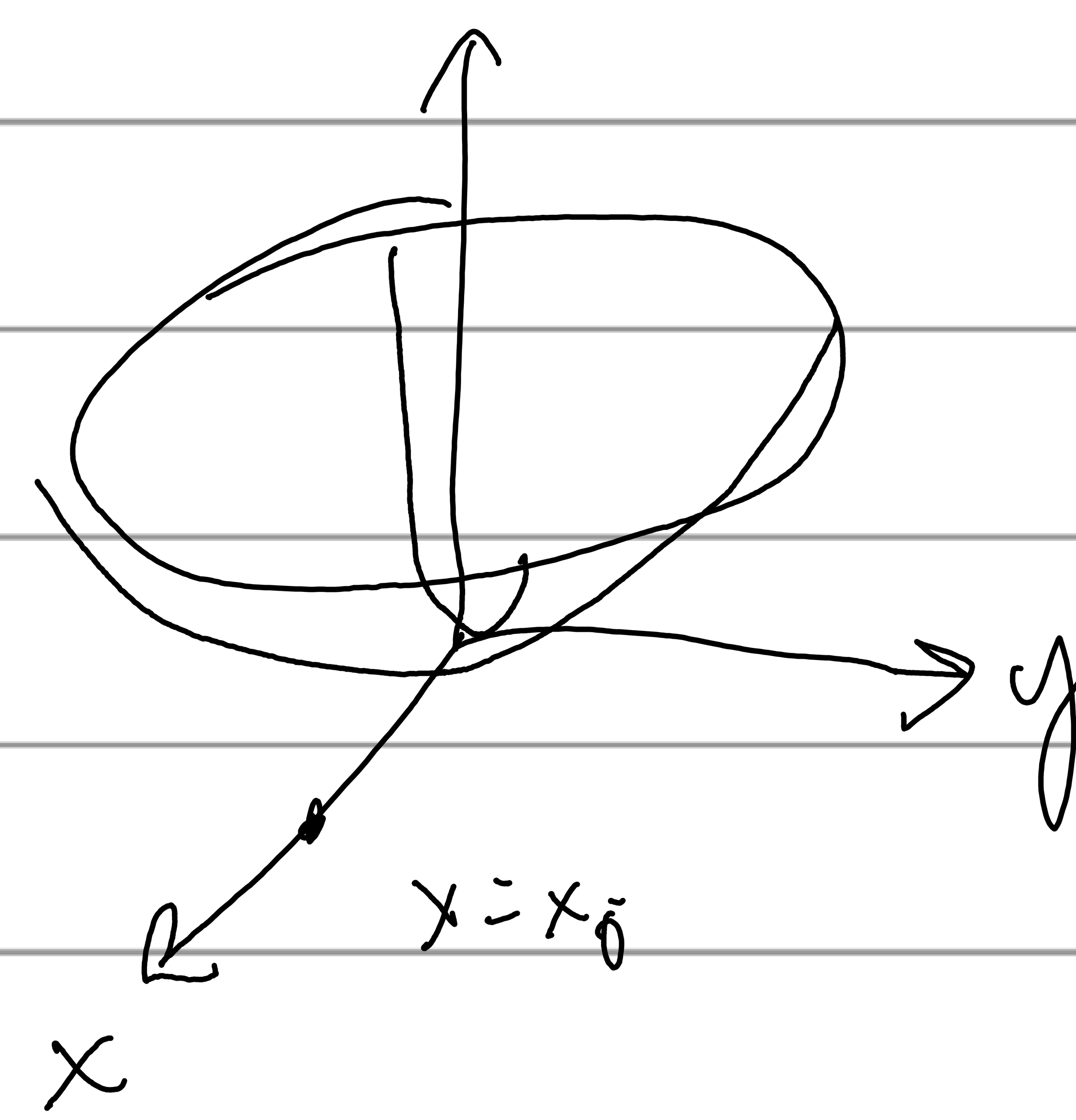
$\frac{\partial f}{\partial y} = 1 \sin(\cancel{x}y) + xy \cos(xy)$

Ü $f(x,y) = x^2 + 3xy + y - 1$

$$\frac{\partial f}{\partial x} = 2x + 3y \qquad \frac{\partial f}{\partial y} = 3x + 1$$

$f(x,y)$: Funktion

$$\frac{\partial f(x,y)}{\partial x}$$



Ü $z = \arctan \frac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\, y\left(-\frac{1}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

Vb Wat is de rico van de raaklijn in $P(1,2,5)$

met $x = \text{cste}$ $\quad z = x^2 + y^2$

opl met x cst is

$$\frac{\partial z}{\partial y} = 2y$$

$$\left(\frac{\partial z}{\partial y}\right) = 2 \cdot 2 = 4$$

$=$

vergelijk met $z = 1 + y^2$

$$z' = 2y \Rightarrow (z')_P = 2 \cdot 2 = 4$$

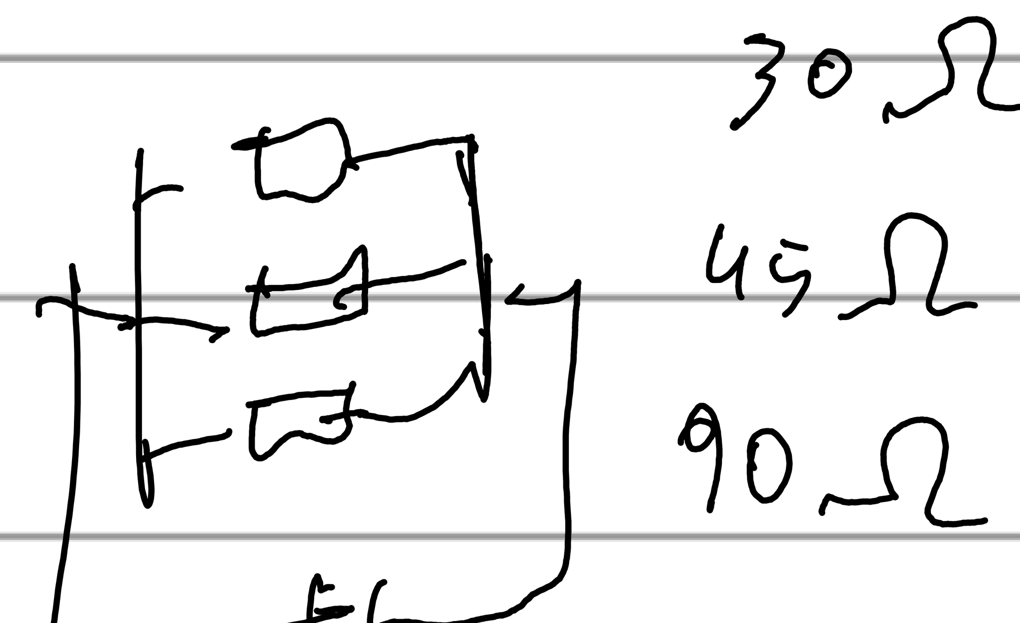
Vb $f(x,y,z) = x \sin(y+3z)$

$$\frac{\partial f}{\partial x} = \sin(y+3z)$$

$$\frac{\partial f}{\partial y} = x\cos(y+3z)$$

$$\frac{\partial f}{\partial z} = 3x\cos(y+3z)$$

Vb Stel elektrisch net



Beschrijf verandering R_{tot} als enkel R_2 wijzigt

$R_{totaal} \Rightarrow \underbrace{\frac{1}{R_1} + \frac{1}{R_2}}_{\text{Partiel}} + \frac{1}{R_3} = \frac{1}{R_{totaal}}$.

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R_{tot}}\right) = \boxed{\frac{\partial R_{totaal}}{\partial R_2}} \rightarrow -\frac{1}{R_2^2} \quad \text{dus } \frac{\partial}{\partial R_2}\left(\frac{1}{R_{tot}}\right) = -\frac{1}{R_2^2}$$

dus $\frac{\partial}{\partial z_2} = \frac{-R_{totaal}}{R_2^2}$

$$\frac{\partial R}{\partial R_2} = \frac{-R_{\text{tot}}^2}{R_2^2} = \frac{15^2}{45^2}$$

dus factor $\frac{1}{9}$

Bewijs niet kennen maar weten wat is

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Young schwartz

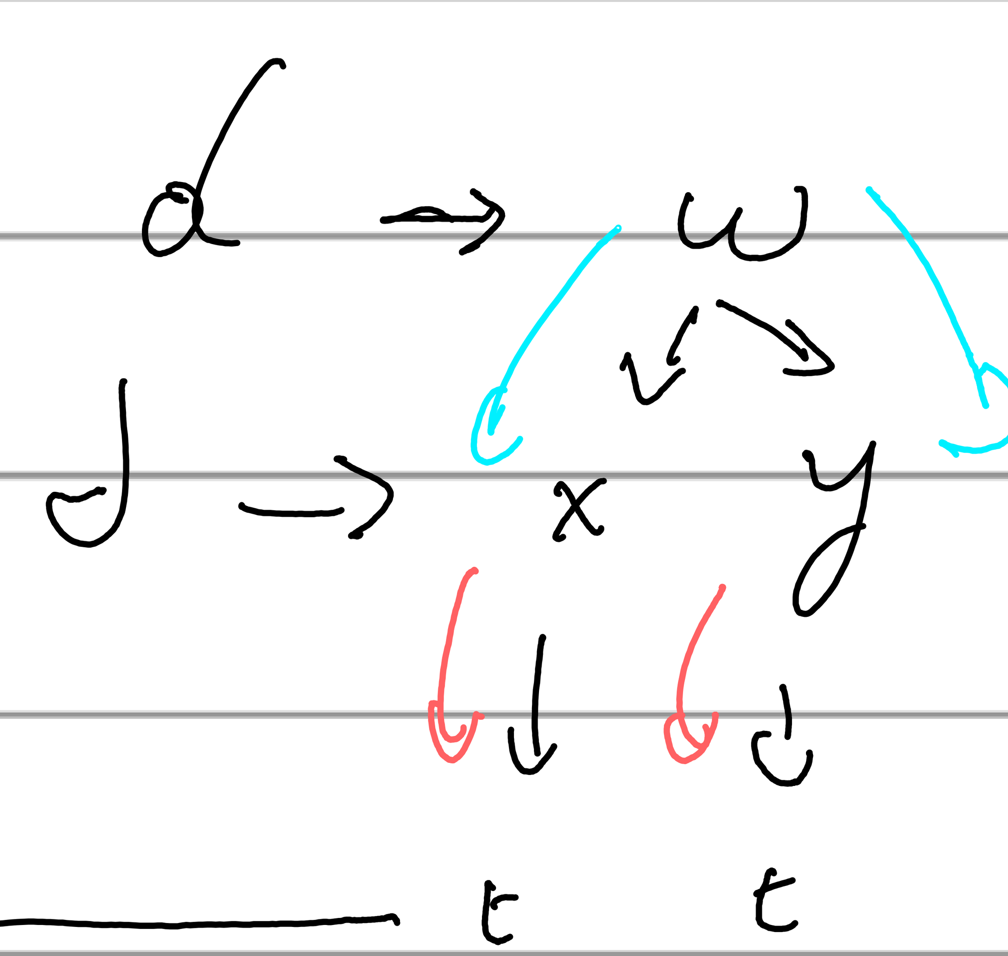
Bewijs niet kennen maar wel snappen.

$$\frac{\partial^3 f}{\partial x \partial y} = \frac{\partial^3 f}{\partial y \partial x}$$

Kettingregel → totale afgeleide

$$w = f(x(t), y(t))$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$



$\frac{dw}{dt}$ = wat mor. t.

dus $\frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = \frac{dw}{dt}$

(beide afhankelijk van t)

Young Schwarz

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$=$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial \cancel{x} \partial y}$$

Volgorde maakt niet uit.

vb: $z = x\cos y + y e^{x}$

$$f_x = \cos y + y e^{\cancel{x}x} \rightarrow f_{\cancel{yx}} = -\sin y + e^{x}$$

$$f_y = -x\sin y + e^{x} \quad f_{xy} = -\sin y + e^{x}$$

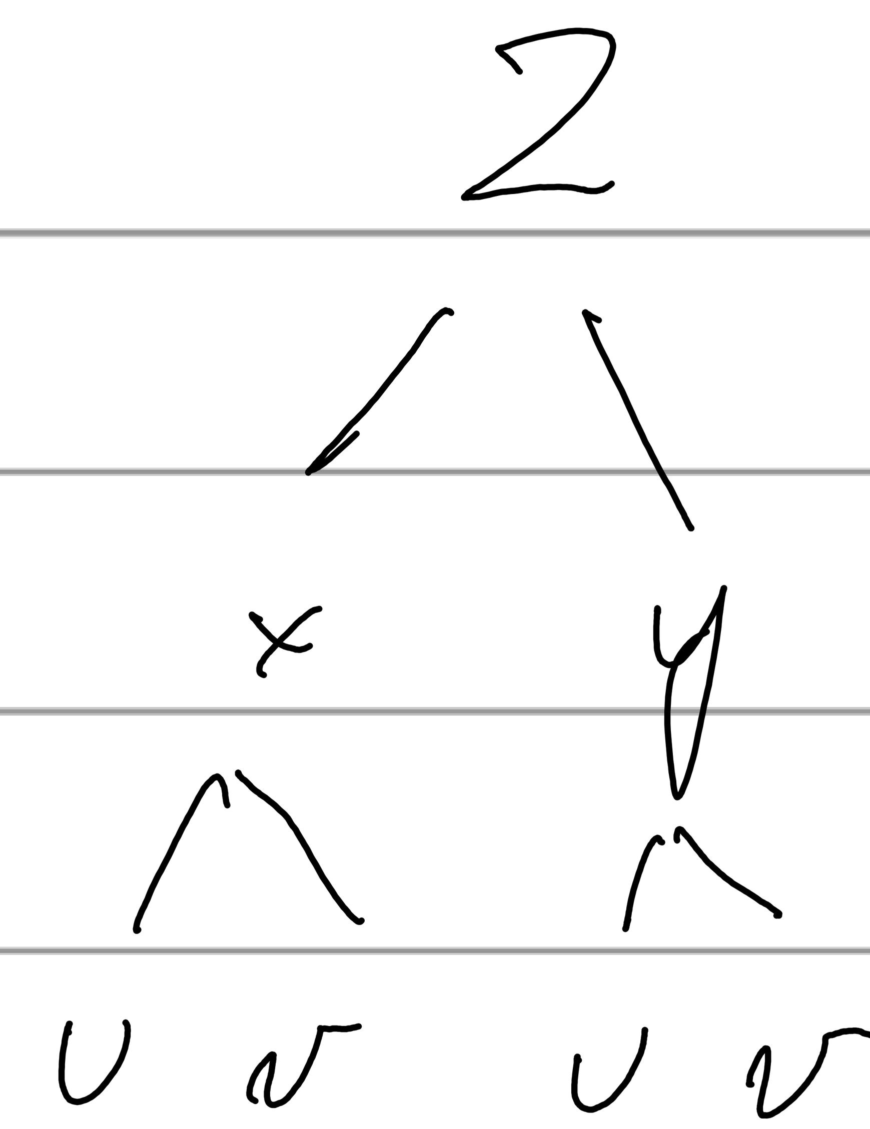
$$\frac{\partial^2 f}{\partial x \partial x} = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

Vb $w = x^2 + y^2$

$$\begin{cases} x = 2t \\ y = t^2 \end{cases}$$

$$\frac{dw}{dt} = 2x \cdot 2 + 2y \cdot 2t = 8t + 4t^3$$

$z = f(x(u,v), y(u,v))$



Toepassing

Partiële afgeleide van een impliciete functie

$F(x, y, z(x,y)) = 0$ (impliciete functie)

$\frac{\partial z}{\partial x} = ?$ of $\frac{\partial z}{\partial y} = ?$

(d) $w = xy + z$

$x = \cos t,\ y = \sin t,\ z = t$

wat is de verandering in T op ~~t~~ $t = 0$

$$
\begin{array}{ccc}
 & w & \\
x & y & z \\
\parallel & \parallel & \parallel \\
t & t & t
\end{array}
$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= y(-\sin t) + x(\cos t) + 1(1)$$

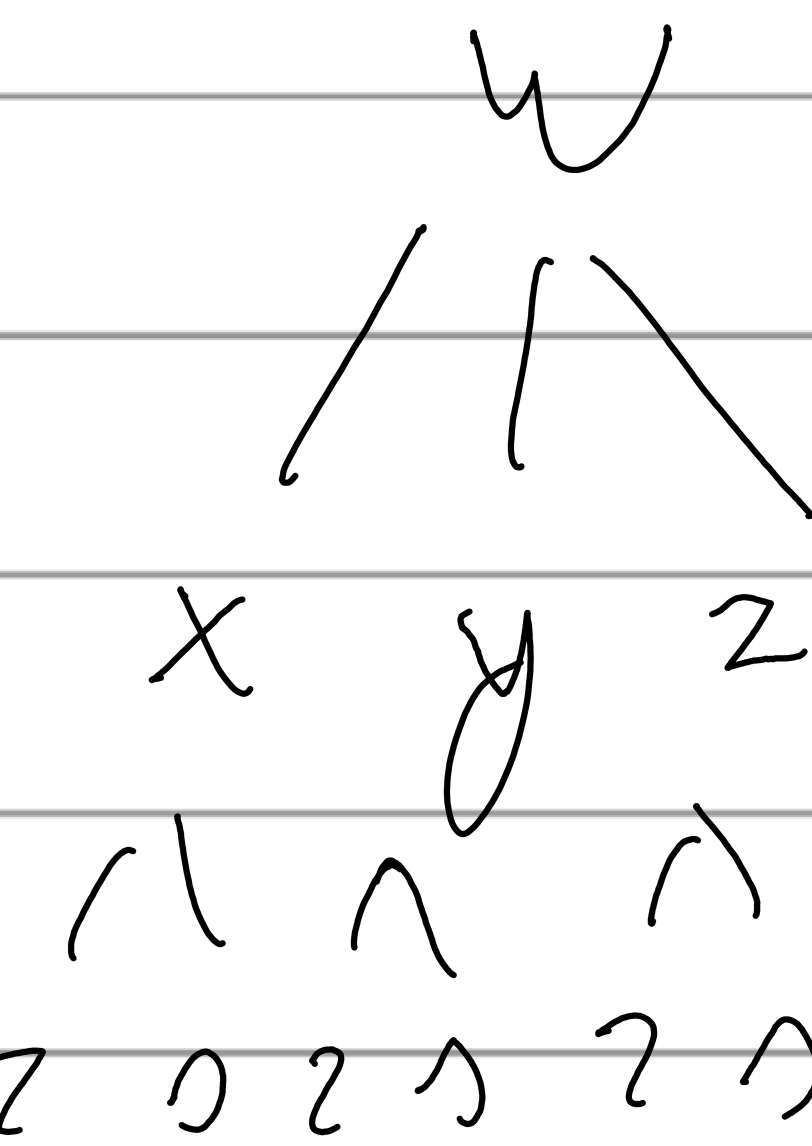
$$= \underbrace{(-\sin^2 t) +}_{1-\cos^2 t} (\cos^2 t) + 1 = 2\cos^2 t$$

$$\left(\frac{dw}{dt}\right)\bigg|_{t=0} = 2$$

Vb $w = f(x, y, z)$

$x = x(r, s)$, $y = y(r, s)$, $z = z(r, s)$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

Stel net opgave

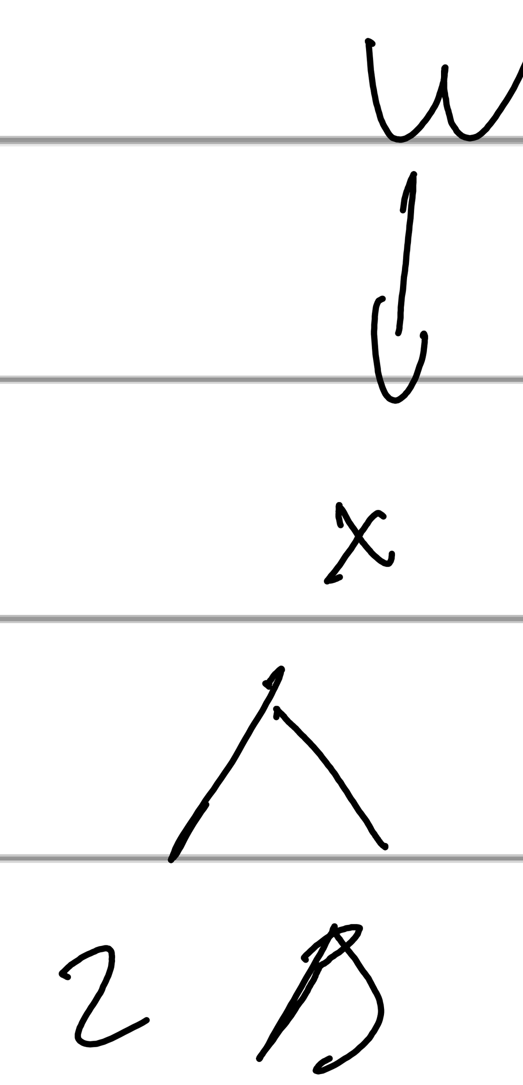
$w = x + 2y + z^2$

$x = \frac{r}{s}$, $y = r^2 \cdot \ln(s)$ $z = 2r$

$$\frac{\partial w}{\partial r} = 1 \cdot \frac{1}{s} + 2 \cdot 2r + 2z \cdot 2 = \frac{1}{s} + 4r + 8r$$

$$\frac{\partial w}{\partial s} = \frac{2}{s} - \frac{r}{s^2}$$

$w = f(x)$ $\quad x = x(z, s)$



$\frac{\partial w}{\partial z} =$

$$\frac{\partial w}{\partial z} = \frac{dw}{dx}\frac{\partial x}{\partial z} + \frac{dw}{dy}\frac{\partial y}{\partial z}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s} + \frac{dw}{dy}\frac{\partial y}{\partial s}$$

Üb

$$x^2 y z^2 + 2 x y^3 = 0$$

$$\begin{array}{ccc} & F & \\ x & y & z \end{array}$$ hängt ab von

$$\begin{array}{cc} \multicolumn{2}{c}{z} \\ x & y \end{array}$$ hängt ab von

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x} \underbrace{\cancel{\frac{\partial x}{\partial x}}}_{=1} + \cancel{\frac{\partial F}{\partial y} \underbrace{\frac{\partial y}{\partial x}}_{=0}} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x}$$

$$\to = 0$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} = 0$$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= \frac{-\partial F / \partial x}{\partial F / \partial z} \\ \frac{\partial z}{\partial y} &\;\; \frac{-\partial F / \partial x}{\partial F / \partial z} \end{aligned}}$$

ableiten erkennen

vi $F(x,y,z) = x^2yz^2 + 2xy^3 = 0$

$$\frac{\partial z}{\partial x} = \frac{-(xyz^2 + 2y^3)}{2x^2yz + \cancel{x}y^3}$$

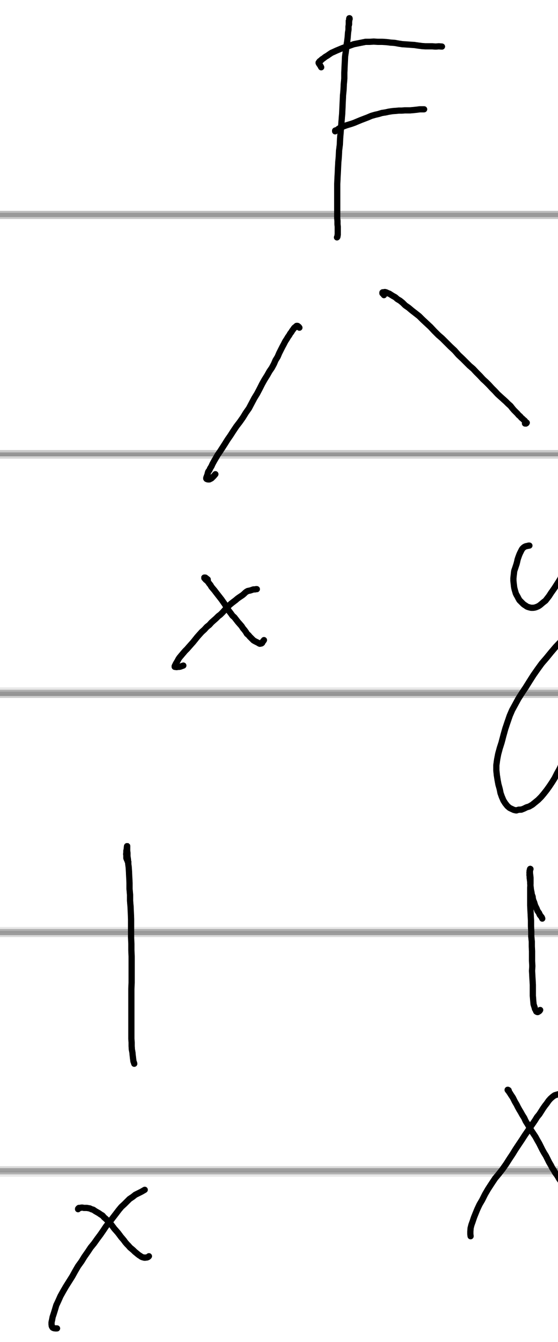
$$\frac{\partial z}{\partial y} = \frac{-(x^2z^2 + 3\cancel{x}y^2z)}{2x^2y^2 + xy^3}$$

Impliciet afleiden

$n = 2$

$$y = f(x)$$

$$F(x,y) = 0$$



$$\frac{dF}{dx} = \frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y}}\ !$$

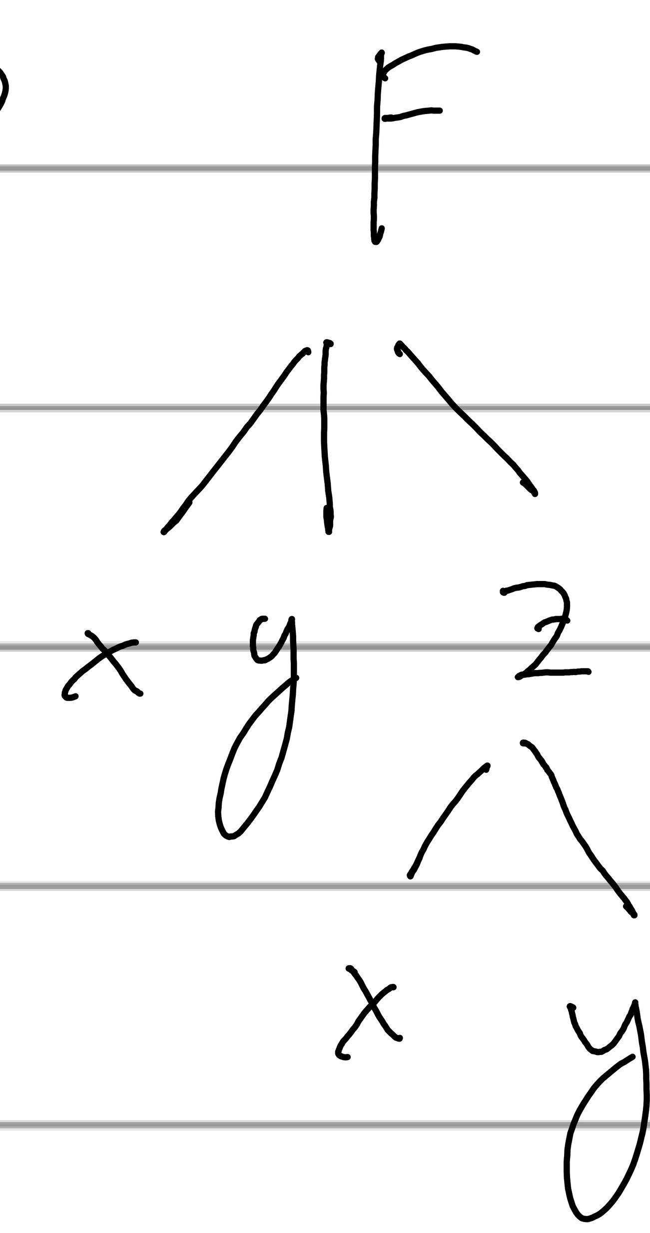
Vb

$$F(x,y) = y^2 - x^2 - \sin xy = 0$$

$$\frac{dy}{dx} = \frac{+2x + y\cos(xy)}{2y - x\cos(xy)}$$

Implicit afleiden $n=3$

$$z = f(x,y) \text{ of } F(x,y,z(x,y)) = 0$$



$$\rightarrow \frac{\partial z}{\partial x} \text{ of } \frac{\partial z}{\partial y}$$

$$\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial x}\underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y}\underbrace{\frac{dy}{dx}}_{=0} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\partial F/\partial z}}$$

$$\frac{\partial F}{\partial y} = 0 =$$

$$0 + \frac{\partial F}{\partial y}\underbrace{\frac{dy}{dy}}_{=1} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\partial F/\partial z}}$$

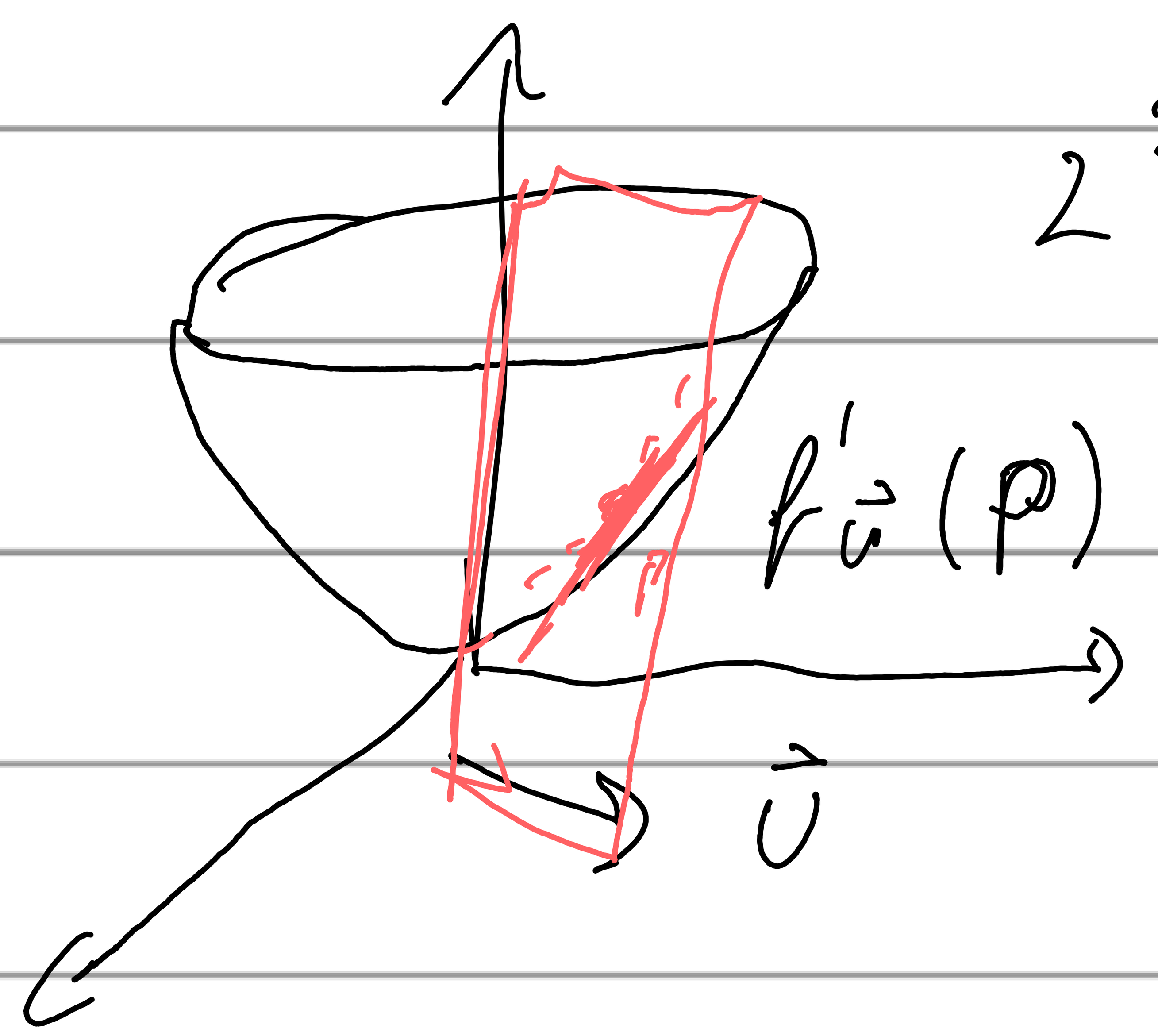
4b $\quad x^3 + y^3 + z^3 + ye^{xz} + z\cos y = 0$

Verandering van

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} = \frac{-(3x^2 + e^{xz} \cdot y \cdot z)}{3z^2 + e^{zx} xy + \cos(y)}$$

Richtingsafgeleide

de afgeleide van een functie in een bepaalde richting



$$z^2 = x^2 + y^2$$

Niet bekijken hoe de functie evolueert volgens assen, maar volgens willekeurige richting

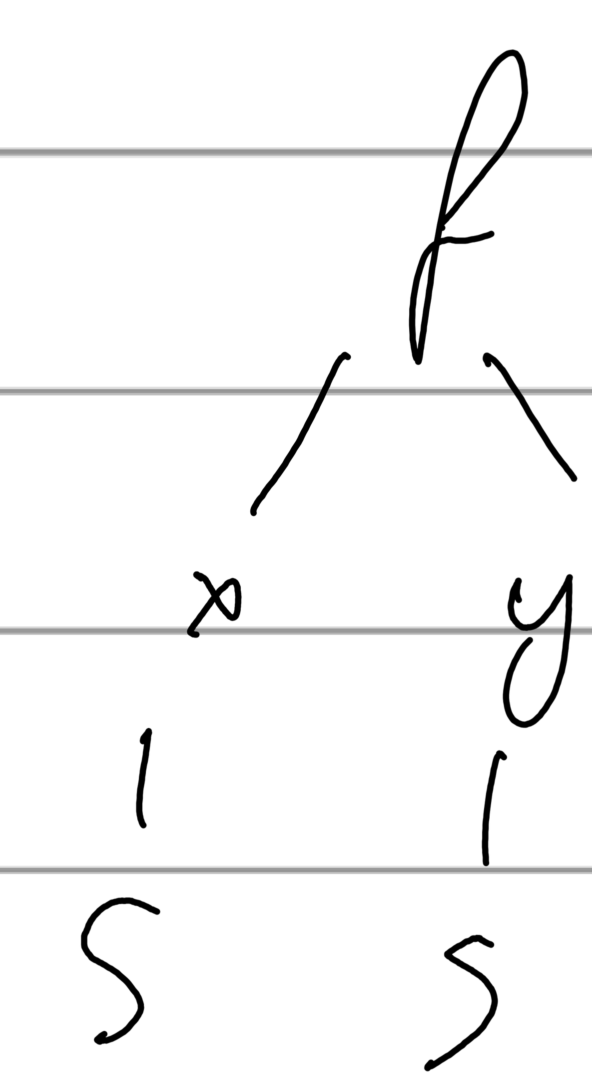
Partieel volgens richting x, y of z

$\vec{u} = (u_1, u_2)$ → normaliseren $\frac{\vec{u}}{|\vec{u}|}$

$$f'_{\vec u}(p) = (D_{\vec u} f)_p = \lim_{s \to 0} \frac{f(x+su_1, y+su_2) - f(x_0, y_0)}{s}$$

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\underbrace{\frac{dx}{ds}}_{=u_1} + \frac{\partial f}{\partial y}\underbrace{\frac{dy}{ds}}_{=u_2}$$

alles in P_0



$$\left(\frac{df}{ds}\right)_{\vec{u}, P_0}$$

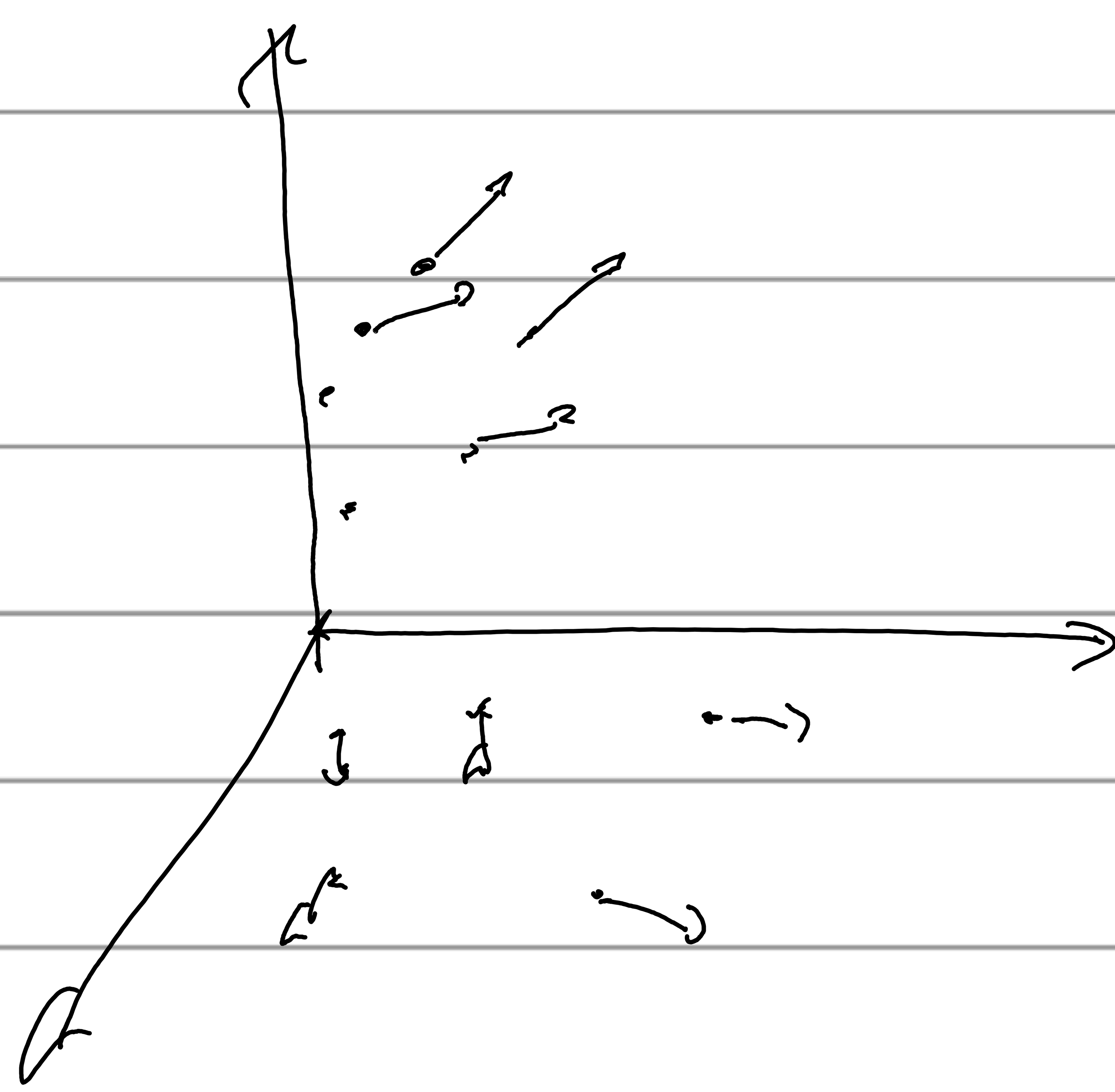
$$= \underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (u_1, u_2)}_{} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

$$= \vec{\nabla} f \cdot \vec{u}$$

(gradiënt)

↳ Nabla operator $\quad \vec{\nabla} f : \mathbb{R}^2 \to \mathbb{R}^2$

afbeelding van $\mathbb{R}^2$ naar $\mathbb{R}^2$



elk punt van de ruimte heeft je vector

$\vec{z}(x, y, z)$

Vb $\quad 2 = x^2 + y^2$

$\vec{v} = (1,1)$ → moet grootte 1 hebben dus normaliseren

$P = (1,0,1)$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (delen door grootte)

$(D_{\vec{u}} f)_p = ?$

$$= (2x, 2y)_p \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

we nullen $P_0 = O(x,y)$ $= \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

Op vector niveau

$$(D_{\vec{u}} f)_p = \|\vec{\nabla} f\| \cdot \cancel{\|\vec{v}\|} \cdot \cos(\vec{\nabla} f, \vec{u})$$

$\cancel{\|\vec{v}\|} = 1$

↳ max waarde als $\cos 1 = 1$ is

dus als $\underset{\text{richting}}{\vec{v}} = \vec{\nabla} f$ $\parallel$

Vl $\quad z = xe^{y} + \cos(xy) \qquad \vec{u} = (3, -4)$

$P = (2, 0)$

$$\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\right)\left(\frac{\vec{u}_1}{|\vec{u}_1|} + \frac{\vec{u}_2}{|\vec{u}_2|}\right)$$

$$\underbrace{\left(e^{y} - \sin(xy)\cdot y\right)}_{=1}\ \underbrace{\left(xe^{y} - x\sin(yx)\right)}_{=2}$$

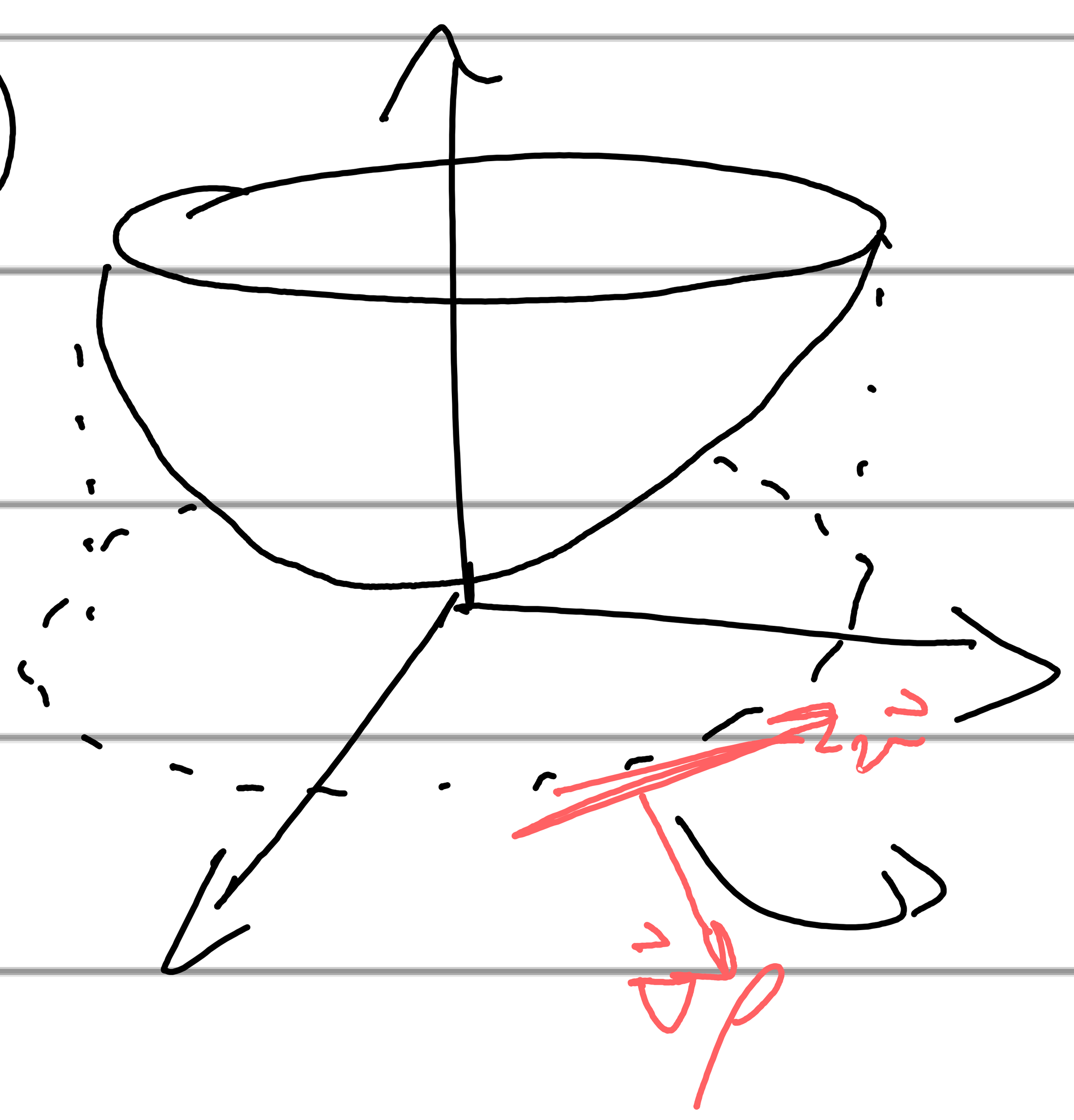
$$\frac{\vec{u}}{|\vec{u}|} = \frac{(3,-4)}{\sqrt{3^2+4^2}} = \frac{(3,-4)}{\sqrt{25}} = \left(\frac{3}{5}, \frac{-4}{5}\right)$$

$$1\cdot\frac{3}{5} + 2\,\frac{-4}{5} = \frac{-5}{5} = -1$$

Gradiënt $= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \dots\right) \dots$

↳ min als $\cos = 0$ of $\vec{v} \perp \vec{\nabla} f$

vb



$z = f(x,y)$

↳ $\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

Projectie op xy vlak

$$f(x(t), y(t)) = 0$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = 0$$

$$\vec{\nabla} f \cdot \vec{v} = 0$$

$\mathbb{R}^n \to \mathbb{R}^n$ = Vectorveld

$$\vec{\nabla}(fg) = g\vec{\nabla} f + f\vec{\nabla} g$$

vb

$$\vec{g}(r) := \frac{Gm}{r^3}\vec{r} \qquad \vec{r}(x,y,z)$$

(zwaartekracht) $\qquad r=\sqrt{x^2+y^2+z^2}$

$$\vec{g}(r) = -\vec{\nabla}V(r)$$

$$V(r) := \frac{Gm}{r}$$

$$\frac{Gm}{r^3}\vec{r} = -\vec{\nabla}\frac{Gm}{r}$$

$$\frac{\vec{r}}{r^3} = -\vec{\nabla}\left(\frac{1}{r}\right) = -\left(\frac{\partial}{\partial x}\left(\frac{1}{r}\right), \frac{\partial}{\partial y}\left(\frac{1}{r}\right), \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\right)$$

$$\frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{-1/2}\right) = -\frac{1}{\cancel{2}}\cancel{2}x\,(x^2+y^2+z^2)^{-3/2}$$

$$= \frac{-x}{\underbrace{\sqrt{(x^2+y^2+z^2)^3}}_{r^3}}$$

$$\vec{r} = x\vec{1}_x + y\vec{1}_y + z\vec{1}_z$$

Verband richting afgeleide en gradiënt

$$D_{\vec{u}} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (u_1 ; u_2)$$

$$= \vec{\nabla} f \cdot \vec{u}$$

$$= \|\vec{\nabla} f\| \cdot \underbrace{\|\vec{u}\|}_{1} \cdot \underbrace{\cos(\vec{\nabla} f, u)}_{-1 \leq \dots \leq 1}$$

$D_{\vec{u}} f$ = is max als $\cos = 1$

↳ Als //

$\vec{\nabla} f$ wijst naar de richting waar functie maximaal stijgt

$D_{\vec{u}} f = 0$ als $\cos = 0$

Als $\perp$

in elke richting die $\perp$ op gradiënt van F is dan nulle richting die zoeken van hoogte

$\vec{\nabla} f$ moet loodrecht aan de hoogtelijn of staan
$\perp$ op het vlak van die functie

Vb $z = x^2 + xy$ $P(1,2)$

In welke richting is de wijziging max

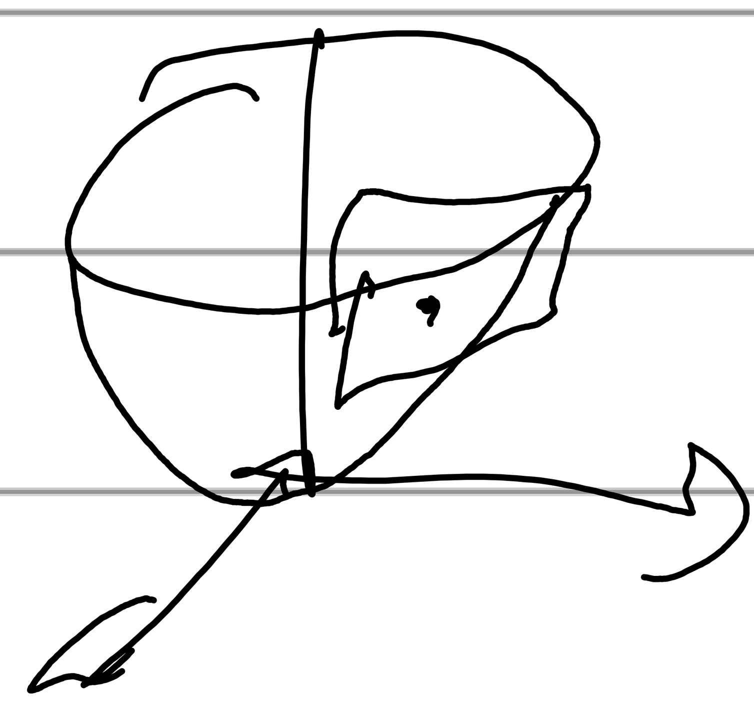
$$\left(\vec{\nabla} f\right)_P = (2x+y, x)_{(1,2)} = (4, 1)$$

? vector normaliseren
lengte

Genormaliseerd: $\left(\frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}}\right)$ max

$= (-1, 4)$: richting zonder wijziging

Raakvlakken, differentiaal, linearisatie



Raakvlak wordt bepaald

$\vec{n} \cdot \vec{r}_p = 0$

alle punten die loodrecht staan liggen op vlak

$\vec{n} \cdot (x - x_0, y - y_0, z - z_0)$

$\vec{n} \parallel \vec{\nabla} f$

$\hookrightarrow \vec{\nabla} f = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$

Impliciet: $\underbrace{\left(\frac{\partial F}{\partial x}\right)_{P_0}}_{-\partial F/\partial z}(x - x_0) + \underbrace{\left(\frac{\partial F}{\partial y}\right)_{P_0}}_{-\partial F/\partial z}(y - y_0) + \underbrace{\left(\frac{\partial F}{\partial z}\right)_{P_0}}_{-\partial F/\partial z}(z - z_0) = 0$

$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}$

Explicit

$z = f(x, y) = z(x, y) - z = 0$

$\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z}$

$$\vec{\nabla} F = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$$

① $$\left(\frac{\partial z}{\partial x}\right)_P (x - x_0) + \left(\frac{\partial z}{\partial y}\right)(y - y_0) - (z - z_0) = 0$$

Vb $$z = x^2 + y^2$$

$$P(1, 0, 1)$$

Expl: $$\underbrace{(2x)_P}_{=2}(x-1) + \underbrace{(2y)_P}_{=0}(y-0) - (z-1) = 0$$

$$2(x-1) - (z-1) = 0$$

Implicit. $$-z + x^2 + y^2 = 0$$

$$\underbrace{(2x)_P}_{=2}(x-1) + \underbrace{(2y)_P}_{=0}(y-0) - (z-1) = 0$$

VII met 2 element

$$y = f(x) \Rightarrow y - f(x) = 0$$

$$F(x,y) = 0 = \frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y}$$

$$\vec{n} \cdot \vec{P_0 P_0} = 0$$

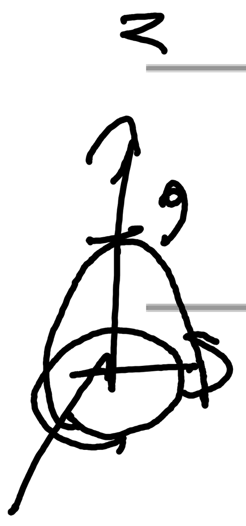
$$\left(\left(\frac{\partial F}{\partial x}\right)_{P_0}, \left(\frac{\partial F}{\partial y}\right)_{P_0} \right) \cdot (x - x_0, y - y_0) = 0$$

$$\frac{\left(\frac{\partial F}{\partial x}\right)_{P_0}}{-\partial F/\partial y}(x - x_0) + \frac{\left(\frac{\partial F}{\partial y}\right)_{P_0}}{\partial F/\partial y}(y - y_0)$$

$$= -\left(\frac{\partial y}{\partial x}\right)_{P_0}(x - x_0) + (y - y_0) = 0 \qquad f'(x)$$

$$y - y_0 = \frac{\partial y}{\partial x}(x - x_0)$$

Vl Bepaal het raakvlak in Punt P(1,2,4)



Impliciet

$$x^2+y^2+z-9=0$$

$$\left(\frac{\partial F}{\partial x}\right)(x-x_0)+\left(\frac{\partial F}{\partial y}\right)(y-y_0)+\left(\frac{\partial F}{\partial z}\right)(z-z_0)$$

$$(2x)_{P_0}(x-1)+(2y)_{P_0}(y-2)+(1)_{P_0}(z-4)$$

$$2(x-1)+4(y-2)+(z-4)=0$$

$$2x+4y+z=14$$

Expliciet

$$9-x^2-y^2=z$$

$$\left(\frac{\partial z}{\partial x}\right)_{P_0}(x-x_0)+\left(\frac{\partial z}{\partial y}\right)_{P_0}(y-y_0)+\left(\frac{\partial z}{\partial z}\right)(z-z_0)$$

$$(-2x)_{P_0}(x-1)+(-2y)_{P_0}(y-2)+(1)(z-4)=0$$

$$-2(x-1)-(4)(y-2)+(z-4)=0$$

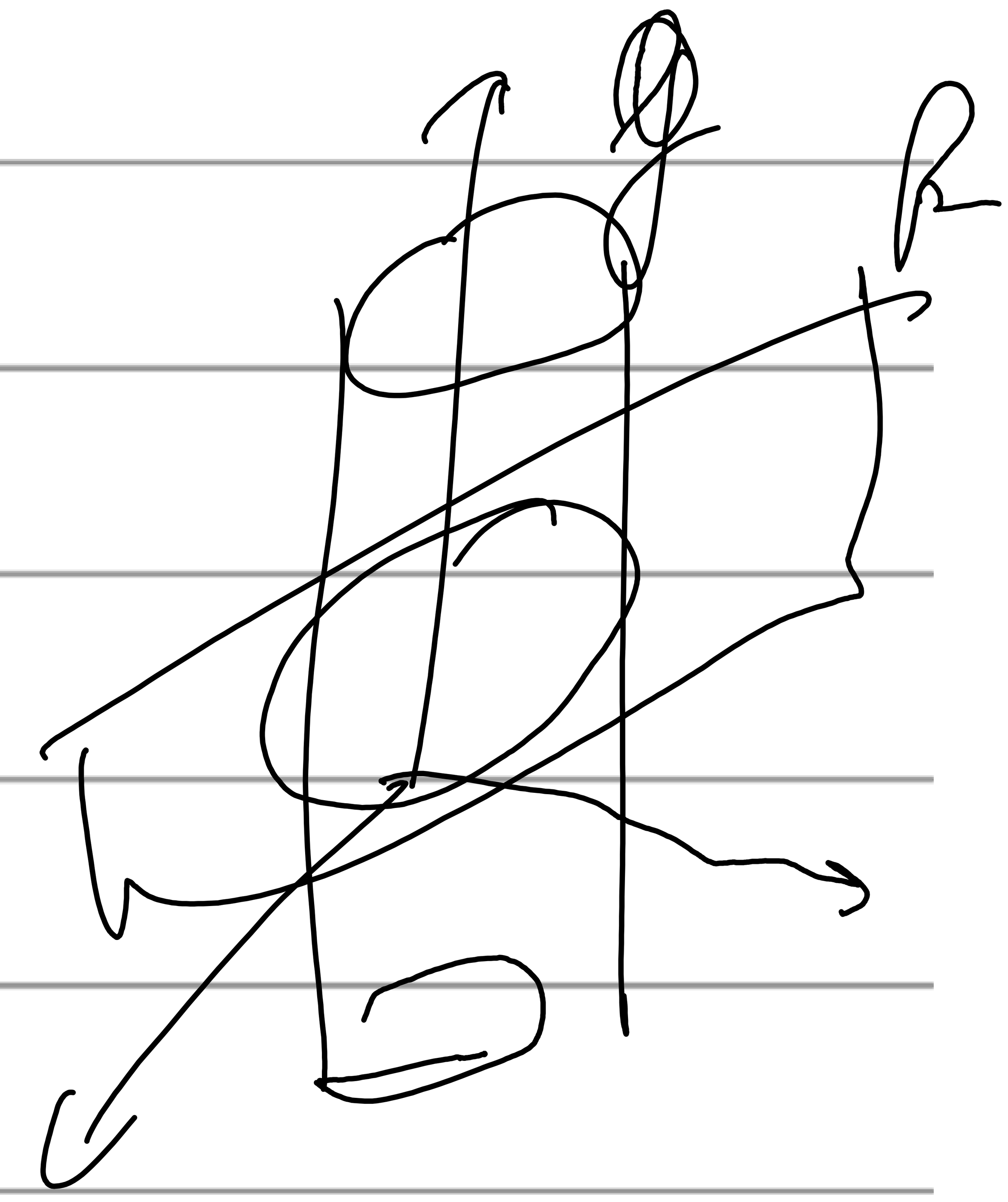
$$2(x-1)+4(y-2)+(z-4)=0$$

$$2x-2+4y-8+z-4=0$$

$$2x+4y+z=14$$

VI raaklijn in $(1,1,3)$ aan snijlijn f en g

$$f \Leftrightarrow x + z - 4 = 0$$
$$g \Leftrightarrow x^2 + y^2 - 2 = 0$$



$$\left.\begin{matrix}\vec{\nabla} f \\ \vec{\nabla} g\end{matrix}\right\} \text{richting } \vec{\nabla} f \times \vec{\nabla} g$$

$(1,0,1)$ $\qquad$ $(2,2,0)$

$$\begin{vmatrix} 1_x & 1_y & 1_z \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \underbrace{(2,-2,-2)}_{\vec{v}}$$

$x = 1 + 2t$

$y = 1 - 2t$

$z = 3 - 2t$

Schatting

wijziging van f in $\vec{u}$ met afstand ds

$$(df)_{\vec{u}, ds} = (D_{\vec{u}} f)_{P_0} \, ds \qquad \vec{u} = 1$$

Vb $f(x,y,z) = y \sin x + 2yz$

$$P_0(0,1,0) \rightarrow P_1(2,2,-2) \qquad ds = 0,1$$

Hoe verandert een functie van $P_0 \rightarrow P_1$ over afstand ds

$$\vec{u} = (2-0, 2-1, -2) = \frac{(2,1,-2)}{\sqrt{9}} = \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$$

$$(D_{\vec{u}} f) = \vec{\nabla} f \cdot \vec{u}$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$$

$$\left(y \cdot \cos(x), 2z + \sin x, 2y\right)\left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$$

$$= (\cos(0), \sin(0)+0, 2\cdot 1)$$
$$\quad\ \ \underset{1}{\shortparallel} \qquad\quad \underset{0}{\shortparallel} \qquad\ \ \underset{2}{\shortparallel}$$

$$(1, 0, 2)\cdot\left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right) = \frac{2}{3} - \frac{4}{3} = \frac{-2}{3} = (D_u f)_{P_0}$$

$$(Df)_{\vec{u}, ds} = -\frac{2}{3}\cdot 0{,}1 = -\frac{1}{15}$$

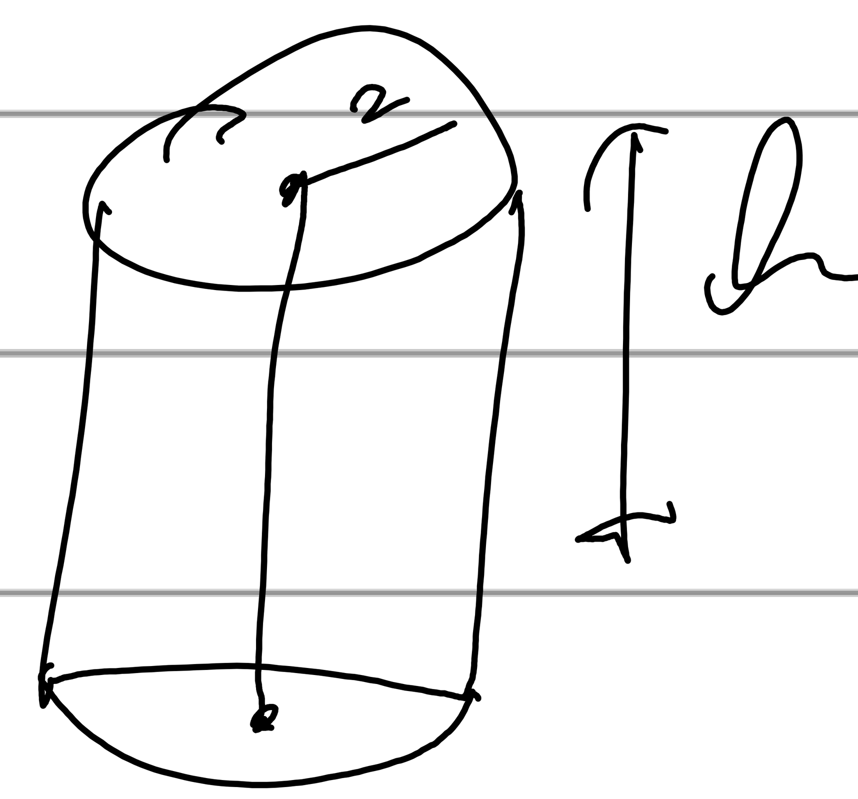
Kleine daling.

Diff.

$$df \approx f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Vb $\quad V = \pi r^2 h$



$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$$

$$\pi 2 r h\, dr + \pi r^2\, dh$$

$$2\cancel{\pi}\cancel{r}h > \cancel{\pi}r^{\cancel{2}}$$

$h > \frac{r}{2}$ dus verandering in straal heeft meer impact als h groter is dan $r/2$

Vd $P = \dfrac{RT}{V}$

$$dP = \frac{\partial P}{\partial V} dV + \frac{\partial P}{\partial T} dT = \frac{-RT}{V^2} dV + \frac{R}{V} dT$$

Vd $z = x^2 + 3xy - 2y^2$

$x: 3 \Rightarrow 3{,}2 = dx = 0{,}3$

$y: 5 \Rightarrow 4{,}9 = dy = 0{,}1$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= (2x + 3y)_{(3,5)} \cdot 0{,}3 + (3x - 4y)_{(3,5)} \cdot 0{,}1$$

$dz = 5{,}26$

Linearisatie

1 Variabele

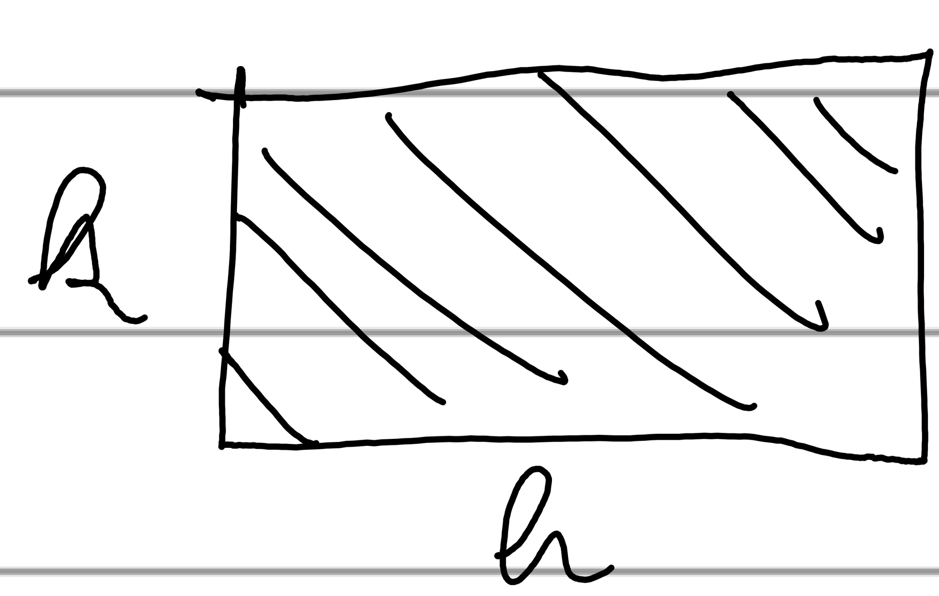
$$= f(x) = f(a) + (x-a) f'(a) + R_a$$

$$|R_a| = \frac{f''(c)}{2!}(x-a)^2$$

$$a < c < x$$

Functie = def.

2 Variabel $\approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + E$



$|x - x_0| \leq h$

$|y - y_0| \leq k$

$$E \leq \frac{1}{2} M \left(|x - x_0| + |y - y_0|\right)^2$$

$$M \geq f''_{xx}$$

$$f''_{xy}$$

$$f''_{yy}$$

Ue

$$z = f(x,y,z) = x^2 - xy + 3\sin z$$

$|x-2| \le 0{,}01$ $\quad (2,1,0)$

$|y-1| \le 0{,}02$

$|z| \le 0{,}01$

$f_x = 2x - y \to 3$

$f_y = -x \to -2$

$f_z = 3\cos z \to 3$

$$2 + 3(x-2) - 2(y-1) + 3z$$

$f_{xx} = 2 \qquad f_{xy} = -1$

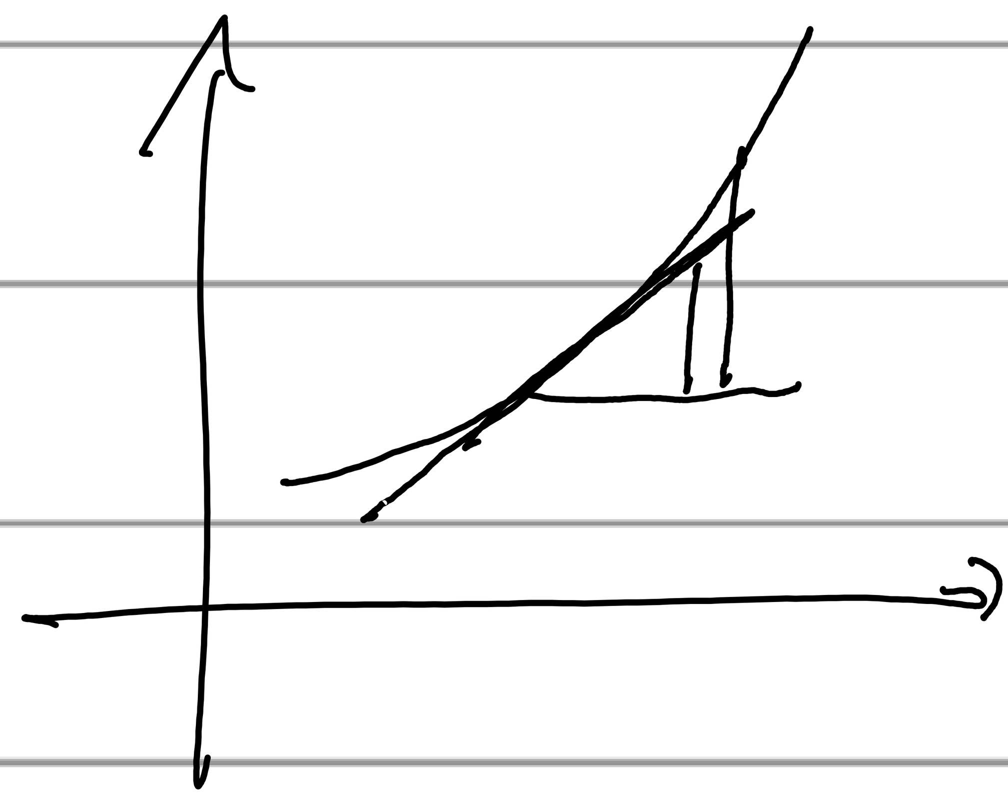
$f_{yy} = 0 \qquad f_{xz} = 0$

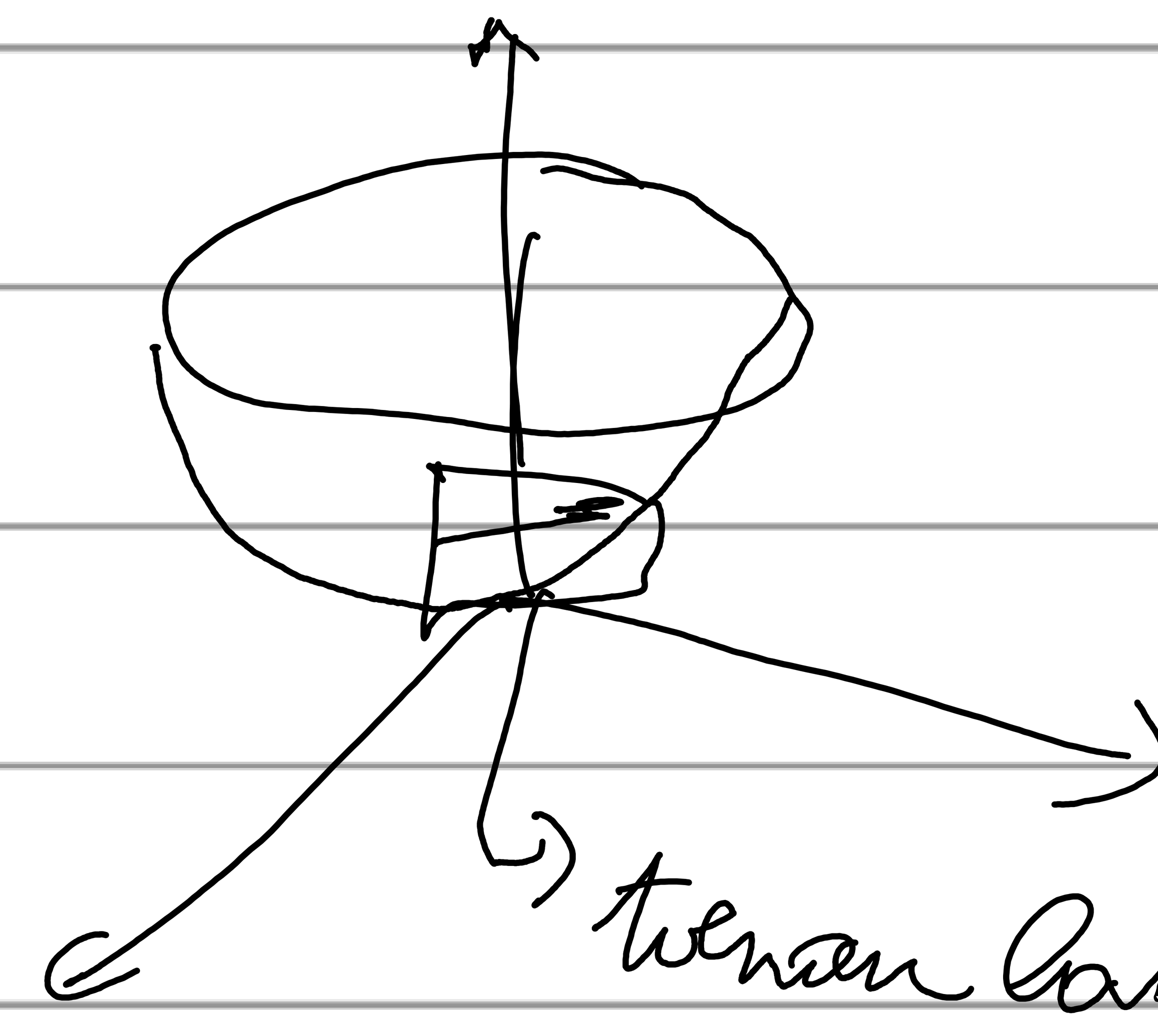
$f_{zz} = -3\sin z \qquad f_{yz} = 0$

$E \le \frac{1}{2}\, 2\,(0{,}01 + 0{,}02 + 0{,}01)^2$

$\le 0{,}0016$

Totale differentiaal





→ tenen langs zadelvlak ipv één van de assen

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dx$$

$$df \approx \Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$$

Toep: $V = \pi r^2 h = f(r,h)$

wat heeft het meeste impact op volume: r of h.

$$dV = \frac{\partial V}{\partial r} \partial r + \frac{\partial V}{\partial h} dh$$

$$= 2r\pi h \, dr + \pi r^2 \, dh$$

Rodwloh lepol

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}$$

$$\vec{n} \cdot \overrightarrow{P_0 P}$$

$$\frac{\partial z}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z}$$

$$\vec{\nabla} F \cdot \overrightarrow{P_0 P}$$

Explicit

$$\left(\frac{\partial F}{\partial x}\right)_{P_0}(x - x_0) + \left(\frac{\partial F}{\partial y}\right)_{P_0}(y - y_0) + \left(\frac{\partial F}{\partial z}\right)_{P_0}(z - z_0) = 0$$

$$\frac{-\partial F}{\partial z} \qquad \frac{-\partial F}{\partial z} \qquad \frac{-\partial F}{\partial z}$$

Implicit

$$\left(\frac{\partial z}{\partial x}\right)_{P_0}(x - x_0) + \left(\frac{\partial z}{\partial y}\right)_{P_0}(y - y_0) - (z - z_0) = 0$$

$=$ ciel !

$$F(x, y, z) = z - f(x, y) = 0$$

$$A z \geq A h \iff z \cancel{z} \cancel{\pi} h \geq \cancel{\pi} z^{\cancel{2}}$$

$$h \geq \frac{z}{2}$$

<u>Linearisatie</u>:

Taylor and Mclaurin reeks: geen theorie, wel oef.

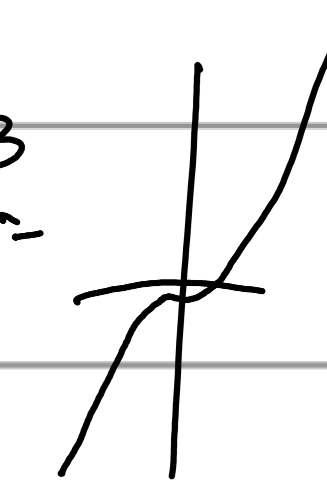
4. Parlee

Recording1
00:00:13 | 97.83KB

Extreme waarde en zadelpunten.

f lokaal $\frac{\text{max}}{\text{min}} \underset{\not\Leftarrow}{\Rightarrow} \frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0$

$y = x^3$



Kritiek punt als $\left\{ \begin{array}{l} f'_x = 0 \\ f'_y = 0 \end{array} \right.$ of bestaan niet.

$f_x(a,b) = f_y(a,b) = 0$

Bestaat niet

$\Rightarrow$ 2 mogelijkheden $\to$ lokaal ext

$\to$ zadelpunt.

- Als f lokaal extrem $\to f_x = 0, \ f_y = 0$

-

(1) Bepaal waar $f_x = f_y = 0$

(2) - $f_{xx} < 0$, $f_{xx} f_{yy} - f_{xy}^2 > 0$: lokaal max

- $f_{xx} > 0$, $f_{xx} f_{yy} - f_{xy}^2 > 0$: lokaal min

- $f_{xx} f_{yy} - f_{xy}^2 < 0$ zadelpunt

- $f_{xx} f_{yy} - f_{xy}^2 = 0$ verder testen

Def : ① Bepaal alle nulpunten (a,b) $\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}$

② Voor elk van Nulpunten

$f''_{xx}(a,b) < 0$	en $f''_{xx}(a,b)\, f''_{yy}(a,b) > 0$	lokaal Max
$f''_{xx}(a,b) > 0$		lokaal min
zadelpunt	" < 0	
onbeslist	$= 0$	

Vb) $x^2+y^2-4y+9=2$

zijn er lokale max of min

$f_x = 2x \rightarrow x=0$ nulpunt

$f_y = 2y-4 \Rightarrow y=2$ nulpunt (0,2)

$$\left.\begin{aligned} f_{xx} &= 2 \\ f_{yy} &= 2 \\ f_{xy} &= 0 \end{aligned}\right\} f_{xx}f_{yy}-f_{xy}^2>0$$

lokaal minimum

Vb $z = y^2 - x^2$

1. Kritische Punten

$$\frac{\partial z}{\partial x} = -2x = 0 \text{ nulpunt} \rightarrow (0,0)$$

$$\frac{\partial z}{\partial y} = 2y = 0 \text{ nulpunt} \rightarrow$$

$$\left.\begin{array}{l} z_{xx} = -2 \\ z_{\cancel{y}yy} = 2 \\ z_{xy} = 0 \end{array}\right\} \quad -2 \cdot 2 - 0 < 0 \rightarrow \text{zadelpunt}$$

Ub) $z = xy - x^2 - y^2 - 2\cancel{x} - 2y + 4$

$f_x = y - 2x - 2 \Rightarrow = 0 \quad y = 2x + 2$

$f_y = x - 2y - 2 \Rightarrow = 0 \rightarrow x - (4x+4) - 2 = 0$

$-3x = 6$

$x = -2$

$f_{xx} = -2$

$y = -2$

$f_{xy} = 1$

$(-2; -2)$

$f_{yy} = -2$

$\underbrace{-2 \cdot -2 - 1}_{3} > 0$

$f_{xx} = -2 > 0$; lokal Max

VI $\quad z = 3y^2 - 2y^3 - 3x^2 + 6xy$

$z_x = -6x + 6y \Rightarrow x = y$

$z_y = 6y - 6y^2 + 6x \Rightarrow$ Nullpunkte

$0 = 6y - 6y^2 - 6y$

$(0,0)$, $(2,2) \to$ Nullpunkte

$z_{xx} = -6$

$z_{yy} = 6 - 12y$

$z_{xy} = 6$

$(0,0) \to z_{yy} = 6 \Rightarrow -72 < 0$: Sattelpunkt

$(2,2) \to z_{yy} = -18 < 0$:

$\Rightarrow (-6)\cdot(-18) - 6^2 > 0$ lokal Max

Absolute Max, min

→ lokale max en min

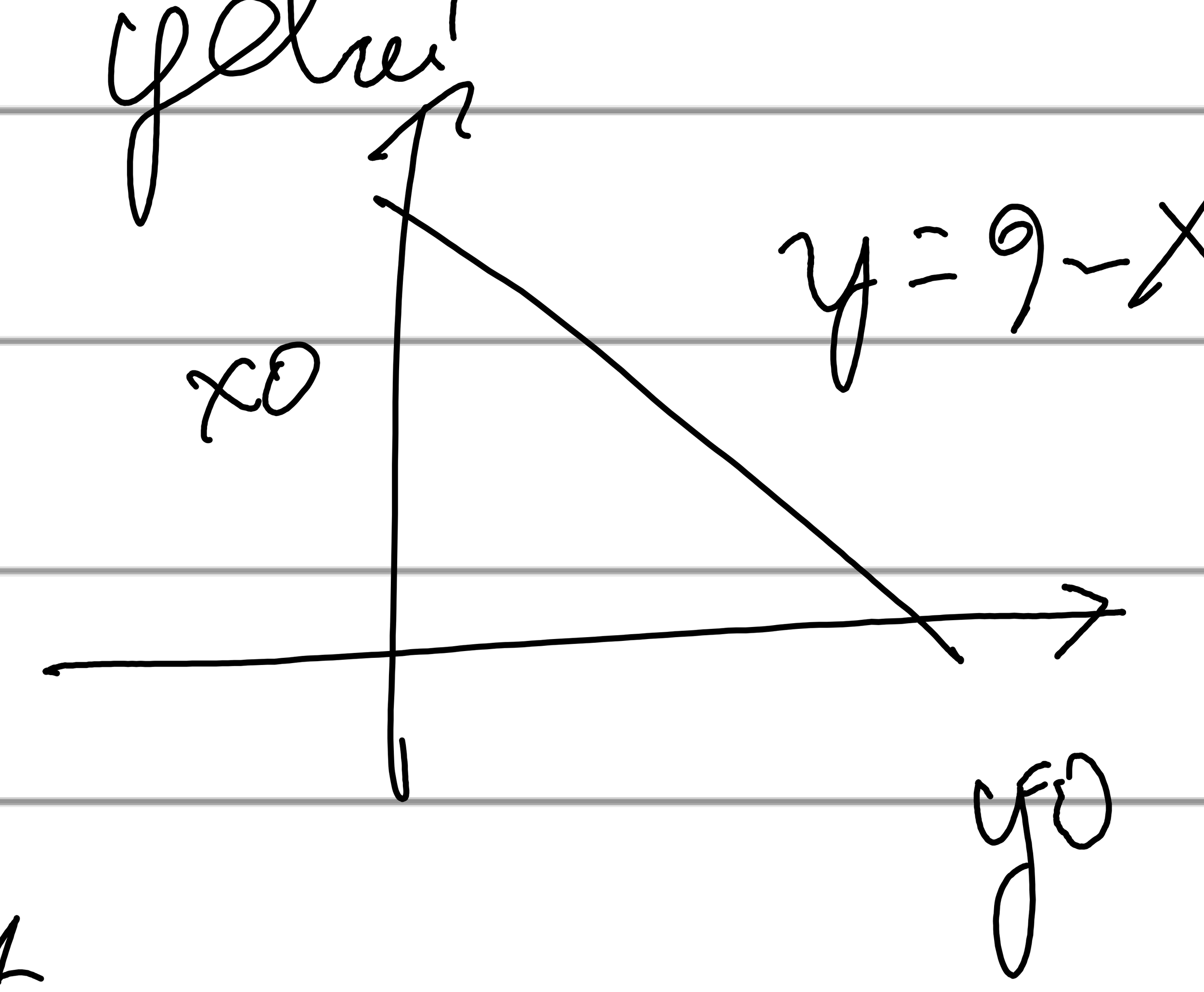
→ grenswaarde zoeken

Def, niet theorie

Extrema

1) lokale extrema
2) grenspunten
3) ob dire vergelijken

Vb $z = 2 + 2x + 2y - x^2 - y^2$



$z_x = 2 - 2x$

$z_y = 2 - 2x$

$(1,1)$ Nulpunt

$z_{xx} = -2$

$z_{yy} = -2$

$z_{xy} = 0$

> 0 lokaal max $\quad f(1,1) = 4$

x-as $\Rightarrow y = 0 \Rightarrow z = 2 + 2x - x^2$

$z' = 2 - 2x \Leftrightarrow x = 1 \quad z(1) = 3$

$z(0) = 2$

$z(9) = -61$

y-as $\Rightarrow$ $x=0$ $\Rightarrow$ $z = 2+2y-y^2$

$z' = 2-2y \Leftrightarrow y = 1$

$z\,\cancel{f}(0) = -$

$z(0) = -61$

$y = 9-x \Rightarrow z = 2+2x+2(9-x)-x^2-(9-x)^2$

$= -61+18x-2x^2$

$z' = 18-4x \Rightarrow x = \frac{9}{2} = 0$ punt

$z\left(\frac{9}{2}\right) = -\frac{41}{2}$

Lagrange Multiplicator: extrema met vw.

Lokale extrema van een functie f dat aan voorwaarde g voldoet.

→ hulpvariabele, moet je niet berekenen.

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$g(x, y, z) = 0$$

Bv

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g = 0 \end{cases}$$

(VI) Zoek de punten die dichtst bij oorsprong en die gelegen zijn op het oppervlak $x^2 - z^2 = 1$ (Hyperboloïde)

We zoeken een functie die de afstand tot de oorsprong geeft.

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$ en daarna det minim

⟹ even goed $x^2+y^2+z^2$ kwadraat minimaliseren

$f(x,y,z) = x^2+y^2+z^2$ eerstens zoek

$\vec{\nabla} = \lambda \vec{\nabla} g$

$g = ?$

$x^2 - z^2 - 1 = 0$

$\vec{\nabla} f = (2x, 2y, 2z)$

$\vec{\nabla} g = (2x, 0, -2z)$

gradiënt van vlak

$$\begin{cases} \cancel{2}x = \cancel{\lambda}\, 2x \rightarrow x = 1 \\ 2y = 0 \quad y = 0 \\ 2z = \lambda(-2z) \leftarrow \text{enkel als } z = 0 \\ x^2 + y^2 + z^2 = 0 \end{cases}$$

$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

Oplossing $(1,0,0)$ en $(-1,0,0)$

VII Vind ob extreme $f(x,y) = 3x + 4y$ of cirkel $x^2 + y^2 = 1$

$\vec{\nabla} f = \lambda \vec{\nabla} g$ 1 $\vec{\nabla} f(3, 4)$

$g = 0$ $\vec{\nabla} g(2x, 2y)$

2 $3 = \lambda 2x$

$4 = \lambda 2y$

$x^2 + y^2 = 1$

$x = \frac{3}{\cancel{\lambda}2\lambda}$

$y = \frac{4}{2\lambda}$

$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 1$ $-\frac{5}{2}$ of $\frac{5}{2}$

3 $x = \frac{3}{5 \cdot \frac{\cancel{2}}{\cancel{2}}}$

$y = \frac{4}{\frac{\cancel{2}5}{\cancel{2}}}$ $\Rightarrow \left(\frac{3}{5}, \frac{4}{5}\right), \left(\frac{-3}{5}, \frac{-4}{5}\right)$

Def extreme waarden

Stel zoek kritische punten en duid of ze lokale min of max zijn)

$$z = 3y^2 - 2y^3 - 3x^2 + 6xy$$

$z'_x = 0 \qquad -6x - 6y = 0 \Rightarrow x = y$

$z'_y = 0 \qquad 6y - 6y^2 + 6\cancel{x} = 0 \Rightarrow$

$$6y + 6y - 6y^2 = 0$$

$$12y - 6y^2 = 0$$

$$6y(2 - y) = 0$$

$$\hookrightarrow y = 2 \text{ en } 0$$

dus $(2,2)$ en $(0,0)$ want $x = y$

$z''_{xx} = -6$

$z''_{yy} = 6 - 12y$

$z''_{xy} = 6$

$$z''_{xx}\, z''_{yy} - {z''_{xy}}^2$$

$(0,0)$ $\quad -6 \cdot 6 - 36 = -72$: onder 0 dus zadelpunt

$(2,2)$ $\quad (-6)\cdot(-18) - 36 > 0$ lokaal Max

Taylor Veelterm in 2 variabelen.

$$f(x,y) = f(a,b) + (x-a) f_x + (y-b) f_y + \frac{1}{2!}\Big[(x-a)^2 f_{xx} + 2(x-a)(x-b) f_{xy} + (y-b)^2 f_{yy}\Big] + \frac{1}{3!} \ldots$$

Samenvatting

① Meetkunde van

$$z = f(x,y) \quad \mathbb{R}^2 \to \mathbb{R}$$

② Partiële afgeleiden

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} = \text{Analytisch} \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Meetkundig

③ Kettingregel

$$\frac{\partial f}{\partial t} \text{ of } \frac{df}{dt}$$

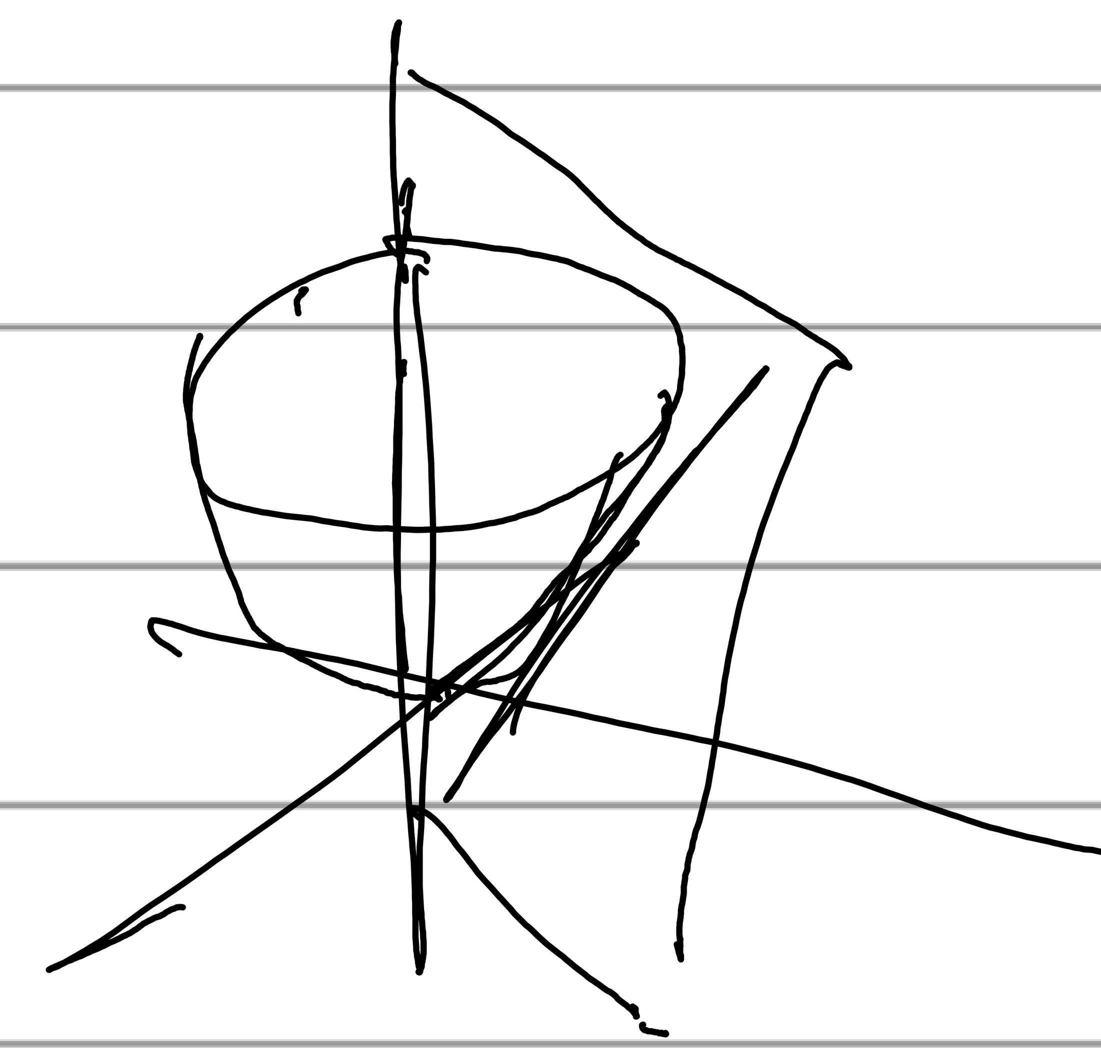
$\mapsto$ Impliciet $\quad \dfrac{\partial z}{\partial x} = \dfrac{-\partial F/\partial x}{\partial F/\partial z}$

$F(x, y, z(x,y))$

(4) richtingsafgeleide + gradiënt

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$$

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$



$\vec{\nabla} f \parallel \vec{z}$ grootste verandering

(5) Raakvlak

Impliciet $\left(\frac{\partial F}{\partial x}\right)_{P_0}(x-x_0)+\left(\frac{\partial F}{\partial y}\right)_{P_0}(y-y_0)+\left(\frac{\partial F}{\partial z}\right)_{P_0}(z-z_0)=0$

Expliciet

$$\left(\frac{\partial z}{\partial x}\right)_{P_0}(x-x_0)+\left(\frac{\partial z}{\partial y}\right)_{P_0}(y-y_0)-(z-z_0)=0$$

(6) Schatting $(D_{\vec{u}} f)_{P_0} \cdot ds$

differentiaal: $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \vec{\nabla} \cdot d\vec{z}$

Linearisatie

$$f(x,a) = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)_{P_0}(x - x_0) + \left(\frac{\partial f}{\partial y}\right)_{P_0}(y - y_0) + E$$

rest

(6) Extreme waarden

Kritieke punt : $f_x = f_y = 0$

lokaal min, max / zadel

voorwaarde

$f_{xx} < 0$: max

$f_{xx} > 0$: min

$f_{xx} f_{yy} - f_{xy}^2 < 0$: zadel

$= 0$: apart

(7) Extremum vraag

f lokaal extremum zoek

$\vec{\nabla} f = \lambda \vec{\nabla} g \rightarrow g = 0$ en gegeven voorwaarde