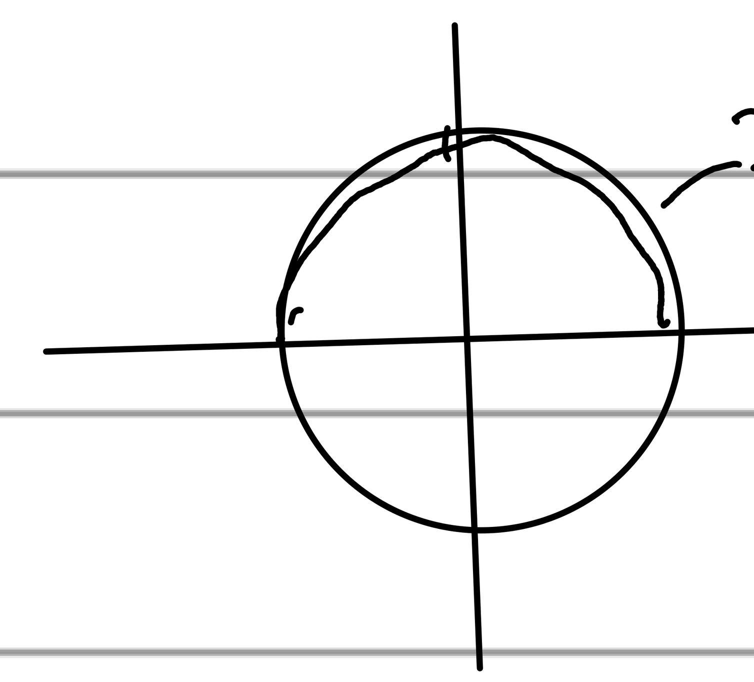
Hfst 2: Parameter Voorstelling en Poolcoördinaten

functie heeft 3 voorstelling:

$y = f(x) \Rightarrow$ expliciet

$F(x,y) = 0 \rightarrow$ impliciet

$y = \sqrt{1-x^2}$



$\rightarrow x^2 + y^2 + z^2 = 0$

$$\boxed{\begin{array}{l} x = \cos t \\ y = \sin t \\ 0 \leq t \leq \pi \end{array}}$$

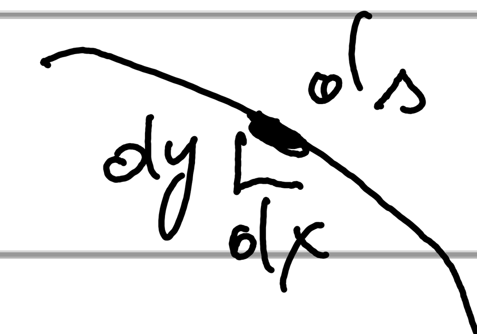
$\left.\begin{array}{l} x = x(t) \\ y = y(t) \end{array}\right\} \rightarrow$ Parameter voorstelling

cycloïde $x = a(t - \sin t)$
$y = a(1 - \cos t)$

van Parameter en dan $\boxed{\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{y'}{x'}}$

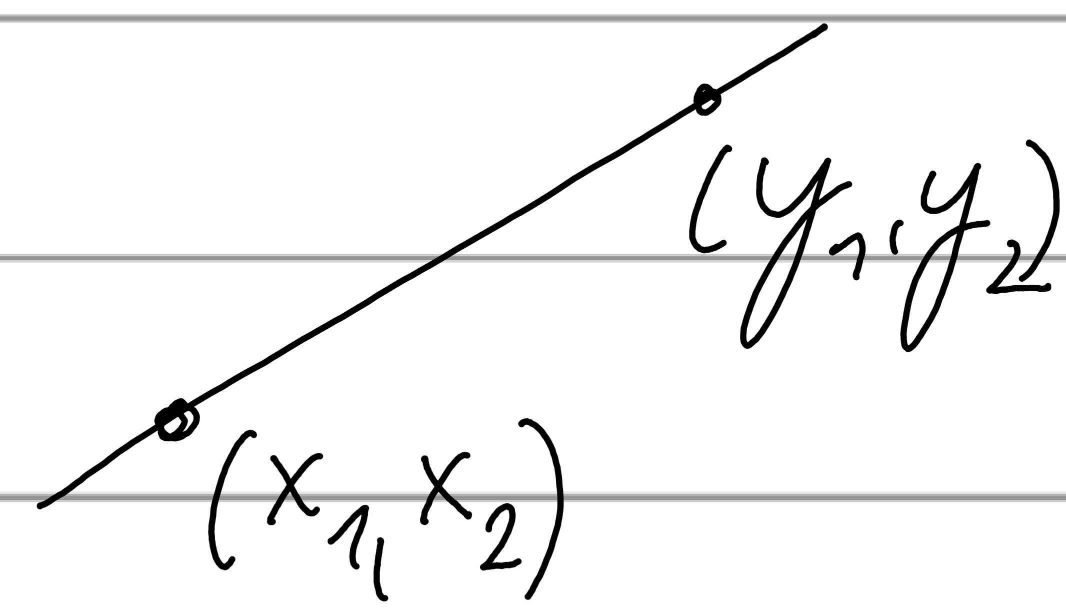
$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$$

ds : stukske van booglengte



$$ds^2 = dx^2 + dy^2$$

Def Rechte in parametervoorstelling



$$x = x_1 + t(y_1 - x_1)$$

$$y = x_2 + t(y_2 - x_2)$$

Berekenen $x = x(t)$

$y = y(t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Vb met cirkel $x = 2\cos t$

$y = 2\sin t$

$$\frac{dy}{dt} = \frac{2\cos t}{-2\sin t} = \cot g\, t$$

2de orde

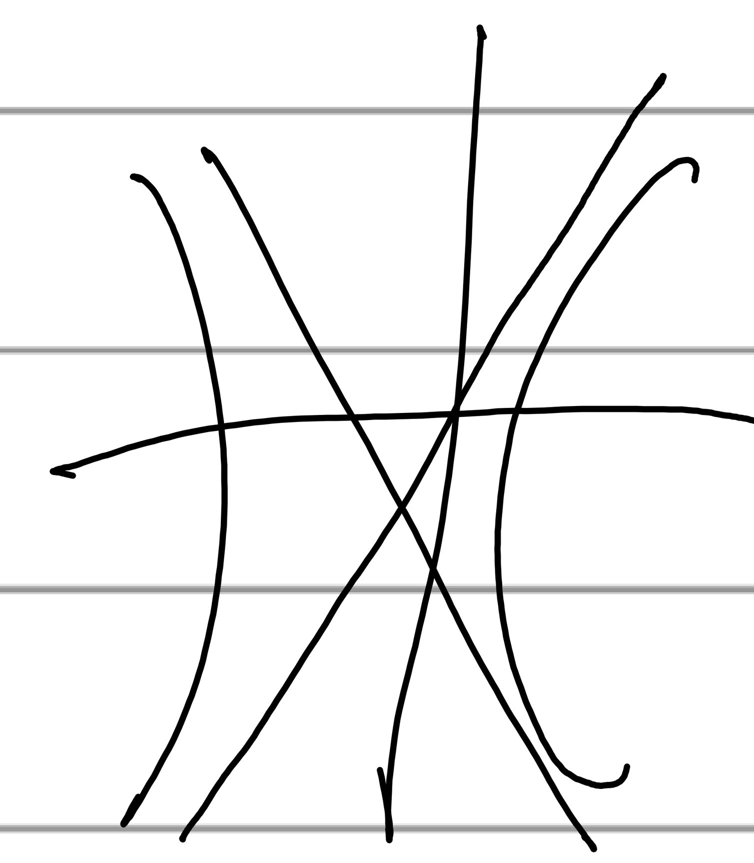
$$\frac{d^2y/dt}{dx^2/dt} = \frac{-2\sin t}{-2\sin t} = 1$$

④ hyperbol

$$x = \sec t = \frac{1}{\cos t}$$

$$y = \operatorname{tg} t$$

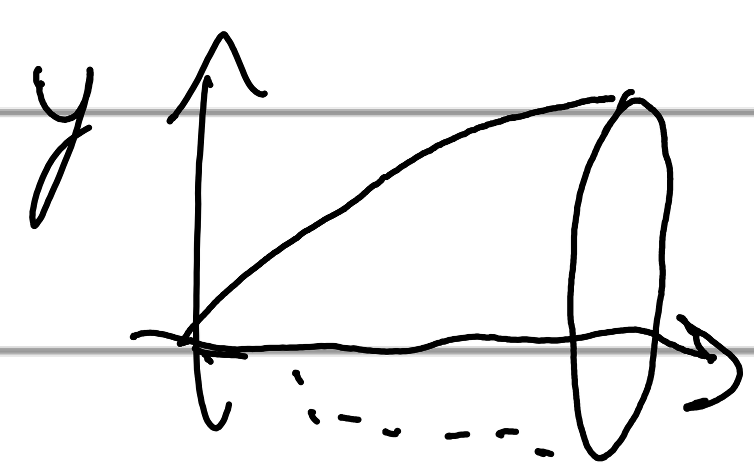
$$x^2 - y^2 = 1 \iff \frac{1}{\cos^2 t} - \operatorname{tg}^2 t = 1$$



Om lengte te bepalen $ds^2 = \underline{dx^2 + dy^2}$ dt^2

$$\boxed{ds = \sqrt{\overline{\underline{x'^2 + y'^2}}}\; dt}$$

Manteloppervlak



$\text{oppervlak} = \int 2\pi y \, ds$

Vb

$x = 1/\cos t$

$y = \tan t$

Bepaal raaklijn aan $T = \frac{\pi}{4}$ $\left[\; y - y_0 = y'(x_0)\,(x - x_0)\right.$

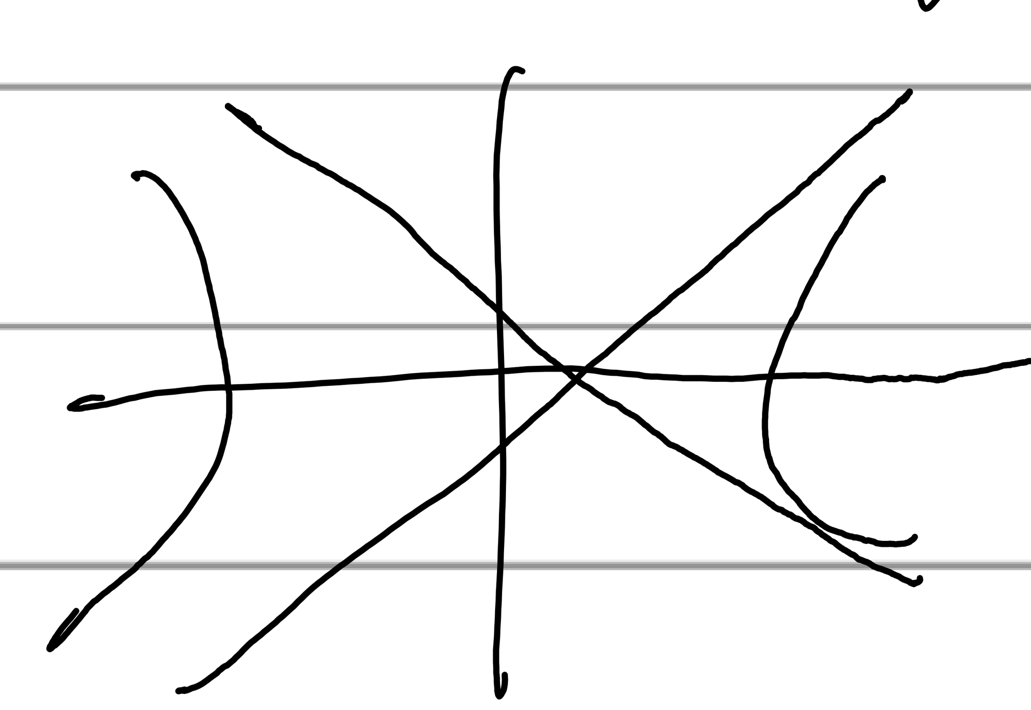
$P(x_0, y_0)$

$y'(x_0)$

$$x^2 - y^2 = \frac{1}{\cos^2 t} - tg^2 t = 1$$

$y = \frac{b}{a} x$

$$\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$



$$x_0 = \frac{1}{\cos(\frac{\pi}{4})} = \sqrt{2}$$

$$y_0 = \operatorname{tg}\frac{\pi}{4} = 1$$

$y'(x_0)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1/\cos^2 t}{-\frac{\sin t}{\cos^2 t}} = \frac{1}{-\sin(t)} = \sqrt{2}$$

afgeleid $\operatorname{tg} t$

afgeleid $\frac{1}{\cos(t)}$

Vb met booglengte van Astroide

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}} \cdot dt$$

$$\frac{1}{4} \cancel{L} = \int_0^{\pi/2} \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}\, dt = \frac{1}{4} \int \sqrt{3\cos^4 t \sin^2 t + \sin^4 t \cdot \cos^2 t}\, dt$$

$$\frac{3}{2}$$

Vb Hyperbool

$x = \sec t = \dfrac{1}{\cos t}$

$y = tg\, t$

$\dfrac{dy}{dx} = ?$ en bepaal raaklijn van punt voor $t = \dfrac{\pi}{4}$

$$\frac{dy}{dx} = \frac{\frac{1}{\cancel{\cos^2 t}}}{\frac{1}{\cancel{\cos^2 t}} + \sin t} = \frac{1}{\sin t}$$

$y - y_0 = f'(x_0)(x - x_0)$ $\quad \Big| \dfrac{dy}{dx}\left(\dfrac{\pi}{4}\right) = \dfrac{2}{\sqrt{2}}$

$x = \dfrac{1}{\cos \frac{\pi}{4}} = \dfrac{2}{\sqrt{2}}$

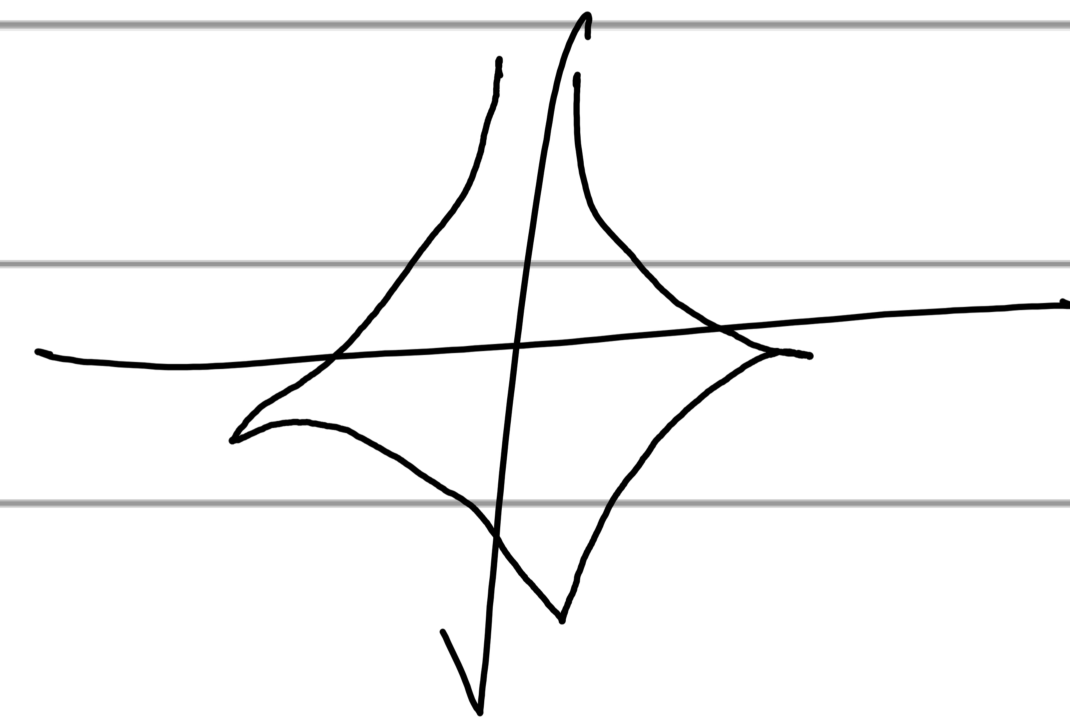
$y - 1 = \dfrac{2}{\sqrt{2}}\left(x - \dfrac{2}{\sqrt{2}}\right)$ $\quad y = tg\, \dfrac{\pi}{4} = 1$

⇝ Vgl van raaklijn

(VI) Asteroide

$x = \cos^3 t$

$y = \sin^3 t$



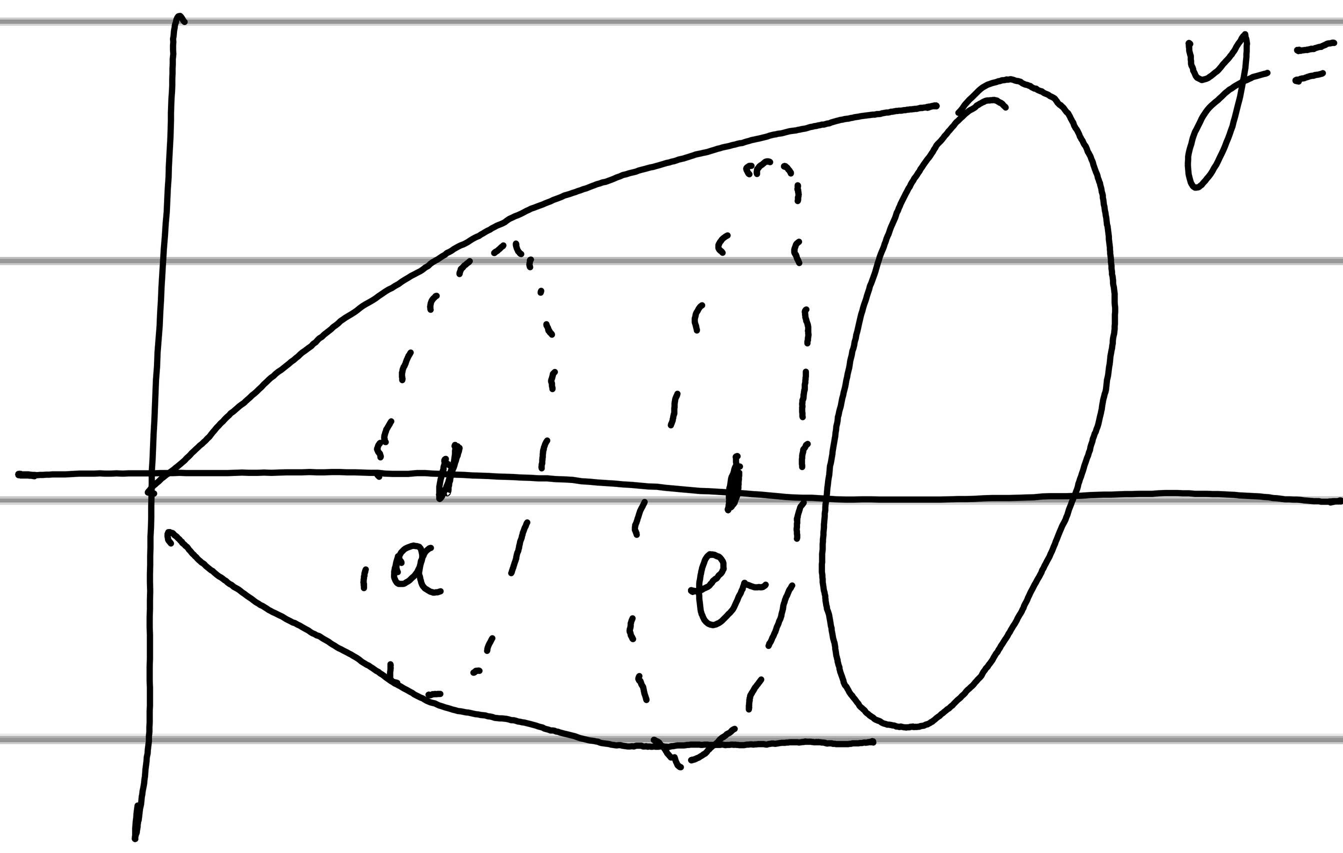
$$\frac{dy}{dx} = \frac{\cancel{3} \sin^2 t \cancel{\cos t}}{\cancel{3} \cos^2 t \, (\cancel{-}\sin t)} = \frac{-\sin t}{\cos t} = -\cancel{tg}\, t$$

wanneer bestaat afgeleide niet of $= 0$ (wanneer // x as

Bestaat niet als $\cos t = 0$

$= 0$ als $\sin t = 0$

Manteloppervlak



$$= \int_a^b 2\pi y \, dS$$

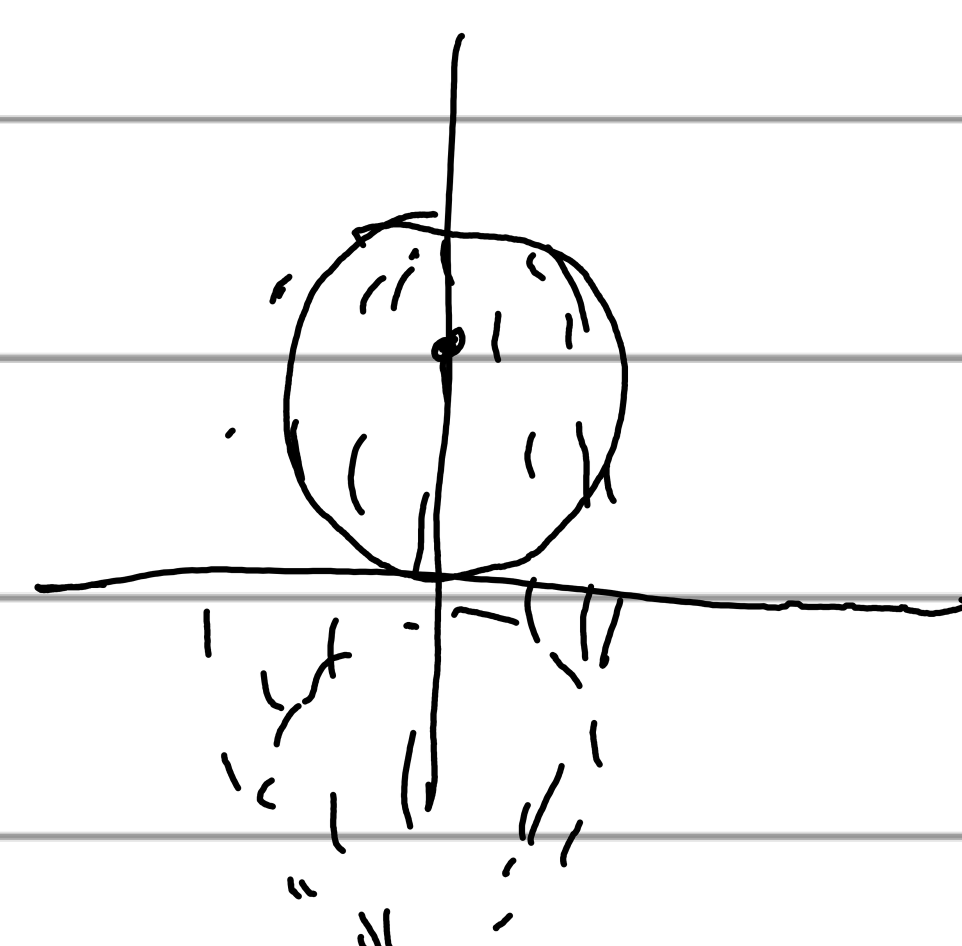
$$= \int_a^b 2\pi f(x) \, dS$$

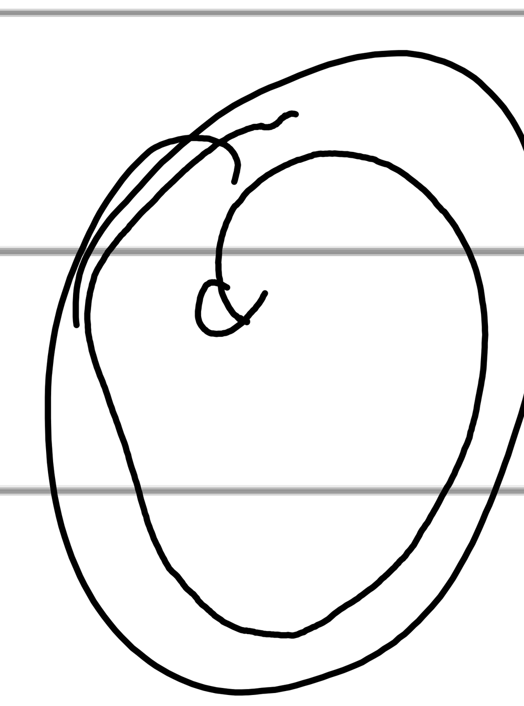
$$= \int_\alpha^a 2\pi y \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}} \, dt$$

Vb

$x = \cos t$

$y = 1 + \sin t$

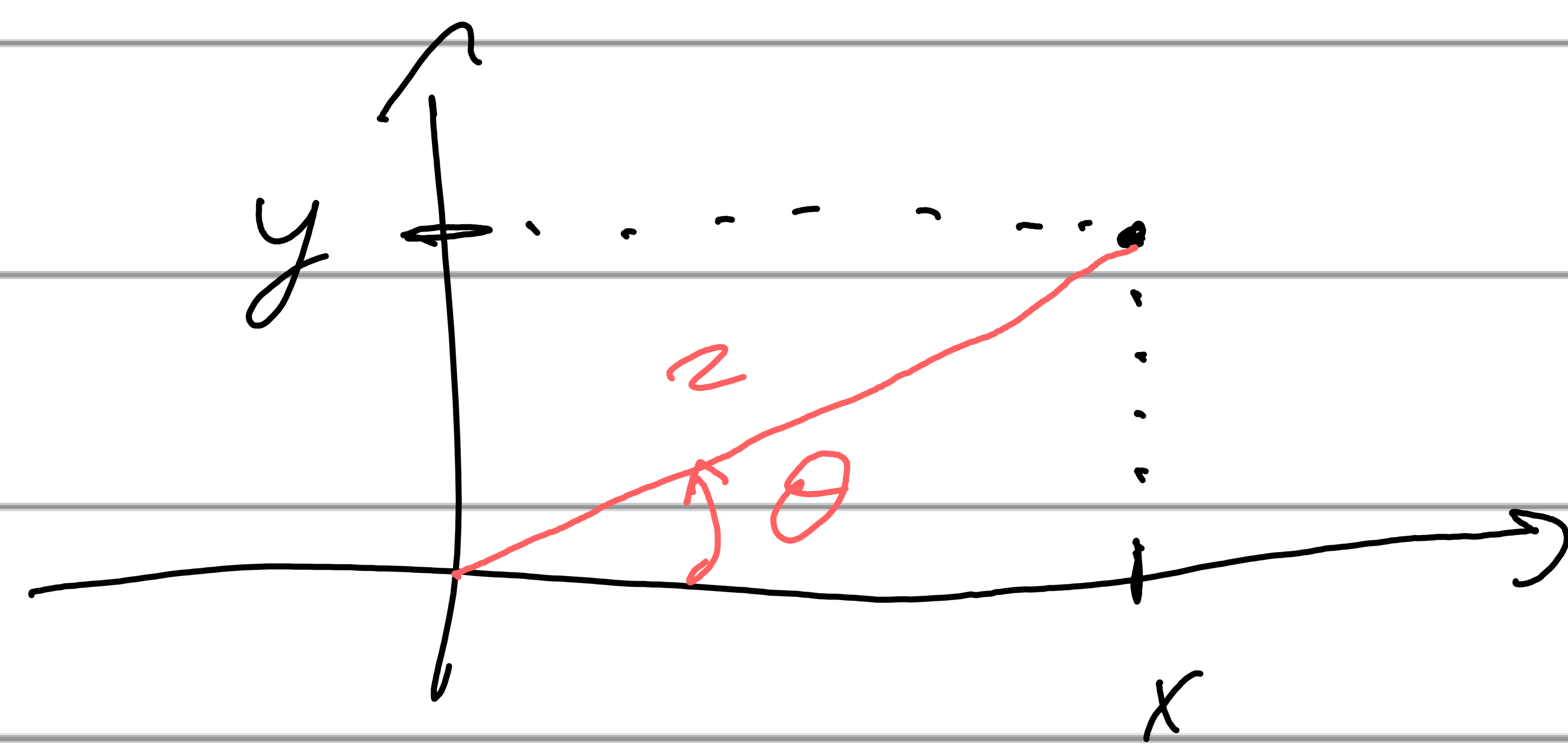




$$M = \int_0^{2\pi} 2\pi (1 + \sin(t)) \underbrace{\sqrt{\sin^2 t + \cos^2 t}}_{=1} \, dt$$

$$= 4\pi^2$$

Poolcoördinaten $(x, y) \to (r, \theta)$



$x = r\cos\theta$

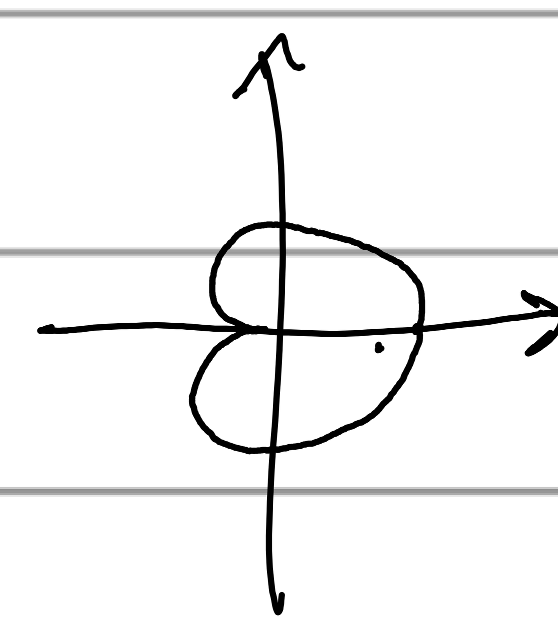
$y = r\sin\theta$

$r = \sqrt{x^2 + y^2}$

$\text{tg}\,\theta = \frac{y}{x}$

$r = r(\theta)$

Bv $r = 1 + \cos\theta$



Hoe $\frac{dy}{dx}$ van Bepalen?

! $\boxed{r = r(\theta) \text{ schrijven als functie } \begin{array}{l} x = r(\theta)\cos(\theta) \\ y = r(\theta)\sin(\theta) \end{array}}$

en dan $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'(\theta)\sin\theta + r(\theta)\cos(\theta)}{r'(\theta)\cos\theta - r(\theta)\sin(\theta)}$

Ve l'ortocoordinate non punto cost coordinate

Es1 $r\cos(\theta) = 1 \rightarrow x = 1$

Es2 $r = r\sin(\theta) + 1$ quadrato

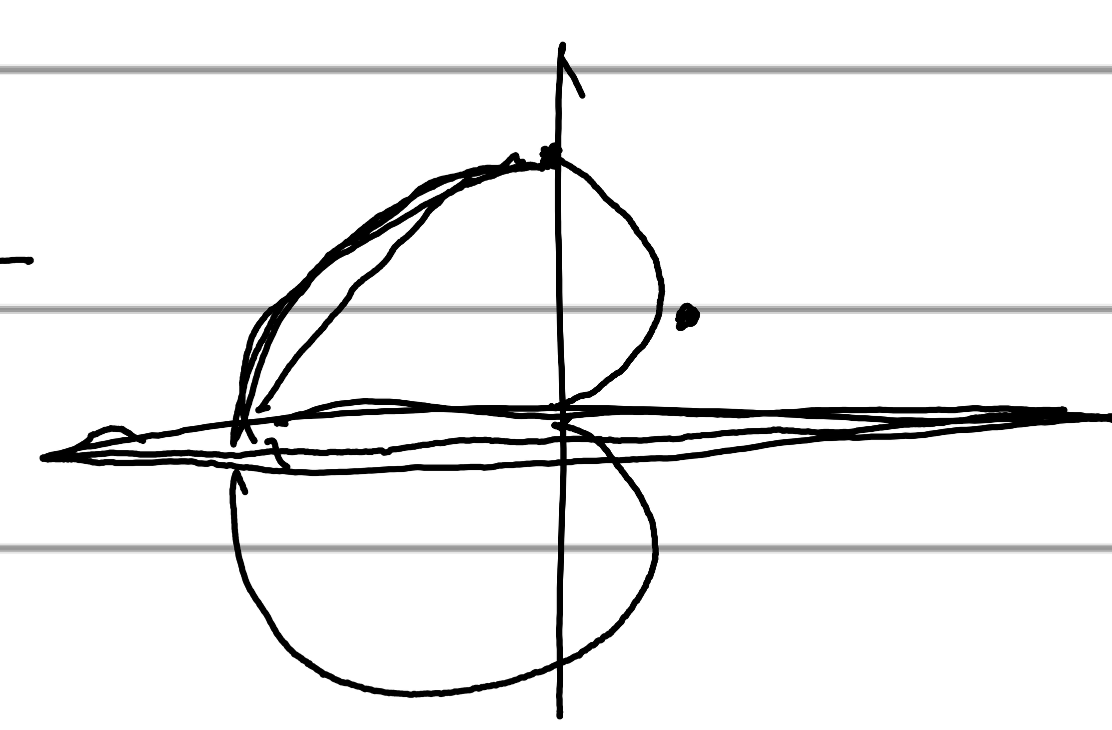
$r^2 = r^2\sin^2\theta + 1 + 2r\sin\theta$ a^2+b^2+2ab

$x^2+y^2 = y^2 + 2y + 1$

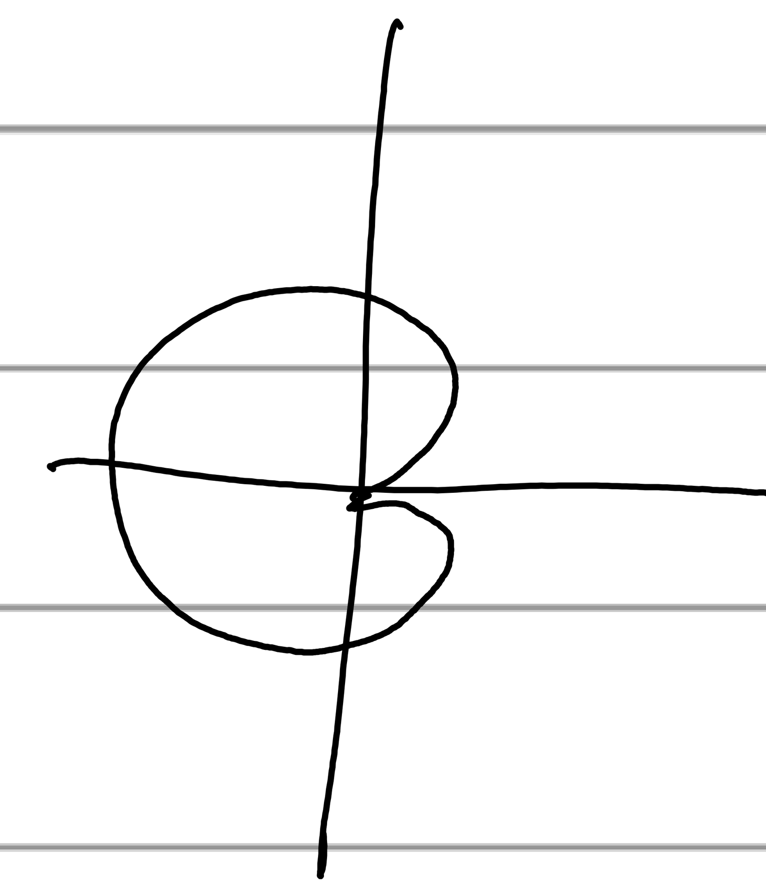
Ve Atelevem?

$r = 1 - \cos\theta$

θ	0	$\pi/4$	$\frac{\pi}{2}$	$\frac{\pi 3}{4}$
r	0	$1-\frac{\sqrt{2}}{2}$		



Vb $\quad r = 1 - \cos\theta$



Vraag: waar // met x-as

$$x = 1 - \cos\theta \cdot \cos\theta$$
$$y = 1 - \cos\theta \cdot \sin\theta$$

$$\frac{dy}{dx} \Rightarrow \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta\sin\theta + (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta) - \sin\theta}$$

$$= \frac{\sin^2\theta + \cos\theta - \cos^2\theta}{2\sin\theta\cos\theta - \sin\theta}$$

Evenwijdig met de y-as = als

(12) Bepaal de punten waar de raaklijn // met x of y as

$$\frac{dy}{dx} = 0 \rightarrow // x\text{-as}$$

$$\frac{dy}{dx} = \infty \rightarrow // y\text{-as}$$

Stel $r = 1 - \cos\theta$ en we willen $\frac{dy}{dx}$ bepalen

$$x = (1 - \cos\theta)\cdot\cos\theta$$

$$y = (1 - \cos\theta)\cdot\sin\theta$$

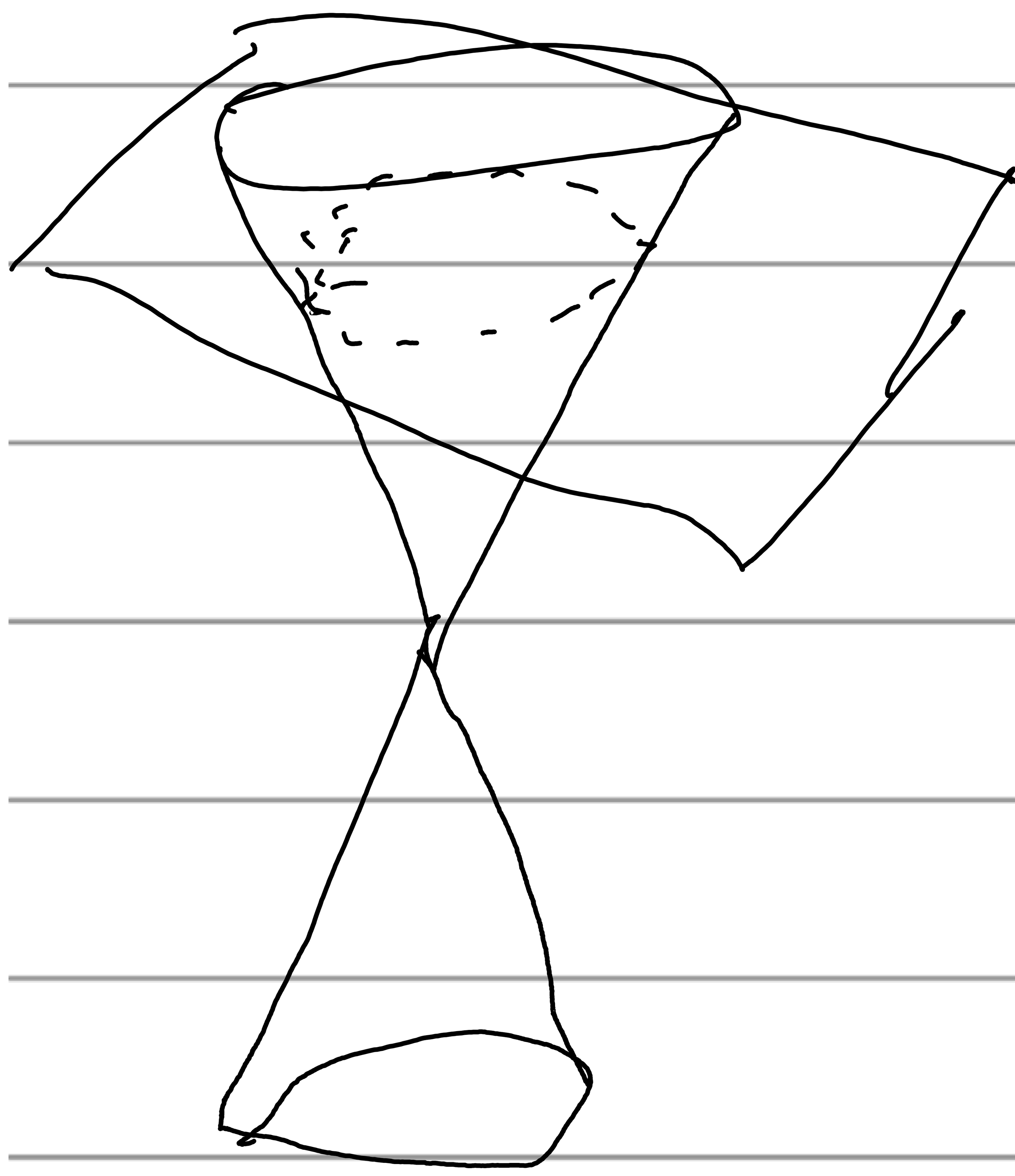
$$\frac{dy/d\theta}{dx/d\theta} = \frac{\sin\theta\sin\theta - (1-\cos\theta)\cos\theta}{\sin\theta\cos\theta - (1-\cos\theta)\sin\theta}$$

↗ teller 0 = // met x

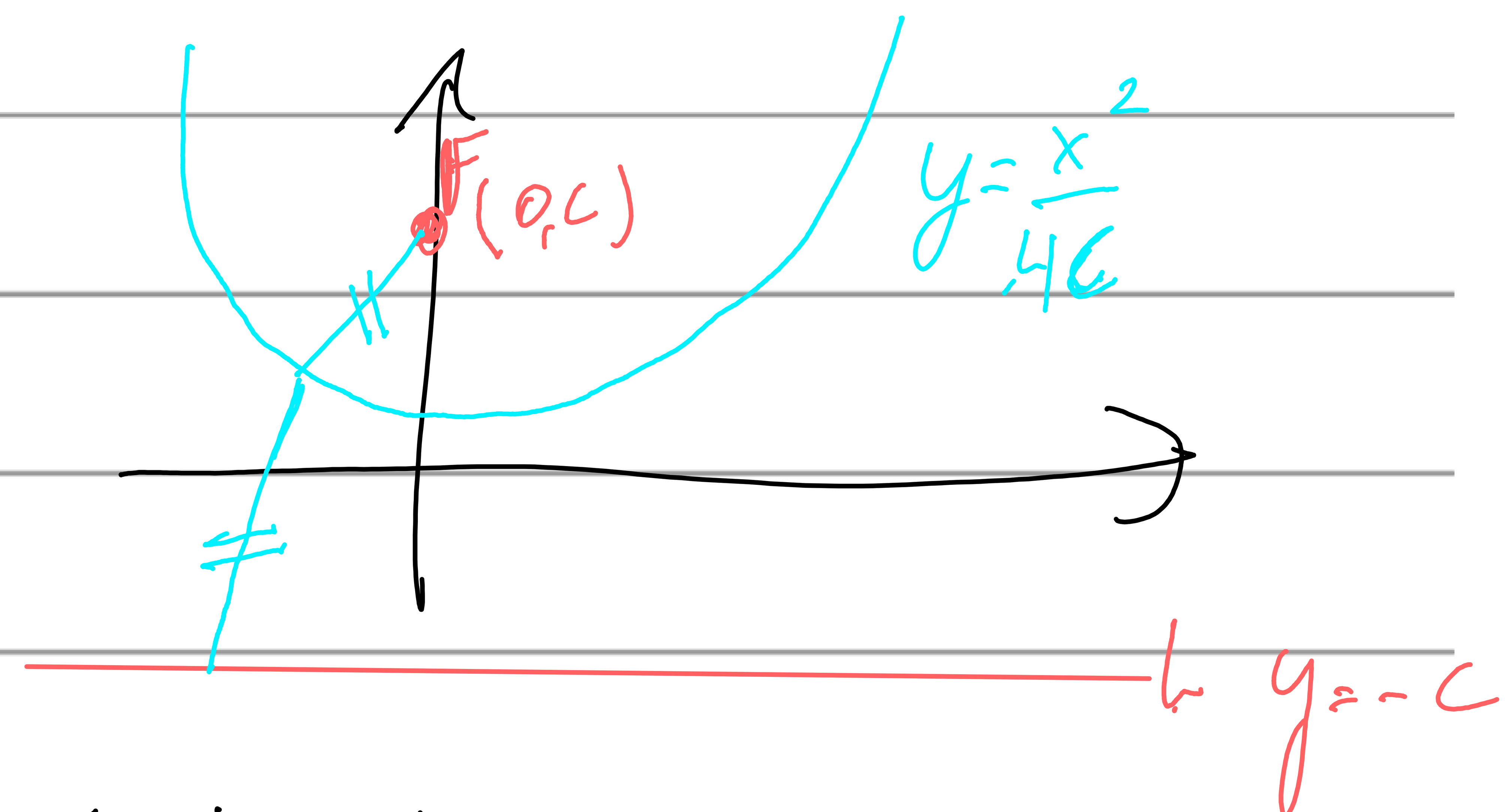
↘ noemer 0 = // met y

Kegelsneden

1) Parabool



Snijd als parabool

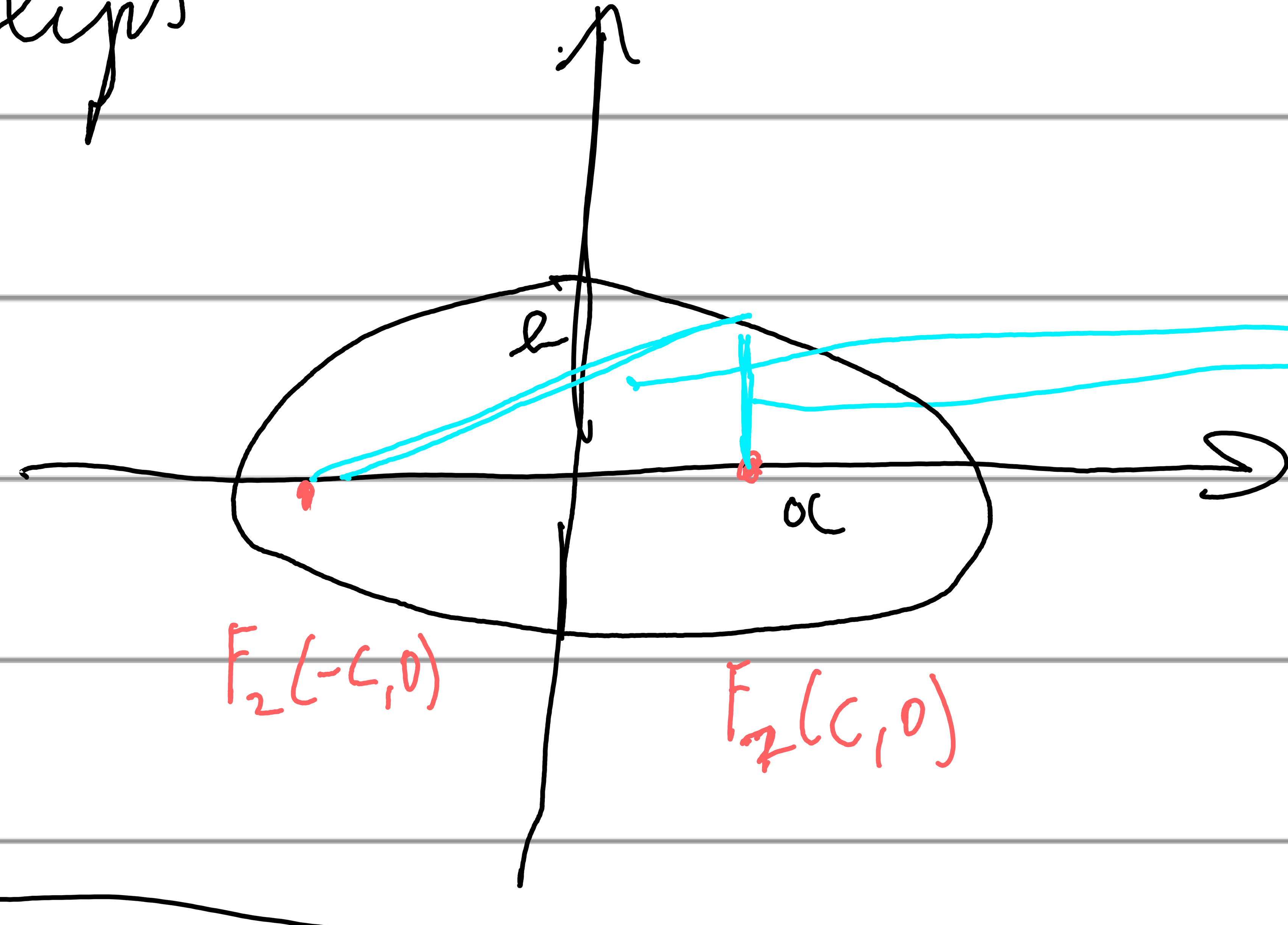


$\forall P \ |PF| = d(P,L)$

$x = t$

$y = \frac{t^2}{4c}$

2) Elips



$\forall \ |PF_1| + |PF_2|$ is constant

$a^2 - c^2 = b^2$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

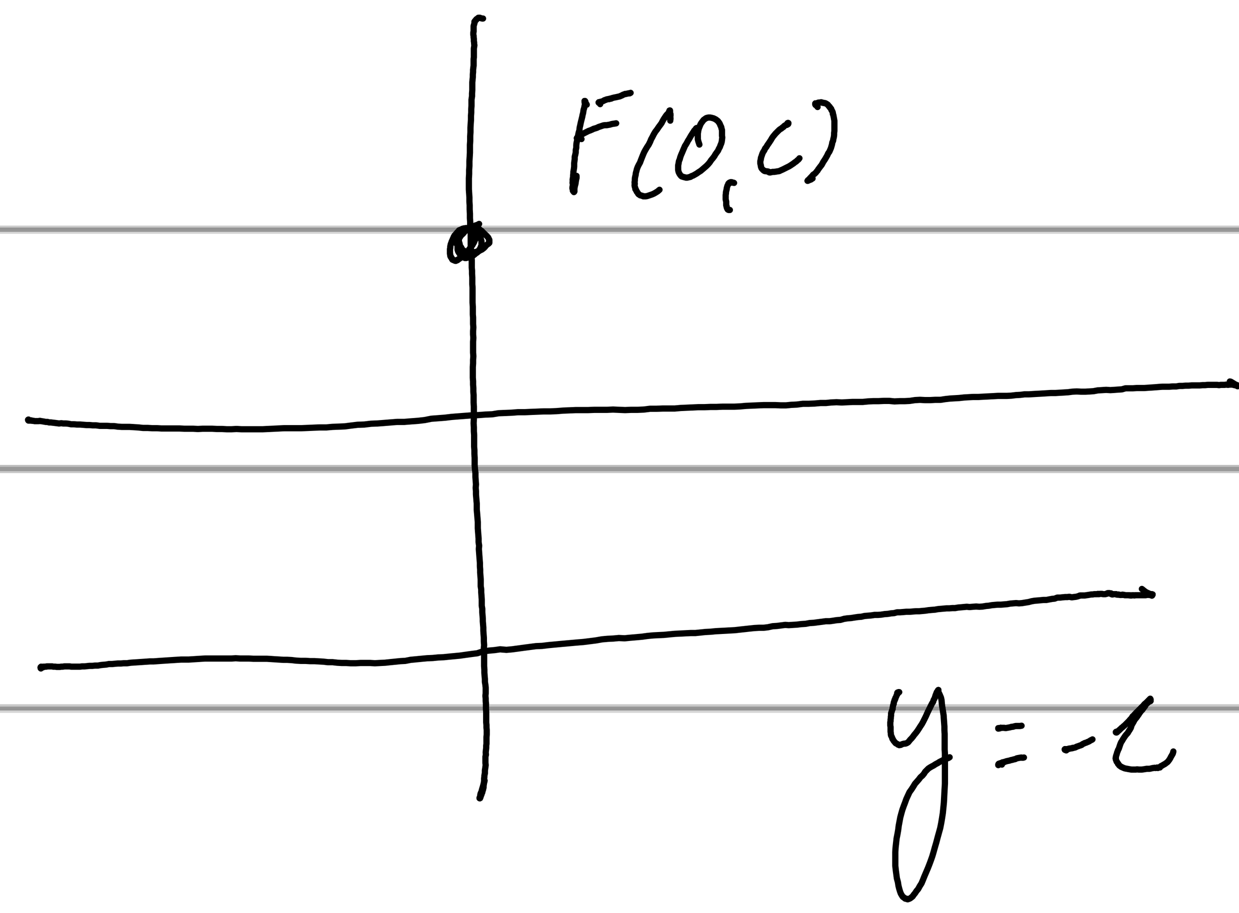
$x = a\cos t$

$y = b\sin t$

$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2} =$

$y = \frac{b}{a}\sqrt{a^2 - x^2}$

Parabel



$|PF| = |PL|$

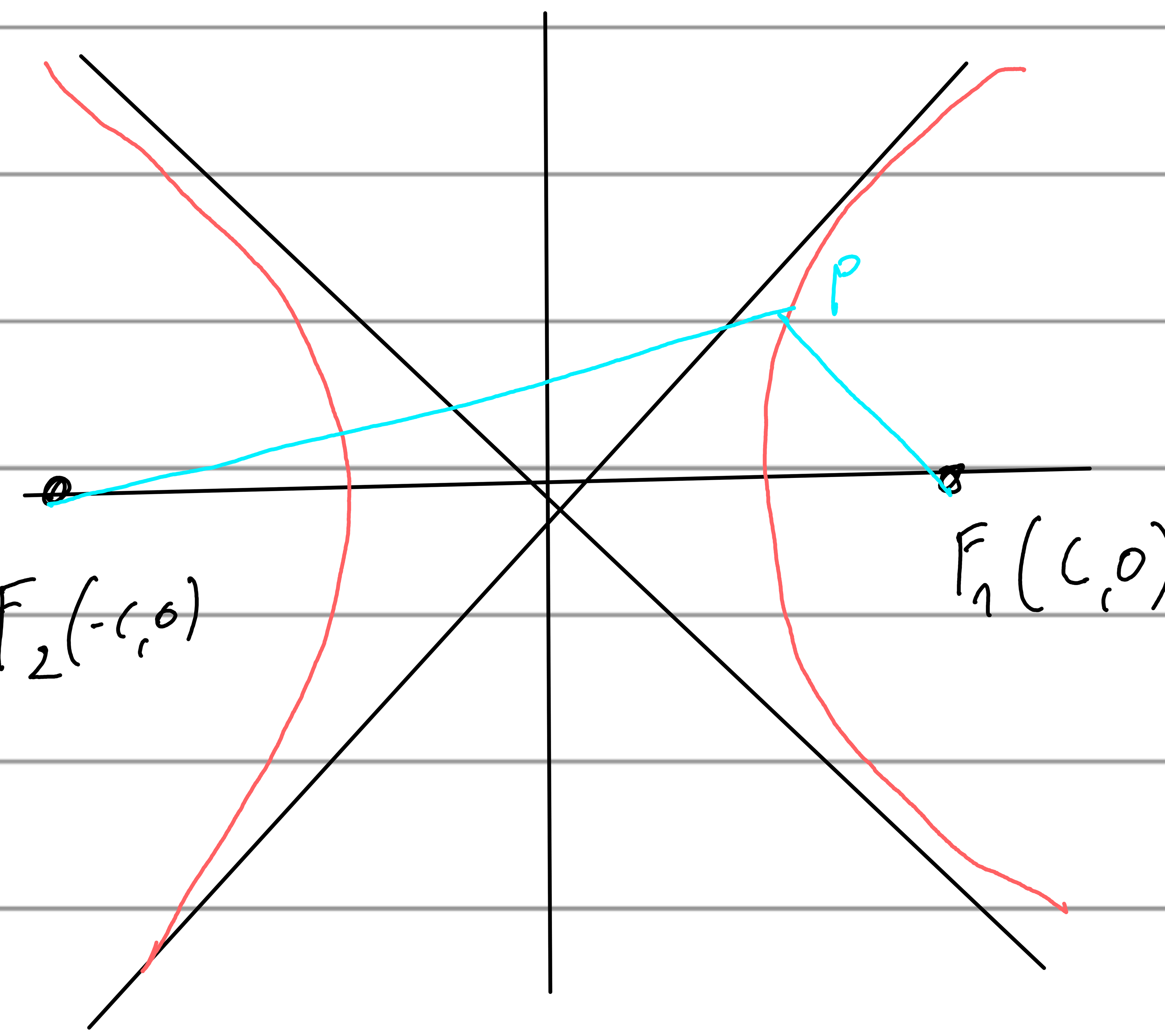
$x = t$

$y = \frac{t^2}{4c}$

$x^2 = 4cy$

$x^2 - 4cy = 0$

3) Hyperbool



$\forall P \quad \big| |PF_1| - |PF_2| \big|$ is constant

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2} \Rightarrow \begin{cases} x = a/\cos t \\ y = b \operatorname{tg} t \end{cases}$$

$\boxed{\text{Examen: 3 voorstellingen bepaal in elk van de vgl } \frac{dy}{dx}}$

Samenvatting:

3 voorstellingswijzes — Expliciet / impliciet / Parameter

Parameter

↳ $\frac{Dy}{dx}$

↳ booglengte

↳ manteloppervlak

=> Poolcoördinaten:

↳ omzetting

↳ $\frac{dy}{dx}$

=> Verschillende kegelsneden.

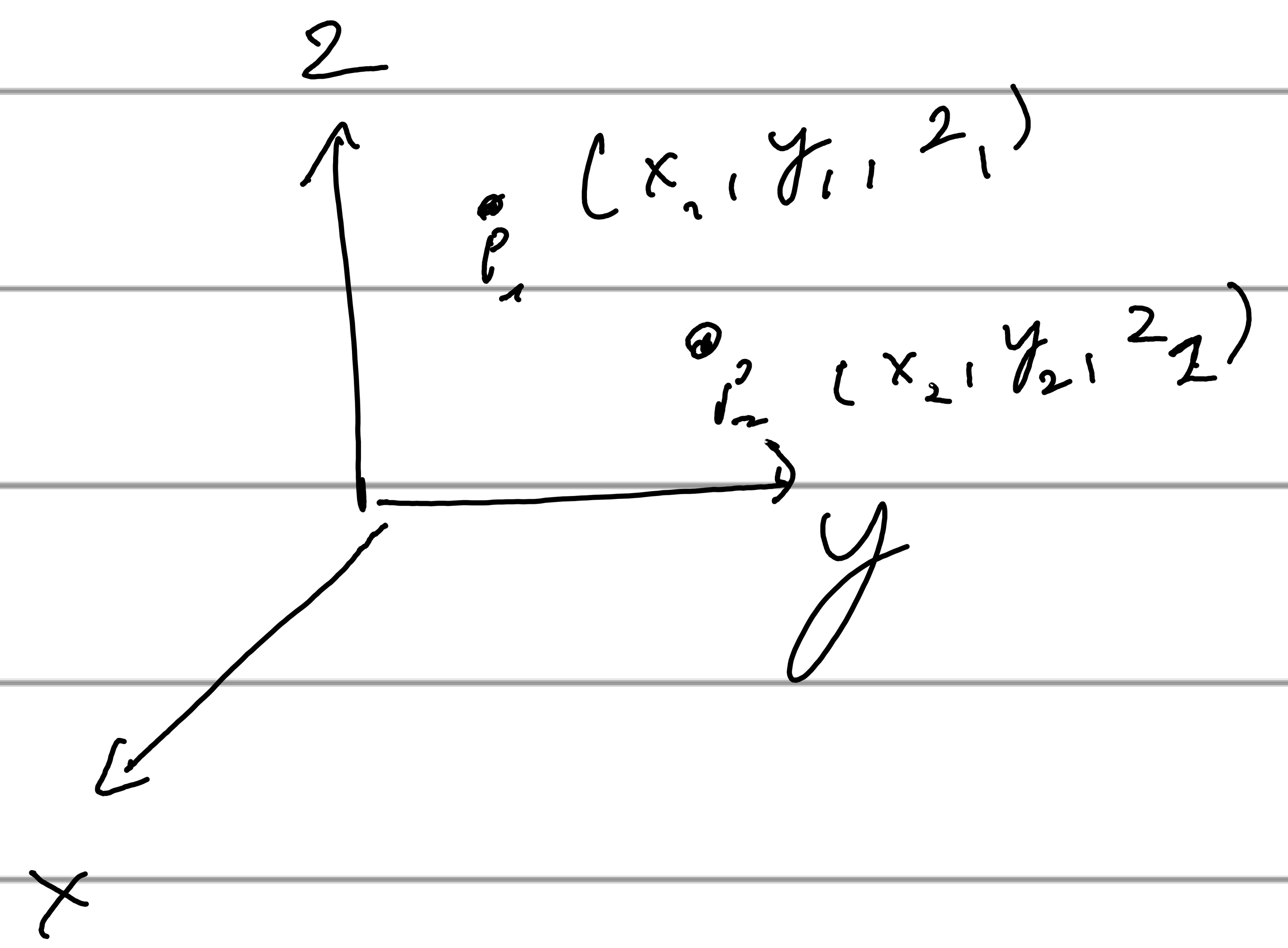
Hfst 3 : Vectorfuncties

$\mathbb{R} \to \mathbb{R}^2$ of $\mathbb{R}^3$ (in ruimte)

(in vlak)

$$\vec{r}(t) = (x(t), y(t), z(t))$$
$$= x(t)\vec{1}_x + y(t)\vec{1}_y + z(t)\vec{1}_z$$

Vb) rechte par voorstelling

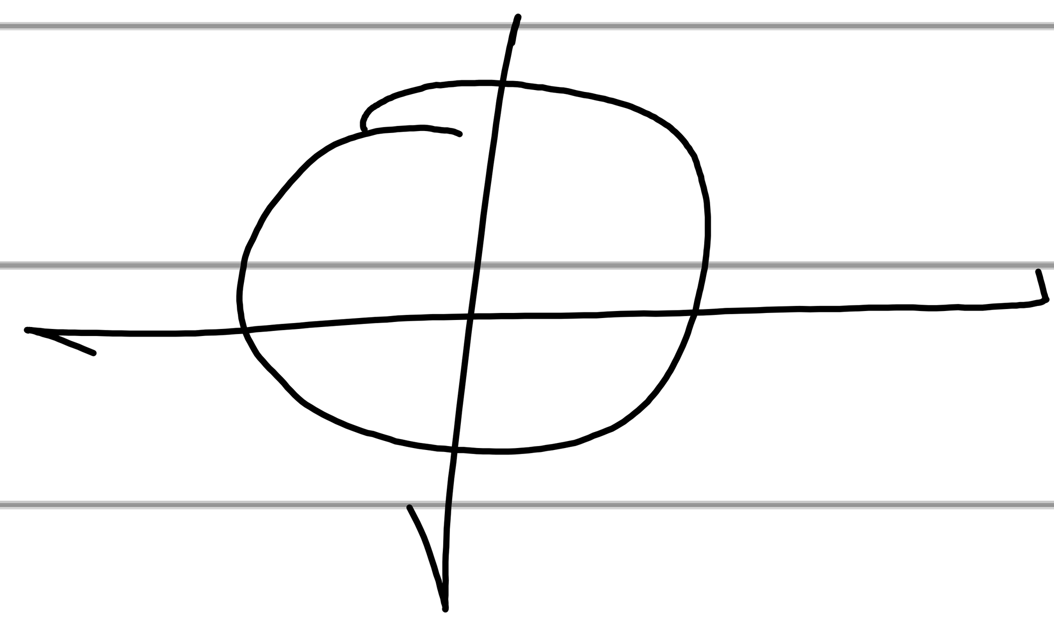


$$x(t) = x_1 + t(x_2 - x_1)$$
$$y(t) = y_1 + t(y_2 - y_1)$$
$$z(t) = z_2 + t(z_2 - z_1)$$

Vb) rechte door $P(x_0, y_0, z_0)$ en richting $\vec{v}(v_1, v_2, v_3)$

$$\left.\begin{aligned} x &= x_0 + t v_1 \\ y &= y_0 + t v_2 \\ z &= z_0 + t v_3 \end{aligned}\right\} \to \vec{r}(t) =$$

Krommen en raaklijn



$x = 2\cos t$

$y = 2\sin t$

vectorfunctie $\quad r(t) = 2\cos(\vec{1_x}) + 2\sin(\vec{1_y})$

$$= x(t)\,\vec{1}_x + y(t)\,\vec{1}_y$$

$$= (x(t), y(t))$$

$$= (2\cos t, 2\sin t)$$

Limiet

$$\lim_{t\to t_0} r(t) = L = (-,-,-) = \lim_{t\to t_0} x(t), \lim_{t\to t_0} y(t) \text{ etc.}$$

Helix $(\cos t, \sin t, t) = r(t)$

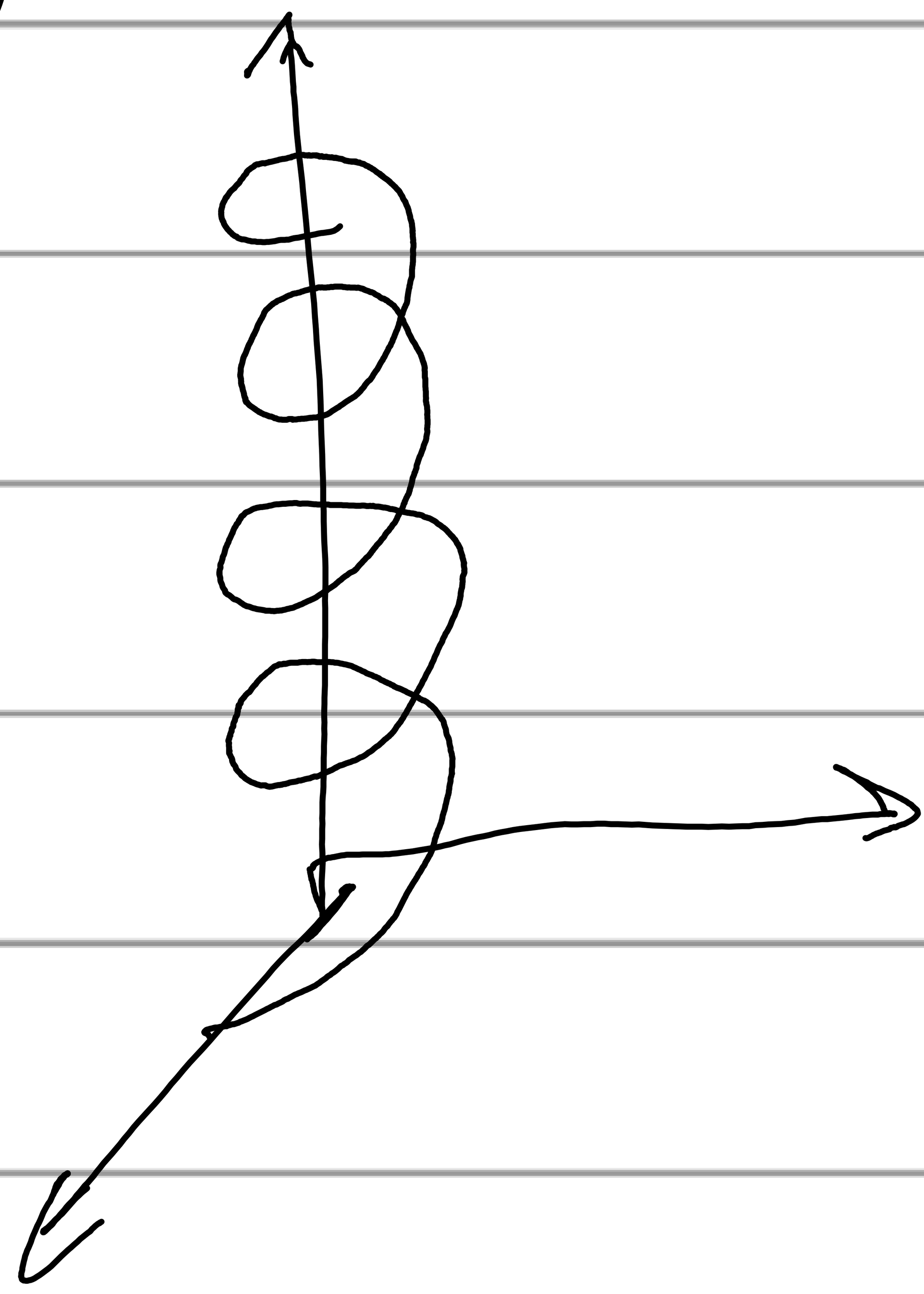
$$\lim_{t\to\frac{\pi}{4}} \Big(\underbrace{\lim_{t\to\frac{\pi}{4}} \cos t}_{=\frac{\sqrt{2}}{2}}, \underbrace{\lim_{t\to\frac{\pi}{4}} \sin\frac{\pi}{4}}_{=\frac{\sqrt{2}}{2}}, \underbrace{\lim_{t\to\frac{\pi}{4}} \frac{\pi}{4}}_{=\frac{\pi}{4}}\Big)$$

Continuïteit

→ continue als limiet in elk punt bestaat

→ $\lim\limits_{t \to t_0} \vec{r}(t) = \vec{r}(t_0)$: continue functie

Ve

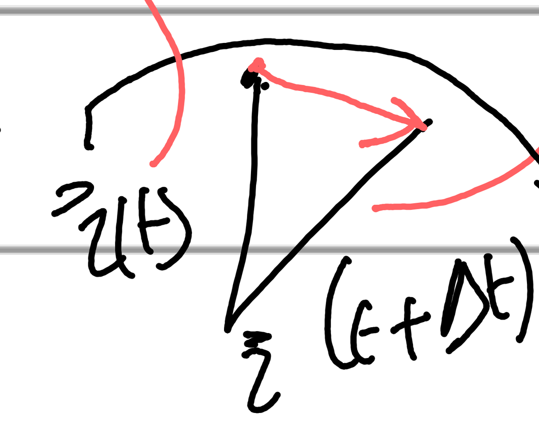


$x = 2\cos t$

$y = 2\sin t$

$z = t$

Vaste def $\dfrac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$



afgeleide

$\vec{r}\,'(t) = \dfrac{d\vec{r}}{dt} = \vec{v} = (x'(t), y'(t), z'(t))$ $\quad r'(t) \perp r(t)$

$\boxed{|\vec{v}| = \sqrt{x'^2 + y'^2 + z'^2}}$ (grootte snelheidsvector)

$\int \vec{r}(t)\,dt = \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt + C$

Nodig om Booglengte te berekenen

s is integraal over gans stuk

Vb integraal

$$\int (\cos t, 1, -2t)\, dt = (\sin t + c_1, t + c_2, -t^2 + c_3)\, dt$$

stel $\int_0^{\pi} \Rightarrow (0, \pi, -\pi^2)$

Fysisch

$\vec{r}(t)$ = afgelegde weg

$\vec{v}\, t = \frac{d\vec{r}}{dt}$

$|\vec{v}(t)| = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} \Leftrightarrow \vec{r}(t) = \int \vec{v}(t)\, dt$

$\vec{a} = \frac{d^2 r}{dt^2}$ = Versnelling

ider in naa a naar v

Vb $\vec{a}(t) = (-3\cos t - 3\sin t, 2)$

$\vec{v}(0) = (0, 3, 0)$

$\vec{r}(0) = (3, 0, 0)$

Gevr. $\vec{r}(t)$

$\vec{v}(t) = \int \vec{a}(t) \cdot dt = (-3\sin t, 3\cos t, t) + (C_1, C_2, C_3)$

$\vec{v}(0) = (0, 3, 0)$ $(C_1, 3 + C_2, C_3)$

$= 0 = C_3$

$= 0 = C_2$

$= 0 = C_1$

$\vec{v}(t) = (-3\sin t, 3\cos t, 2t)$

$\vec{r}(t) = \int \vec{v}(t)\, dt = (3\cos t, 3\sin t, t^2) + (C_1, C_2, C_3)$

$\vec{r}(0)$ constantes bepalen ... $= 0$

dus $\vec{r}(t)$ staat hierboven ...

$$\boxed{s = \int \underbrace{\sqrt{x'^2+y'^2+z'^2}}_{|\vec{v}|}\, dt}$$

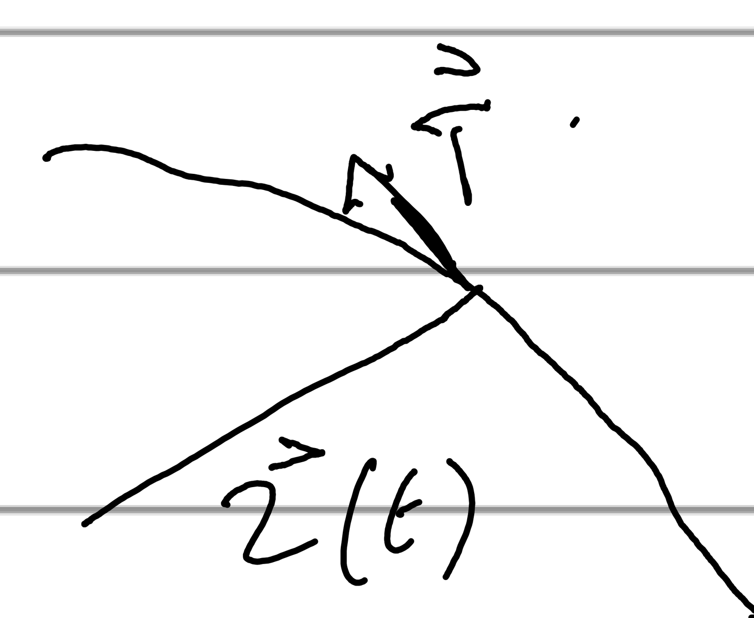
$$= \int |\vec{v}|\, dt \iff |\vec{v}| = ds/dt$$

$$\boxed{\vec{T} = \vec{1}_t \text{ eenheidsvector} = \frac{\vec{v}}{|\vec{v}|}} \quad (\text{Praktijk})$$

$$\vec{T} = \frac{d\vec{r}/dt}{ds/dt} = \frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|} = \vec{1}_t = \vec{T} \quad (\text{theorie})$$

$$\text{Kromming } K = \left|\frac{d\vec{T}}{ds}\right| \quad (\text{hoe krom is het})$$

$$= \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \boxed{\frac{1}{v}\left|\frac{d\vec{T}}{dt}\right|}$$



Kromming rechte $= 0$

Kromming cirkel $= \dfrac{1}{r}$

(Ur) helix $r(t) = (\cos t, \sin t, t)$

$\vec{T}$?

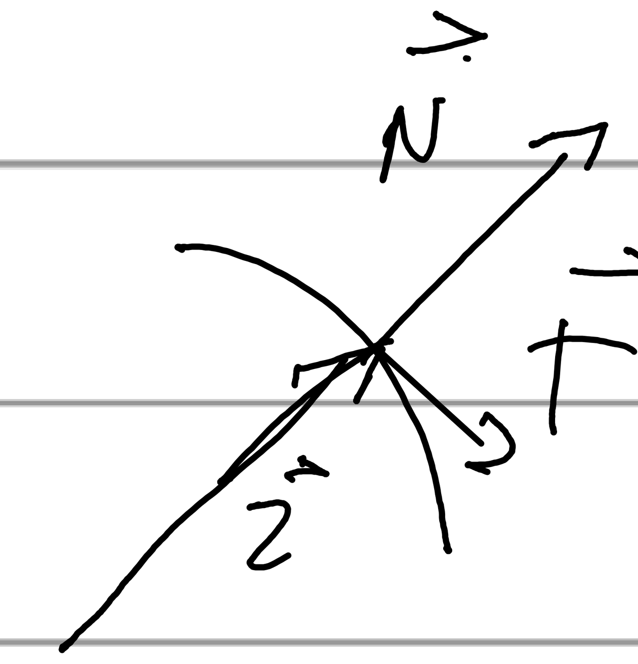
$\vec{v}(t) = (-\sin t, \cos t, 1))$

$|\vec{v}t| = \sqrt{\underbrace{-\sin^2 t + \cos^2 t}_{1} + 1^2} = \sqrt{2}$

$\vec{T} = \left(\frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Eenheidsnormaalvector

$\vec{N} = \frac{d\vec{T}}{ds}\,\frac{1}{K}$ (theorie)



$$\frac{\dfrac{d\vec{T}/dt}{\cancel{ds/dt}}}{\left|\dfrac{d\vec{T}/dt}{\cancel{ds/dt}}\right|} = \boxed{\frac{d\vec{T}/dt}{|d\vec{T}/dt|}} = \frac{d\vec{T}/ds}{\left|d\vec{T}/ds\right|}$$

Vb

$\vec{r} = (a\cos t, a\sin t, \cdot$

$\vec{v} = (-a\sin t, a\cos t, b)$

$|\vec{v}| = \sqrt{(a^2\sin^2 t + a^2\cos^2 t + b^2)} = \sqrt{a^2+b^2}$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t, a\cos t, b)$

$\frac{dT}{dt} = \frac{1}{\sqrt{a^2+b^2}}(-a\cos t, -a\sin t, 0)$

$\left|\frac{dT}{dt}\right| = \frac{a}{\sqrt{a^2+b^2}}$

Kromming: κ mate van kromming

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}/dt}{\underbrace{ds/dt}}\right| = \boxed{\frac{1}{|\vec{V}|}\left|\frac{d\vec{T}}{dt}\right|}$$

$\hookrightarrow ds/dt = |\vec{V}|$

Vb

rechte

$$\vec{r} = (x_0 + v_1 t, y_0 + v_2 t, z_0 + v_3 t)$$

$$\vec{v} = (v_1, v_2, v_3)$$

$$T = \frac{(v_1, v_2, v_3)}{\sqrt{v_1^2 + v_2^2 + v_3^2}}$$

$$\frac{dT}{dt} = (0, 0, 0) \longrightarrow = 0 \text{ dus } \kappa = 0$$

$\hookrightarrow$ allemaal constanten

(Vb) cirkel met straal a

$$\vec{r}(t) = (a\cos t, a\sin t)$$

$$\vec{v}(t) = (-a\sin t, a\cos t)$$

$$\|\vec{v}(t)\| = \sqrt{(\ )^2 + (\ )^2} = a$$

$$T = \frac{\vec{v}}{|\vec{v}|} = (-\sin t, \cos t)$$

$$\frac{dT}{dt} = (-\cos t, -\sin t) \Rightarrow \left|\frac{dT}{dt}\right| = \sqrt{(\ )^2 + (\ )^2} = 1$$

$$\kappa = \underset{\overset{\|}{a}}{\frac{1}{|\vec{v}|}} \underset{\overset{\|}{1}}{\left|\frac{dT}{dt}\right|} = \kappa = \frac{1}{a} \Rightarrow \text{constant in elk punt}$$

$$\varkappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{\sqrt{a^2+b^2}} \cdot \frac{a}{\sqrt{a^2+b^2}}$$

$$\vec{N} = \frac{dT/dt}{\left|dT/dt\right|} = \frac{\frac{1}{\cancel{\sqrt{a^2+b^2}}}(-\cancel{a}\cos t, -\cancel{a}\sin t, 0)}{\frac{\cancel{a}}{\cancel{\sqrt{a^2+b^2}}}}$$

Bij cirkel

$$\vec{r} = (a\cos t, a\sin t)$$

$$\vec{v} = (-a\sin t, a\cos t)$$

$$|\vec{v}| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$$

$$T = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{a}(-a\sin t, a\cos t) - (-\sin t, \cos t)$$

$$\varkappa = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{a}\,1$$

$$\vec{N} = \frac{d\vec{T}/dt}{|dT/dt|} = (-\cos t, -\sin t)$$

Vb

$$x = 1 + 2t$$
$$y = -1 - t$$
$$z = 2 + 3t$$

$$\vec{r} = (1+2t, -1-t, 2+3t)$$

$$\vec{v} = (2, -1, 3)$$

$$|\vec{v}| = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$$

$$\vec{T} = \frac{1}{\sqrt{14}} (2, -1, 3)$$

$$\frac{d\vec{T}}{dt} = (0, 0, 0) \rightarrow \kappa = 0$$: kromming overal = 0

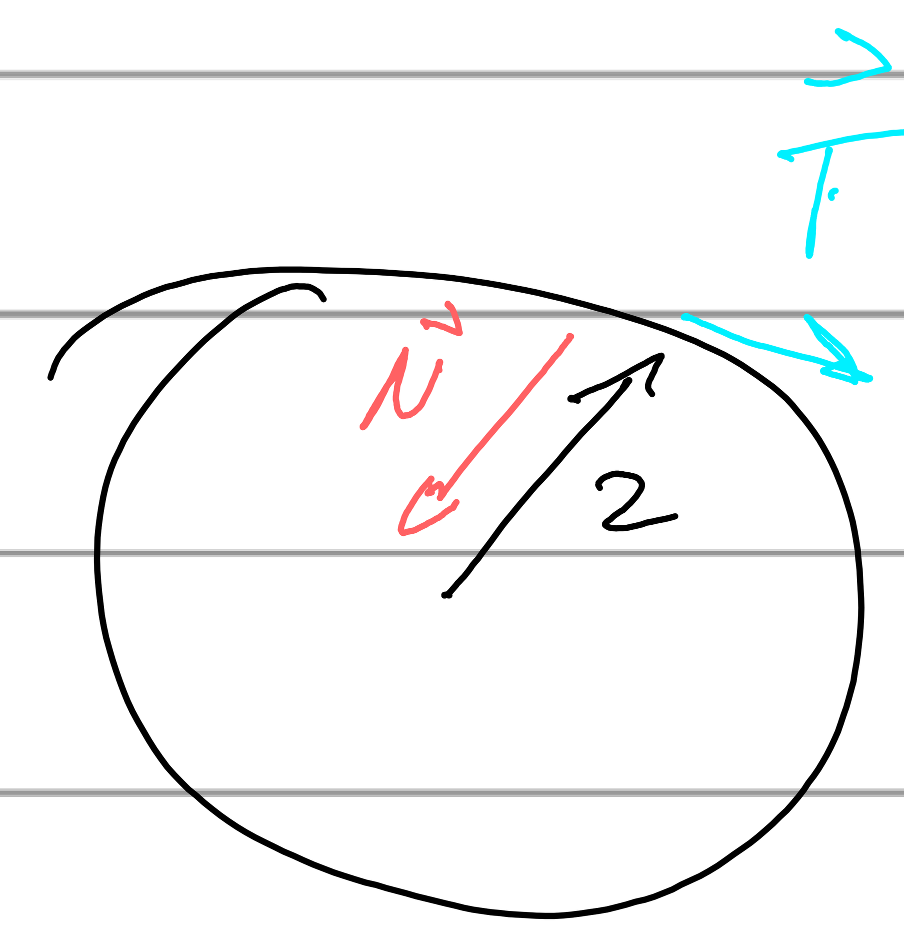
Cirkel: normalvect

$$\vec{r}(t) = (a\cos t, a\sin t)$$

$$\vec{T}(t) = (-\sin t, \cos t)$$

$$\frac{d\vec{T}}{dt} = (-\cos t, -\sin t) \qquad \left|\frac{d\vec{T}}{dt}\right| = 1$$

$$\vec{N} = (-\cos t, -\sin t)$$



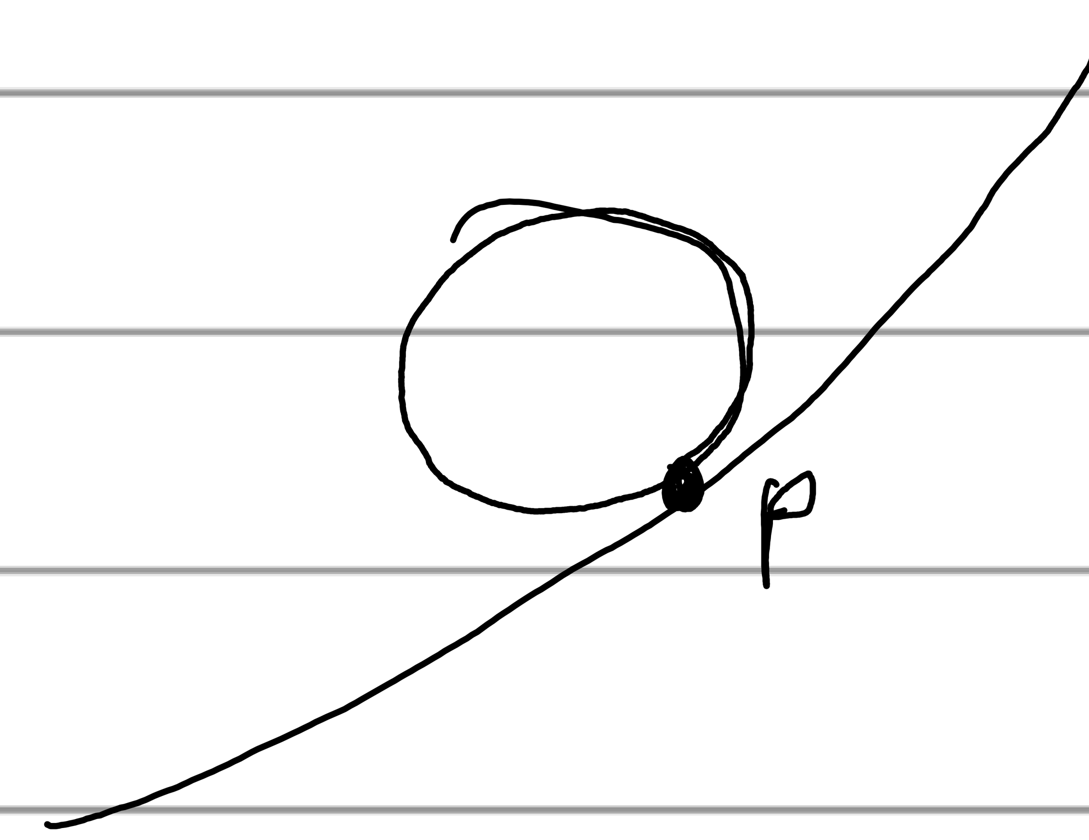
Kromming cirkel in P aan kromme $\vec{r}(t)$

→ raakt aan kromme in punt P

→ ligt langs binnenkant

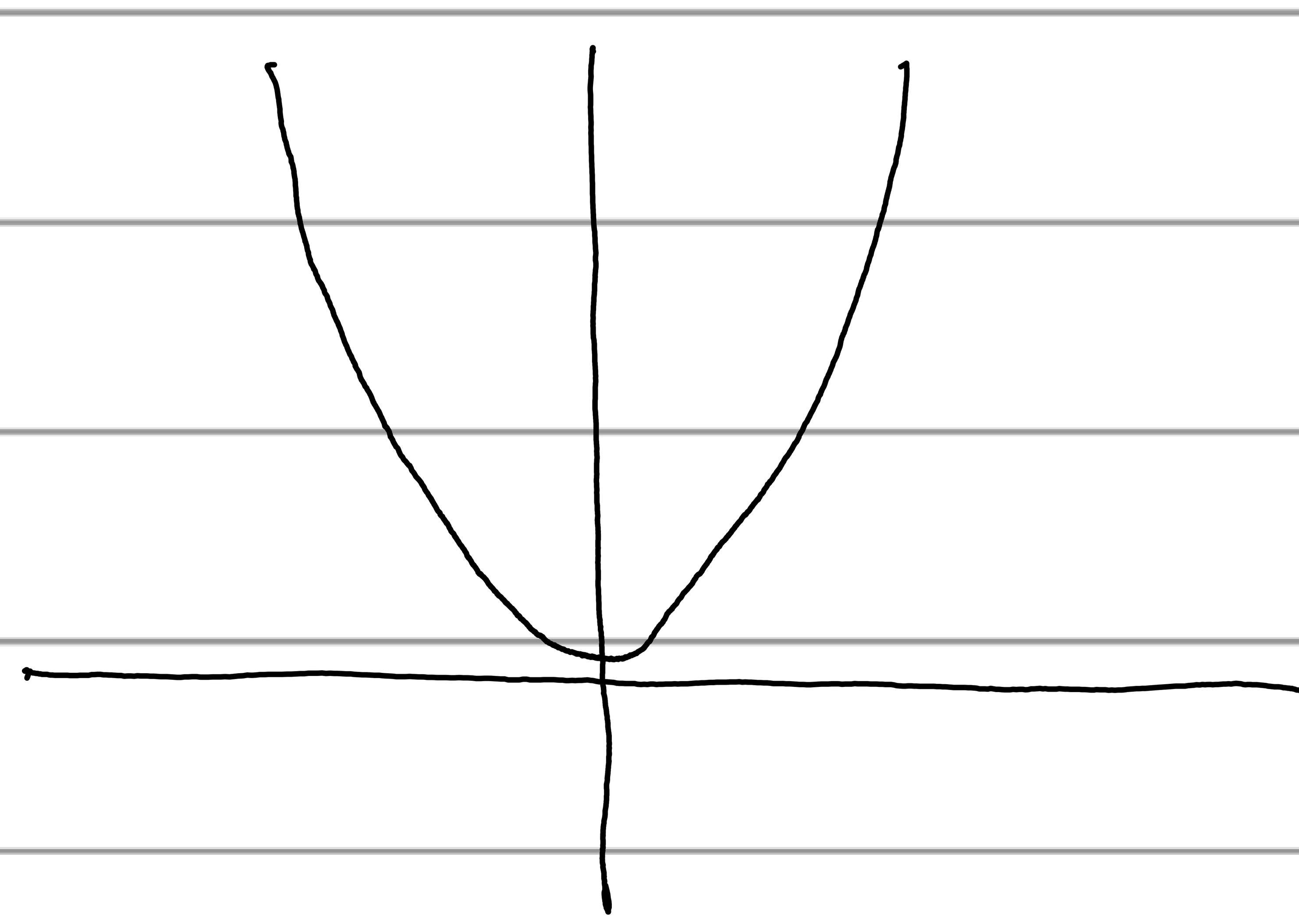
→ cirkel heeft zelfde kromming als kromme

$\Rightarrow$ Straal $= \dfrac{1}{K}$



Vb $y = x^2 \xrightarrow{\text{naar parameter}} \begin{cases} x = t \\ y = t^2 \end{cases}$

Bepaal $\vec{r}, \vec{v}, \vec{T}, K, N$



$\vec{r} = (t, t^2)$

$\vec{v} = (1, 2t)$

$|\vec{v}| = \sqrt{1+4t^2}$

$\vec{T} = \dfrac{(1, 2t)}{\sqrt{1+4t^2}}$

K voor $\begin{cases} t = 0 \\ t = 1 \\ t = 2 \end{cases}$

$t = 0 \rightarrow$ $|v| = 1$

$$\frac{dT}{dt} = (0, 2)$$

$$\kappa = \frac{1}{1} \cdot 2 = 2$$

$t = 1 \rightarrow$ $|v| = \sqrt{5}$

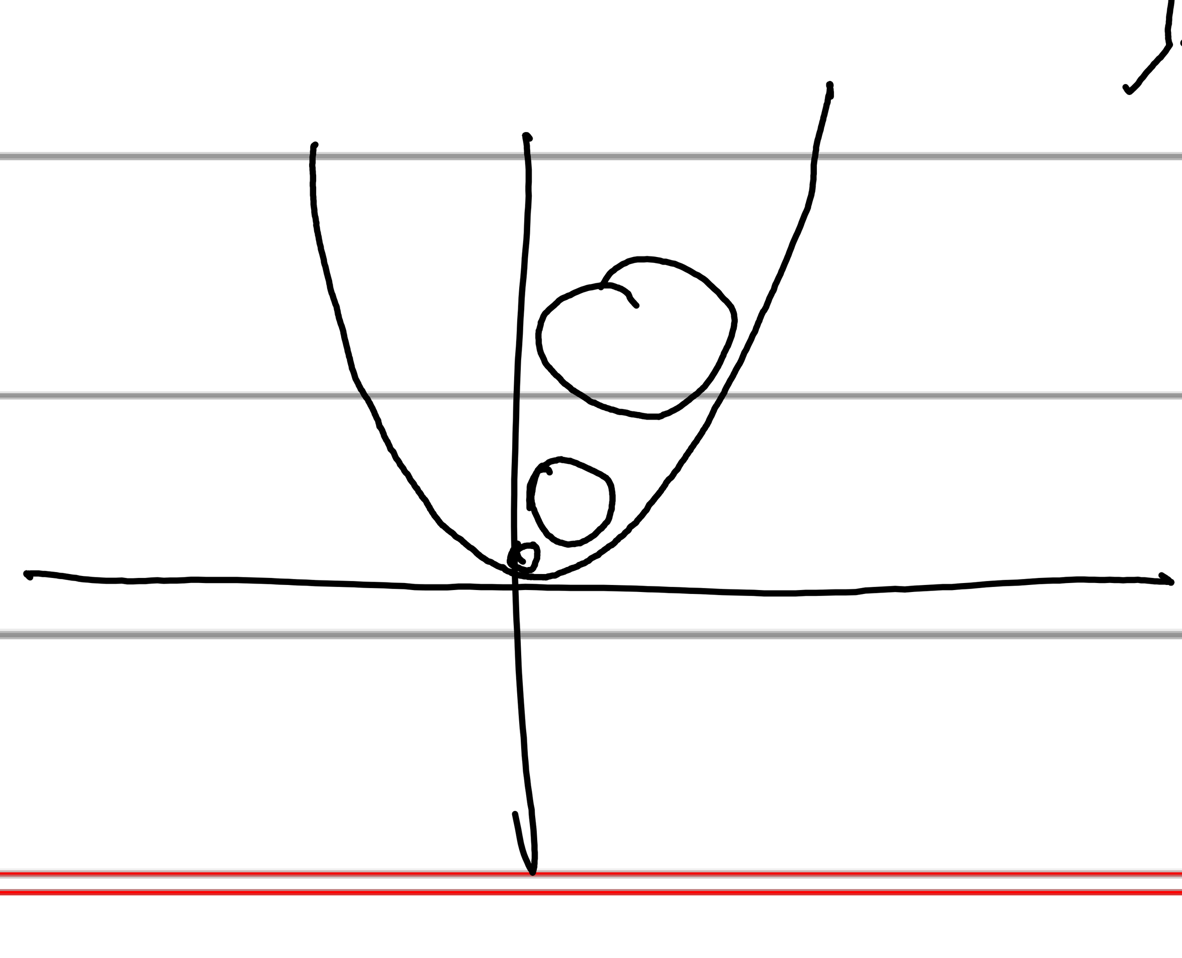
$$\frac{dT}{dt} = (0, 2)$$

$\kappa = \frac{2}{\sqrt{5}}$ of straal $= \frac{\sqrt{5}}{2} \approx 1$

$t = 2$

$|v| = \sqrt{17}$

$\kappa = \frac{2}{\sqrt{17}}$ dus straal $\frac{\sqrt{17}}{2} \approx 2$



(Vb) $\vec{N}$ bij helix

$$z(t) = (a\cos t, a\sin t, bt)$$

$$v(t) = (-a\sin t, a\cos t, b)$$

$$\sqrt{-a\sin t^2 + a\sin t^2 + b^2} = \sqrt{a^2+b^2}$$

$$T = \frac{(-a\sin t, a\cos t, b)}{\sqrt{a^2+b^2}}$$

$$\frac{d\vec{T}}{dt} = \frac{(-a\cos t, -a\sin t, b)}{\sqrt{a^2+b^2}}$$

$$\left|\frac{d\vec{T}}{dt}\right| = \frac{a}{\sqrt{a^2+b^2}}$$

$$\vec{N} = \frac{\frac{1}{\cancel{\sqrt{a^2+b^2}}}(-\cancel{a}\cos t, -\cancel{a}\sin t, 0)}{\frac{\cancel{a}}{\cancel{\sqrt{a^2+b^2}}}} = (-\cos t, -\sin t, 0)$$