Scaling laws, from Perceptrons to Deep networks

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Outline of the talk

1. Review on neural scaling law
   - Empirical findings on neural scaling laws
   - Two models to predict power-laws exponents
   - Discussion (1^o part)

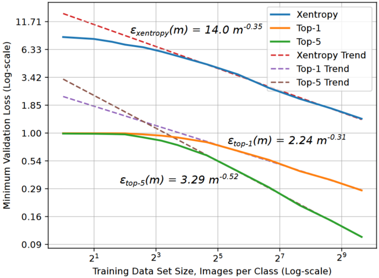
2. Our results (with Dario Bocchi and Matteo Negri)
   - Simple perceptron model
   - Experiments on deep networks
   - Discussion (2^o part)

Part IA: Empirical findings

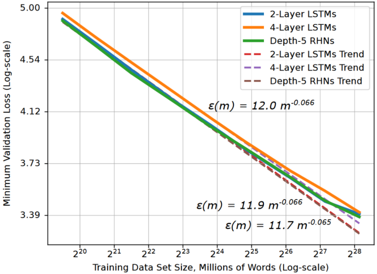
- What is meant by "neural scaling laws"
- Why they motivated large scale LLMs like GPT-3/4
- How can be used to optimize compute cost

Hestness et al (2017):
Deep Learning Scaling is Predictable, Empirically

ResNet, image classification



LLM, next word prediction



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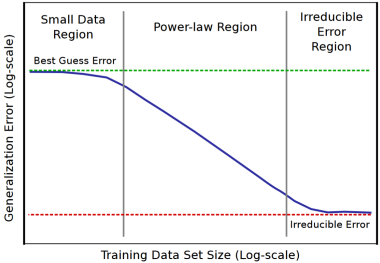

Power law region in the intermediate regime:

$$\mathscr{L} \sim cP^{-\gamma}$$

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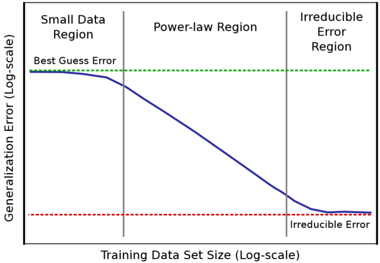

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Empirical properties of curves for model tested:

- Power laws in all domains tested
- Within each domain, model architectures mainly changes the constant not the exponent
- Same for optimizers (SGD, Adam ..)

Rosenfeld et al. (2020): A Constructive Prediction of the Generalization Error Across Scales

With P number of data and N number of parameters, two separate scaling laws:

$$\varepsilon(N,P) \approx \begin{cases} aP^{-\alpha} + c_P(N) & \text{(data scaling at fixed model)} \\ bN^{-\beta} + c_N(P) & \text{(model scaling at fixed dataset)} \end{cases}$$

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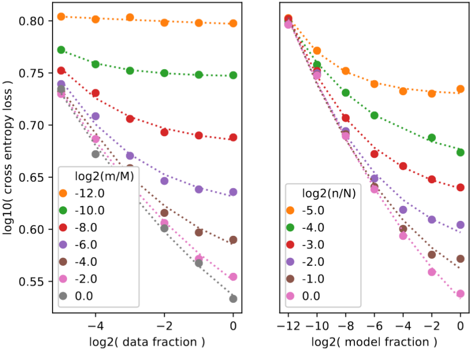

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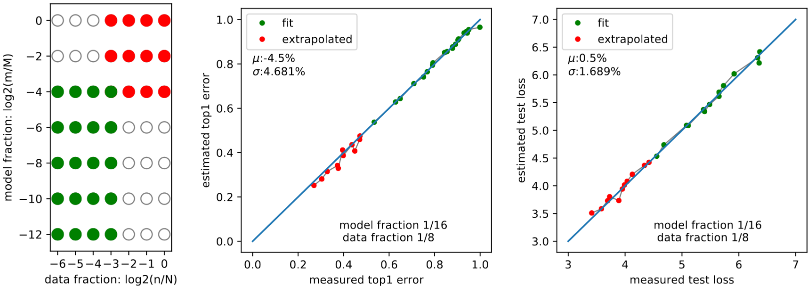

(a) Illustration. (b) Extrapolation on ImageNet (c) Extrapolation on WikiText-103.

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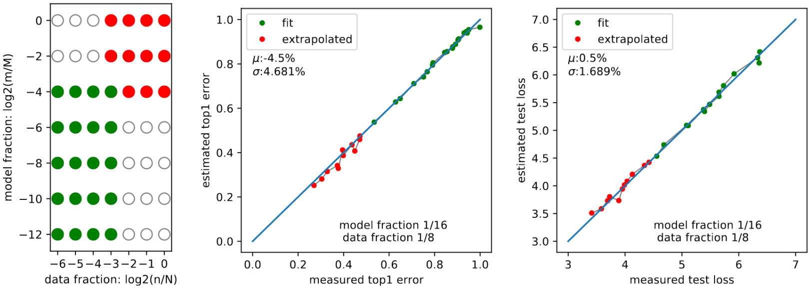

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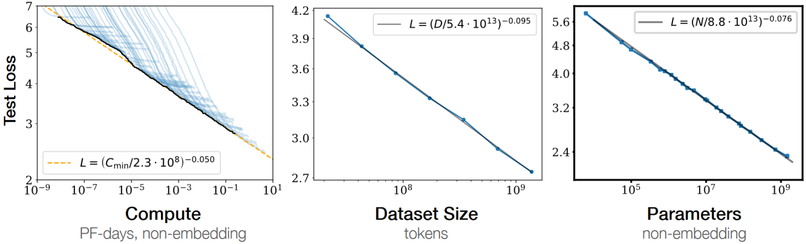
$\Rightarrow$ small P, N models capable of predicting large P, N models

Kaplan et al (2020): Scaling laws for neural language models

Almost perfect scaling laws in GPT models across many magnitudes



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- Large models are more sample-efficient than small models: same performance with fewer datapoints
- Given a fixed compute budget C, best strategy $\Rightarrow$ very large model stopped very short of convergence

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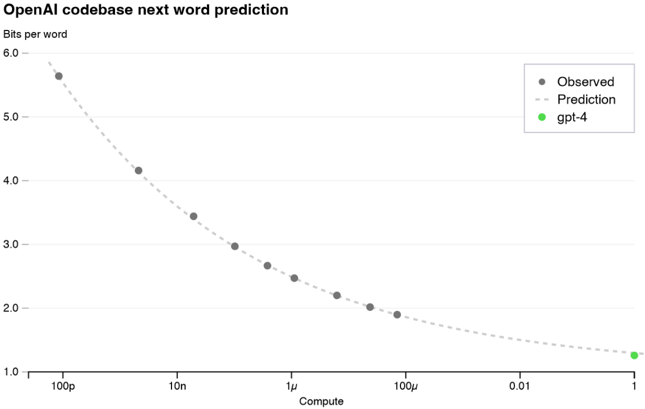
All those results motivated extreme P, N scaling $\Rightarrow$ GPT-3/4 models



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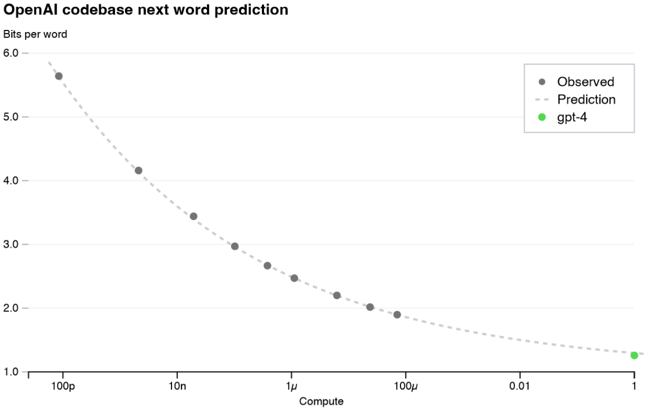

Smaller models fit predicted GPT-4 loss

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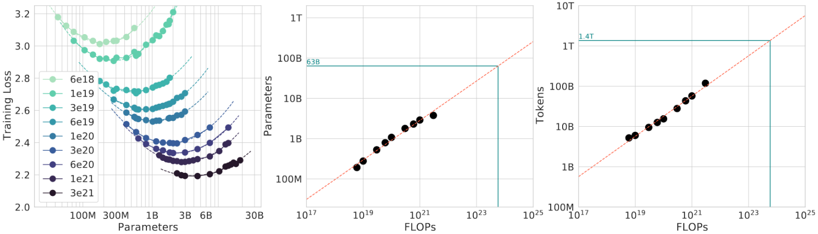

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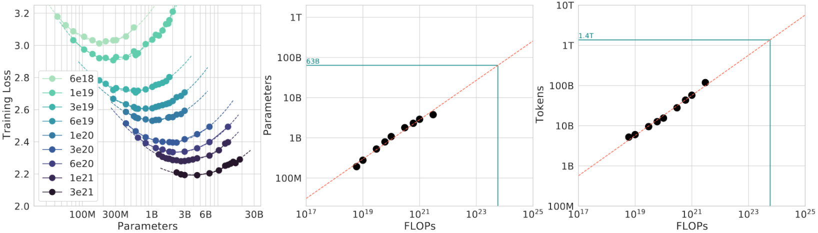

They found $P_{opt}(C), N_{opt}(C)$ both $\sim C^{0.5}$

Summary of empirical results

1. Loss/error scales as $\varepsilon(N,P) = aP^{-\alpha} + bN^{-\beta} + c_\infty$
2. Exponents robusts wrt most of details of training and architectures
3. Exponents found $\in [0.05, 0.5]$
4. Best strategy given a compute C to scale $P, N \sim C^{0.5}$

Part IB: Two attempts to explain exponents: geometric bounds and DMFT models

Idea:

Bahri et al. (2021): Explaining Neural Scaling Laws

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Case $\mathscr{L}(P) - \mathscr{L}(\infty) = \Delta(P)$:

1. **Underparametrized** ($P \gg N \gg 1$): **variance** dominates
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Case $\mathscr{L}(N) - \mathscr{L}(\infty) = \Delta(N)$:

1. **Overparametrized** ($N \gg P \gg 1$): **variance** dominates
   $\Delta(N) \sim c_{\text{var}} N^{-1}$ ($N^{-1/2}$ deep case) (infinite limit + corrections)
2. **Underparametrized** ($P \gg N \gg 1$): **bias** dominates
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Assuming:

- Compact d-dimensional hidden manifold of data
- Teacher-student case: $y = F(x)$ and $\hat{y} = f(x)$
- Both F, f are smooth

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Why $-1/d$ exponents? Arguments for *bounds*

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- Student features $f_\mu \in P$-dimensional subspace of teacher features

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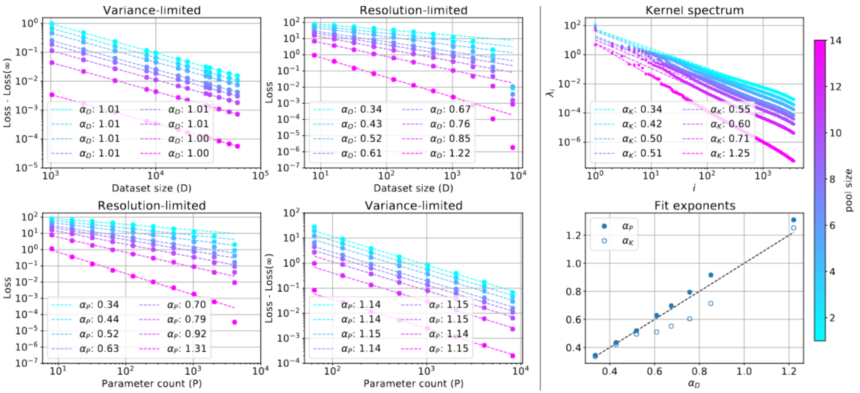
3. They show $\alpha_K \sim 1/d$

Bahri et al. (2021): Explaining Neural Scaling Laws

Result: linear random features



Bordelon et al. (2024): A Dynamical Model of Neural Scaling Laws

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$$y(\mathbf{x}) = \frac{1}{\sqrt{M}} \mathbf{w}^* \cdot \psi(\mathbf{x}) + \sigma \varepsilon(\mathbf{x})$$

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- Student is a lower-dimensional projection of features $\mathbf{A}\psi(\mathbf{x})$ where $\mathbf{A} \in \mathbb{R}^{N \times M}$, A_{ij} i.i.d.

$$f(\mathbf{x}) = \frac{1}{\sqrt{N}} \mathbf{w} \cdot \mathbf{A}\psi(\mathbf{x})$$

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Assumption: power-law features + data

1. Given $\langle \psi_k(\mathbf{x}) \psi_l(\mathbf{x}) \rangle_{\mathbf{x} \sim p(\mathbf{x})} = \delta_{kl} \lambda_k$

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   - $(\omega_k^*)^2 \lambda_k$ controls generalization error per mode
   - Large a $\Rightarrow$ target error concentrated in first modes $\Rightarrow$ easy task

Bordelon et al. (2024): A Dynamical Model of Neural Scaling Laws

DMFT results

(1) *Bottleneck scalings*

$$\mathscr{L}(t,P,N) \approx \begin{cases} t^{-\frac{a-1}{b}}, & P,N \to \infty \quad \text{(Time)}, \\ P^{-\min\{a-1,2b\}}, & t,N \to \infty \quad \text{(Data)}, \\ N^{-\min\{a-1,2b\}}, & t,P \to \infty \quad \text{(Model)}. \end{cases}$$

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(2) Compute optimal

- Using typical cases $\min\{a-1,\, 2b\} = a-1$, compute optimal

$$\boxed{t \sim C^{\frac{b}{1+b}},\ N \sim C^{\frac{1}{1+b}}}$$

$\Rightarrow$ different scalings for time and data/size

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- $\boxed{\mathscr{L}_{\text{opt}}(C) \sim C^{-\frac{a-1}{1+b}}}$

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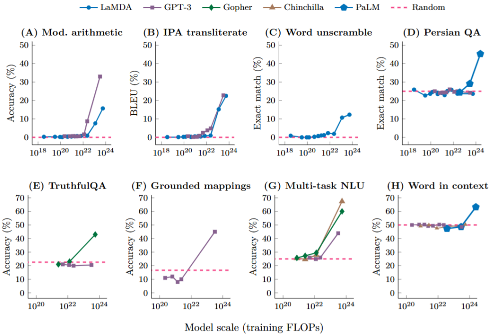
2. Different (complicated) tasks produce "phase-transitions"
   Wei et al., (2022): Emergent Abilities of Large Language Models



References

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2. Rosenfeld et al. (2020): A Constructive Prediction of the Generalization Error Across Scales
3. Kaplan et al (2020): Scaling laws for neural language models
4. Bahri et al. (2021): Explaining Neural Scaling Laws
5. Hoffmann et al. (2022): Training Compute-Optimal Large Language Models
6. Maloney et al. (2022): A Solvable Model of Neural Scaling Laws
7. Wei et al., (2022): Emergent Abilities of Large Language Models
8. Bordelon et al. (2024): A Dynamical Model of Neural Scaling Laws
9. Bordelon et al. (2025) How Feature Learning Can Improve Neural Scaling Laws
10. Defilippis et al. (2025) Scaling Laws and Spectra of Shallow Neural Networks in the Feature Learning Regime

Part II: Our work

Implicit bias produces neural scaling laws in learning curves, from perceptrons to deep networks

Francesco D'Amico[1,2*], **Dario Bocchi**[1,2*], **Matteo Negri**[1,2]
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Outline:

Part II: Our work

Implicit bias produces neural scaling laws in learning curves, from perceptrons to deep networks

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1. We show two new scalings laws in a simple Perceptron model

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1. We show two new scalings laws in a simple Perceptron model
2. These new laws combined reproduce $\varepsilon \sim P^{-\gamma}$ scaling law
3. Valid empirically for Deep Nets in real image classification

Perceptron model

- *Student* perceptron $\boldsymbol{w} \in \mathbb{R}^N$, *Teacher* perceptron $\boldsymbol{w}^* \in \mathbb{R}^N$

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where *margins*

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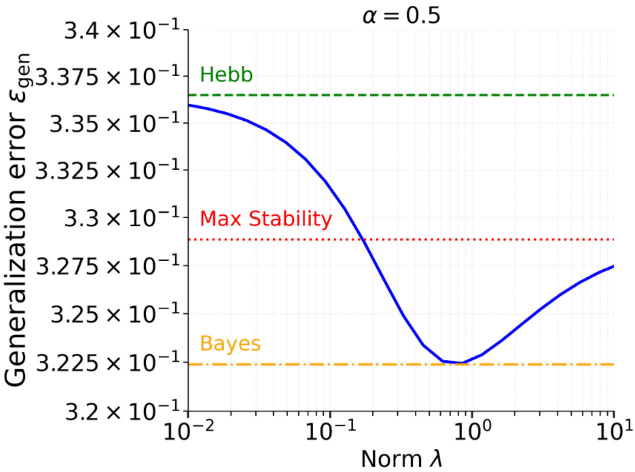
- λ reabsorbed in norm of weights: $\|\boldsymbol{w}^*\|^2 = \|\boldsymbol{w}\|^2 = \lambda N$

Solution at fixed α interpolates known learning rules



Unbounded norm perceptrons $\approx$ fixed-norm

- Norm $\lambda(t)$ increases monotonically for GD, Soudry et al., (2018)

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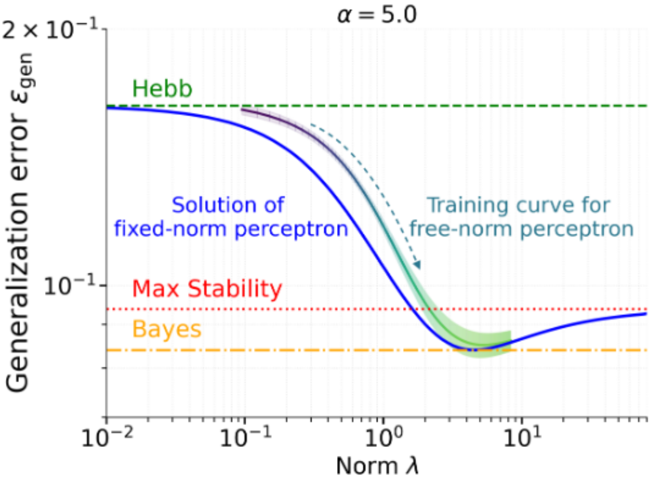
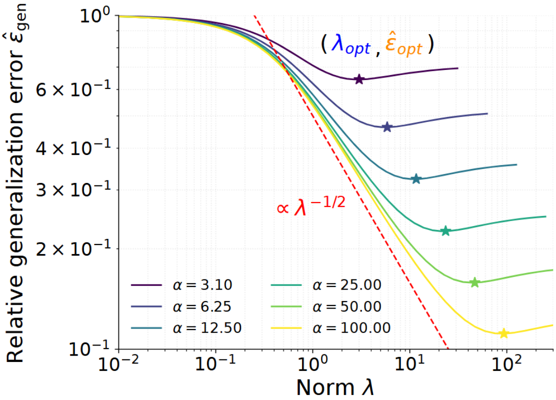


Result (1): Two new scaling laws in λ

Relative error $\hat{\varepsilon}_{\text{gen}} \equiv \varepsilon_{\text{gen}}/\varepsilon_0$, where $\varepsilon_0 = \varepsilon(\lambda = 0)$



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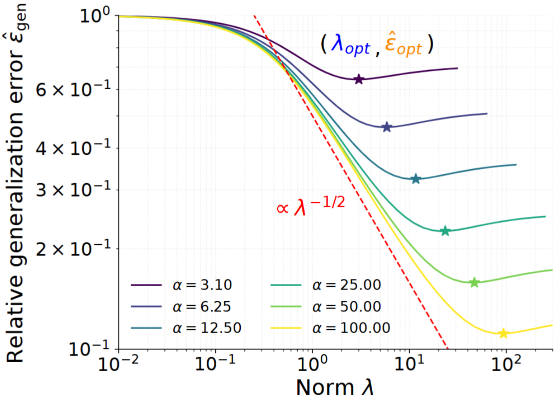

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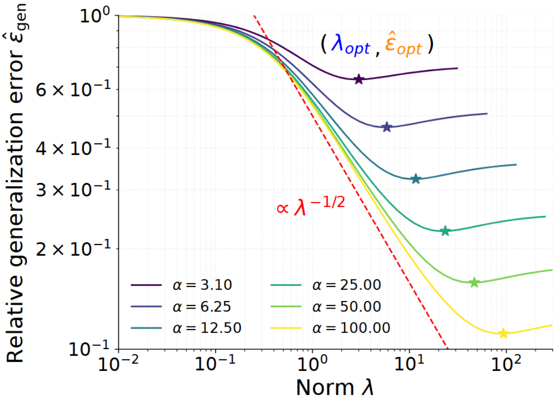
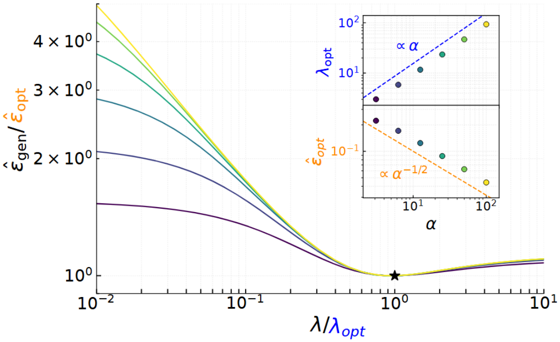


1. Early training ($\lambda < \lambda_{elbow}(\alpha)$) $\rightarrow$ $\hat{\varepsilon}_{\text{gen}} \sim k_1 \lambda^{-\gamma_1}$
2. Optima of curves ($\lambda > \lambda_{elbow}(\alpha)$) $\rightarrow$ $\lambda_{\text{opt}} \sim k_2 \alpha^{\gamma_2}$

Result (2): collapse on a master curve Φ

Define the rescaling $\hat{\varepsilon}_{\text{gen}}/\hat{\varepsilon}_{\text{opt}} = \Phi_\alpha(\lambda/\lambda_{\text{opt}})$

Curves converge to a master curve for $\alpha >> 1$: $\Phi_\alpha \to \Phi$



Result (3): predict neural scaling law

1. $\hat{\varepsilon}_{\text{gen}} \sim k_1 \lambda^{-\gamma_1}$ for $\lambda < \lambda_{elbow}(\alpha)$
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3. $\hat{\varepsilon}_{\text{gen}} / \hat{\varepsilon}_{\text{opt}} = \Phi(\lambda / \lambda_{\text{opt}})$ for $\alpha \gg 1$

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Does the theory also apply to deep networks?

Architectures:

- Convolutional Neural Networks (CNN)
- Residual Neural Networks (ResNet)
- Vision Transformers (ViT)

Datasets:

- MNIST (greyscale digits, 10 classes)
- CIFAR10 (RGB images, 10 classes)
- CIFAR100 (RGB images, 100 classes)

Norm in deep networks: Bartlett et al. (2017) Spectrally-normalized margin bounds for neural networks

Spectral Complexity norm for a L-layer deep net with matrices A_i:

- ρ_i Lipschitz constant of layer i activation function
- $\|\cdot\|_\sigma$ biggest singular value (spectral norm)
- $\|\cdot\|_{2,1}$ sum of ℓ_2 norms of columns
- M_i reference matrix (can be $= \mathbf{0}$)

$$R_A = \left(\prod_{i=1}^{L} \rho_i \|A_i\|_\sigma\right) \left(\sum_{i=1}^{L} \frac{\|A_i^\top - M_i^\top\|_{2,1}^{2/3}}{\|A_i\|_\sigma^{2/3}}\right)^{3/2}$$

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Maximum expansion **Effective rank**

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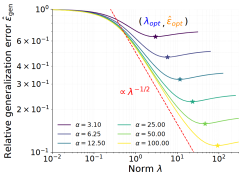

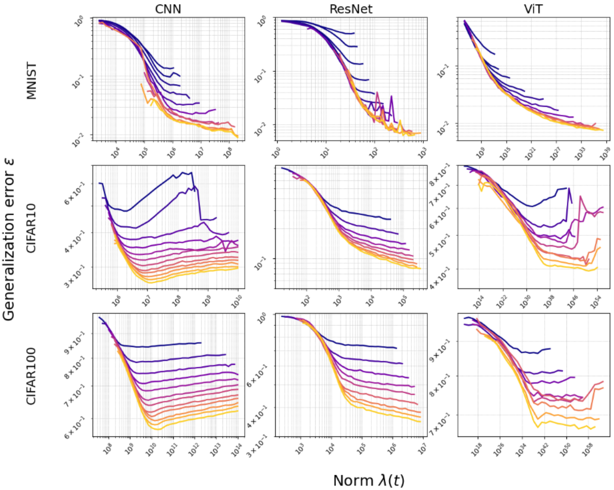

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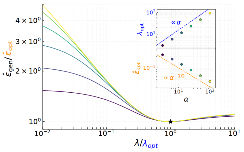

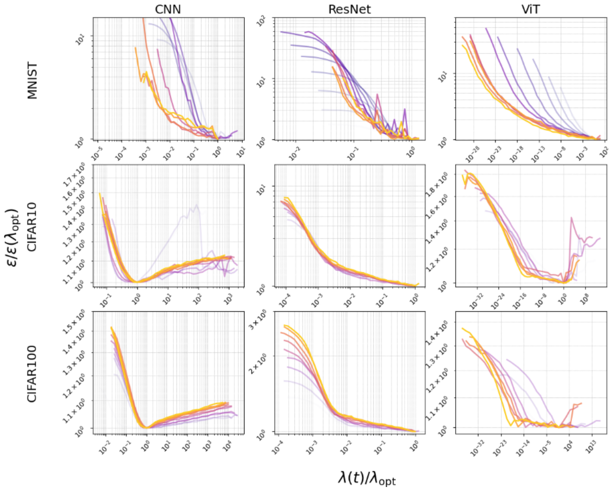

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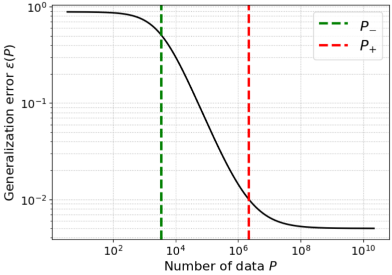

Hestness et al (2017) empirical curve

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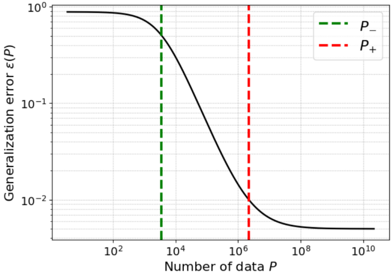

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Model	Dataset	γ_{pred}	γ_{meas}	σ
CNN	MNIST	0.60	0.55	0.09
CNN	CIFAR10	0.28	0.25	0.07
CNN	CIFAR100	0.16	0.16	0.03
ResNet	MNIST	0.57	0.69	0.08
ResNet	CIFAR10	0.54	0.56	0.04
ResNet	CIFAR100	0.31	0.37	0.03
ViT	MNIST	0.47	0.54	0.03
ViT	CIFAR10	0.23	0.21	0.03
ViT	CIFAR100	0.14	0.12	0.04

$\gamma_1 \gamma_2$ **compatible with** γ_{meas}

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3. In (3) $\gamma_1\gamma_2 \neq \gamma \Rightarrow$ Spectral complexity is "special"

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- Only image classification

  Extension: LLMs (i.e. Maloney et al. (2022) A Solvable Model of Neural Scaling Laws)

Thank you for your attention!

Francesco D'Amico

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