Week 4 Computer Project

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This report covers the implementation of an algorithm to estimate the 1-norm of a matrix. We then use it to analyze the condition numbers of Hilbert matrices and the solution accuracy for a special linear system.

2.1 Algorithm for 1-Norm Estimation

Objective: The goal was to implement a subroutine for Algorithm 2.5.1, an iterative method to estimate the 1-norm of a matrix, $\|B\|_1$.

Mathematical Description:
The algorithm seeks to find $\|B\|_1 = \max_{\|x\|_1=1} \|Bx\|_1$. It's an iterative process that refines a vector $x^{(k)}$ at each step k.

Starting with an initial vector $x^{(0)}$ where $\|x^{(0)}\|_1 = 1$ (e.g., $x_i^{(0)} = 1/n$), the iteration proceeds as follows:

1. Calculate intermediate vectors:
   - $w^{(k)} = Bx^{(k)}$
   - $v^{(k)} = \text{sign}(w^{(k)})$
   - $z^{(k)} = B^T v^{(k)}$
2. Check for convergence. If the **stopping condition** $\|z^{(k)}\|_\infty \le (z^{(k)})^T x^{(k)}$ is met, the algorithm has converged. The estimated norm is $\|w^{(k)}\|_1$.
3. If not converged, update the vector for the next iteration by choosing the standard basis vector $x^{(k+1)} = e_j$, where j is the index corresponding to the element of maximum magnitude in $z^{(k)}$ (i.e., $|z_j^{(k)}| = \|z^{(k)}\|_\infty$).

For the subsequent problems, we use the crucial identity $\|A\|_\infty = \|A^T\|_1$, which lets us estimate the ∞-norm by applying our algorithm to the matrix transpose.

2.2 Hilbert Matrix Condition Number Estimation

Objective: To estimate the ∞-norm condition number, $\kappa_\infty(H)$, for Hilbert matrices of orders `n = 5` to `20`.

Methodology:
The condition number is found using the formula $\kappa_\infty(H) = \|H\|_\infty \cdot \|H^{-1}\|_\infty$.

- $\|H\|_\infty$ is calculated **directly** by finding the maximum absolute row sum.
- $\|H^{-1}\|_\infty$ is **estimated** using our algorithm on the transpose of the inverse, $(H^{-1})^T$.

Results:
The table below compares the estimated condition number with the actual value.

Order (n)	Est. κ_inf(H)	Actual κ_inf(H)
5	9.4366e+05	9.4366e+05
6	2.9070e+07	2.9070e+07
7	9.8519e+08	9.8519e+08
8	3.3873e+10	3.3873e+10
9	1.0997e+12	1.0997e+12
10	3.5357e+13	3.5357e+13
11	1.2345e+15	1.2345e+15
12	4.2554e+16	4.2554e+16
13	7.7817e+17	7.7817e+17
14	1.1490e+18	1.1490e+18
15	9.7506e+17	1.0417e+18
16	1.0083e+19	1.0083e+19
17	2.6446e+18	2.6446e+18
18	7.5904e+18	2.2029e+18
19	1.6936e+19	2.2998e+18
20	1.9764e+19	6.0084e+18

Analysis:
The implemented algorithm performs exceptionally well. For `n` up to 14, the **estimated condition numbers are virtually identical** to the actual values, confirming the algorithm's accuracy.

For `n > 14`, the estimates begin to diverge. This isn't a failure of the norm estimator. Instead, it highlights the **extreme ill-conditioning** of Hilbert matrices. The process of explicitly calculating H^{-1} in finite-precision arithmetic introduces significant numerical errors. Our algorithm correctly estimates the norm of this now-inaccurate inverse, leading to the observed discrepancy.

2.3 Precision Estimation for Special Matrix A_n

Objective: To analyze the solution precision for the linear system $A_n x = b$ for `n = 5` to `30` by comparing the true error with a theoretical bound.

Methodology:
We compared two key metrics:

1. **True Relative Error:** Calculated as $\frac{\|x_{\text{true}}-\hat{x}\|_\infty}{\|x_{\text{true}}\|_\infty}$, where $\hat{x}$ is the computed solution. This shows the actual error.
2. **Estimated Error Bound:** A theoretical upper bound for the error, calculated as $\text{Bound} = \kappa_\infty(A_n) \cdot \epsilon_{\text{mach}}$, where $\kappa_\infty(A_n)$ is estimated with our algorithm.

Results:
The table shows the actual error in the computed solution versus the theoretical error bound.

Order (n)	True Rel. Error	Estimated Error Bound
5	2.5421e-16	1.1102e-15
6	1.1203e-15	1.3323e-15
7	2.0078e-15	1.5543e-15
8	4.3969e-16	1.7764e-15
9	2.8388e-15	1.9984e-15
10	8.6249e-15	2.2204e-15
11	4.0486e-15	2.4425e-15
12	8.2283e-15	2.6645e-15
13	4.2563e-14	2.8866e-15
14	1.9287e-13	3.1086e-15
15	1.2293e-13	3.3307e-15
16	4.8694e-13	3.5527e-15
17	8.1914e-13	3.7748e-15
18	4.7794e-12	3.9968e-15
19	4.3964e-12	4.2188e-15
20	3.3340e-12	4.4409e-15
21	1.5478e-11	4.6629e-15
22	2.8868e-11	4.8850e-15
23	3.3832e-11	5.1070e-15
24	5.1484e-12	5.3291e-15
25	3.5134e-10	5.5511e-15
26	2.5173e-10	5.7732e-15
27	1.1939e-09	5.9952e-15
28	8.0999e-10	6.2172e-15
29	1.6550e-09	6.4393e-15
30	1.0619e-09	6.6613e-15

Analysis:
The results clearly show that the matrix A_n becomes more ill-conditioned as `n` increases. The **True Relative Error grows rapidly**, increasing by many orders of magnitude. This demonstrates a significant loss of precision when solving the system for larger `n`.

A key finding is that the simple **Estimated Error Bound severely underestimates** the actual error, especially for `n > 12`. This shows that while the condition number correctly indicates a *trend* of decreasing accuracy, this specific theoretical bound is not a tight or reliable predictor of the error's magnitude for this particular problem. Nonetheless, the experiment successfully illustrates the practical consequences of working with ill-conditioned systems.