Computer Project: Numerical Integration Analysis

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1 Problem Setting

Use the **composite midpoint rule**, **composite trapezoidal rule**, **composite Simpson's rule**, **Romberg integration method**, and **adaptive Simpson method** to compute the following numerical integral:

$$\int_0^1 \frac{4}{x^2+1}\,dx = \pi$$

Then express the error as a function of the step size h, and further analyze the **order of convergence**.

In addition, analyze whether there exists a certain h, such that when h is smaller than this value, further decreasing h will **no longer improve** the computation. Analyze the reason for this.

2 Algorithm

2.1 Composite Midpoint Rule

Divide the interval $[a,b]$ into n equal subintervals

$$a = x_0 < x_1 < \cdots < x_n = b, \quad x_i - x_{i-1} = h = \frac{b-a}{n} \quad (i = 1, \ldots, n).$$

On each $[x_{i-1}, x_i]$, the midpoint rule is

$$I_M(x_{i-1}, x_i) = (x_i - x_{i-1})\, f\Big(\tfrac{x_{i-1}+x_i}{2}\Big).$$

Let $x_{i-\frac{1}{2}} = \frac{x_{i-1}+x_i}{2}$. Summing over i gives

$$M(h) \;=\; h\sum_{i=1}^{n} f(x_{i-\frac{1}{2}}).$$

Since the local error is $O(h^3)$, the global error is

$$E_M = O(h^2) = O(n^{-2}).$$

2.2 Composite Trapezoidal Rule

With the same partition, on $[x_{i-1}, x_i]$ the trapezoidal rule is

$$I_T(x_{i-1}, x_i) = \frac{x_i - x_{i-1}}{2}\big[f(x_{i-1}) + f(x_i)\big].$$

Summing yields

$$T(h) = \frac{h}{2}\sum_{i=0}^{n-1}[f(x_{i-1}) + f(x_i)].$$

Local error $O(h^3)$ implies global error

$$E_T = O(h^2) = O(n^{-2}).$$

2.3 Composite Simpson's Rule

Partition $[a, b]$ into n (even) subintervals. On each $[x_{i-1}, x_i]$,

$$I_S(x_{i-1}, x_i) = \frac{x_i - x_{i-1}}{6}\Big[f(x_{i-1}) + 4f\big(\tfrac{x_{i-1}+x_i}{2}\big) + f(x_i)\Big].$$

Thus

$$S(h) = \frac{h}{6}\sum_{i=1}^{n}\Big[f(x_{i-1}) + 4f(x_{i-\frac{1}{2}}) + f(x_i)\Big].$$

Local error $O(h^5)$ gives global error

$$E_S = O(h^4) = O(n^{-4}).$$

2.4 Romberg Integration

By Euler–Maclaurin,

$$\int_a^b f(x)\,dx = T(h) + \sum_{k=1}^{\infty} c_{2k}^{(1)} h^{2k},$$

where $T(h)$ is the composite trapezoidal rule. Let

$$T_1(h) = T(h), \quad T_{k+1}(h) = \frac{T_k(h/2) - 4^{-k}T_k(h)}{1 - 4^{-k}}.$$

Then

$$\int_a^b f(x)\,dx - T_{k+1}(h) = O\big(h^{2k+2}\big).$$

2.5 Adaptive Simpson

Prescribe a total tolerance ε. Initially subdivide into

$$n = \lfloor \varepsilon^{-4} \rfloor$$

panels. On each panel compute its integral and local error. If the local error exceeds its share, bisect the panel and recompute until the criterion is met. Continue panel by panel until the global error is within ε.

3 Experiment

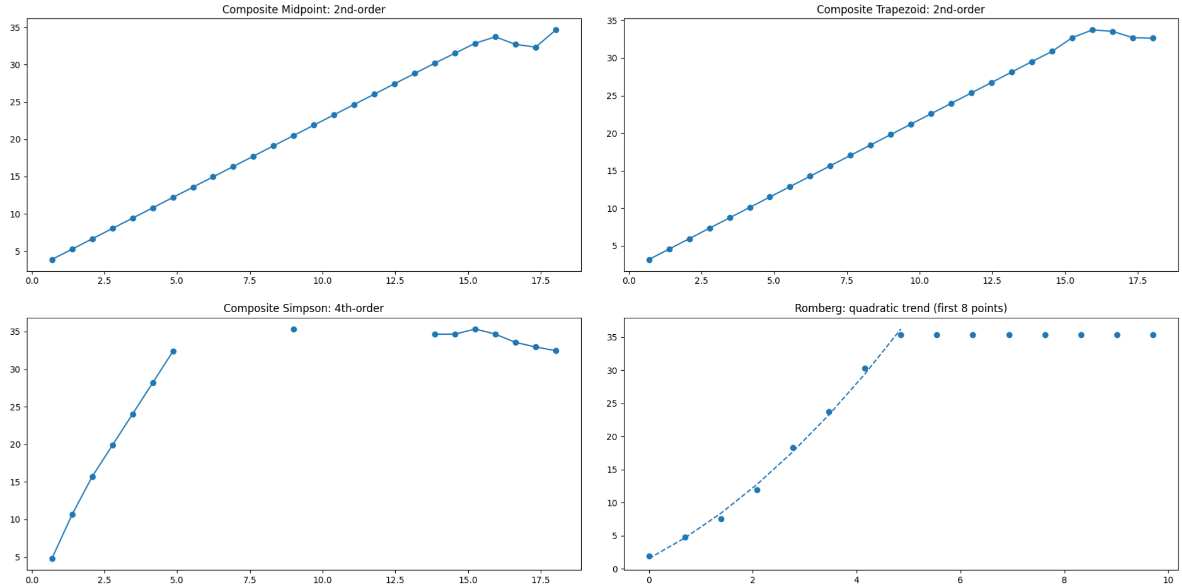

Figure 1: Error Order

Based on the numerical experiments for $\int_0^1 \frac{4}{1+x^2}\,dx = \pi$, we observe the following:

1. **Composite Midpoint Rule:** In the plot of $-\log(E_M)$ versus $\log(n)$ (Upper left), the data points lie on a straight line of slope approximately 2. This confirms the theoretical global error
$$E_M(h) = \bigl|M(h) - \pi\bigr| = O(h^2) = O(n^{-2}).$$

2. **Composite Trapezoidal Rule:** Similarly, trapezoidal rule errors (Upper right) exhibit a slope of about 2 on the same log–log plot, matching
$$E_T(h) = \bigl|T(h) - \pi\bigr| = O(h^2) = O(n^{-2}).$$

3. **Composite Simpson's Rule:** The Simpson errors(Lower left) lie on a line of slope 4, in agreement with the theoretical estimate
$$E_S(h) = \bigl|S(h) - \pi\bigr| = O(h^4) = O(n^{-4}).$$

4. **Romberg Integration:** After k levels of Richardson extrapolation, the error behaves like
$$E_{R,k}(h) = O(h^{2k+2}), \quad n = 2^k,\ h = \tfrac{1}{n}.$$
Hence
$$-\log E_{R,k} \approx \frac{2}{\log 2}(\log n)^2 + 2\log n + O(1),$$
so the $(-\log E)$ vs. $\log n$ curve follows a quadratic profile (Lower right). A polynomial fit confirms the dominant $(\log n)^2$ term.

5. **Machine Precision Saturation:** Across all methods, once the error reaches the order of machine epsilon ($\approx 10^{-16}$), further refinement of h no longer reduces the error. At this point, rounding errors dominate, and total error "saturates."

References