Computer Project: Numerical ODE

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1 Problem Setting

Consider the Lorenz system of ordinary differential equations:

$$\begin{cases} \dfrac{dx}{dt} = \sigma\,(y - x), \\ \dfrac{dy}{dt} = \rho\,x - y - x\,z, \\ \dfrac{dz}{dt} = x\,y - \beta\,z, \end{cases} \tag{1}$$

where $\sigma, \rho, \beta \in \mathbf{R}$ are real parameters. Address the following questions:

1. For the fixed parameter values

$$\sigma = 10, \quad \rho = 28, \quad \beta = \frac{8}{3},$$

choose different initial conditions and observe the numerical solutions. What qualitative features do you observe? In particular, determine whether the solution curves remain bounded, whether they become periodic, or whether they tend toward a fixed point.

2. Within the admissible range of parameters, vary one or more of σ, ρ, and β, then again choose different initial conditions. Observe and record any qualitative changes in the behavior of the solutions. What new phenomena, if any, do you observe?

2 Algorithms

We employ the classical fourth-order Runge–Kutta method (RK4) to integrate the Lorenz system. Denote by

$$w = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad f(w) = \begin{pmatrix} \sigma(y - x) \\ \rho\,x - y - x\,z \\ x\,y - \beta\,z \end{pmatrix}, \quad \frac{dw}{dt} = f(w).$$

Given a time step size $h > 0$ and the approximation $w_n \approx w(t_n)$, the RK4 update from t_n to $t_{n+1} = t_n + h$ is:

$$\begin{aligned}
k_1 &= f\left(w_n\right), \\
k_2 &= f\Big(w_n + \tfrac{h}{2}\, k_1\Big), \\
k_3 &= f\Big(w_n + \tfrac{h}{2}\, k_2\Big), \\
k_4 &= f\big(w_n + h\, k_3\big), \\
w_{n+1} &= w_n + \frac{h}{6}\,\big(k_1 + 2\,k_2 + 2\,k_3 + k_4\big).
\end{aligned}$$

Here, the local truncation error of RK4 is $O(h^5)$ and the global error is $O(h^4)$, making it a fourth-order accurate method.

3 Results Analysis

3.1 Varying Initial Values

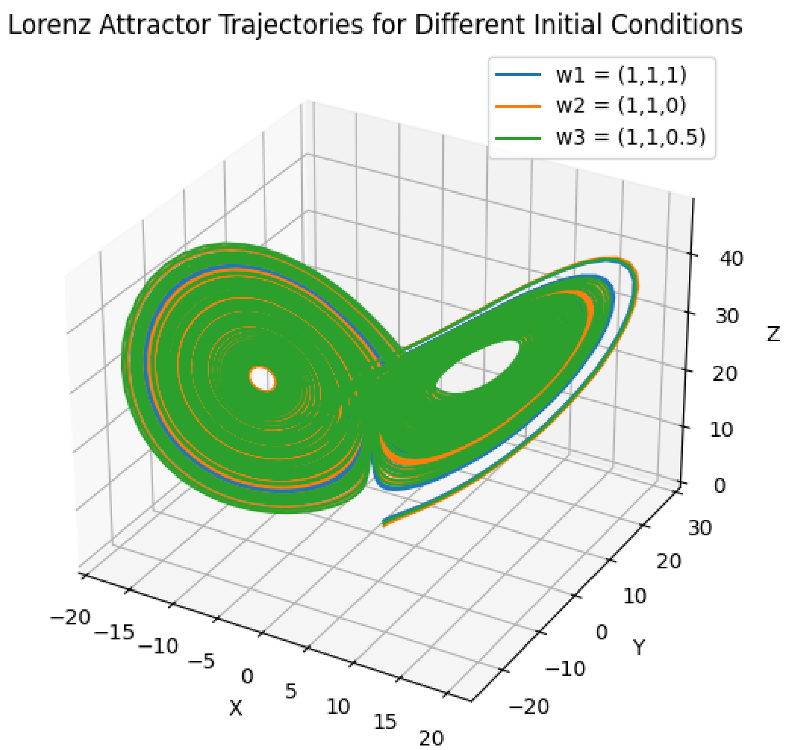

Figure 1: Attractors under different initial values

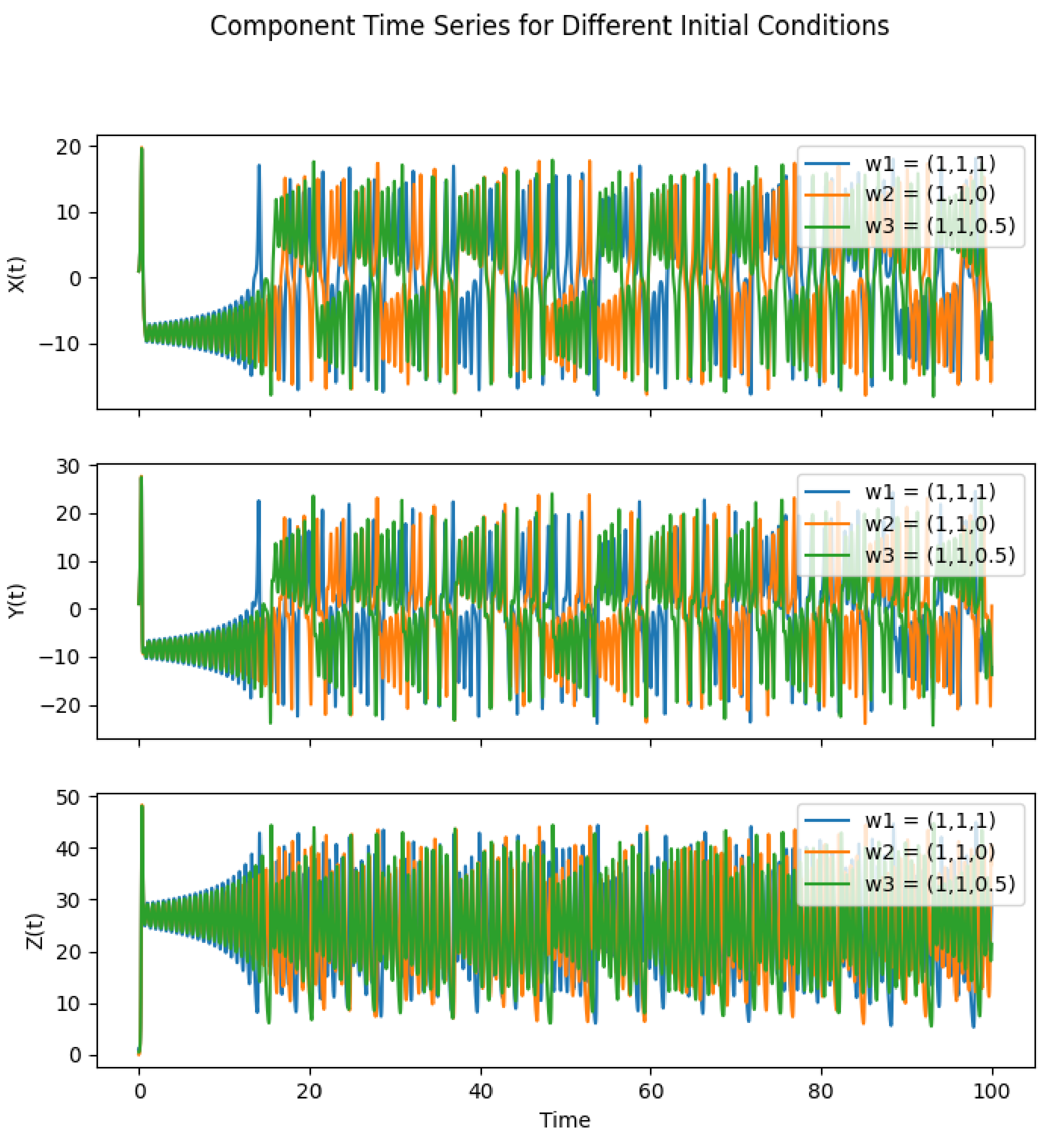

Figure 2: Time series under different initial values

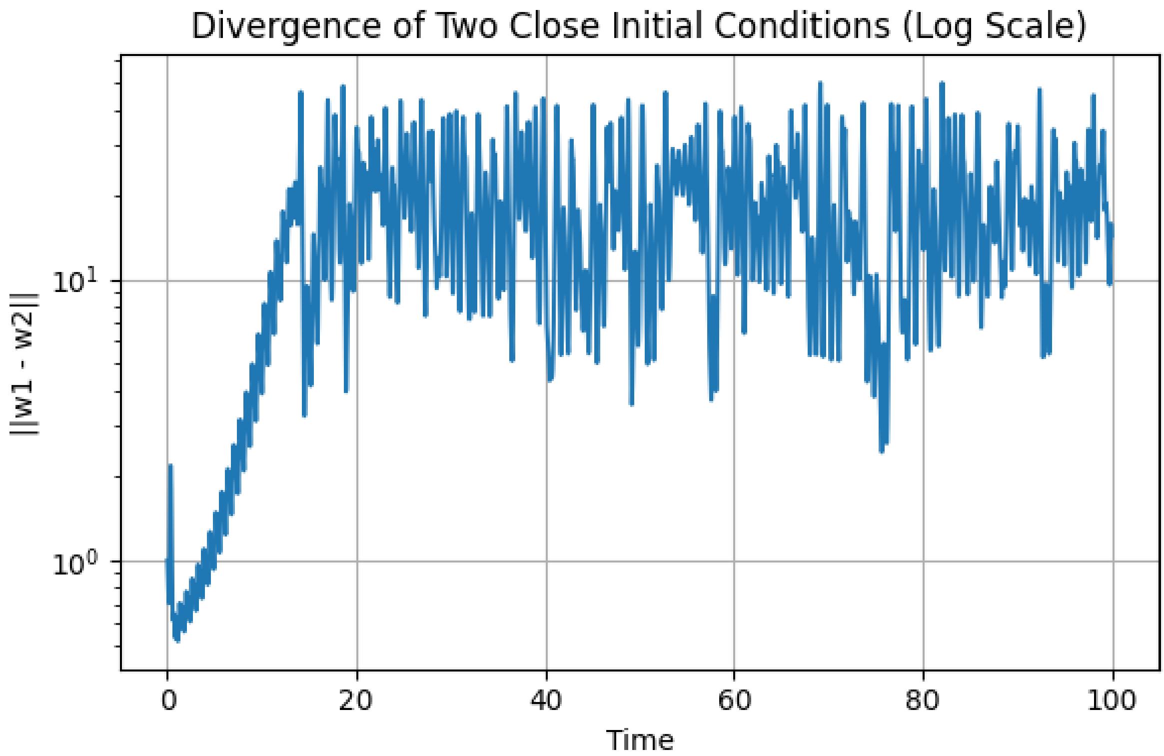

Figure 3: Divergence of the first and the second trajectory

From the figures above, we observed that:

Sensitivity to initial conditions Under fixed parameters $\sigma = 10, \rho = 28, \beta = 8/3$, three different initial states $w_0^{(1)} = (1, 1, 1), \quad w_0^{(2)} = (1, 1, 0), \quad w_0^{(3)} = (1, 1, 0.5)$ produce markedly different trajectories over the same time interval $[0, 100]$. A semilog plot of $\|w^{(1)}(t) - w^{(2)}(t)\|$ confirms exponential divergence after a short transient.

Boundedness Despite the wildly different evolutions, all three trajectories remain confined within a finite region of phase space. They never diverge to infinity.

No convergence to fixed points or simple periodic orbits None of the solution curves settles down to a constant or repeats periodically. Instead, each tends toward the characteristic "butterfly" attractor, a fractal manifold in $\mathbf{R}^3$.

Non-periodicity Even though trajectories stay in a bounded region, they never pass through the same point twice; they remain non-periodic, as is typical for chaotic systems.

3.2 Varying Parameters

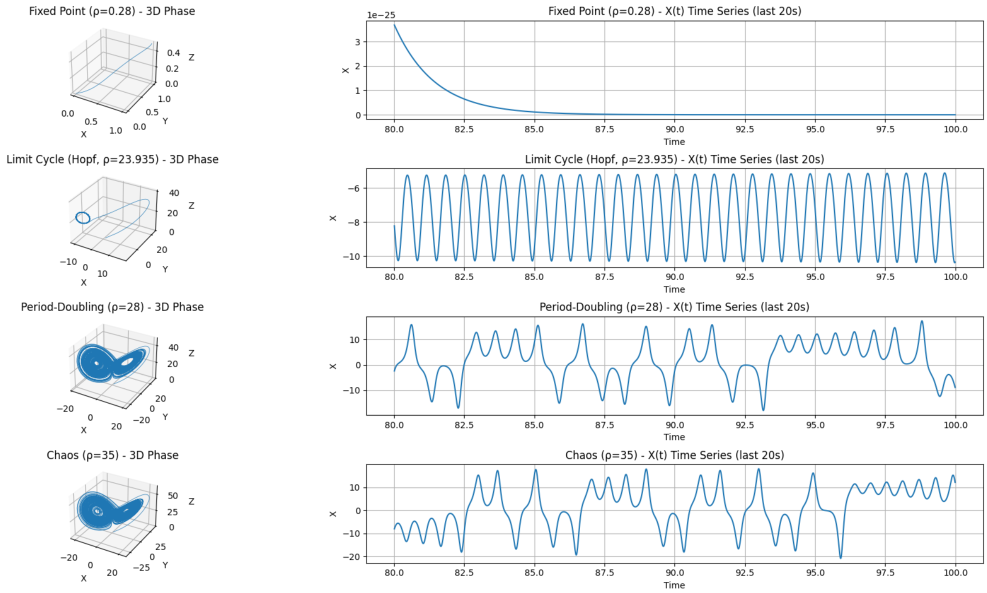

Figure 4: System behaviors under different ρ

As the parameter ρ varies, the Lorenz system undergoes a sequence of bifurcations, producing distinct dynamical regimes:

- **Fixed-point regime** ($\rho = 0.28$)
  - The 3D trajectory quickly spirals into a single equilibrium.
  - The time series decays exponentially toward zero (machine precision near 10^{-20}).
- **Limit-cycle regime (Hopf bifurcation)** ($\rho \approx 23.935$)
  - After the Hopf bifurcation, the solution converges to a stable periodic orbit.
  - The time series of $x(t)$ shows a constant-amplitude sinusoidal oscillation.
- **Period-doubling regime** ($\rho = 28$)
  - The original single-period orbit bifurcates into a two-cycle.
  - In the zoomed time series, two distinct peaks alternate in amplitude, indicating a period-doubling route to chaos.
- **Chaotic attractor regime** ($\rho = 35$)
  - The trajectory no longer repeats and fills out a strange attractor.
  - The time series of $x(t)$ is irregular, non-periodic, and aperiodic.

References