Computer Project: Periodic Cubic Spline Interpolation

Zexi Fan, 2200010816

23rd March 2025

1 Problem Setting

Given $f(x)$ on interval $[a, b]$ with period $T = b - a$ and $n+1$ interpolation points $(x_i, y_i), 0 \leq i \leq n$, where $x_0 = a, x_n = b, y_i = f(x_i), 0 \leq i \leq n$. We want to find a cubic polynomial in pieces $S(x) \in C^2[a, b]$, such that

$$S(x_i) = y_i, \quad 0 \leq i \leq n \tag{1}$$

$$S^{(k)}(x_0) = S^{(k)}(x_n), \quad 0 \leq k \leq 2. \tag{2}$$

2 Algorithm

$S(x)$ is piecewise cubic, therefore $S^{(2)}(x) \in C^0[a, b]$ is piecewise linear. We assume

$$S^{(2)}(x) = M_{j-1} \times \frac{x_j - x}{h_j} + M_j \times \frac{x - x_{j-1}}{h_j}, \quad x \in I_j, \quad j = 1 : n, \tag{3}$$

where

$$I_j = [x_{j-1}, x_j], \quad h_j = x_j - x_{j-1}, \quad j = 1 : n \tag{4}$$

$$M_j = S^{(2)}(x_j), \quad j = 0 : n. \tag{5}$$

Integrate back, and we have

$$S(x) = \frac{1}{6} M_{j-1} \frac{(x_j - x)^3}{h_j} + \frac{1}{6} M_j \frac{(x - x_{j-1})^3}{h_j} + L_j x + K_j, \quad x \in I_j, \quad j = 1 : n. \tag{6}$$

Plugging in the interpolation value at x_j, x_{j-1}, we get

$$\begin{aligned} L_j \cdot x_j + K_j &= y_j - \frac{M_j}{6h_j} h_j^3 \\ L_j \cdot x_{j-1} + K_j &= y_{j-1} - \frac{M_{j-1}}{6h_j} h_j^3. \end{aligned} \tag{7}$$

Hence,

$$L_j x + K_j = \left(y_{j-1} - \frac{1}{6} M_{j-1} h_j^2 \right) \frac{x_j - x}{h_j} + \left(y_j - \frac{1}{6} M_j h_j^2 \right) \frac{x - x_{j-1}}{h_j}. \tag{8}$$

Since $S'(x) \in C^1[a,b]$, for $j = 1 : n-1$, $S'(x_j + 0) = S'(x_j - 0)$, that is

$$\frac{1}{3}M_j h_j + \frac{1}{6}M_{j-1}h_j + \frac{y_j - y_{j-1}}{h_j} = -\frac{1}{3}M_j h_{j+1} - \frac{1}{6}M_j h_{j+1} + \frac{y_{j+1} - y_j}{h_{j+1}} \tag{9}$$

$$\Rightarrow \frac{1}{6}h_j M_{j-1} + \frac{1}{3}(h_j + h_{j+1})M_j + \frac{1}{6}h_{j+1}M_{j+1} = \frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j} \tag{10}$$

$$\Rightarrow \mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = b_j, \tag{11}$$

where

$$\begin{aligned}
\lambda_j &= \frac{h_{j+1}}{h_j + h_{j+1}}, \quad j = 1 : n-1 \\
\mu_j &= \frac{h_j}{h_j + h_{j+1}}, \quad j = 1 : n-1 \\
b_j &= \frac{6}{h_j + h_{j+1}} \left(\frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j} \right), \quad j = 1 : n-1.
\end{aligned} \tag{12}$$

Note periodic boundary condition

$$S^{(k)}(x_0) = S^{(k)}(x_n), \quad k = 0 : 2. \tag{13}$$

We arrive

$$\begin{cases} M_0 = M_n \\ -\frac{1}{3}h_1 M_0 - \frac{1}{6}h_1 M_1 + \frac{y_1 - y_0}{h_1} = \frac{1}{3}h_n M_n + \frac{1}{6}h_n M_{n-1} + \frac{y_n - y_{n-1}}{h_n}, \end{cases} \tag{14}$$

which guarantees that (12) holds for $j = 1 : n$ under $\mod n$ sense. We observe that for $j = 1 : n$ under $\mod n$,

$$\begin{aligned}
&\lambda_j + \mu_j = 1 \\
&\mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = b_j.
\end{aligned} \tag{15}$$

This gives the following linear system

$$AM = b, \tag{16}$$

where

$$A = \begin{bmatrix}
2 & \lambda_1 & 0 & 0 & \dots & 0 & \mu_1 \\
\mu_2 & 2 & \lambda_2 & 0 & \dots & 0 & 0 \\
0 & \mu_3 & 2 & \lambda_3 & \dots & 0 & 0 \\
0 & 0 & \mu_4 & 2 & \dots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 2 & \lambda_{n-1} \\
\lambda_n & 0 & 0 & 0 & \dots & \mu_n & 2
\end{bmatrix}. \tag{17}$$

$$M = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

Note that A is a tridiagonal cyclic matrix, and $\lambda_j + \mu_j = 1 < 2$, we can see that A is strictly dominant in the main diagonal. Therefore, the system is solvable, and gives the expression of $S(x)$.

3 Experiment

3.1 Setup

Specifically, we consider[1]

$$f(x) = e^{\sin x} + \cos 4x, x \in [0, 2\pi]. \tag{18}$$

Choose interpolation points with equal space

$$x_k = kh, \quad 0 \le k \le n, \quad h = \frac{2\pi}{n}, \tag{19}$$

which yields

$$h_k = h = \frac{2\pi}{n} \tag{20}$$

$$\lambda_k = \mu_k = \frac{1}{2} \tag{21}$$

for $1 \le k \le n$. On the interval $I_j = [x_{j-1}, x_j]$, we have

$$S(x) = \frac{1}{6}M_{j-1}\frac{(x_j - x)^3}{h} + \frac{1}{6}M_j\frac{(x - x_{j-1})^3}{h} \tag{22}$$

$$+ \left(y_{j-1} - \frac{1}{6}M_{j-1}h^2\right)\frac{x_j - x}{h} + \left(y_j - \frac{1}{6}M_j h^2\right)\frac{x - x_{j-1}}{h}, \tag{23}$$

where

$$\begin{bmatrix} 2 & 1/2 & 0 & 0 & \dots & 0 & 1/2 \\ 1/2 & 2 & 1/2 & 0 & \dots & 0 & 0 \\ 0 & 1/2 & 2 & 1/2 & \dots & 0 & 0 \\ 0 & 0 & 1/2 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 1/2 \\ 1/2 & 0 & 0 & 0 & \dots & 1/2 & 2 \end{bmatrix} \cdot \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ \vdots \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}. \tag{24}$$

Thus, it suffices to solve the system to get M, and plug M into the expression of $S(x)$.

3.2 Results Analysis

The table below summarizes the computed maximum error e_h for various numbers of subintervals n:

n	e_h
5	1.909434×10^{0}
10	3.476368×10^{-1}
20	9.527473×10^{-3}
30	1.580524×10^{-3}
50	1.842478×10^{-4}
100	1.091524×10^{-5}
300	1.338389×10^{-7}
500	1.733632×10^{-8}

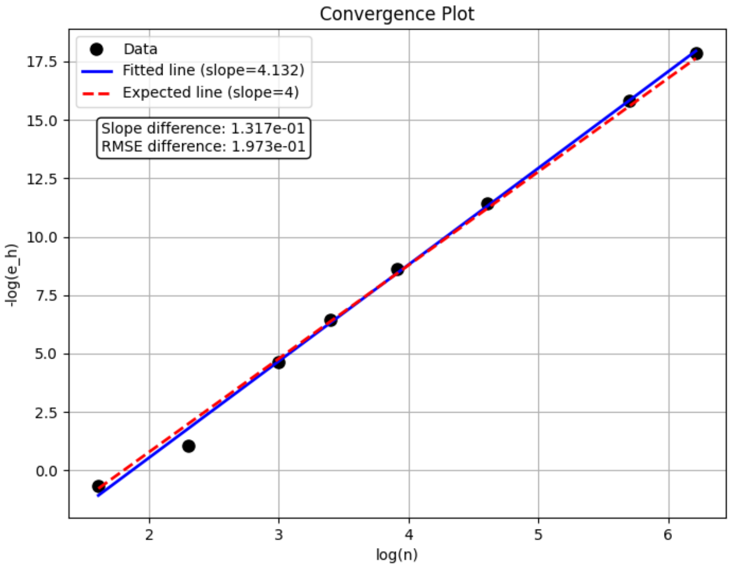

Figure 1: Convergence Plot

The convergence behavior is further illustrated by a log-log plot of the error e_h against the number of subintervals n. A linear regression on the data yields a fitted slope of approximately 4.132, which is very close to the expected slope of 4 for an $O(n^{-4})$ (or equivalently $O(h^4)$) convergence rate. The RMSE between the fitted line and the expected line is 0.1973, indicating a good agreement between the observed convergence and theoretical expectations.

In summary, these results confirm that the periodic cubic spline interpolation implementation achieves the anticipated fourth-order convergence. The error decreases rapidly as the mesh is refined (i.e., as n increases), which demonstrates the efficiency and accuracy of the numerical method.

References

[1] Tiejun Li. Homepage of professor tiejun li, 2025. URL `https://www.math.pku.edu.cn/teachers/litj/`. Accessed: 2025-03-24.