

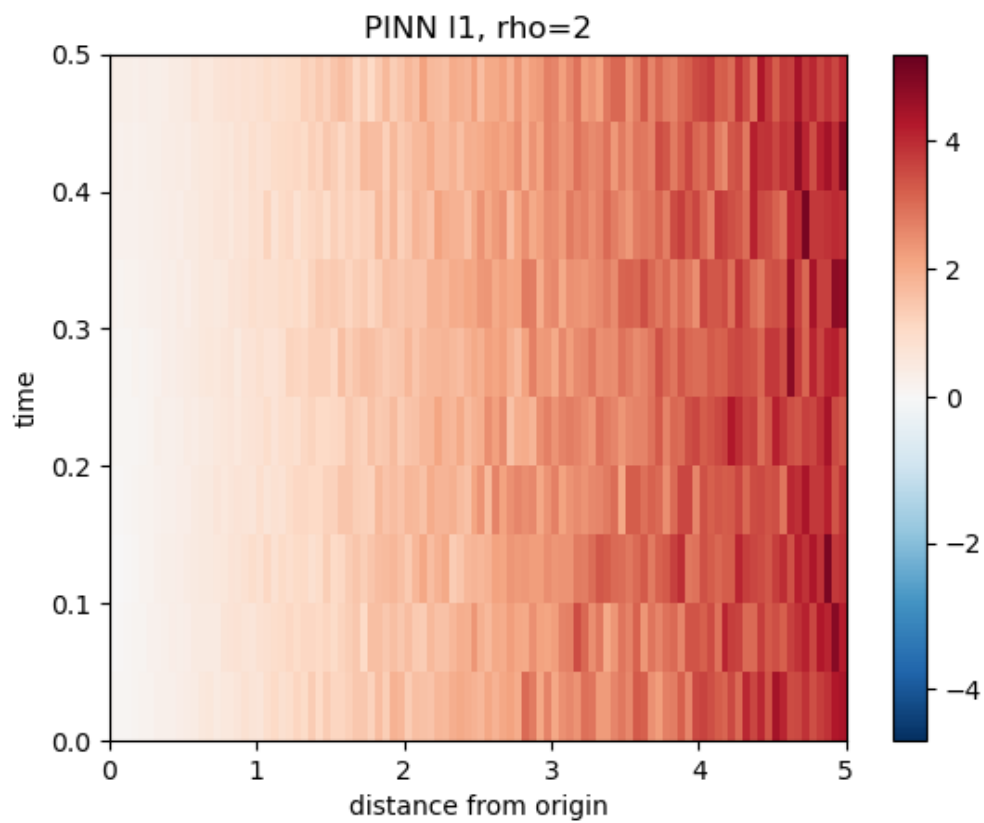
# Summary\_of\_Results(MAE,Full\_History)

## Complicated\_HJB

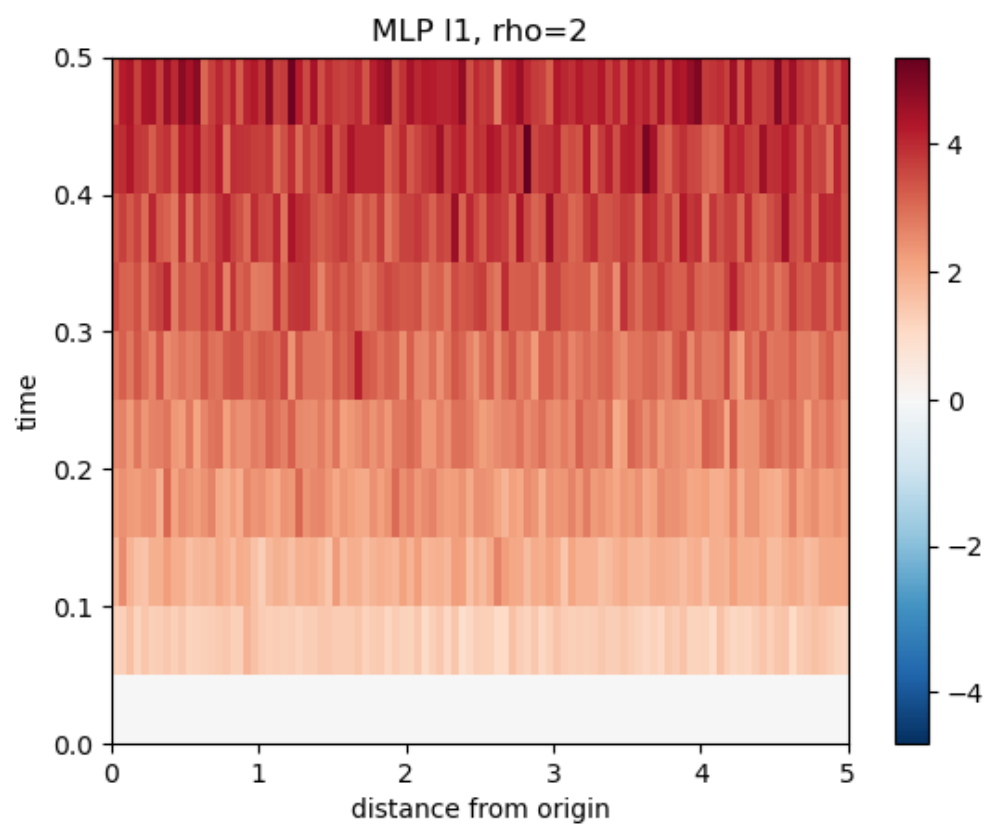
NormalSphere

100d

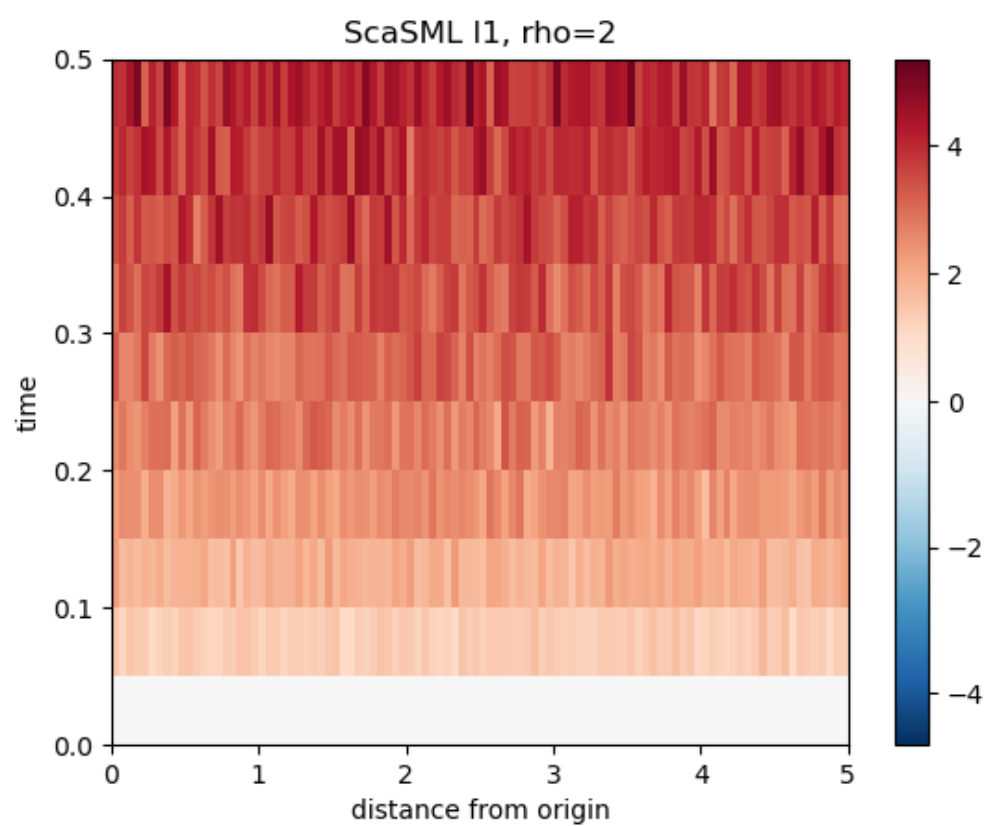
PINN



MLP

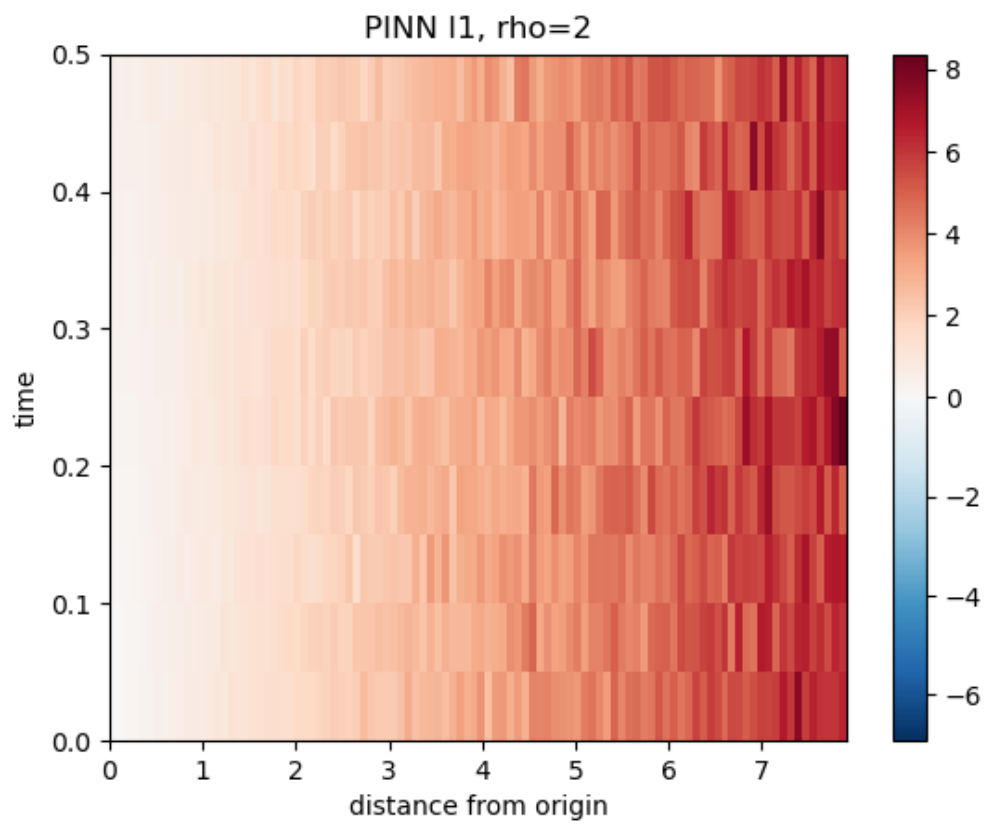


SCaSML

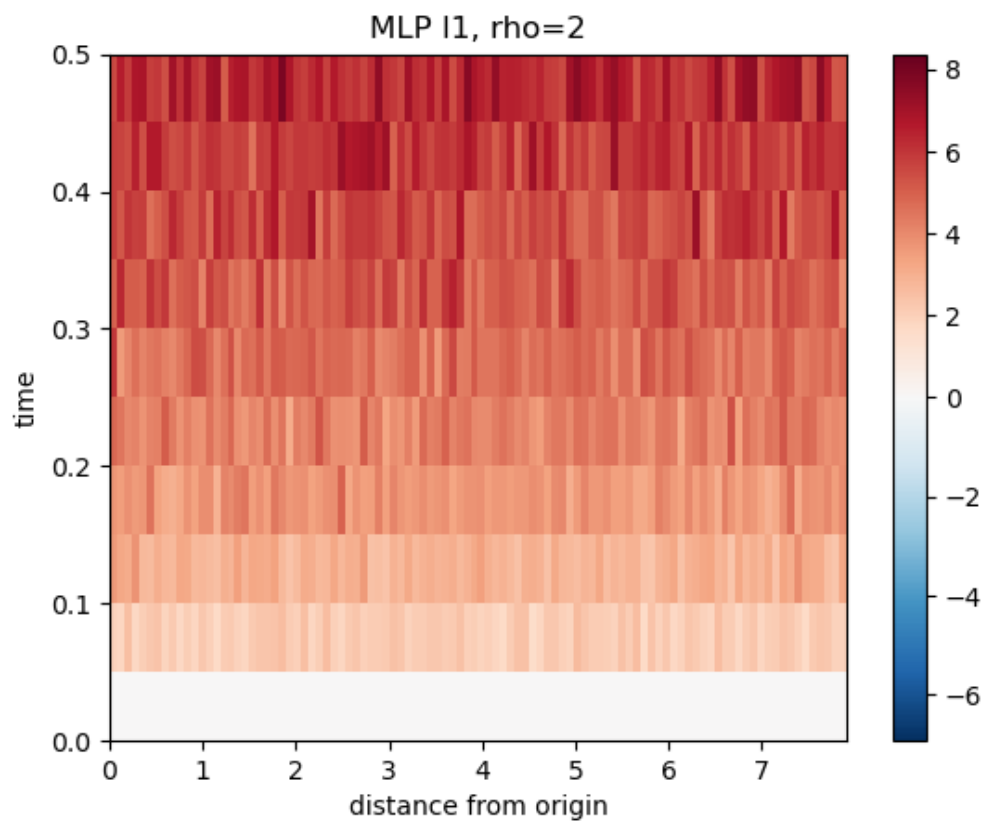


250d

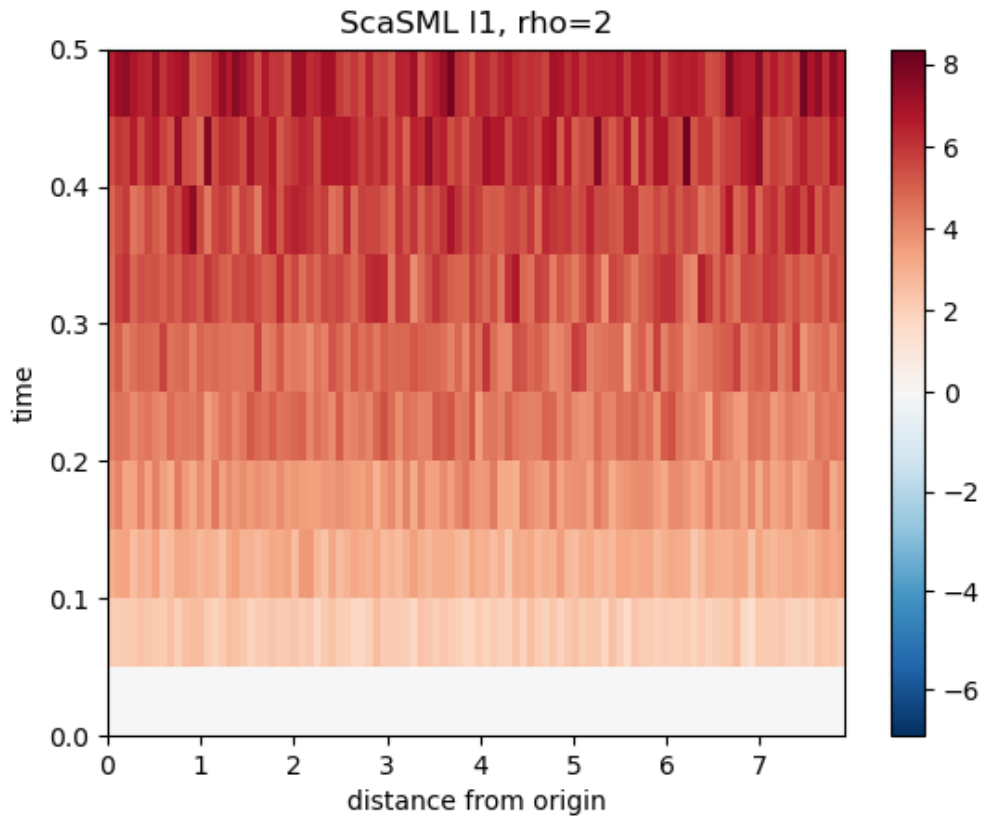
PINN



MLP



## SCaSML



## SimpleUniform

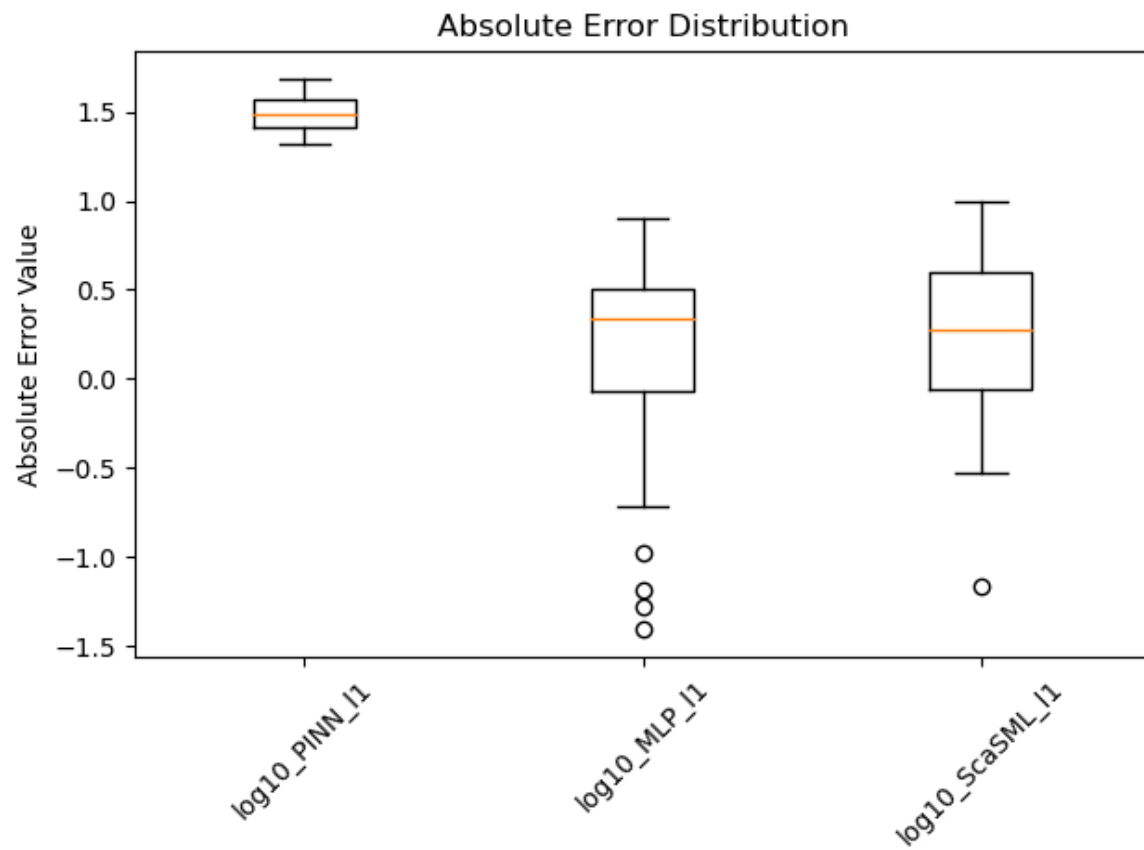
### 100d

Real Solution-> min: 21.08817 max: 47.40346 mean: 31.57981

PINN l1, rho=2-> min: 20.95896 max: 48.12542 mean: 31.836462

MLP l1, rho=2-> min: 0.039306646030652814 max: 8.000174376765138 mean:  
2.553234381201442

ScaSML l1, rho=2-> min: 0.06791453660794389 max: 10.00529483343557 mean:  
2.633477011632597



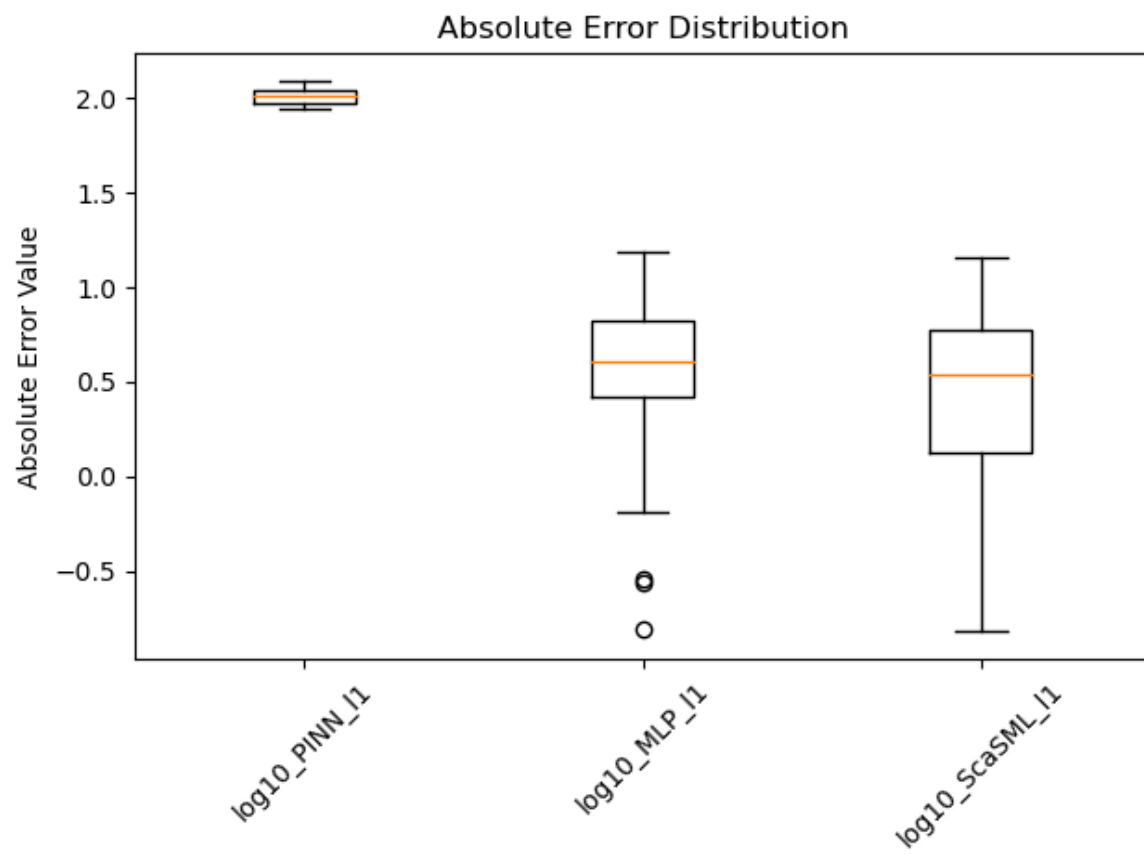
## 250d

Real Solution-> min: 88.26935 max: 122.24708 mean: 102.59221

PINN I1, rho=2-> min: 88.28333 max: 122.26325 mean: 102.6078

MLP I1, rho=2-> min: 0.15548511594090542 max: 15.487591823426683 mean:  
5.0831840414465015

ScaSML I1, rho=2-> min: 0.15286268404449288 max: 14.327133842593568 mean:  
4.199262405912148



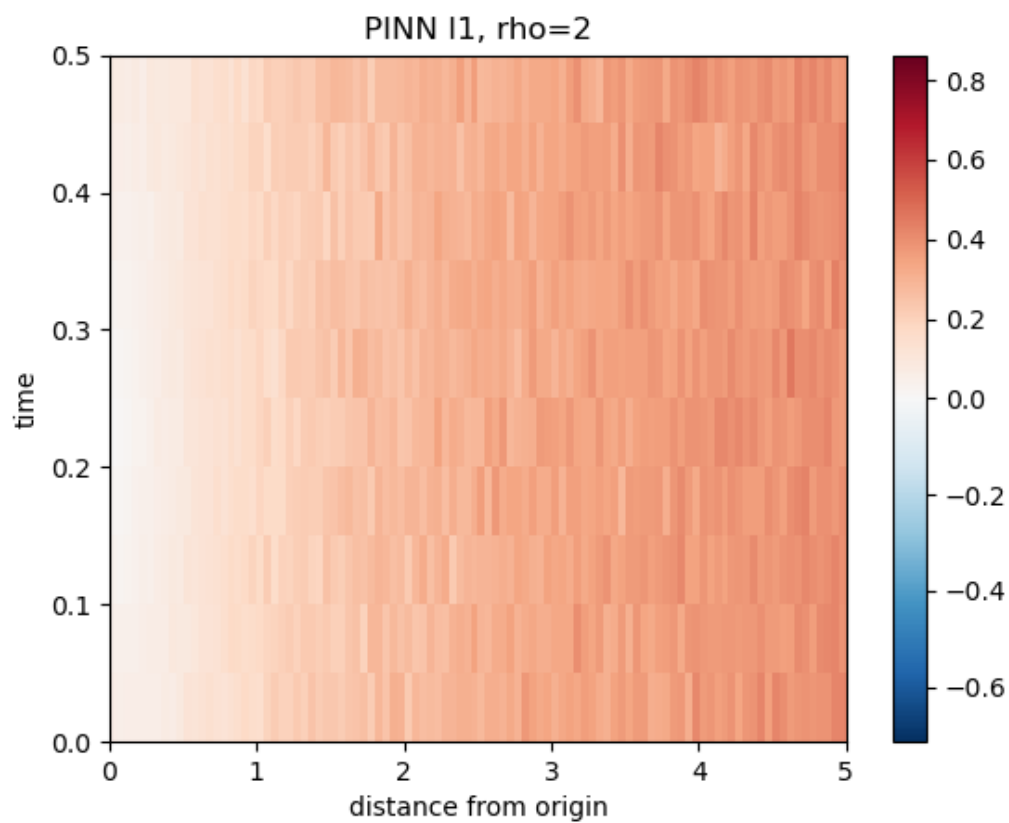
## Explicit\_Solution\_Example

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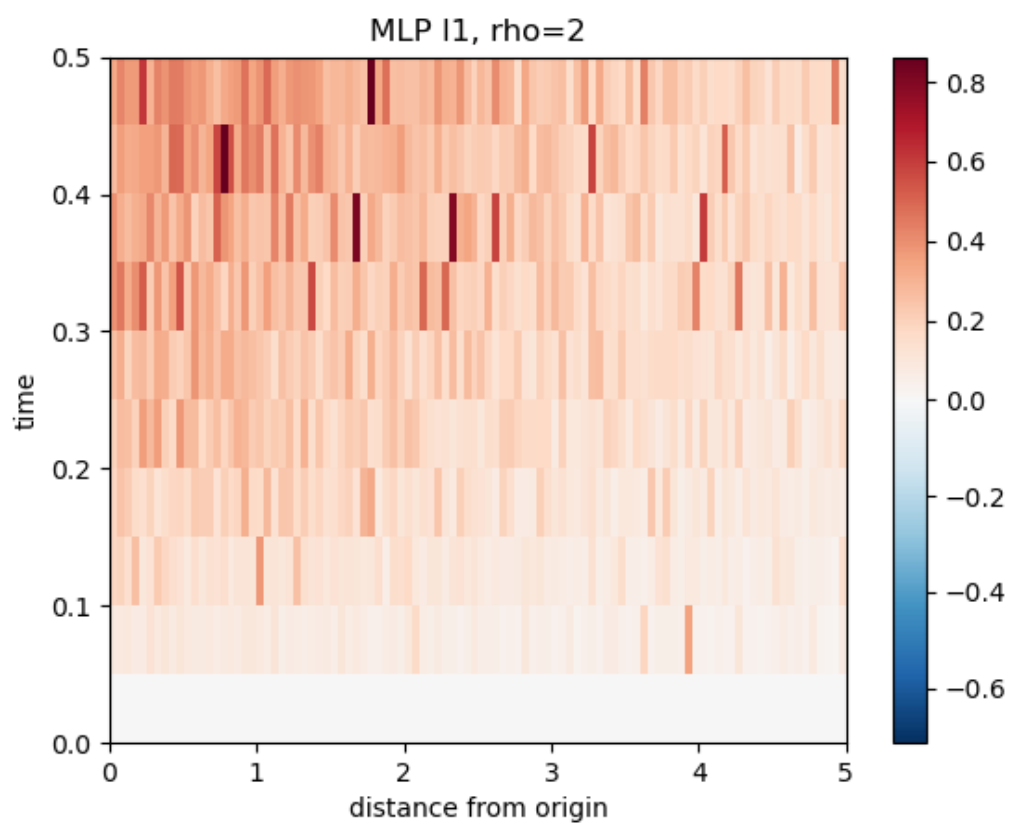
NormalSphere

100d

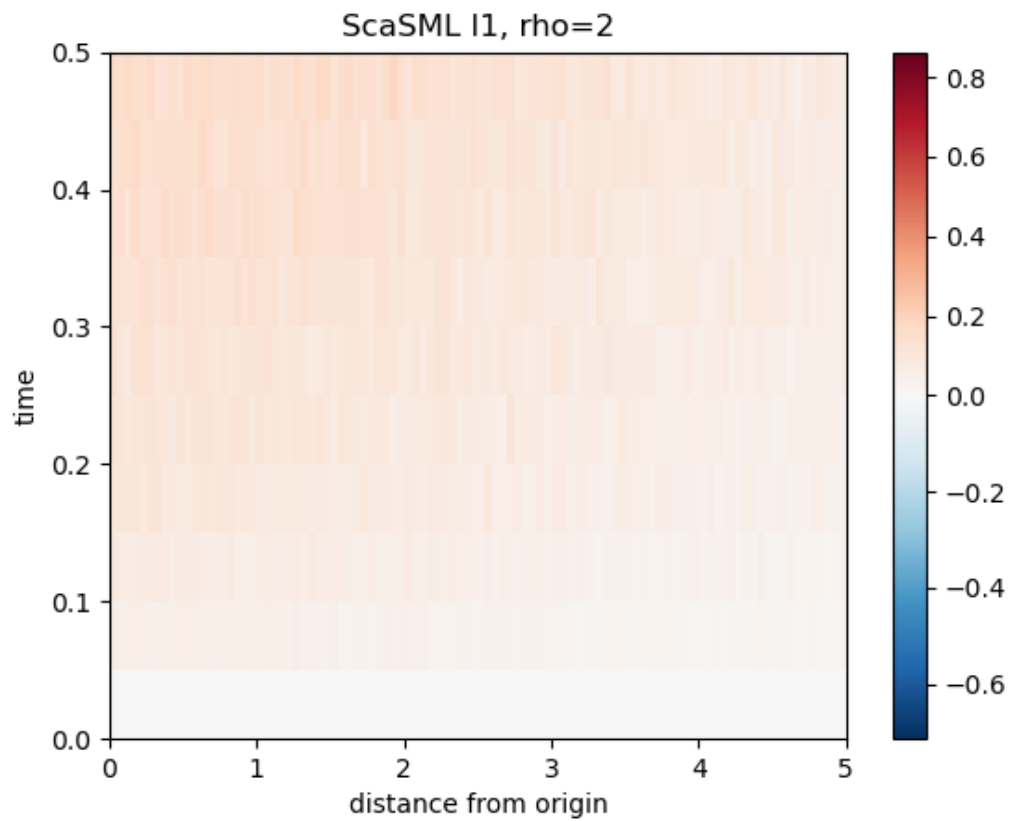
PINN



MLP

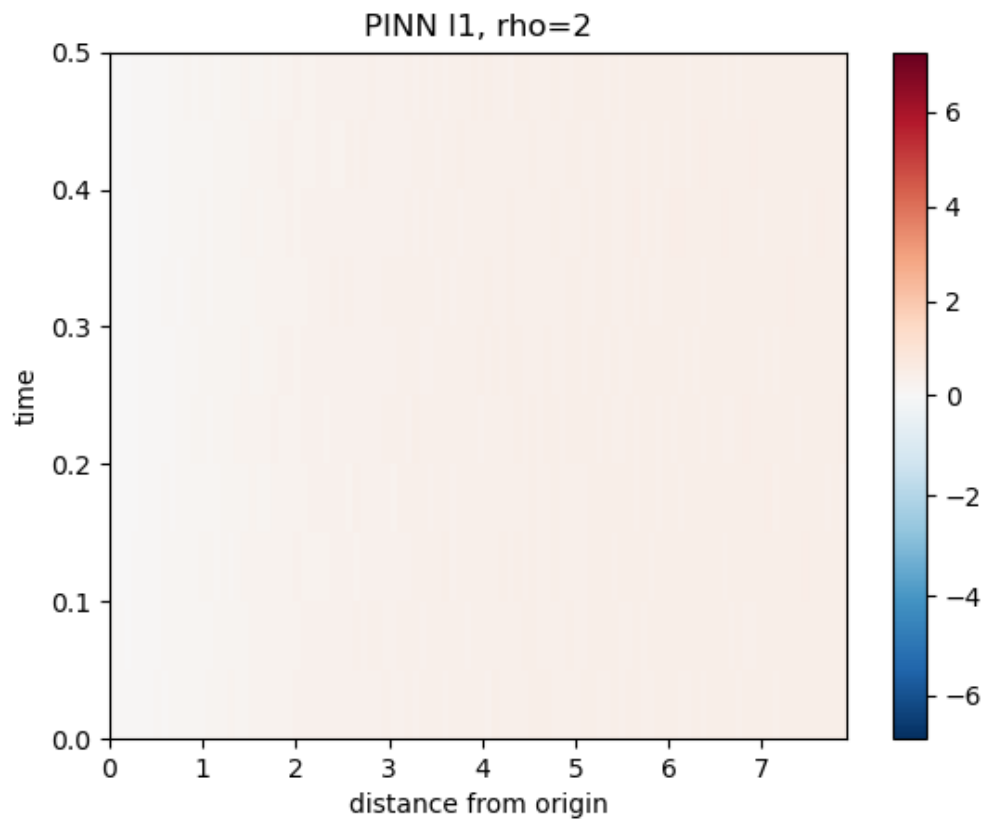


## SCaSML



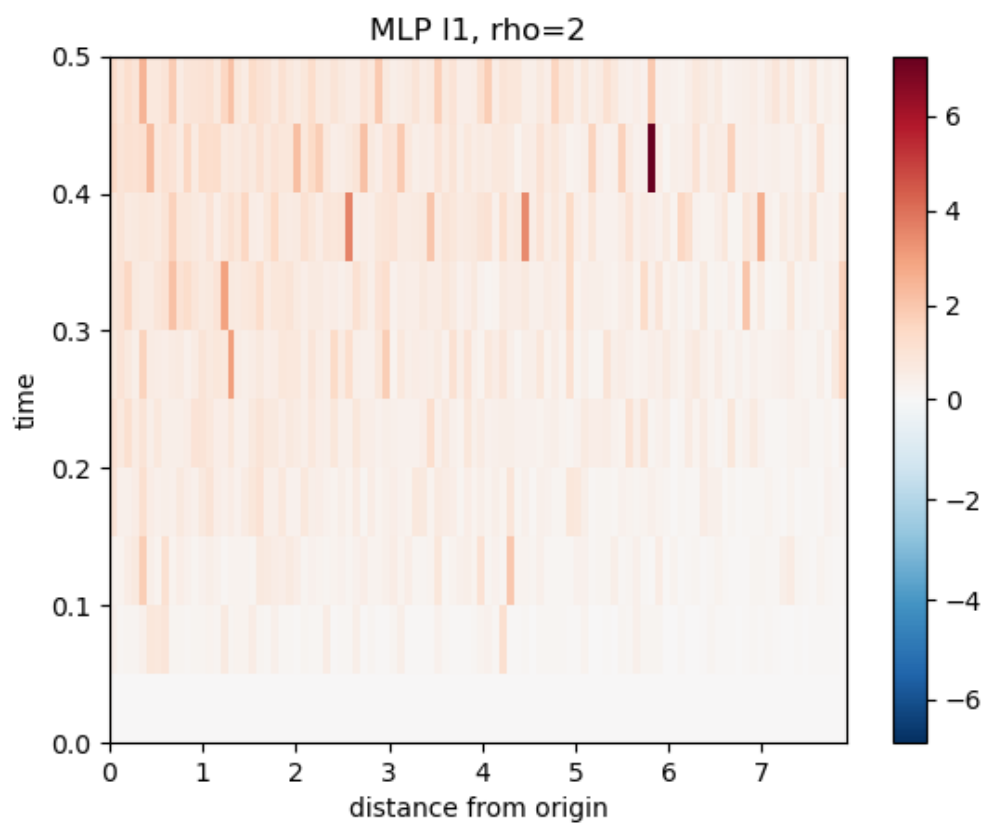
## 250d

### PINN

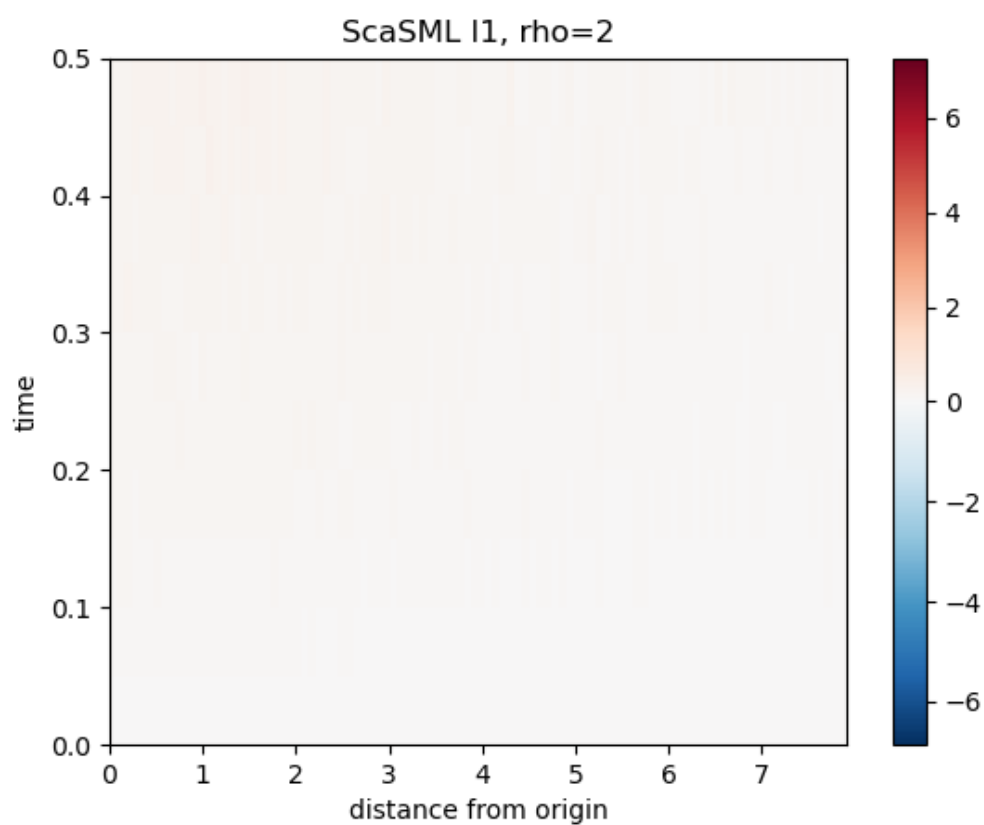




## MLP



## SCaSML



# SimpleUniform

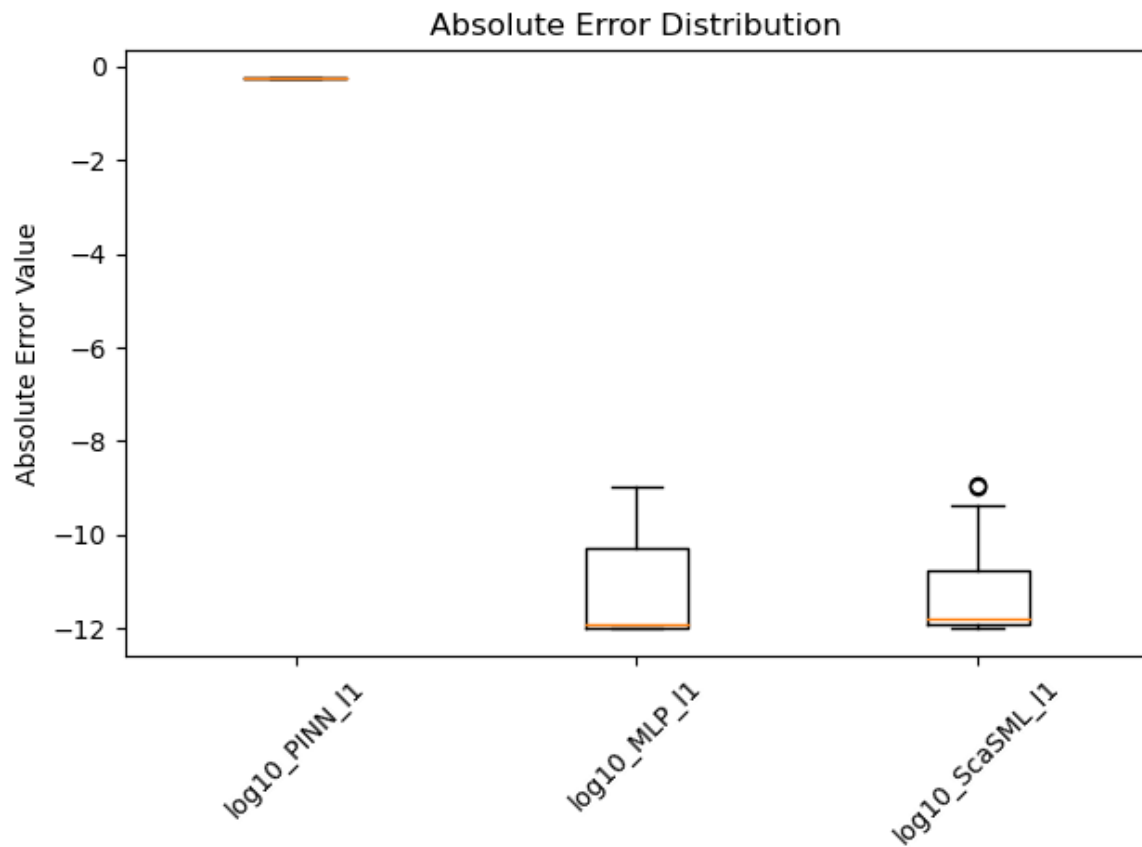
## 100d

Real Solution-> min: 0.0 max: 0.0 mean: 0.0

PINN l1, rho=2-> min: 0.56846434 max: 0.56846434 mean: 0.56846434

MLP l1, rho=2-> min: 0.0 max: 1.0221228874271143e-09 mean: 9.221824070934431e-11

ScaSML l1, rho=2-> min: 9.325873406851315e-15 max: 1.1986187598012066e-09 mean: 8.663899375704887e-11



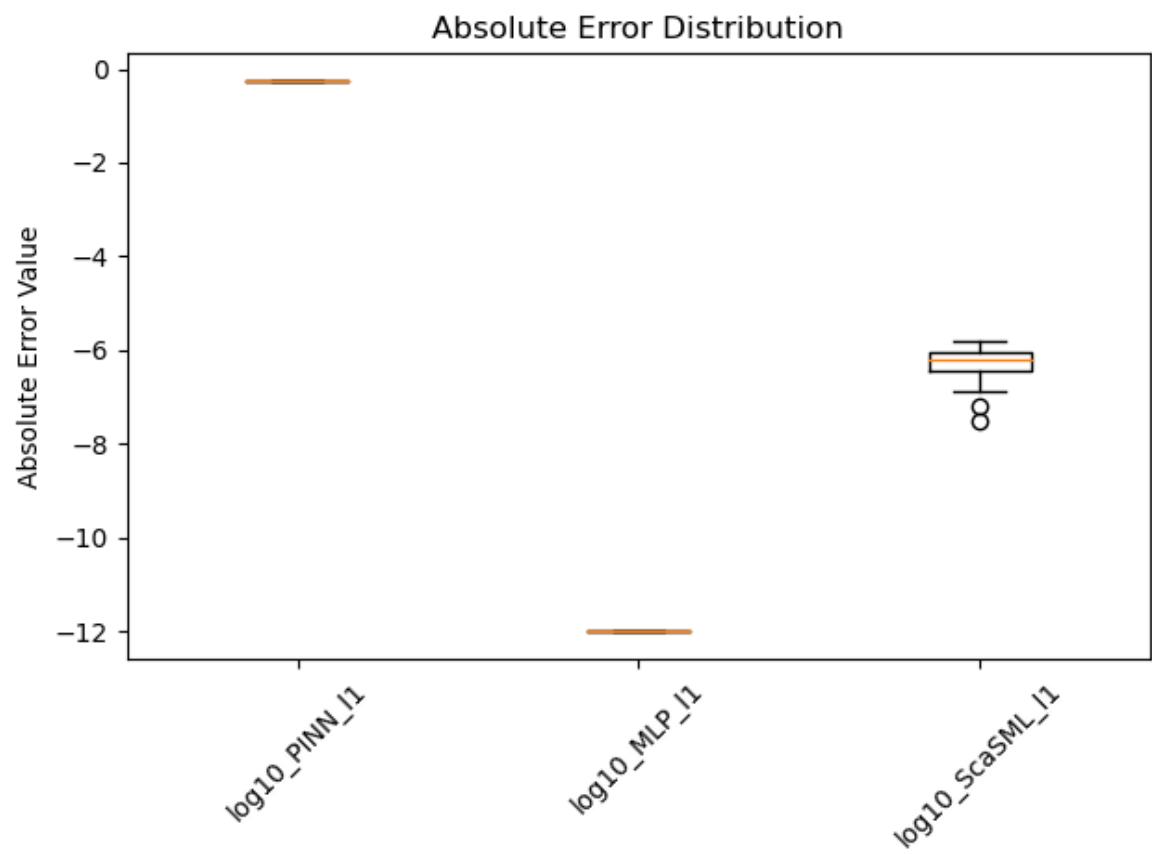
## 250d

Real Solution-> min: 0.0 max: 0.0 mean: 0.0

PINN l1, rho=2-> min: 0.5461806 max: 0.5461825 mean: 0.5461815

MLP l1, rho=2-> min: 0.0 max: 0.0 mean: 0.0

ScaSML l1, rho=2-> min: 3.1806162925640535e-08 max: 1.5431055342407518e-06 mean: 6.294460226086151e-07

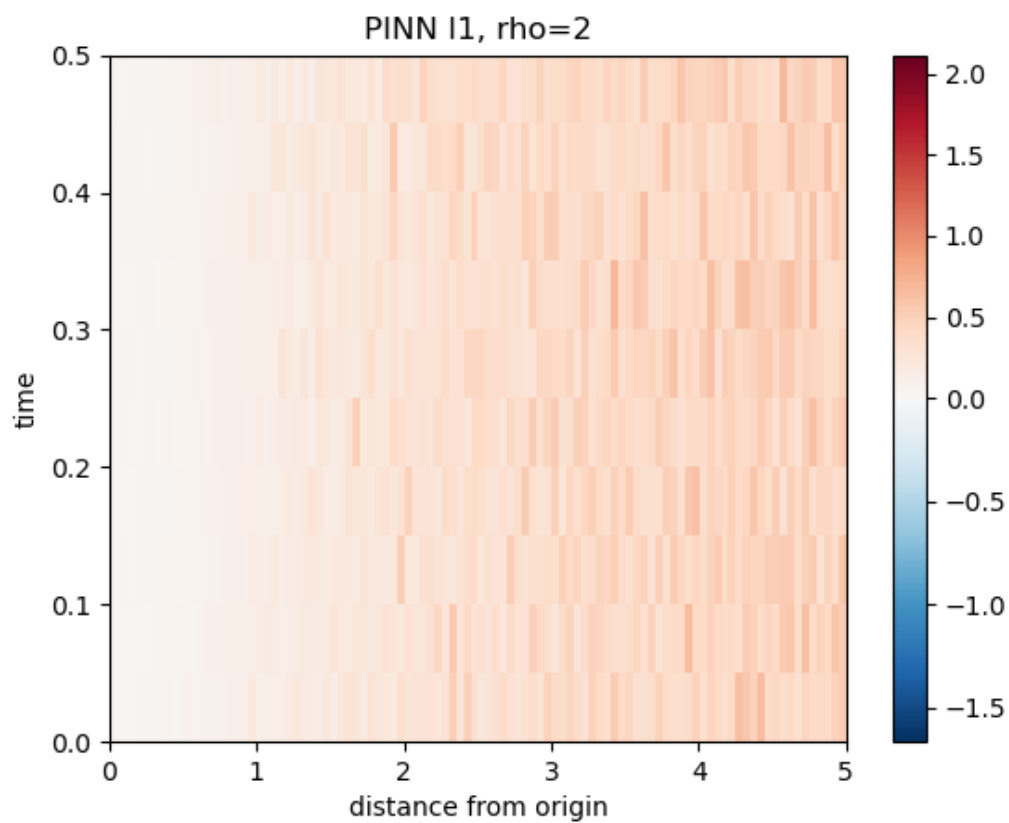


## Neumann\_Boundary

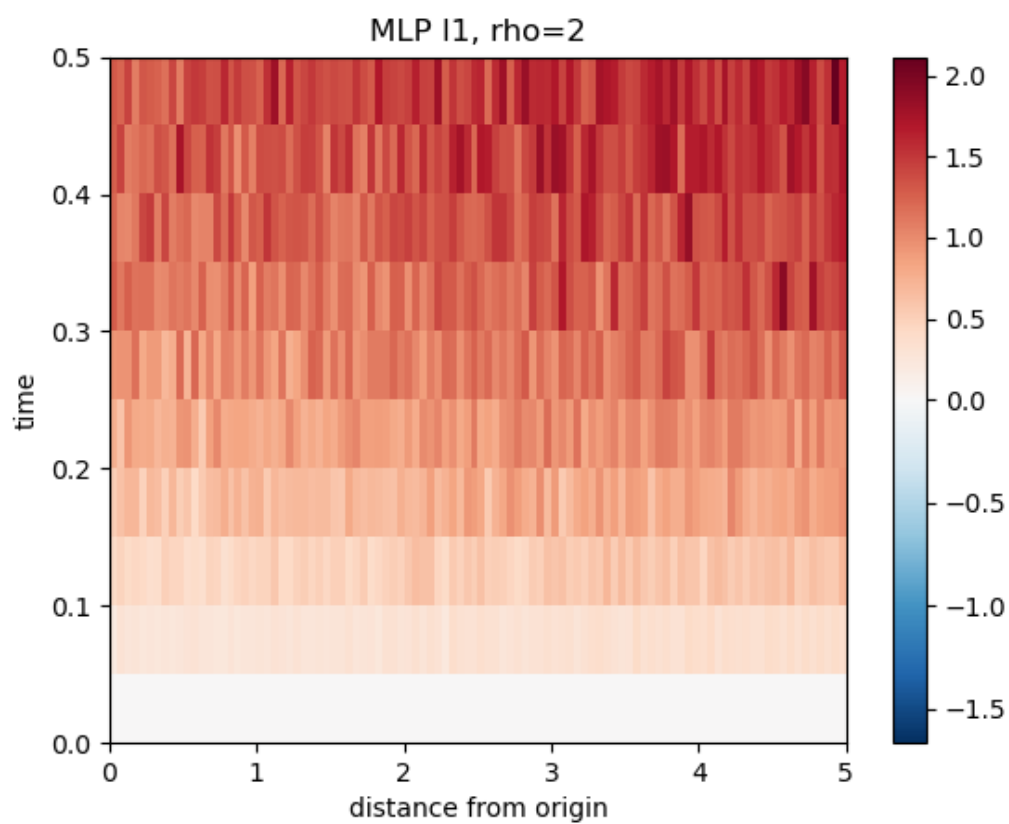
### NormalSphere

100d

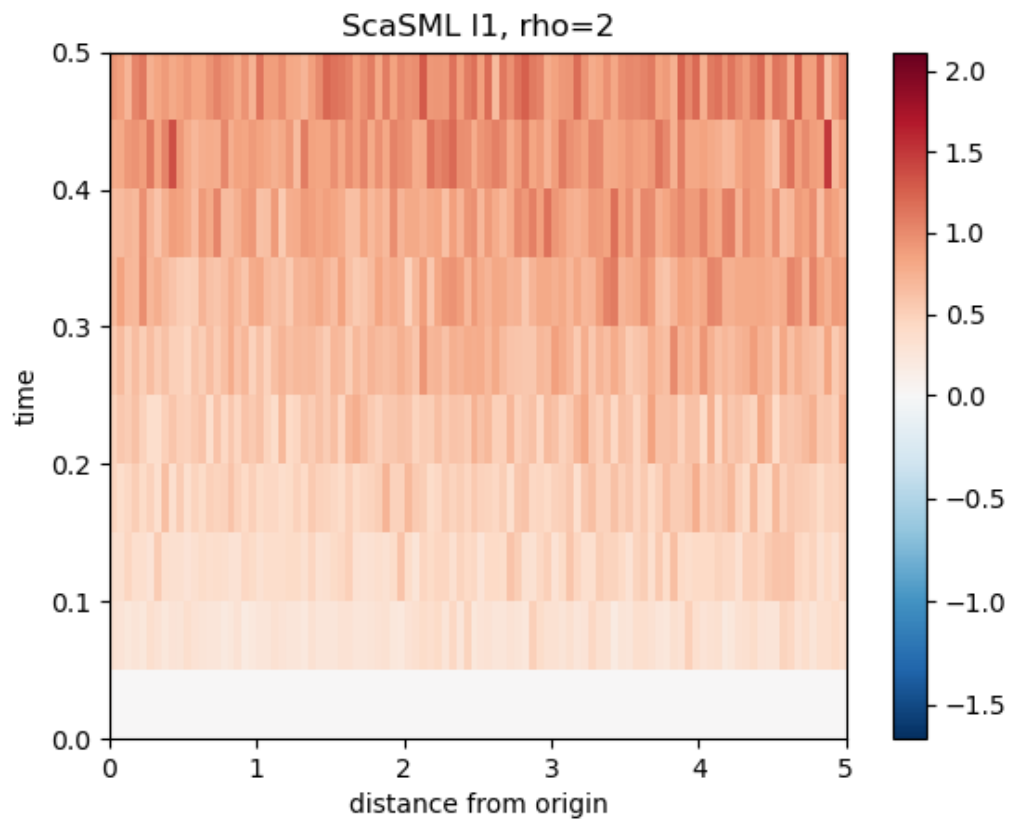
PINN



MLP

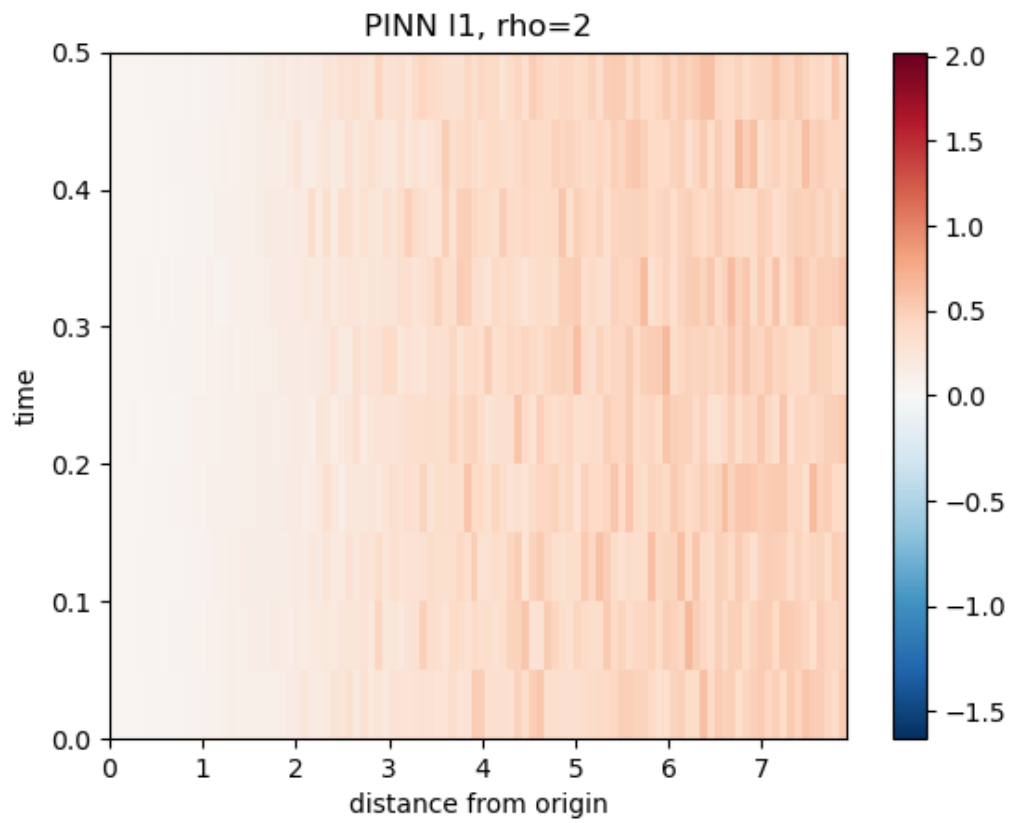


SCaSML

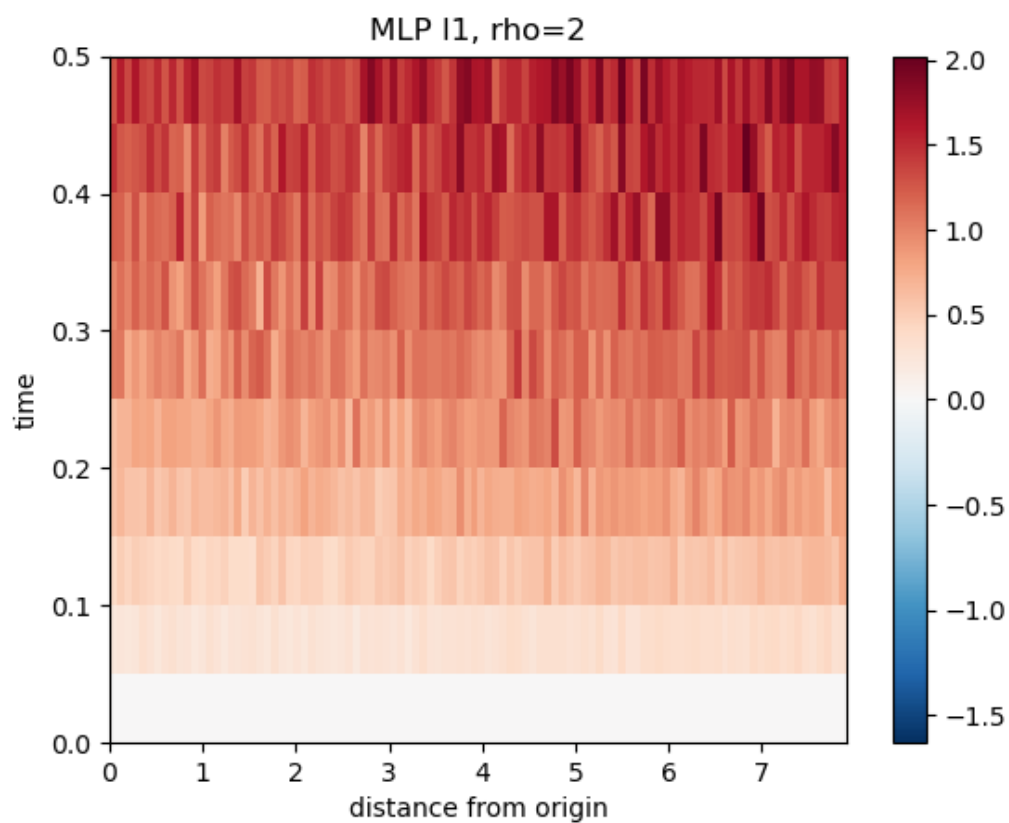


250d

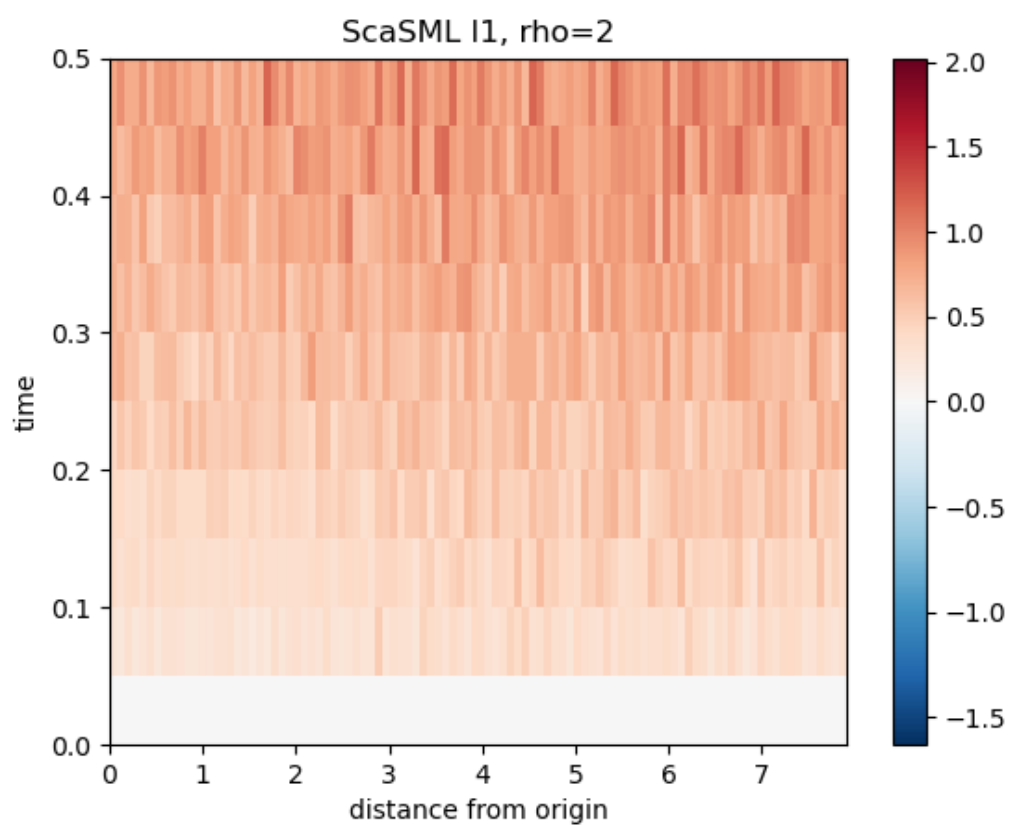
PINN



## MLP



## SCaSML



# SimpleUniform

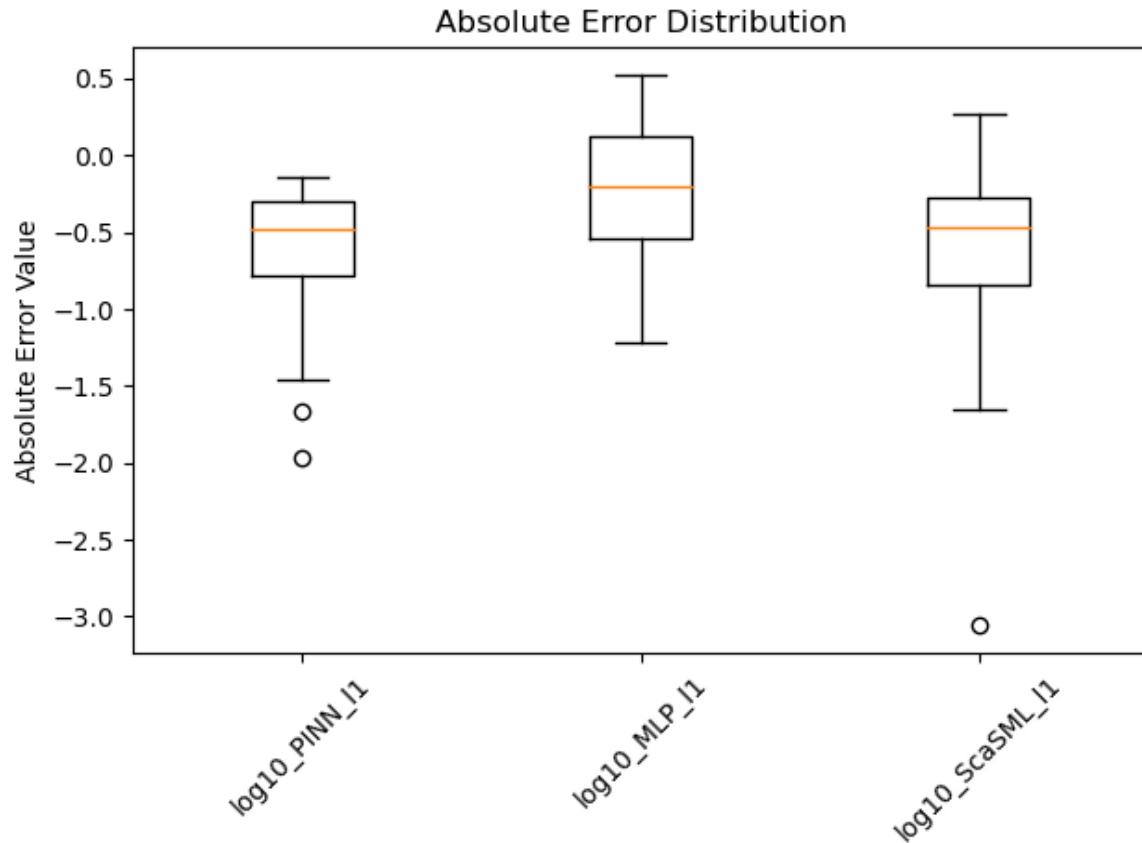
## 100d

Real Solution-> min: 0.0 max: 0.6700127 mean: 0.33071044

PINN l1, rho=2-> min: 0.0108173415 max: 0.72156096 mean: 0.33114284

MLP l1, rho=2-> min: 0.06014029962834201 max: 3.323620538566565 mean:  
0.9010895304055028

ScaSML l1, rho=2-> min: 0.0008780561305484769 max: 1.8739295568279328 mean:  
0.44091610712241114



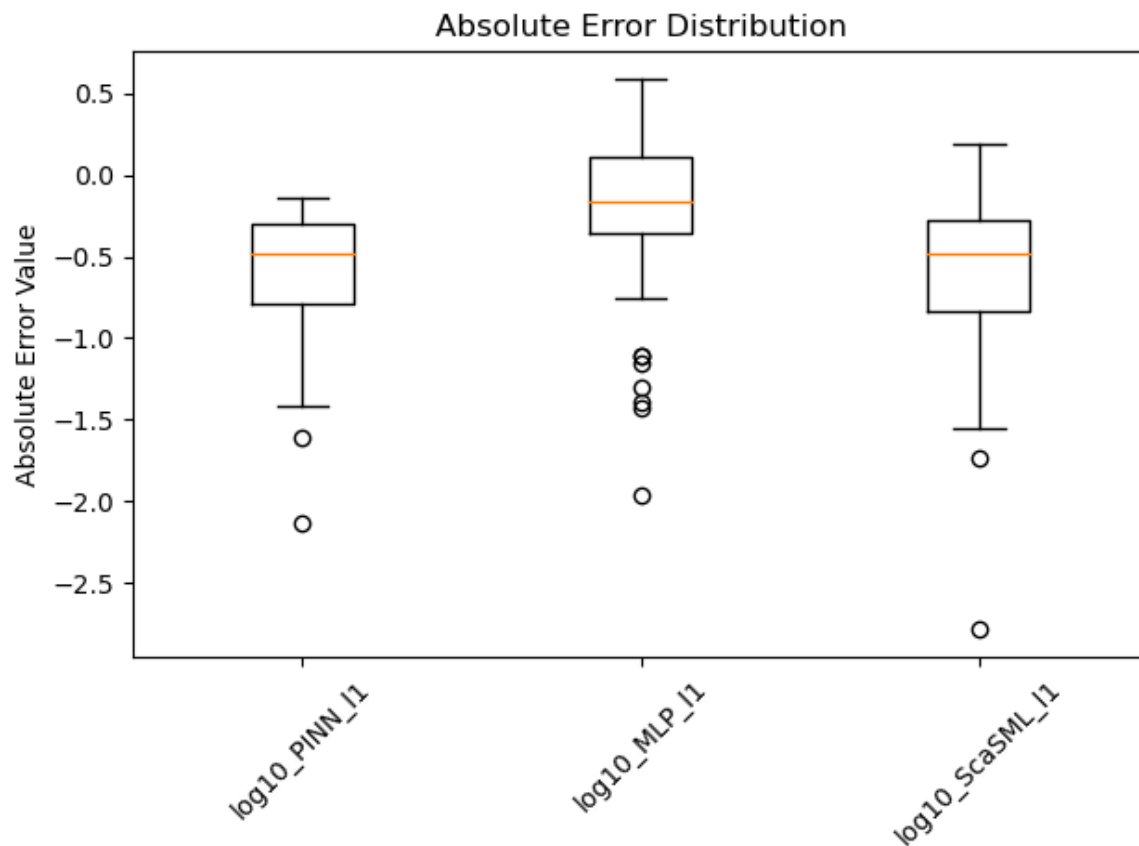
## 250d

Real Solution-> min: 0.0 max: 0.6700127 mean: 0.33071044

PINN l1, rho=2-> min: 0.0073871575 max: 0.71813184 mean: 0.33100587

MLP l1, rho=2-> min: 0.010961447300344873 max: 3.8541148322681673 mean:  
0.8835014103898221

ScaSML l1, rho=2-> min: 0.001643262105699761 max: 1.5398553768754413 mean:  
0.4152441216040941



## Conclusion

### Complicated\_HJB

#### Space-time distribution (NormalSphere):

The images show similar patterns to the Quadrature version, with PINN having a uniform error distribution, while MLP and SCaSML display more localized error patterns. The differences between MLP and SCaSML are subtle in the visualizations.

#### Statistical features (SimpleUniform):

- 100d:
  - PINN: min 20.96, max 48.13, mean 31.84 (unchanged)
  - MLP: min 0.039, max 8.00, mean 2.55 (improved from 0.058, 12.36, 2.95)
  - SCaSML: min 0.068, max 10.01, mean 2.63 (slightly changed from 0.043, 10.53, 2.68)
- 250d:
  - PINN: min 88.28, max 122.26, mean 102.61 (unchanged)
  - MLP: min 0.16, max 15.49, mean 5.08 (slightly worse than 0.076, 14.60, 4.35)
  - SCaSML: min 0.15, max 14.33, mean 4.20 (improved from 0.17, 17.97, 3.46)

In the Full\_History version, MLP shows improvement in 100d, while SCaSML improves in 250d. Both still significantly outperform PINN.

### Explicit\_Solution\_Example



## Space-time distribution (NormalSphere):

The images remain similar to the Quadrature version, with PINN showing uniform error and MLP and SCaSML displaying near-zero errors throughout.

## Statistical features (SimpleUniform):

- 100d:  
PINN: min 0.57, max 0.57, mean 0.57 (unchanged)  
MLP: min 0, max  $1.02 \times 10^{-9}$ , mean  $9.22 \times 10^{-11}$  (slightly higher errors than 0,  $1.08 \times 10^{-9}$ ,  $7.37 \times 10^{-11}$ )  
SCaSML: min  $9.33 \times 10^{-15}$ , max  $1.20 \times 10^{-9}$ , mean  $8.66 \times 10^{-11}$  (slightly higher errors than  $2.26 \times 10^{-14}$ ,  $1.10 \times 10^{-9}$ ,  $6.94 \times 10^{-11}$ )
- 250d:  
PINN: min 0.55, max 0.55, mean 0.55 (unchanged)  
MLP: min 0, max 0, mean 0 (unchanged, perfect accuracy)  
SCaSML: min  $3.18 \times 10^{-8}$ , max  $1.54 \times 10^{-6}$ , mean  $6.29 \times 10^{-7}$  (slightly improved from  $1.05 \times 10^{-8}$ ,  $1.62 \times 10^{-6}$ ,  $6.59 \times 10^{-7}$ )

The Full\_History version shows minor changes in error magnitudes, but the overall performance remains similar, with both MLP and SCaSML vastly outperforming PINN.

## Neumann\_Boundary

### Space-time distribution (NormalSphere):

The images show patterns similar to the Quadrature version, with PINN having a smooth error distribution and MLP and SCaSML displaying more complex patterns. SCaSML appears to have smaller error magnitudes compared to MLP.

### Statistical features (SimpleUniform):

- 100d:  
Real Solution: min 0, max 0.67, mean 0.33 (unchanged)  
PINN: min 0.011, max 0.72, mean 0.33 (unchanged)  
MLP: min 0.060, max 3.32, mean 0.90 (worse than 0.0023, 2.45, 0.62)  
SCaSML: min 0.00088, max 1.87, mean 0.44 (changed from 0.024, 0.97, 0.28)
- 250d:  
Real Solution: min 0, max 0.67, mean 0.33 (unchanged)  
PINN: min 0.0074, max 0.72, mean 0.33 (unchanged)  
MLP: min 0.011, max 3.85, mean 0.88 (worse than 0.0016, 1.89, 0.65)  
SCaSML: min 0.0016, max 1.54, mean 0.42 (changed from 0.00069, 1.02, 0.30)

In the Full\_History version, MLP's performance deteriorates, while SCaSML's results change but remain closer to the real solution mean compared to MLP.

Comparison with Quadrature Case:

1. Complicated\_HJB: The Full\_History version shows improvements for MLP in 100d and SCaSML in 250d. Both methods continue to outperform PINN significantly.
2. Explicit\_Solution\_Example: Results are very similar between versions, with only minor fluctuations in error magnitudes. MLP and SCaSML maintain their substantial advantage over PINN.
3. Neumann\_Boundary: The Full\_History version shows decreased performance for MLP, while SCaSML's results change but remain more accurate than MLP's. PINN's performance is unchanged.

Overall, the Full\_History version demonstrates that the SCaSML algorithm still effectively calibrates the bias of PINN via MLP, although with some variations in performance compared to the Quadrature version:

1. Space-time distribution: The error patterns remain similar across versions, with SCaSML generally showing more localized and smaller error regions compared to PINN.
2. Statistical features: SCaSML maintains lower mean errors or closer approximations to the real solution mean in most cases, especially for complex equations like Complicated\_HJB and Neumann\_Boundary.
3. Relative performance: While there are some changes in absolute error values, SCaSML generally maintains its advantage over PINN and often outperforms MLP, particularly in higher dimensions.
4. Robustness: SCaSML demonstrates consistent performance across different equation types and dimensions, showcasing its ability to adapt to various problem scenarios.



