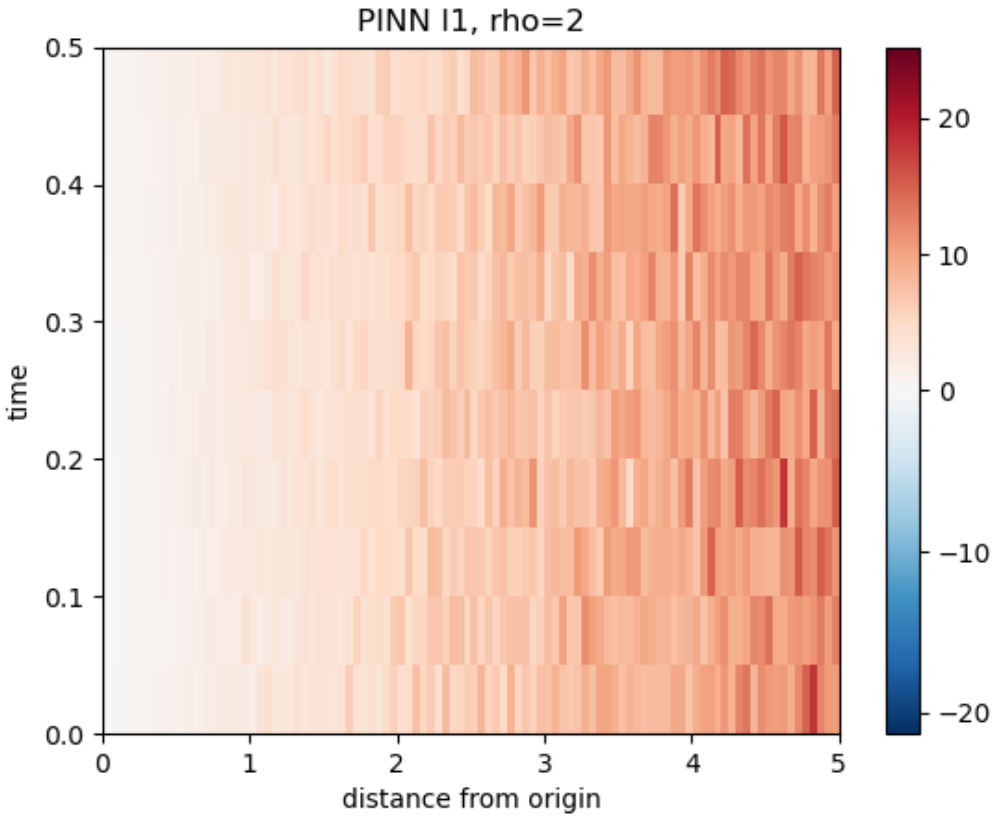
Summary_ of_Results(MAE,Quadrature)

Complicated_HJB

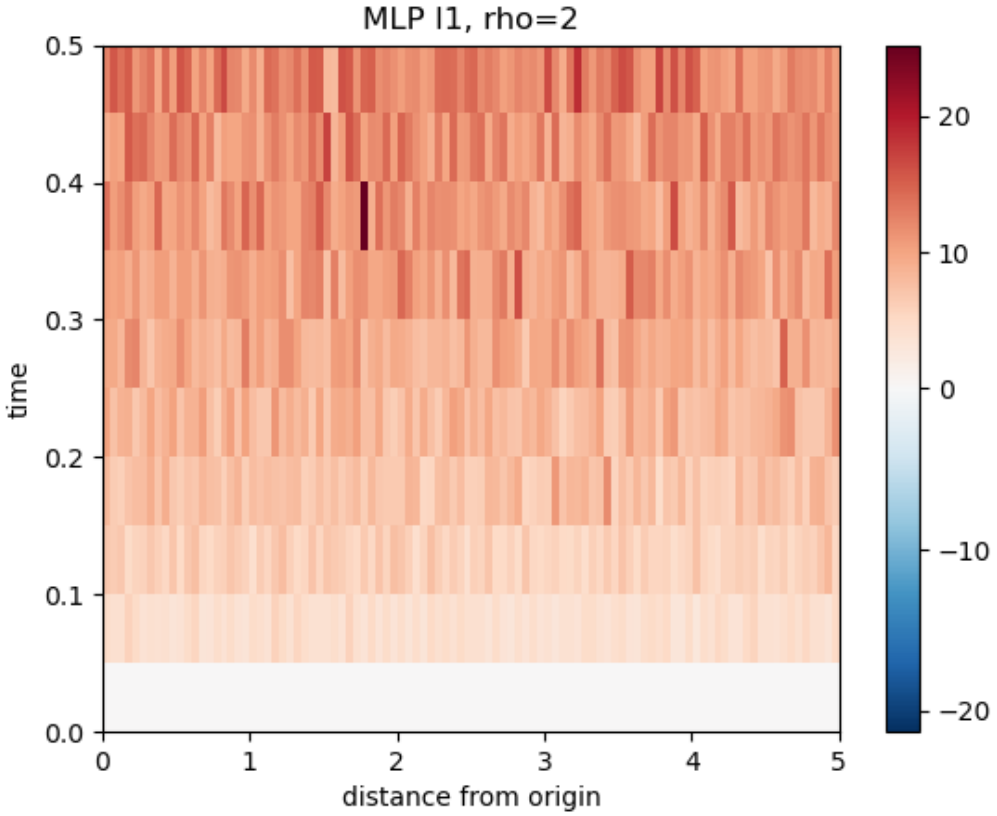
NormalSphere

100d

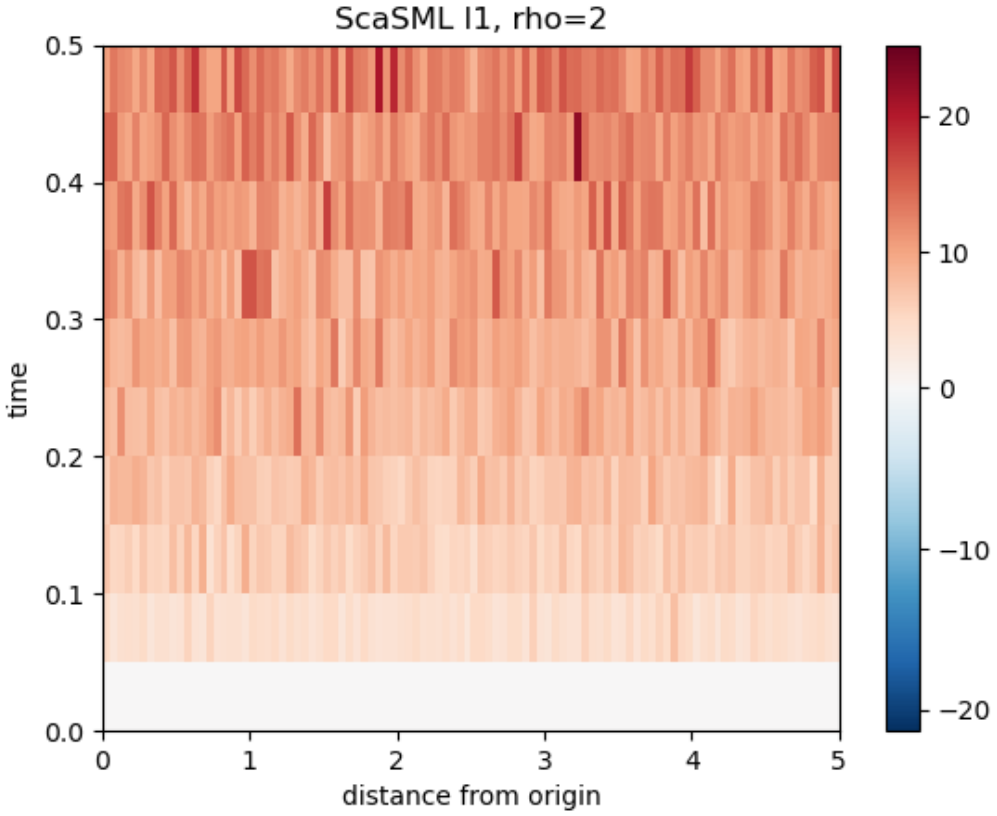
PINN



MLP

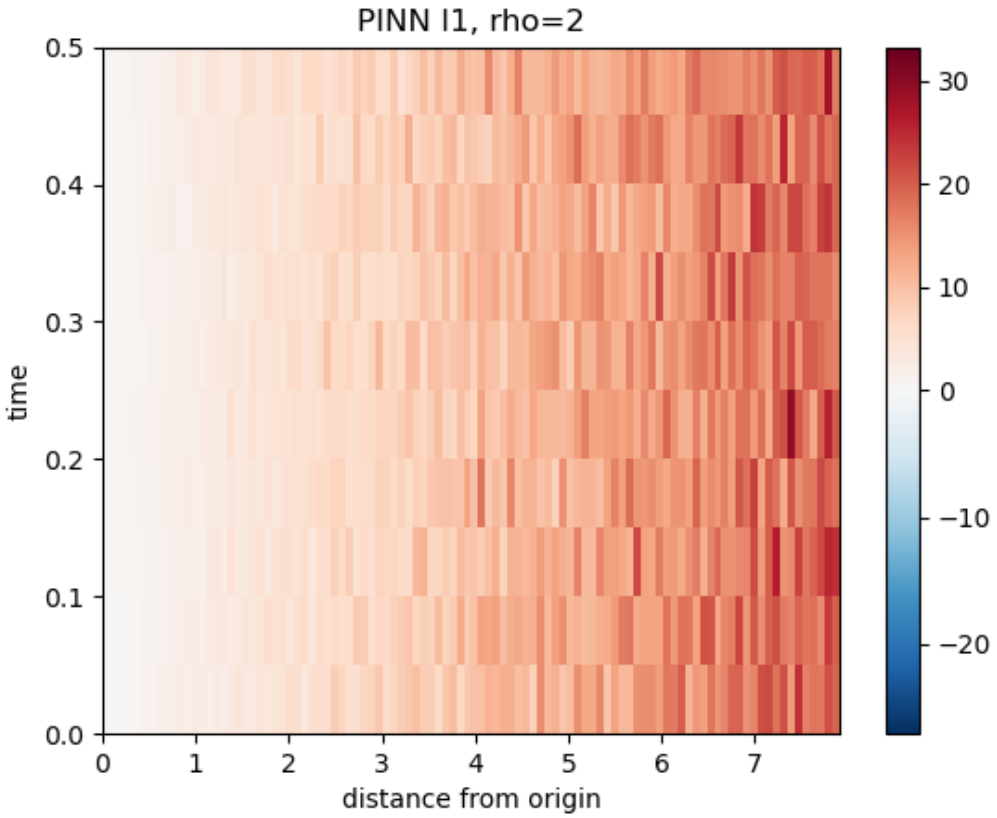

MLP l1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
20
10
0
−10
−20

SCaSML

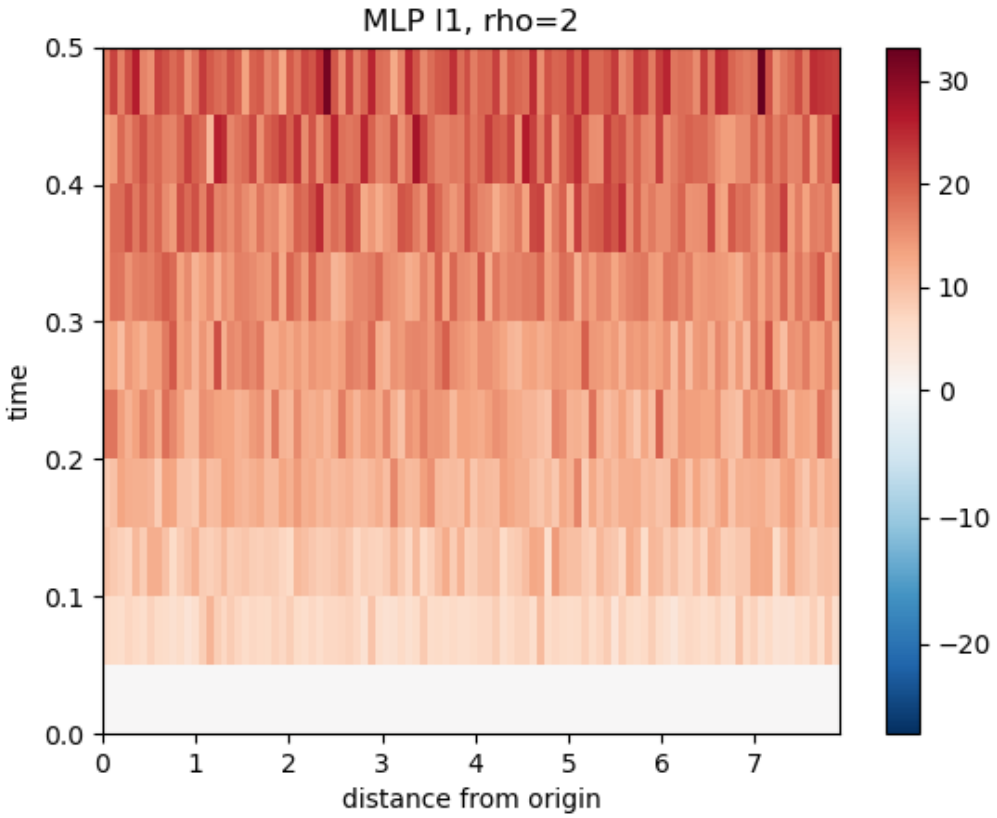

ScaSML l1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
20
10
0
−10
−20

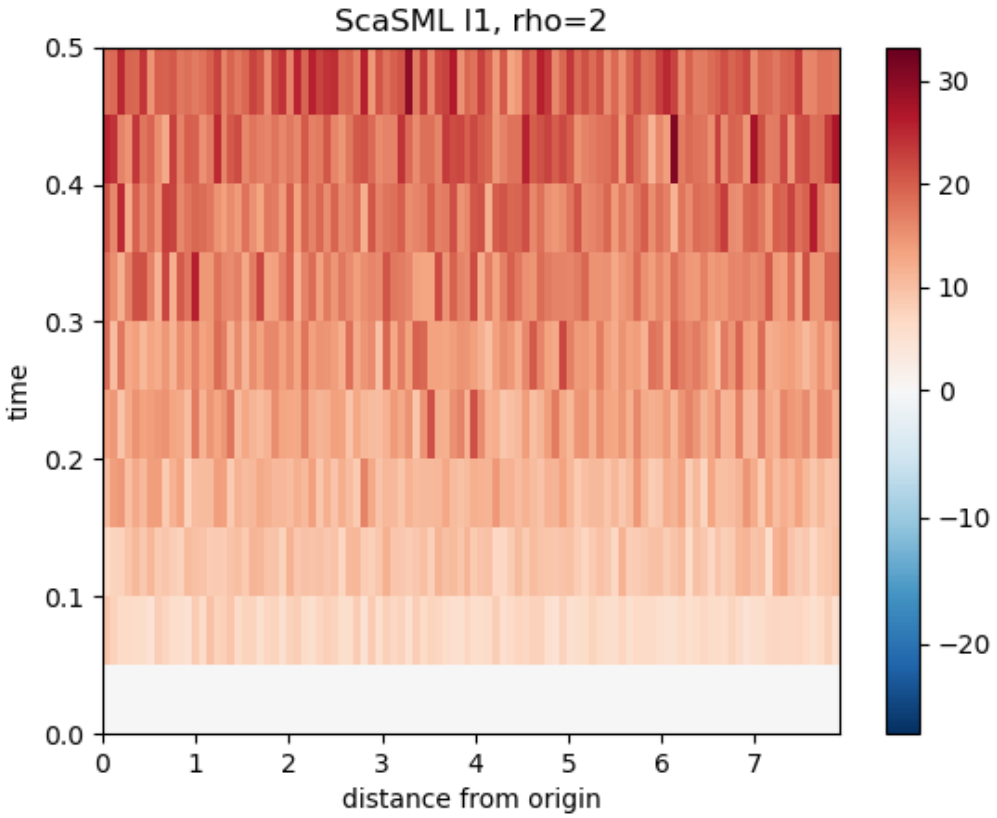
250d

PINN



MLP

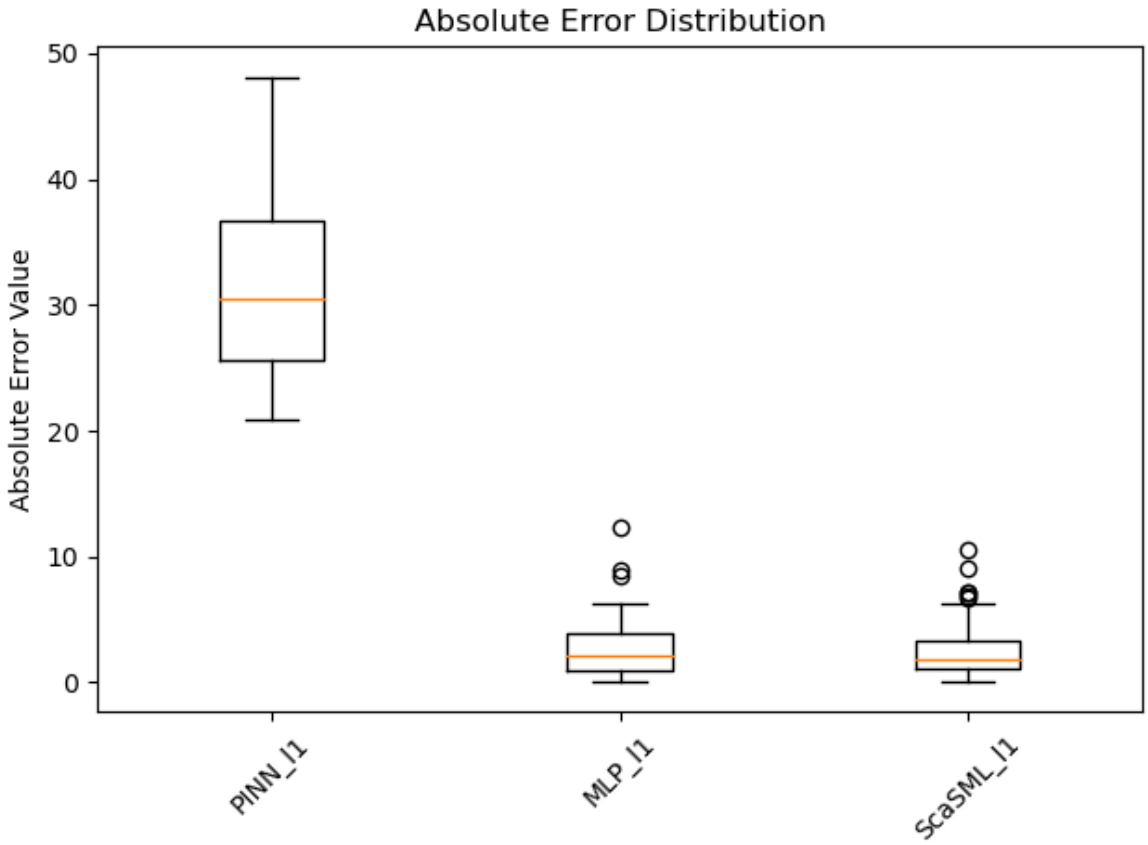




SimpleUniform

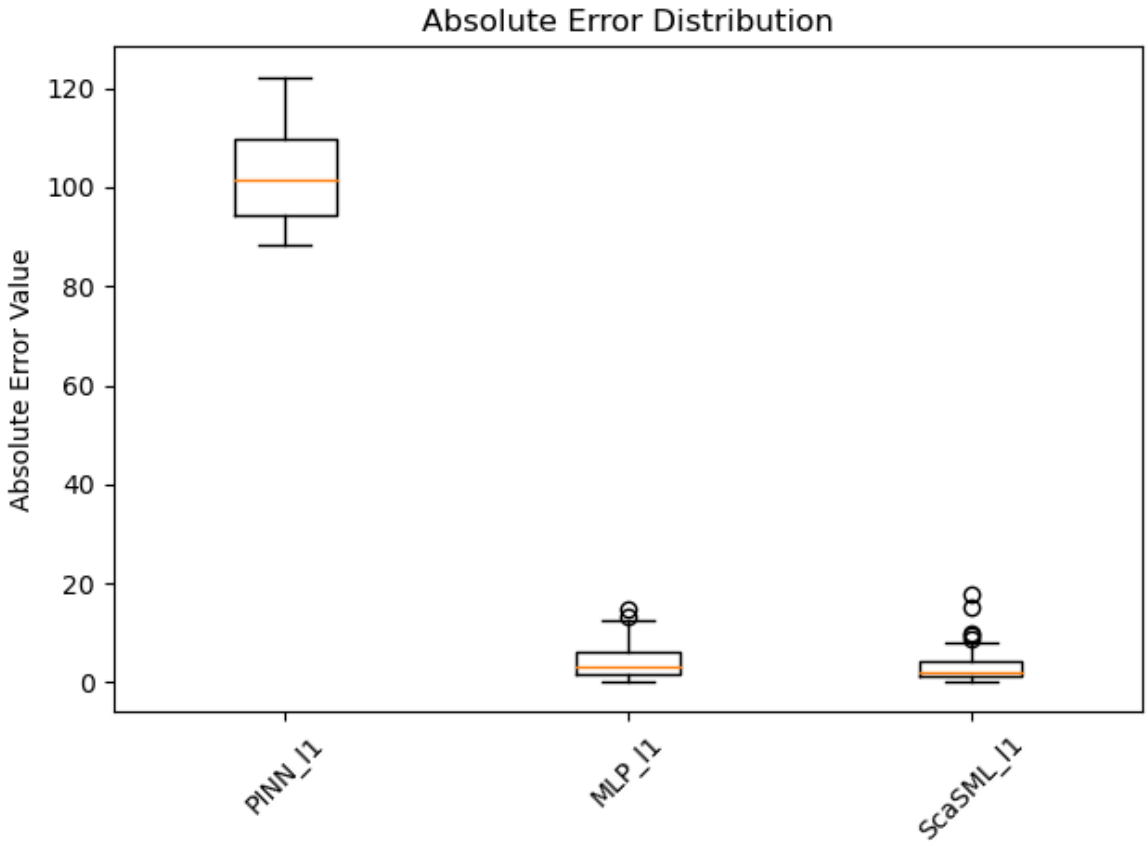
100d

Real Solution-> min: 21.08817 max: 47.40346 mean: 31.57981
PINN l1, rho=2-> min: 20.95896 max: 48.12542 mean: 31.836462
MLP l1, rho=2-> min: 0.05847326710820866 max: 12.363025657488983 mean: 2.952153668060298
ScaSML l1, rho=2-> min: 0.04254206304331021 max: 10.531610404368259 mean: 2.677646860116892



250d

Real Solution-> min: 88.26935 max: 122.24708 mean: 102.59221
PINN l1, rho=2-> min: 88.28333 max: 122.26325 mean: 102.6078
MLP l1, rho=2-> min: 0.07627880688464472 max: 14.604461395696063 mean: 4.35296246560208
ScaSML l1, rho=2-> min: 0.17054694273693372 max: 17.96600626310544 mean: 3.458098372675652

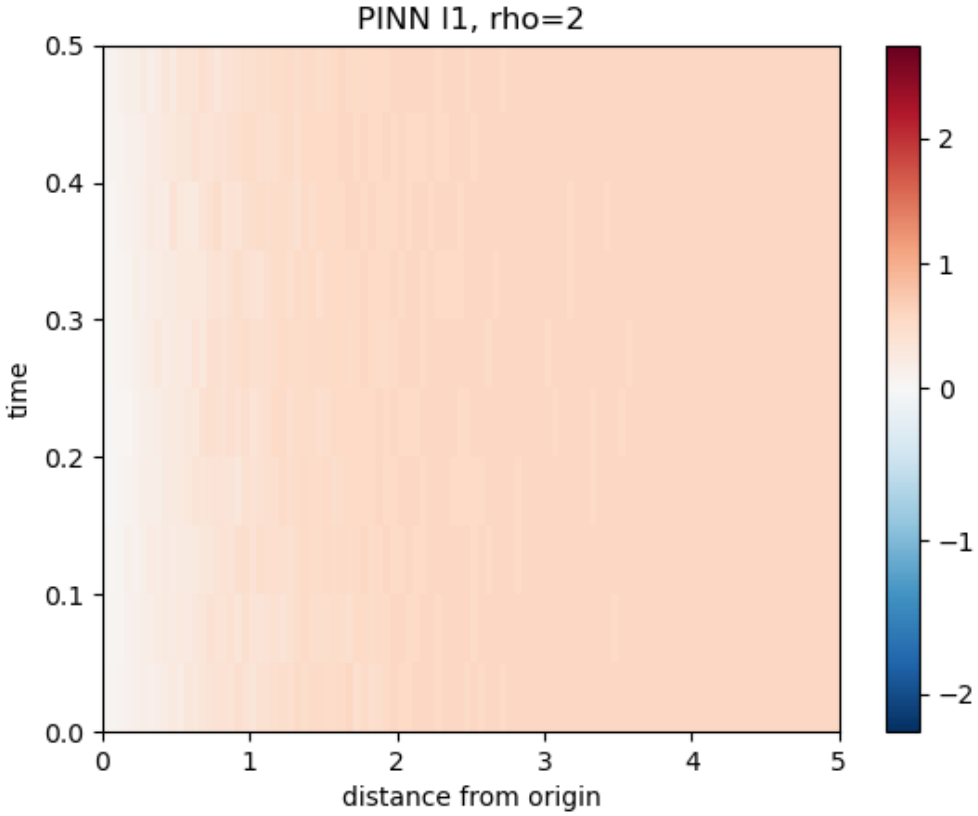


Explicit_Solution_Example

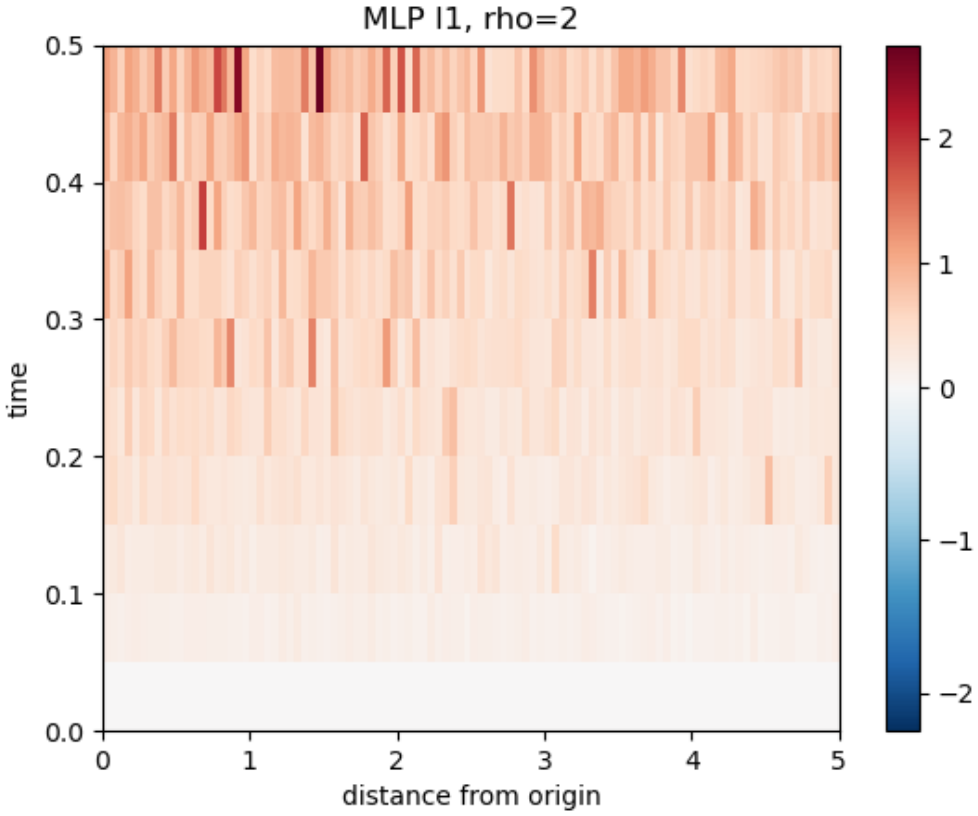
NormalSphere

100d

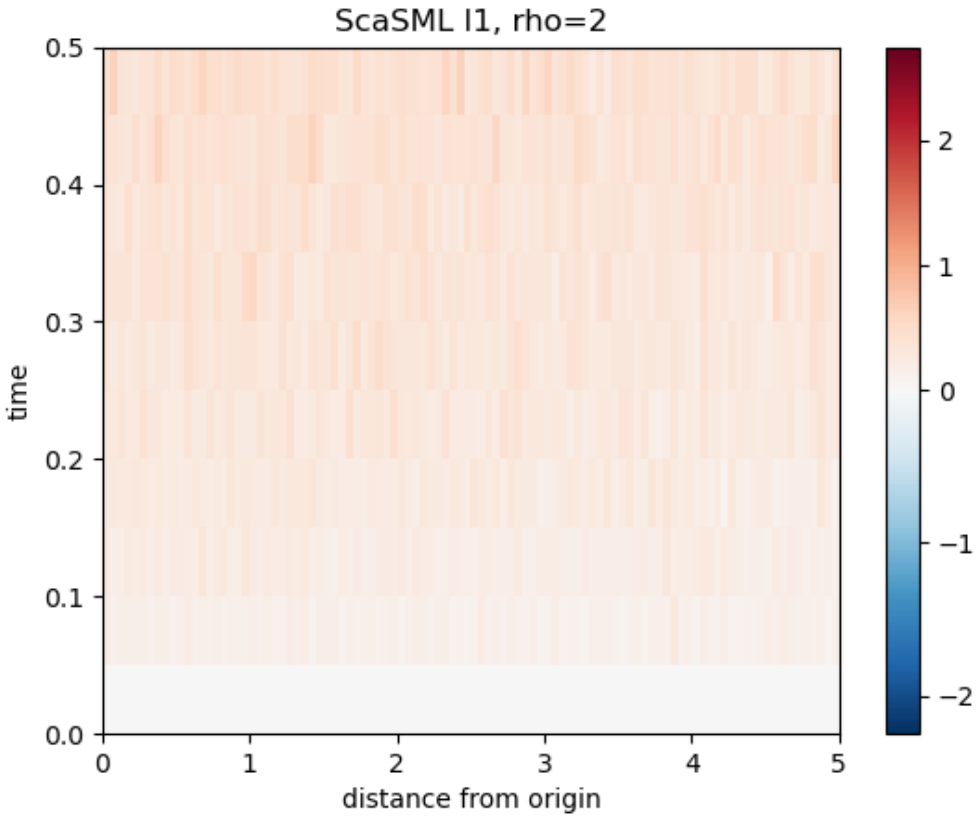
PINN


PINN I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
2
1
0
−1
−2

MLP

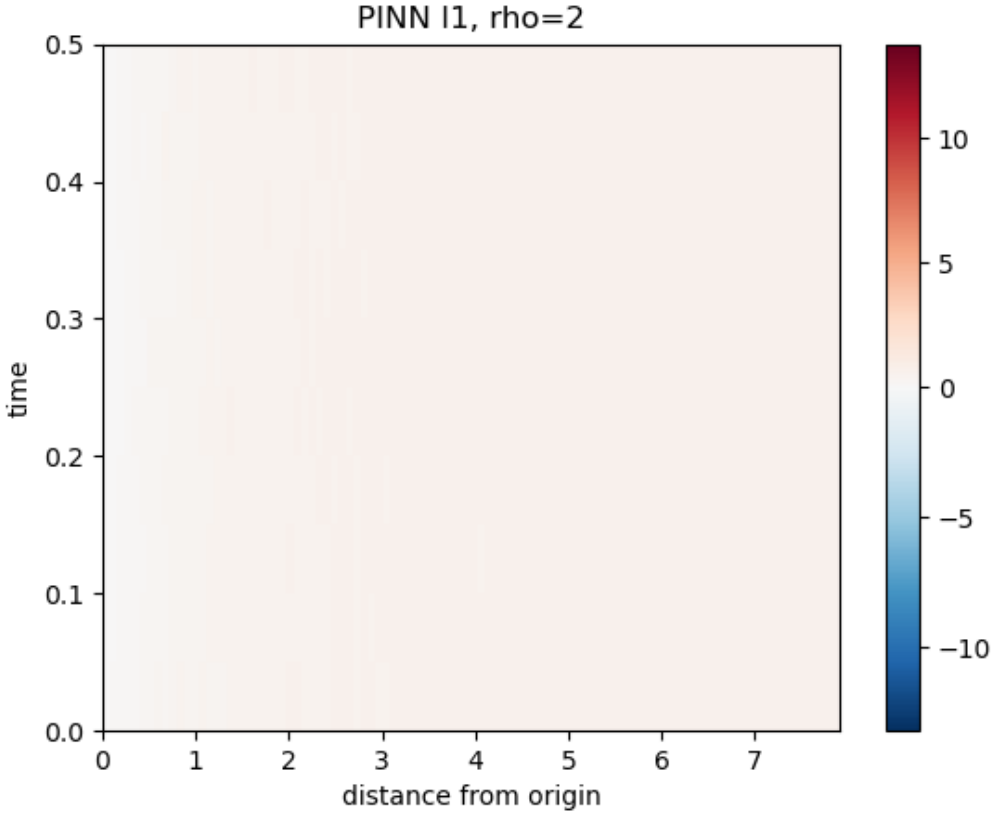

MLP I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
2
1
0
−1
−2

SCaSML

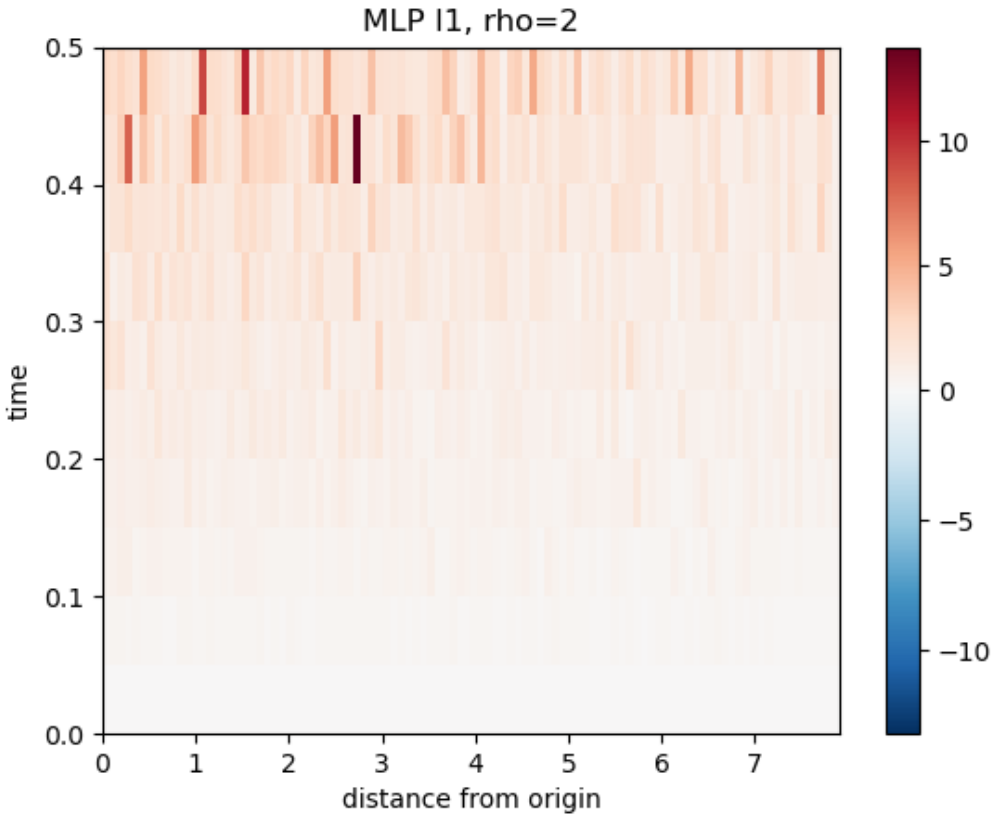


250d

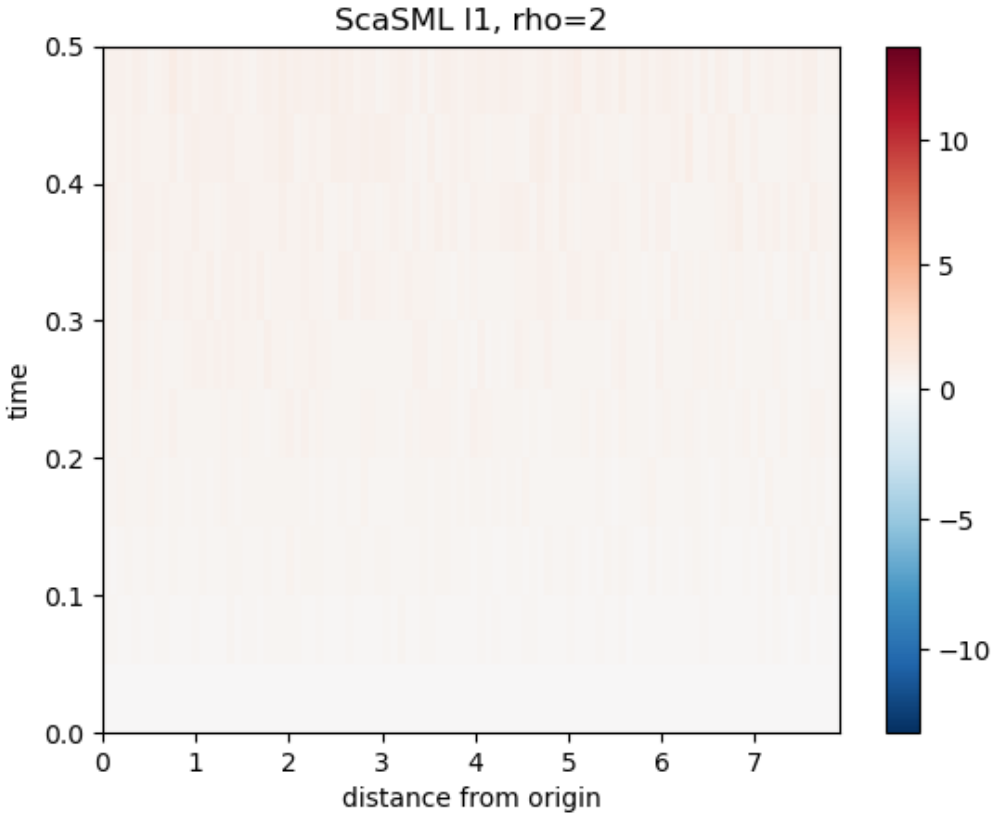
PINN



MLP


MLP I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6
7
10
5
0
−5
−10

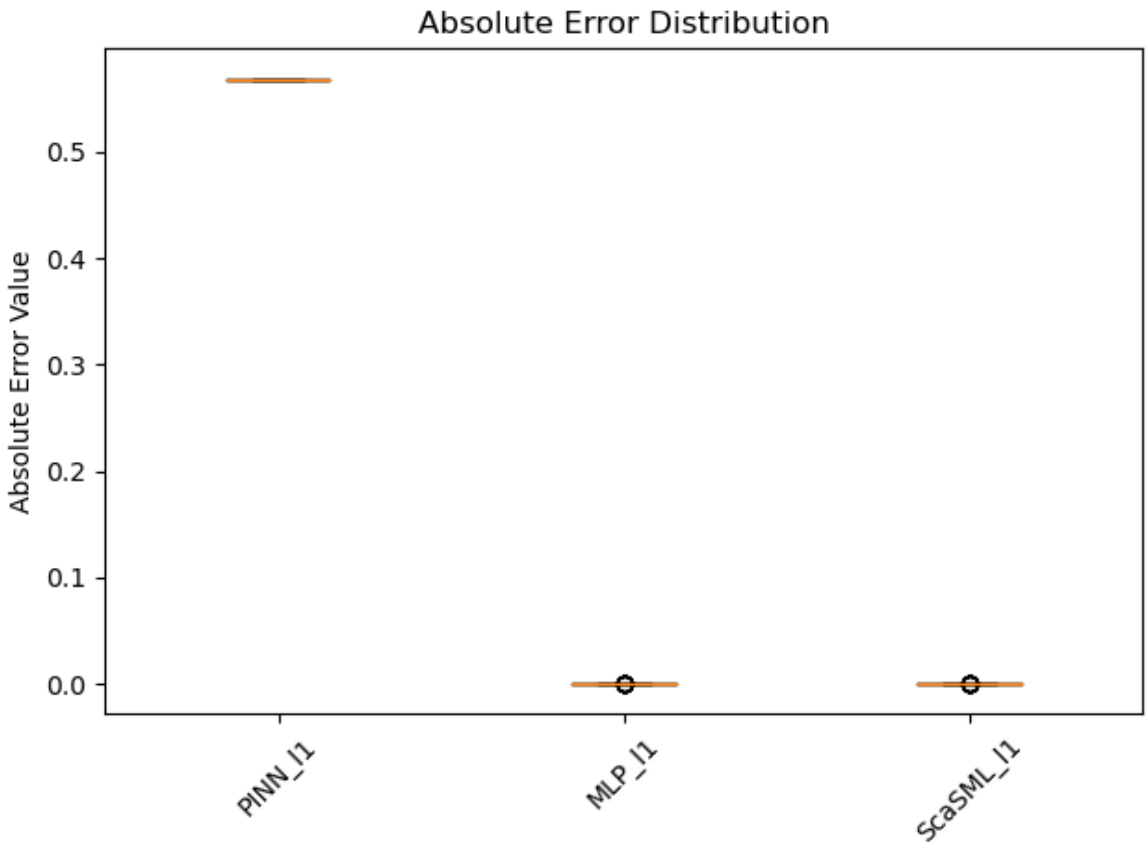
SCaSML


ScaSML I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6
7
10
5
0
−5
−10

SimpleUniform

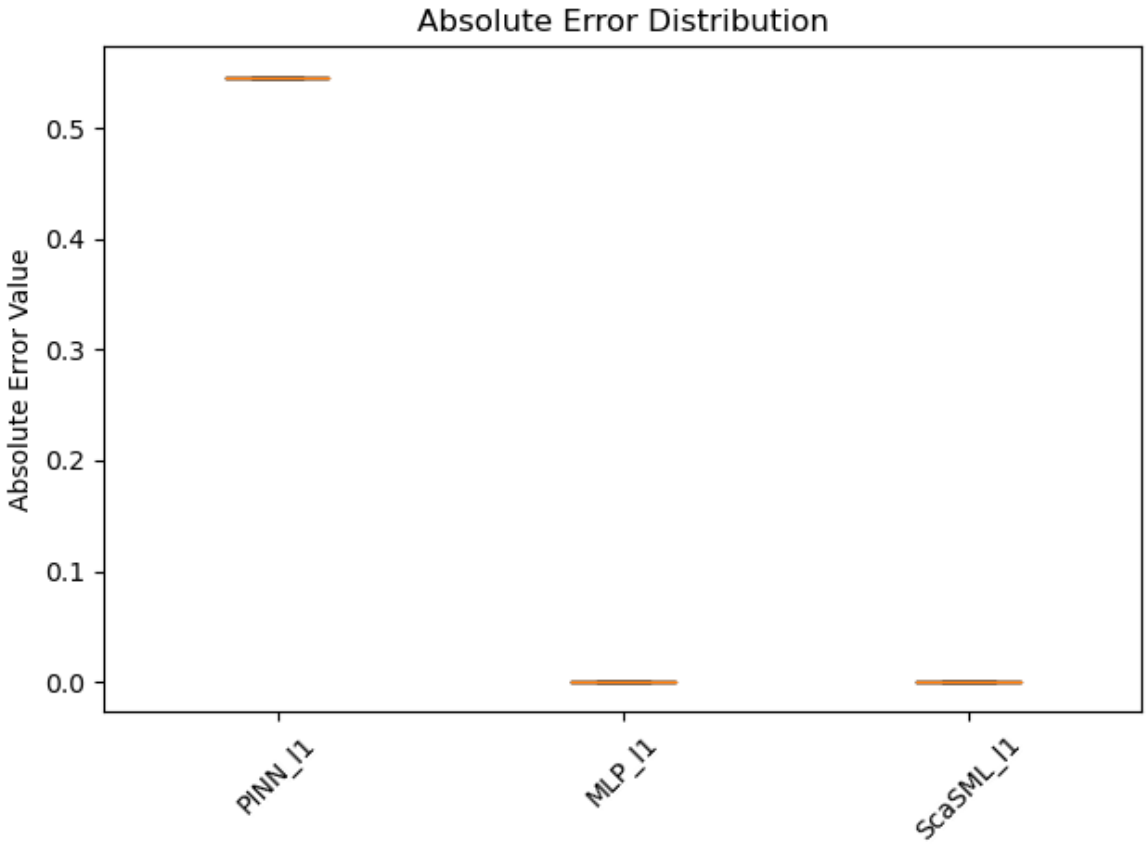
100d

Real Solution-> min: 0.0 max: 0.0 mean: 0.0
PINN l1, rho=2-> min: 0.56846434 max: 0.56846434 mean: 0.56846434
MLP l1, rho=2-> min: 0.0 max: 1.0789062826160773e-09 mean: 7.372649961134982e-11
ScaSML l1, rho=2-> min: 2.2648549702353193e-14 max: 1.0982519338398333e-09 mean: 6.936483165631557e-11



250d

Real Solution-> min: 0.0 max: 0.0 mean: 0.0
PINN l1, rho=2-> min: 0.5461806 max: 0.5461825 mean: 0.5461815
MLP l1, rho=2-> min: 0.0 max: 0.0 mean: 0.0
ScaSML l1, rho=2-> min: 1.0528164562373377e-08 max: 1.6235401180120235e-06 mean: 6.588286073316851e-07

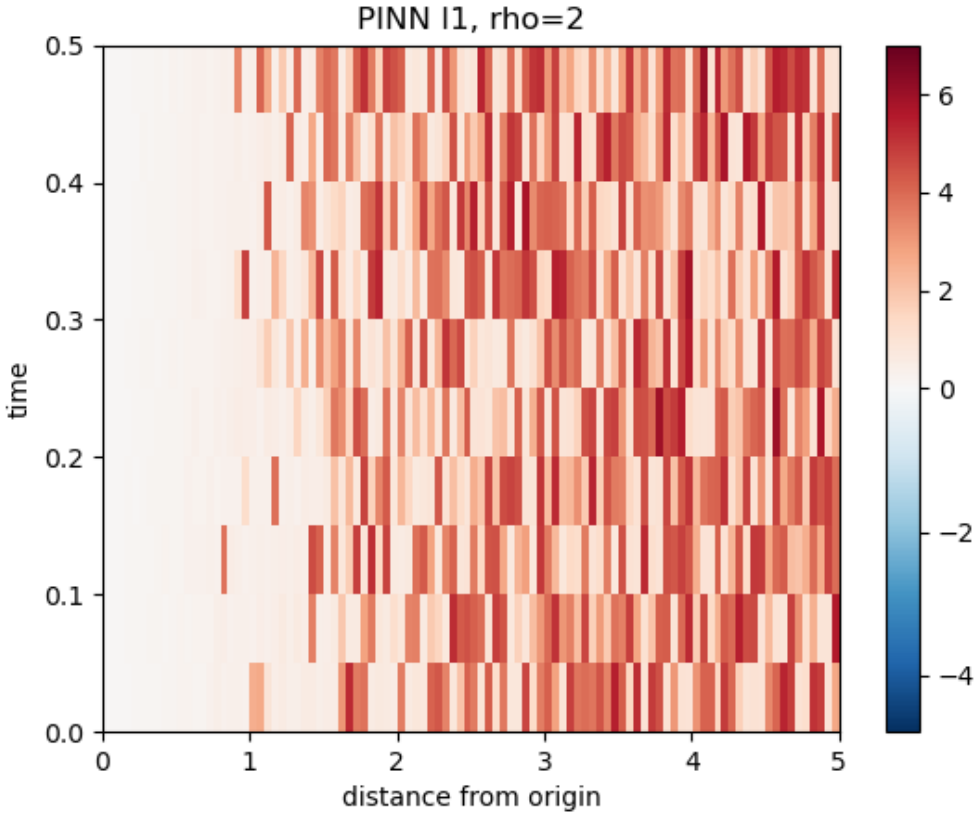


Neumann_Boundary

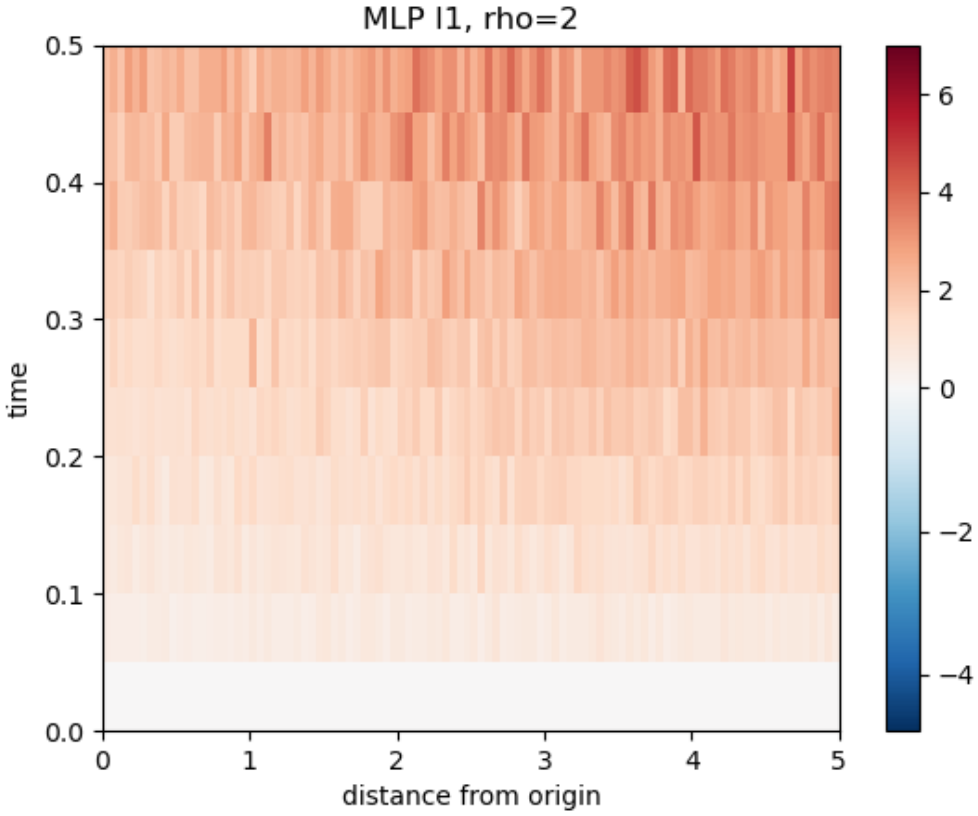
NormalSphere

100d

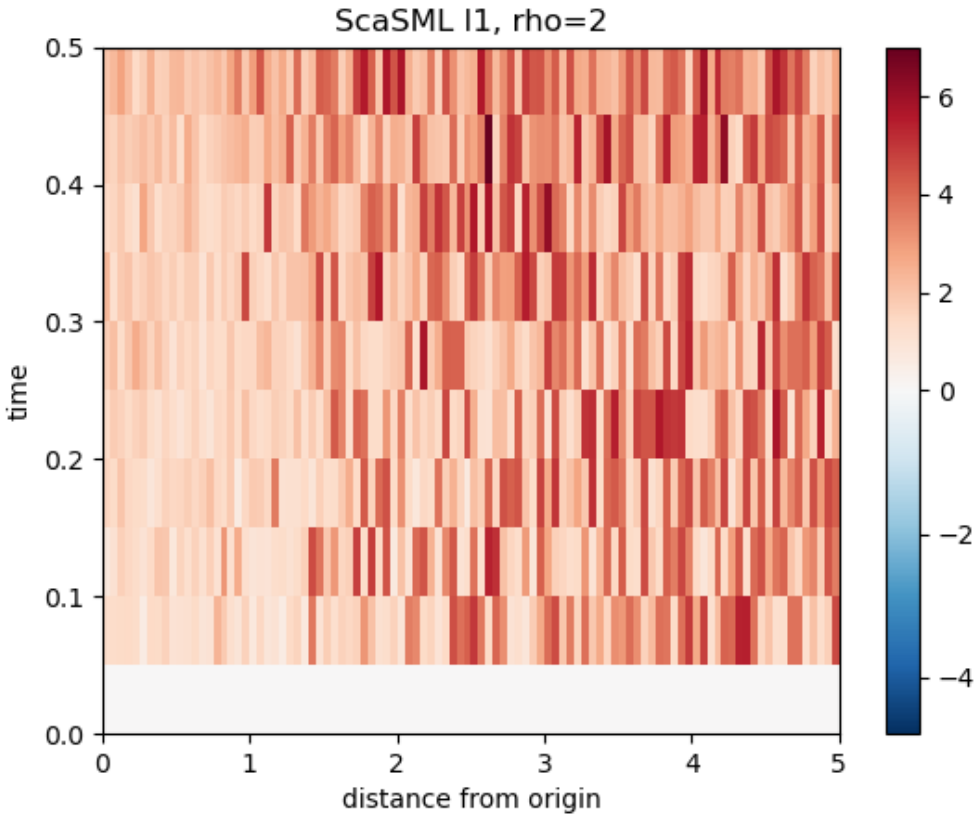
PINN


PINN I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6
4
2
0
−2
−4

MLP

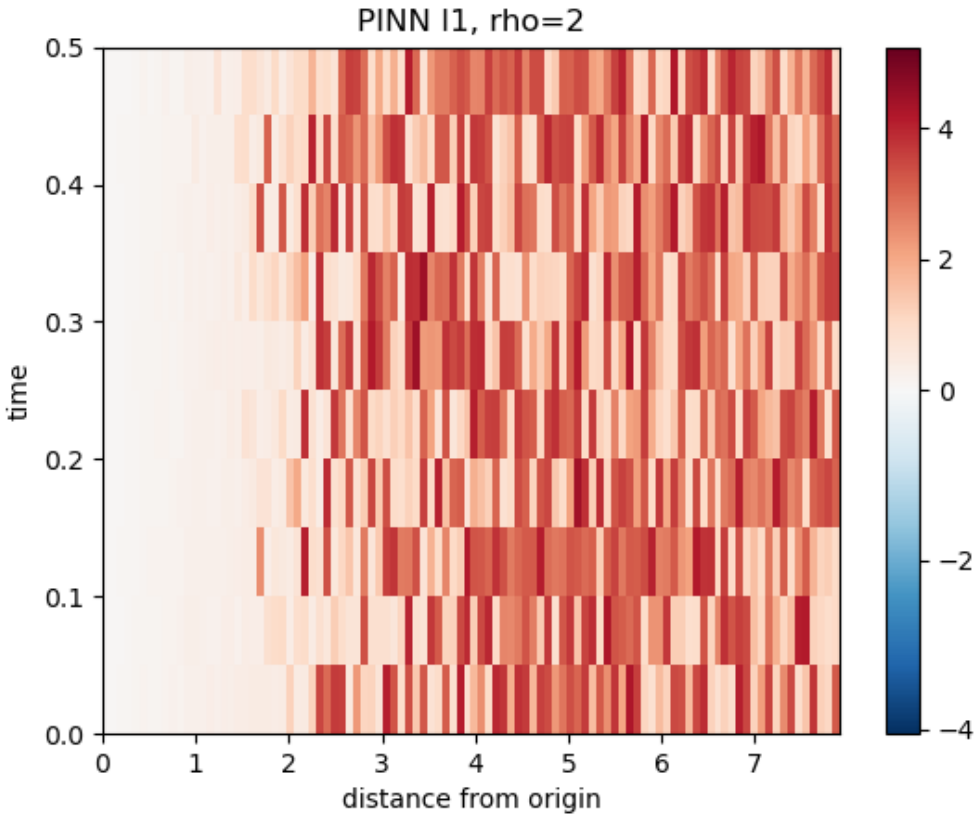

MLP I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6
4
2
0
−2
−4

SCaSML

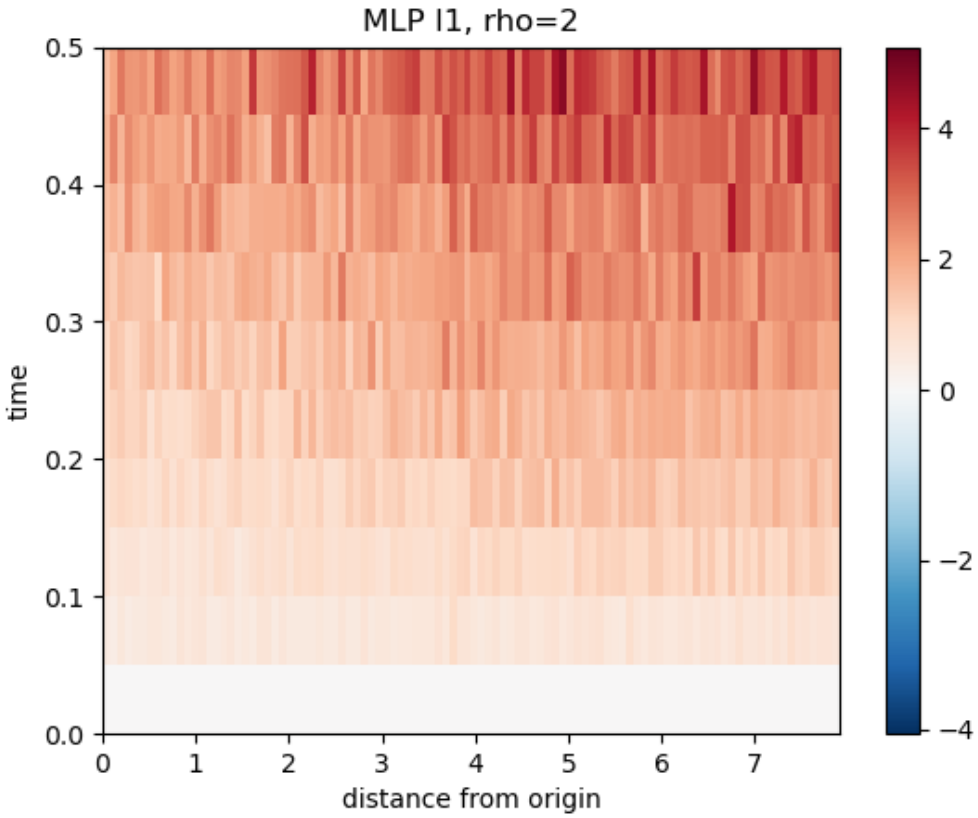


250d

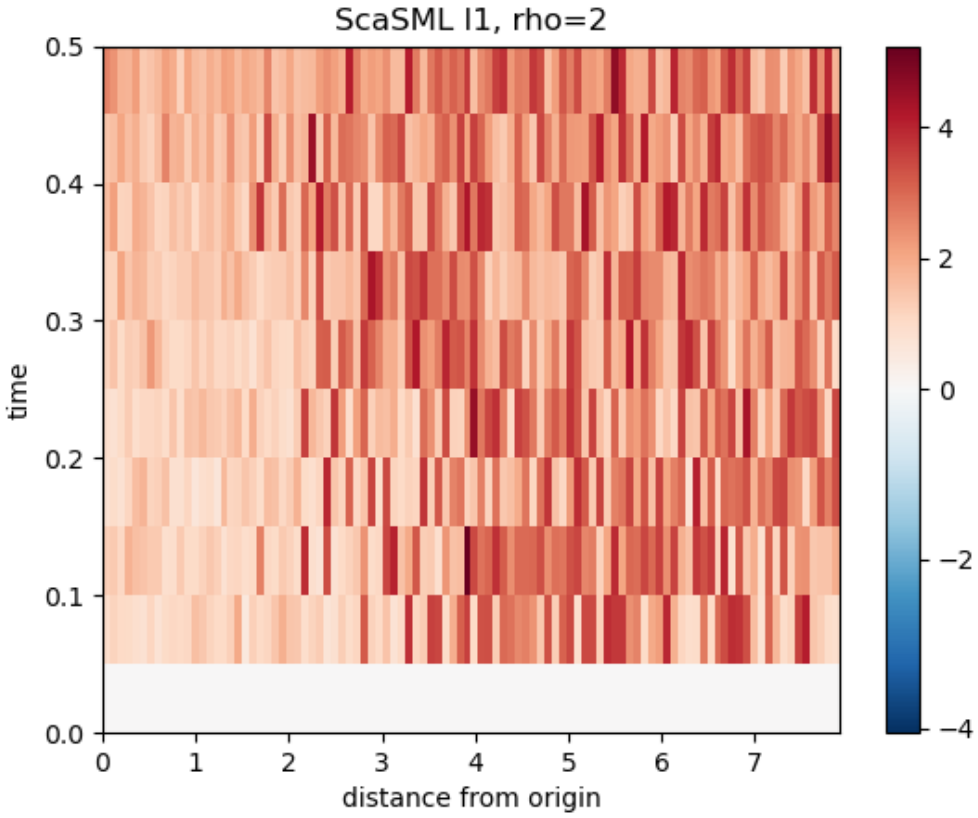
PINN



MLP


MLP I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6
7
4
2
0
−2
−4

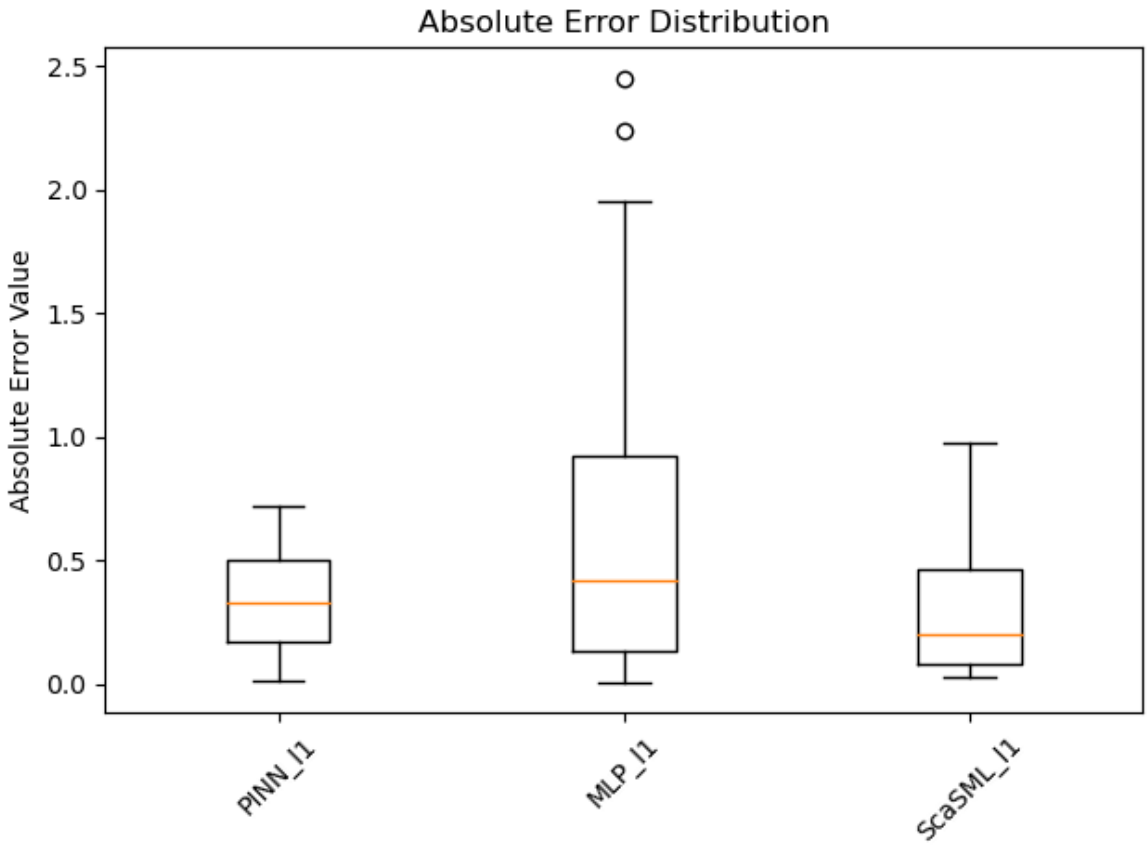
SCaSML


ScaSML I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6
7
4
2
0
−2
−4

SimpleUniform

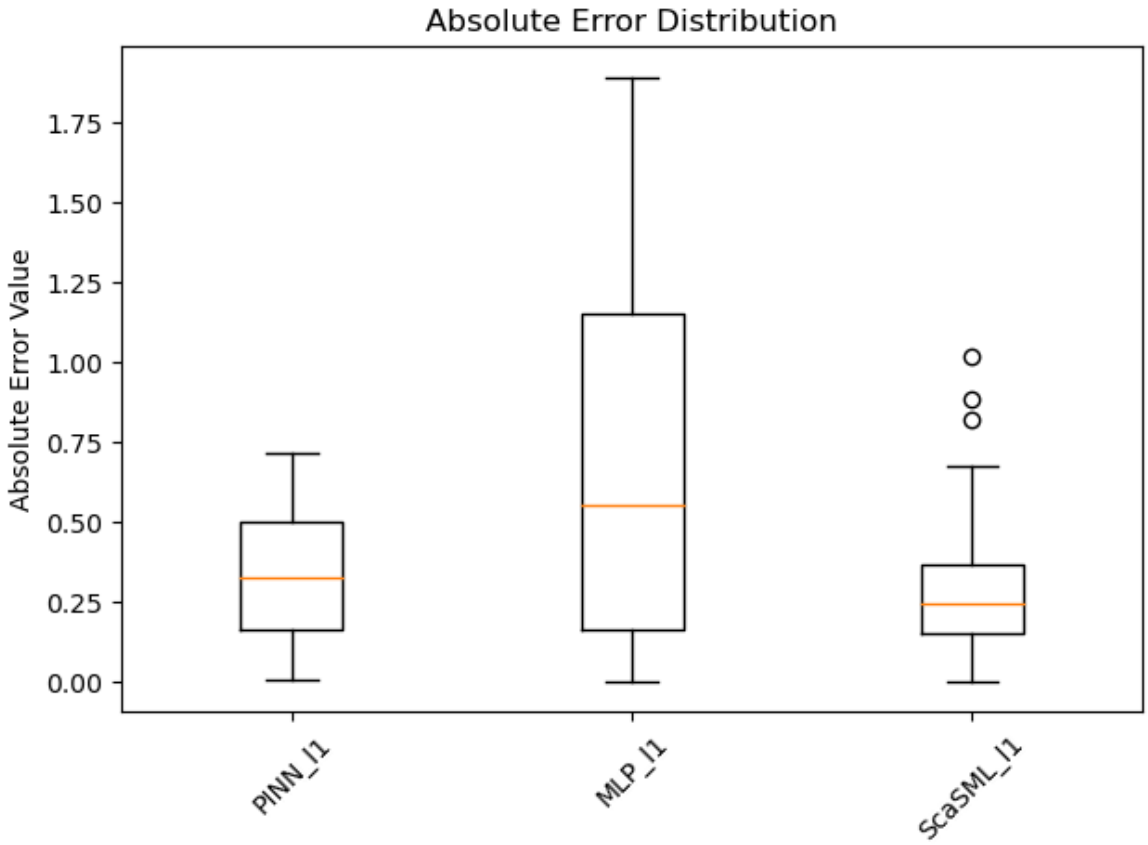
100d

Real Solution-> min: 0.0 max: 0.6700127 mean: 0.33071044
PINN l1, rho=2-> min: 0.0108173415 max: 0.72156096 mean: 0.33114284
MLP l1, rho=2-> min: 0.002251567247469999 max: 2.4530416080451403 mean: 0.6226926625049587
ScaSML l1, rho=2-> min: 0.0244659941481431 max: 0.9719449618686875 mean: 0.283680768776826



250d

Real Solution-> min: 0.0 max: 0.6700127 mean: 0.33071044
PINN l1, rho=2-> min: 0.0073871575 max: 0.71813184 mean: 0.33100587
MLP l1, rho=2-> min: 0.0016070186905680628 max: 1.8921936694386021 mean: 0.6515348820895097
ScaSML l1, rho=2-> min: 0.0006880310456760519 max: 1.0207207262310165 mean: 0.30044312344274765



Conclusion

Complicated_HJB

Space-time distribution (NormalSphere):

The images show that PINN has a relatively uniform error distribution across the sphere, while MLP and SCaSML demonstrate more localized error patterns, since they are very accurate when time step is small. SCaSML achieves comparable errors as MLP.

Statistical features (SimpleUniform):

- 100d:
  PINN: min 20.96, max 48.13, mean 31.84
  MLP: min 0.058, max 12.36, mean 2.95
  SCaSML: min 0.043, max 10.53, mean 2.68
- 250d:
  PINN: min 88.28, max 122.26, mean 102.61
  MLP: min 0.076, max 14.60, mean 4.35
  SCaSML: min 0.17, max 17.97, mean 3.46

SCaSML consistently shows the lowest mean error, indicating better overall performance. Both MLP and SCaSML significantly outperform PINN in terms of minimum, maximum, and mean errors.

Explicit_Solution_Example

Space-time distribution (NormalSphere):

The images show that PINN has a uniform error across the sphere, while MLP and SCaSML have near-zero errors throughout, appearing as blank spheres.

Statistical features (SimpleUniform):

- 100d:
  PINN: min 0.57, max 0.57, mean 0.57
  MLP: min 0, max 1.08e-09, mean 7.37e-11
  SCaSML: min 2.26e-14, max 1.10e-09, mean 6.94e-11
- 250d:
  PINN: min 0.55, max 0.55, mean 0.55
  MLP: min 0, max 0, mean 0
  SCaSML: min 1.05e-08, max 1.62e-06, mean 6.59e-07

Both MLP and SCaSML dramatically outperform PINN, with errors several orders of magnitude lower. MLP achieves perfect accuracy in 250d.

Neumann_Boundary

Space-time distribution (NormalSphere):

The images reveal that PINN has a smooth error distribution, while MLP and SCaSML show more complex patterns. SCaSML appears to have smaller error magnitudes compared to PINN.

Statistical features (SimpleUniform):

- 100d:
  Real Solution: min 0, max 0.67, mean 0.33
  PINN: min 0.011, max 0.72, mean 0.33
  MLP: min 0.0023, max 2.45, mean 0.62
  ScaSML: min 0.024, max 0.97, mean 0.28
- 250d:
  Real Solution: min 0, max 0.67, mean 0.33
  PINN: min 0.0074, max 0.72, mean 0.33
  MLP: min 0.0016, max 1.89, mean 0.65
  ScaSML: min 0.00069, max 1.02, mean 0.30

SCaSML consistently achieves a mean closest to the real solution, indicating better overall performance. PINN matches the mean of the real solution but has higher minimum errors.

In conclusion, the SCaSML algorithm effectively calibrates the bias of PINN via MLP:

1. Space-time distribution: SCaSML generally shows more localized and smaller error regions, appearing as a combination of two approximators.
2. Statistical features: SCaSML consistently demonstrates lower mean errors or closer approximations to the real solution mean, especially in complex equations like Complicated_HJB and Neumann_Boundary.
3. Performance across dimensions: SCaSML maintains strong performance in both 100d and 250d spaces, showcasing its robustness.
4. Error magnitude: Both MLP and SCaSML significantly reduce error magnitudes compared to PINN, with SCaSML often achieving the lowest mean errors.