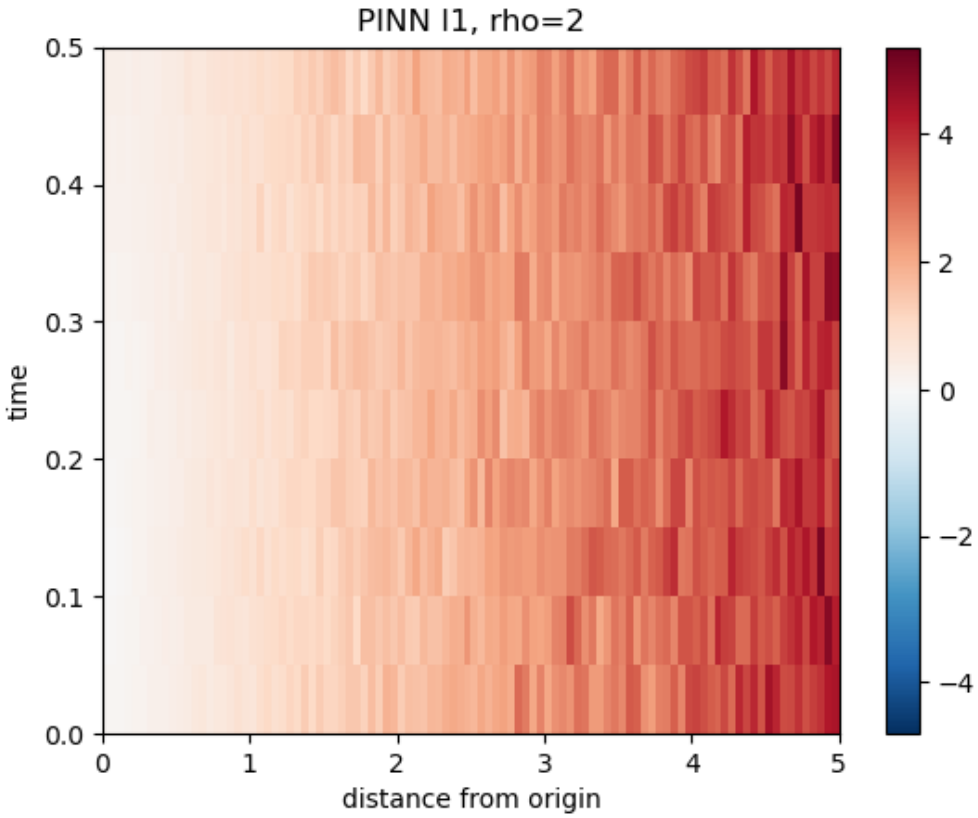
Summary_ of_Results(MAE,Full_History)

Complicated_HJB

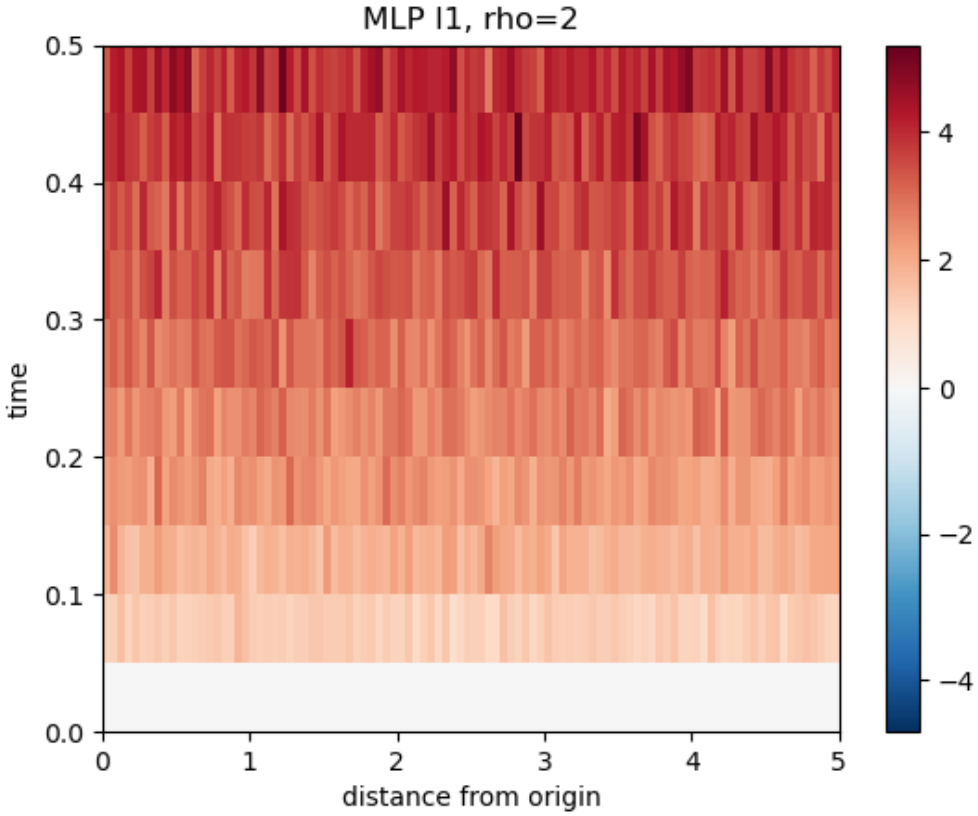
NormalSphere

100d

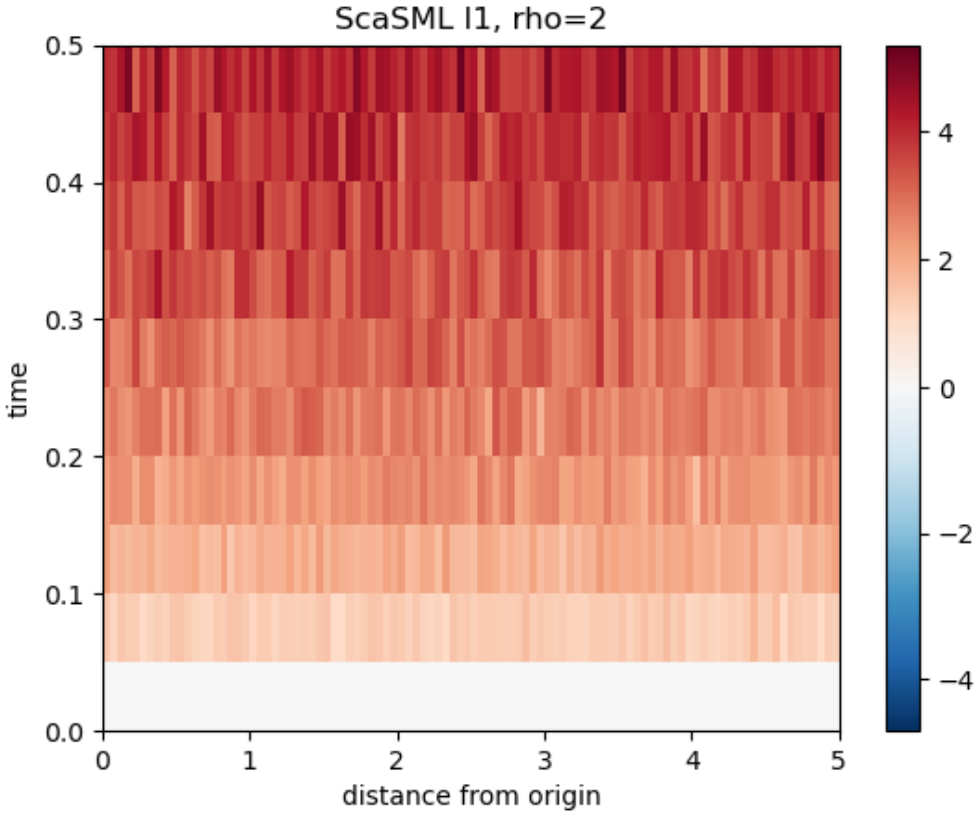
PINN



MLP

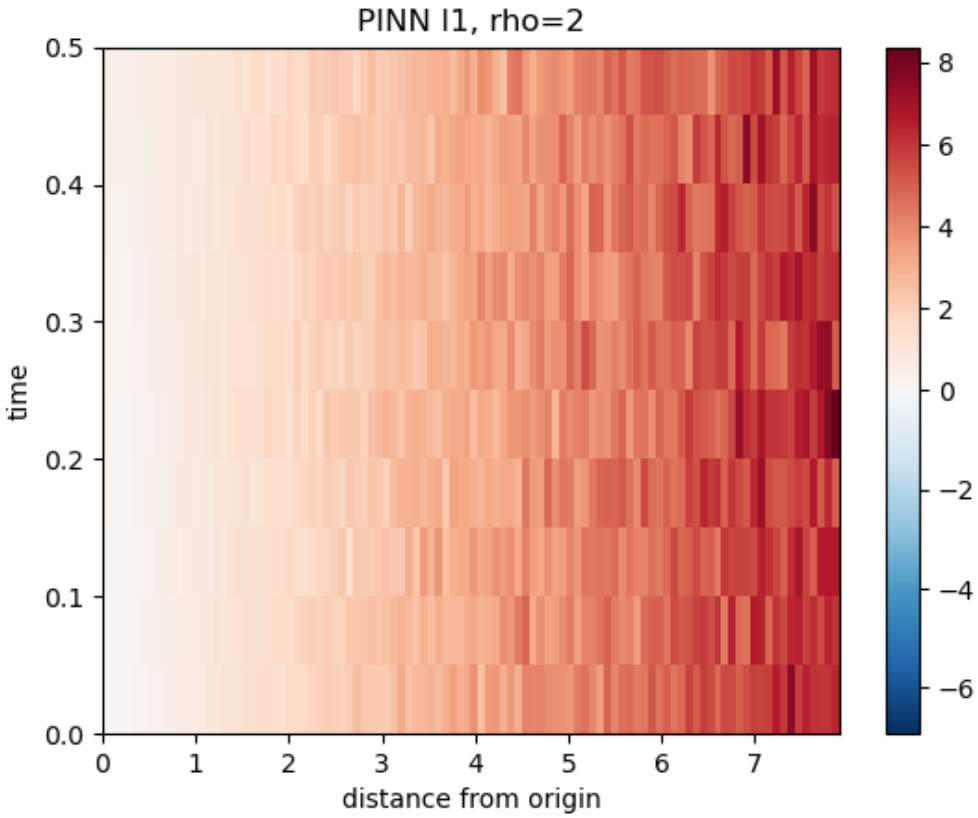

MLP I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
4
2
0
−2
−4

SCaSML

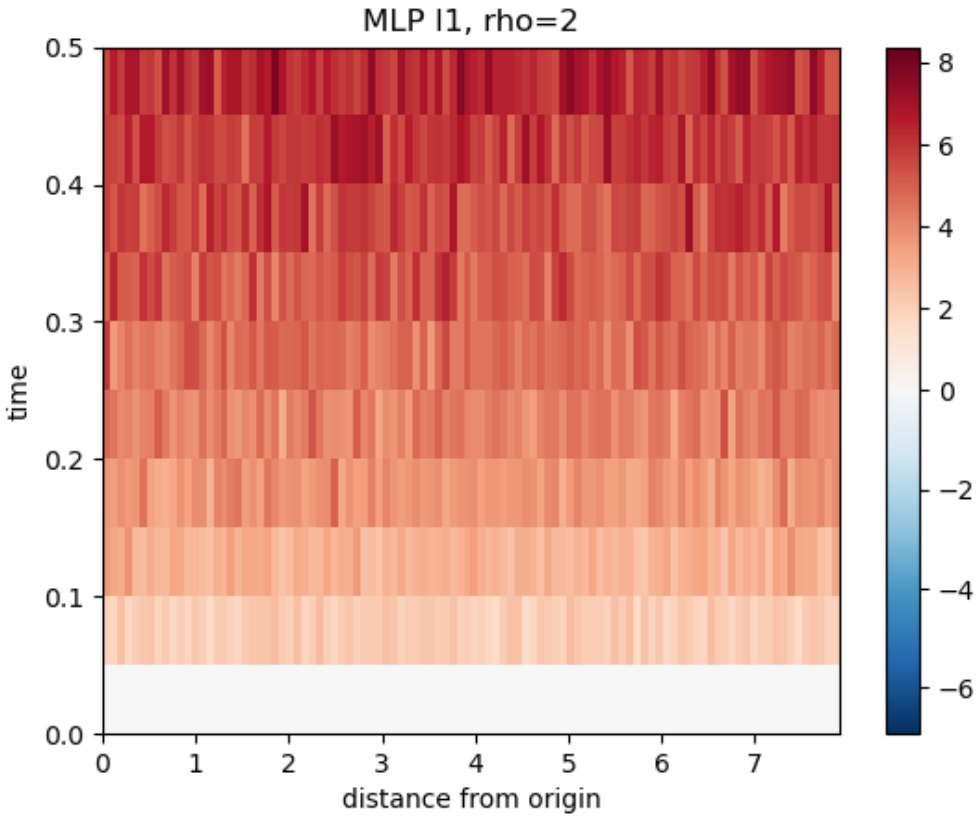

ScaSML I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
4
2
0
−2
−4

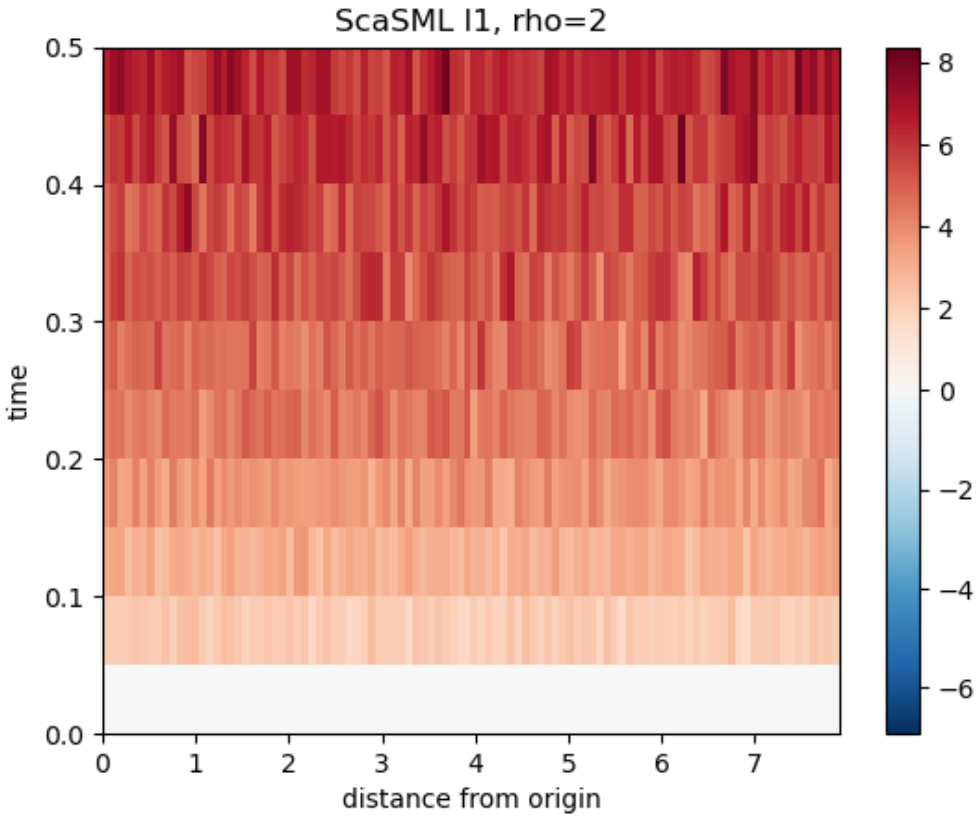
250d

PINN



MLP

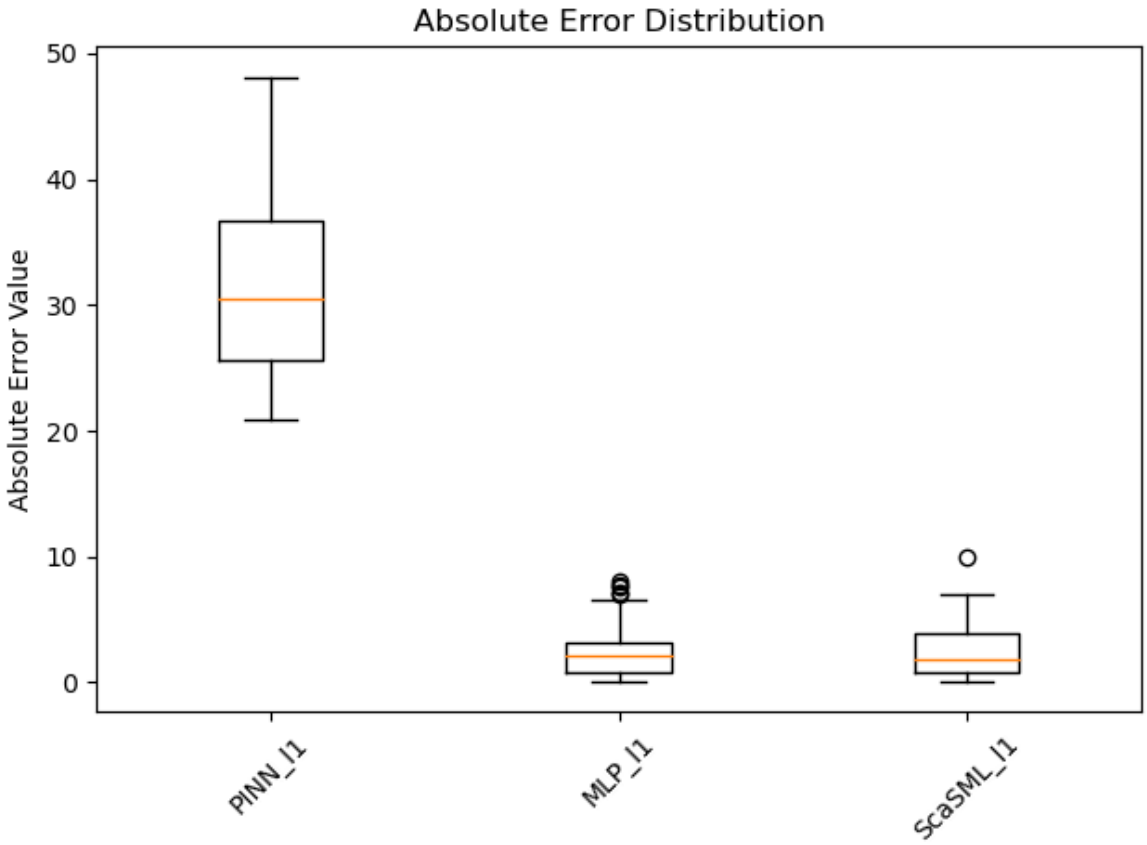




SimpleUniform

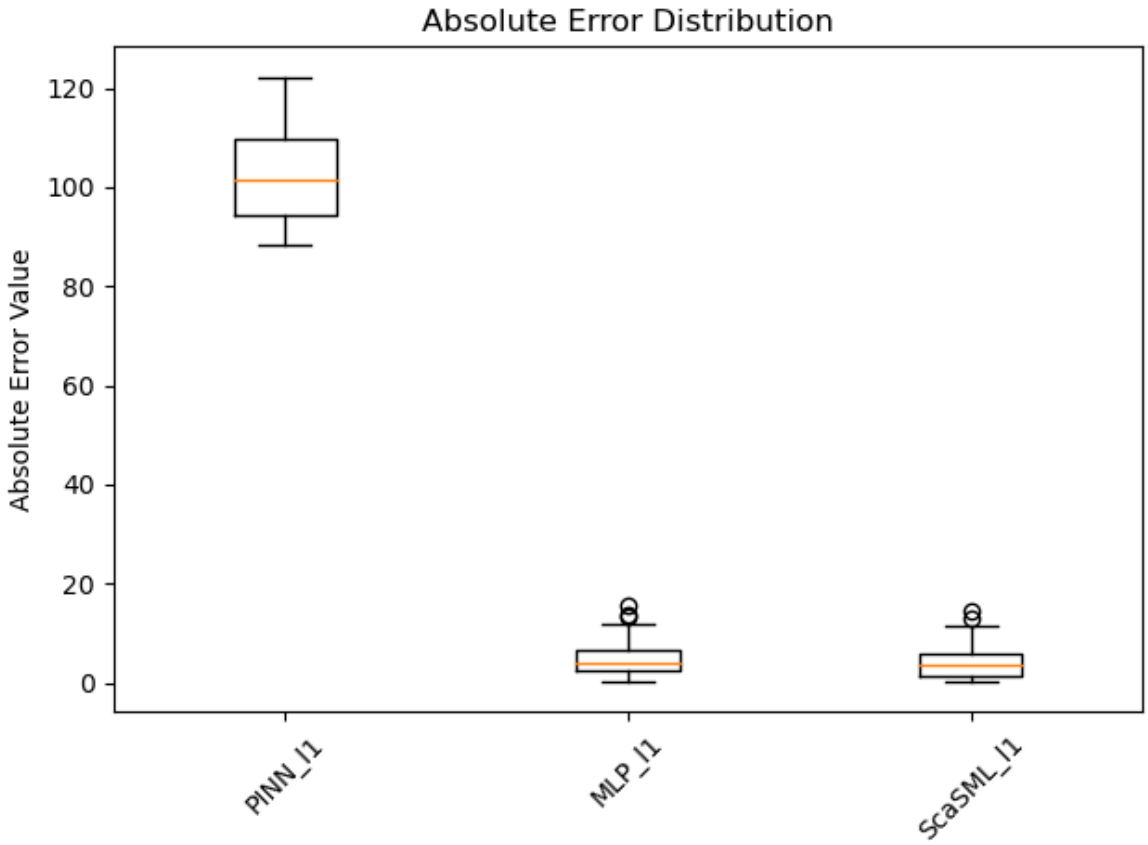
100d

Real Solution-> min: 21.08817 max: 47.40346 mean: 31.57981
PINN l1, rho=2-> min: 20.95896 max: 48.12542 mean: 31.836462
MLP l1, rho=2-> min: 0.039306646030652814 max: 8.000174376765138 mean: 2.553234381201442
ScaSML l1, rho=2-> min: 0.06791453660794389 max: 10.00529483343557 mean: 2.633477011632597



250d

Real Solution-> min: 88.26935 max: 122.24708 mean: 102.59221
PINN l1, rho=2-> min: 88.28333 max: 122.26325 mean: 102.6078
MLP l1, rho=2-> min: 0.1554851594090542 max: 15.487591823426683 mean: 5.0831840414465015
ScaSML l1, rho=2-> min: 0.15286268404449288 max: 14.327133842593568 mean: 4.199262405912148

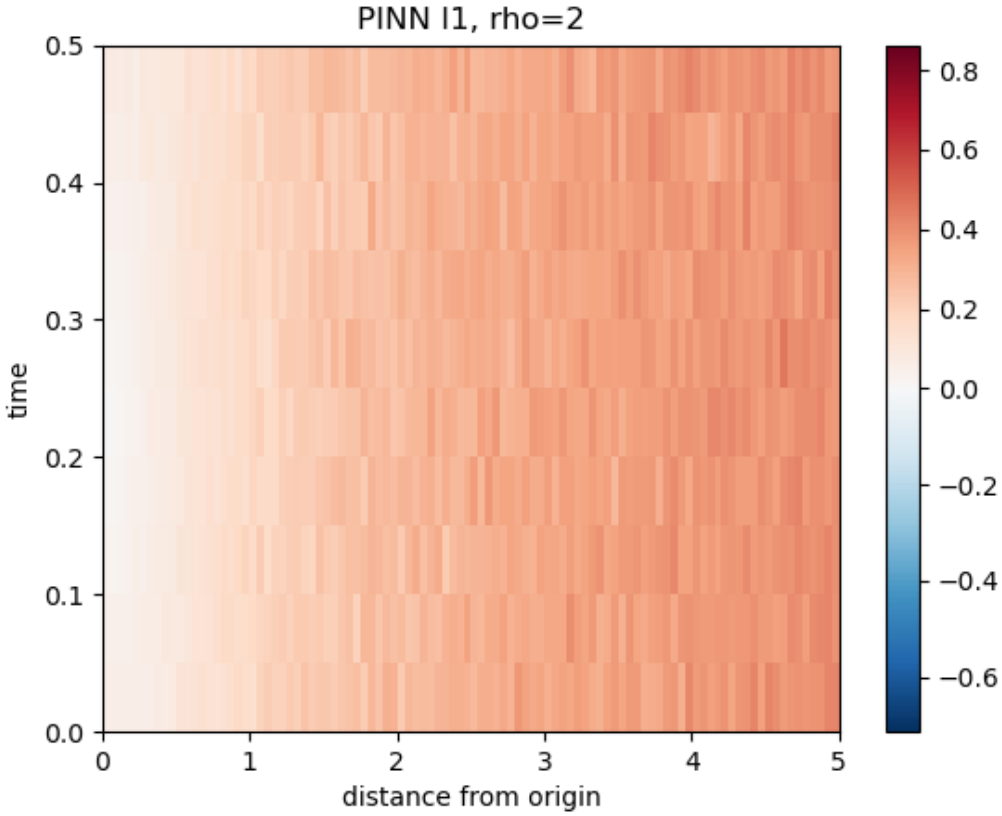


Explicit_Solution_Example

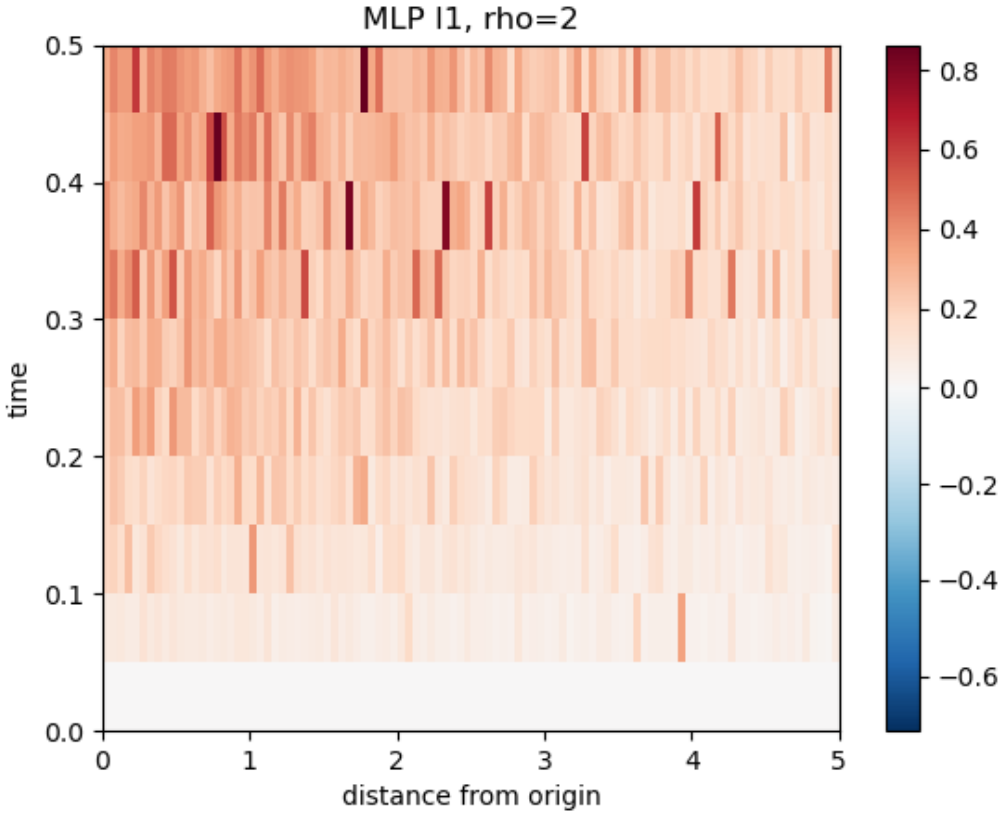
NormalSphere

100d

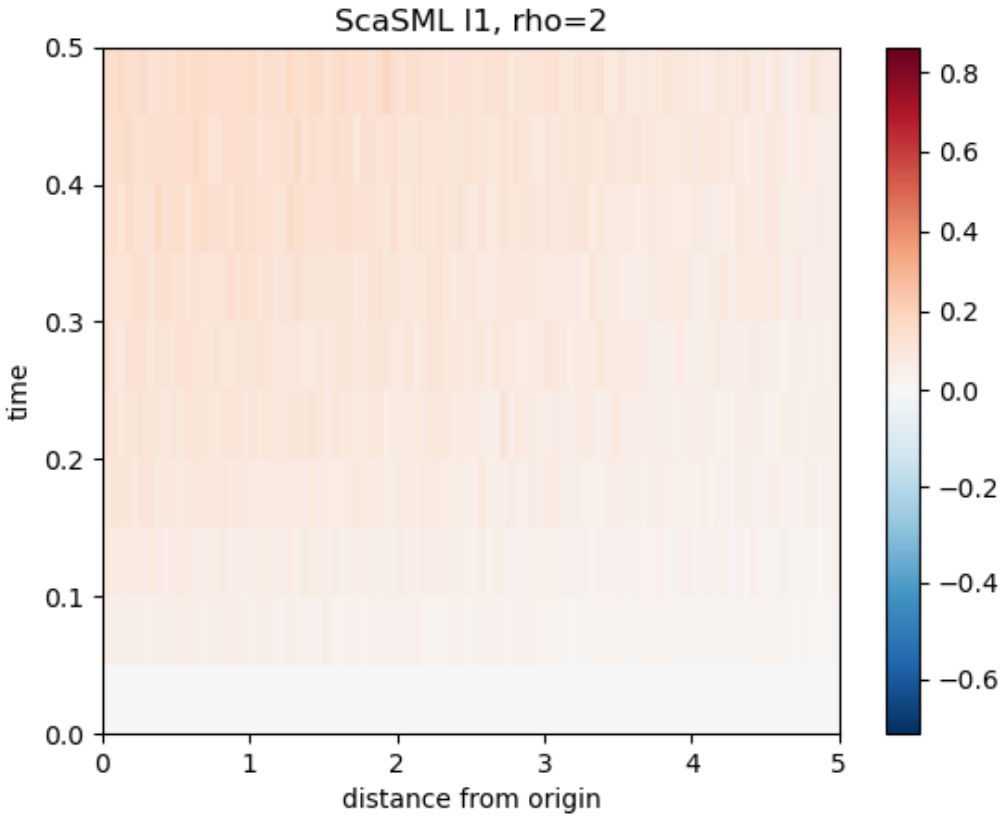
PINN


PINN I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6

MLP

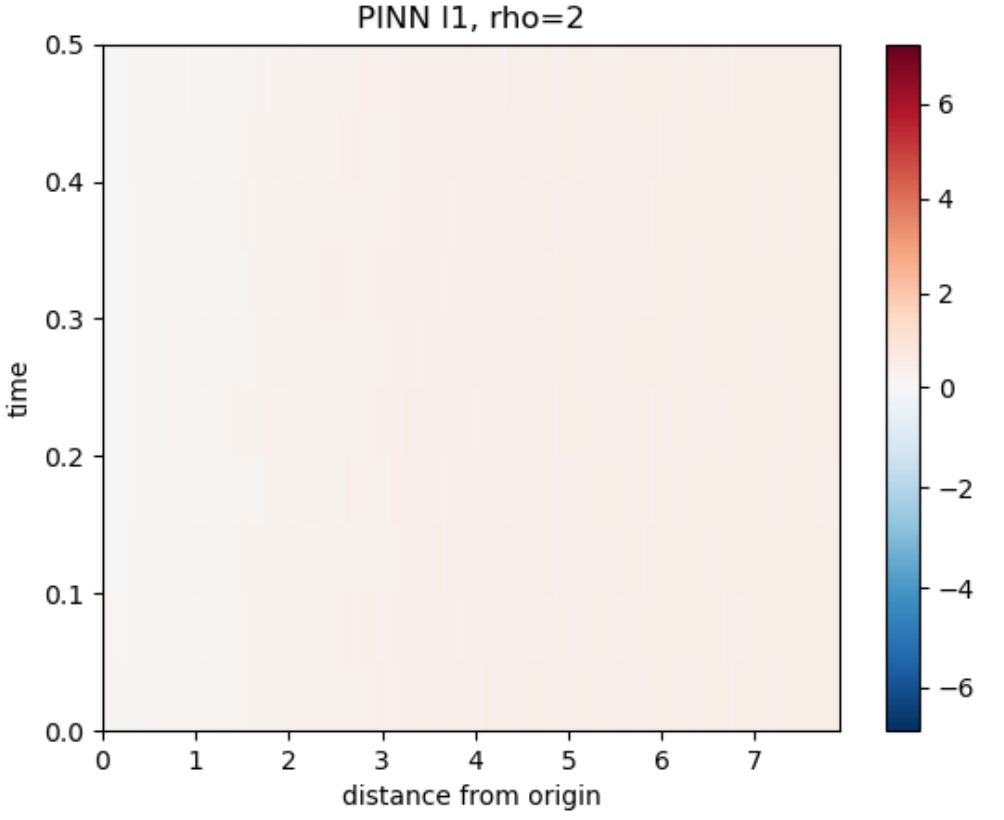

MLP I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6

SCaSML

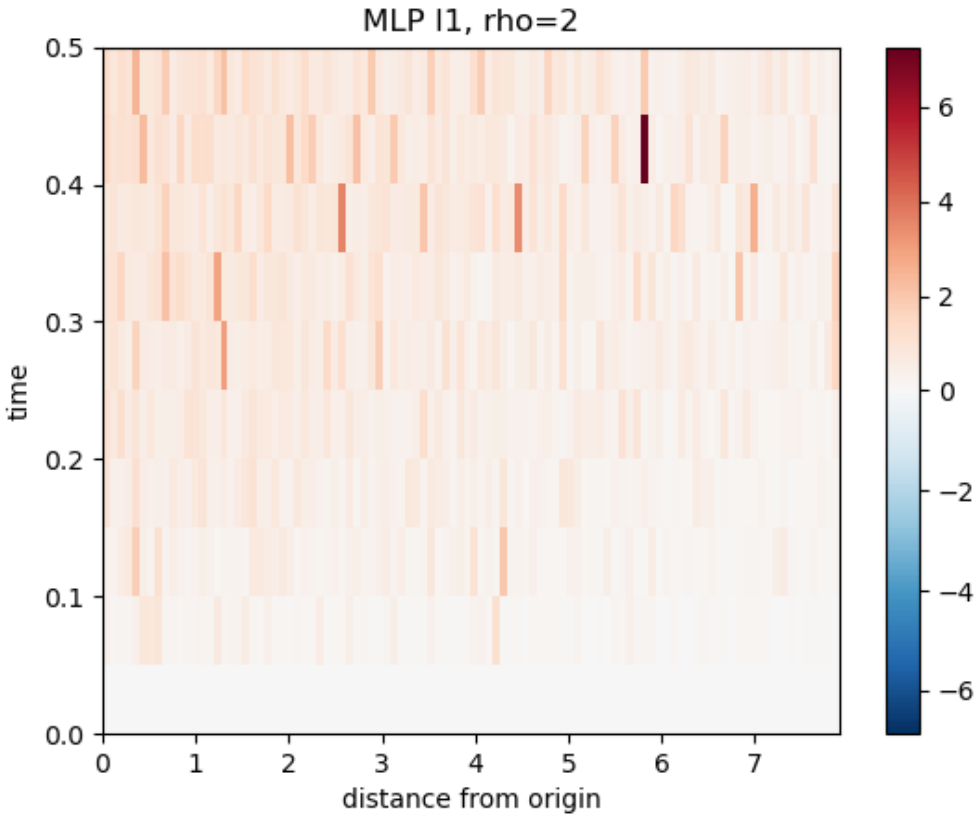


250d

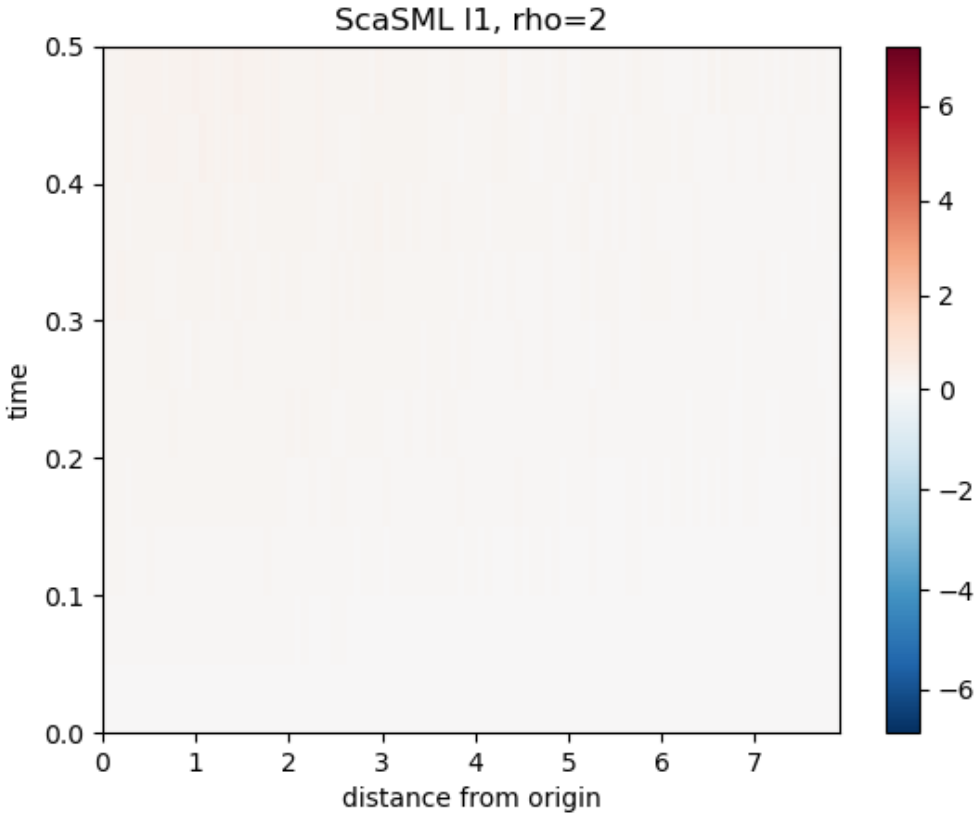
PINN



MLP


MLP I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6
7
6
4
2
0
−2
−4
−6

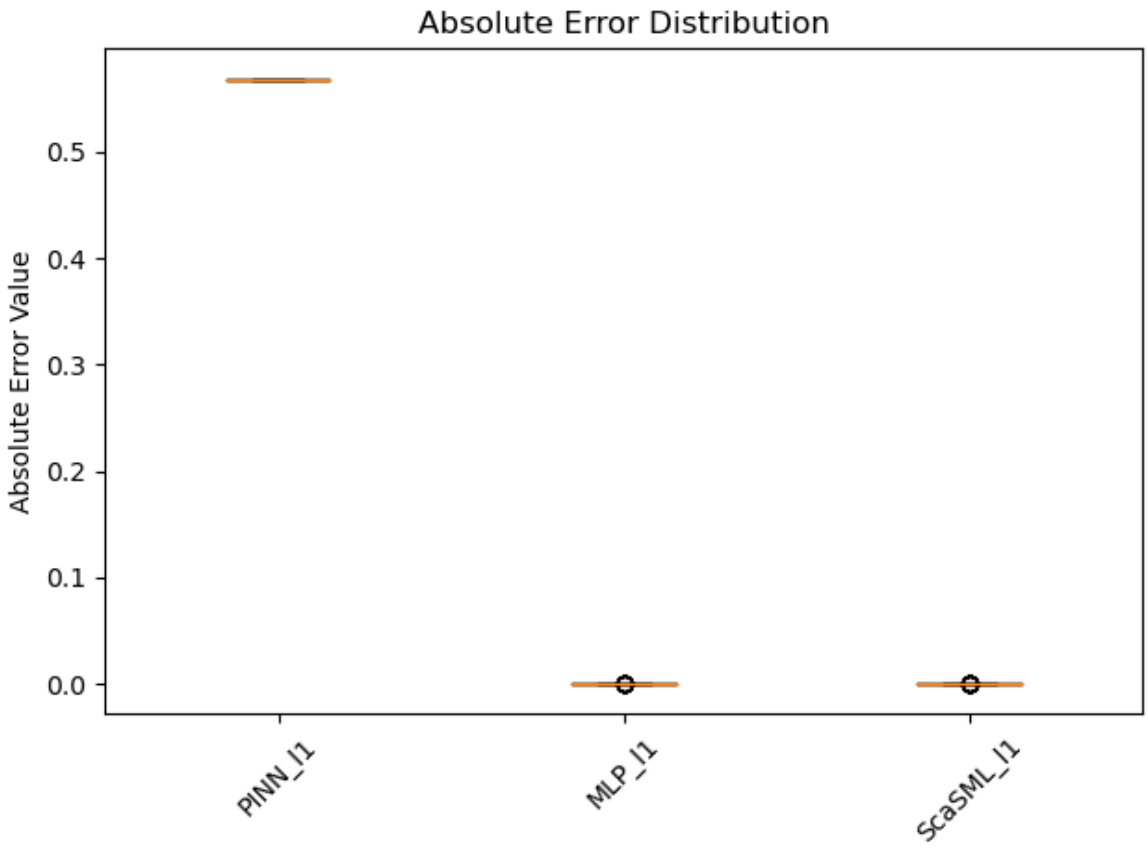
SCaSML


ScaSML I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6
7
6
4
2
0
−2
−4
−6

SimpleUniform

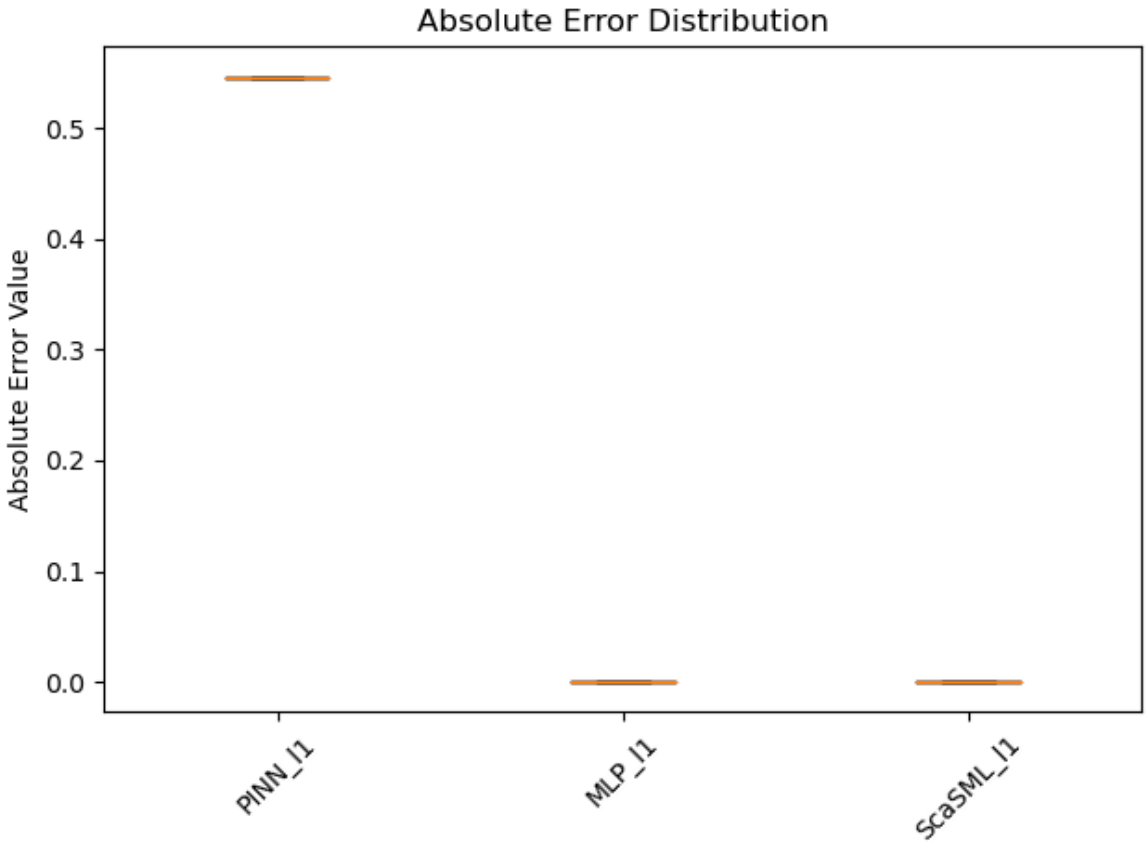
100d

Real Solution-> min: 0.0 max: 0.0 mean: 0.0
PINN l1, rho=2-> min: 0.56846434 max: 0.56846434 mean: 0.56846434
MLP l1, rho=2-> min: 0.0 max: 1.0221228874271143e-09 mean: 9.221824070934431e-11
ScaSML l1, rho=2-> min: 9.325873406851315e-15 max: 1.1986187598012066e-09 mean: 8.663899375704887e-11



250d

Real Solution-> min: 0.0 max: 0.0 mean: 0.0
PINN l1, rho=2-> min: 0.5461806 max: 0.5461825 mean: 0.5461815
MLP l1, rho=2-> min: 0.0 max: 0.0 mean: 0.0
ScaSML l1, rho=2-> min: 3.1806162925640535e-08 max: 1.5431055342407518e-06 mean: 6.294460226086151e-07

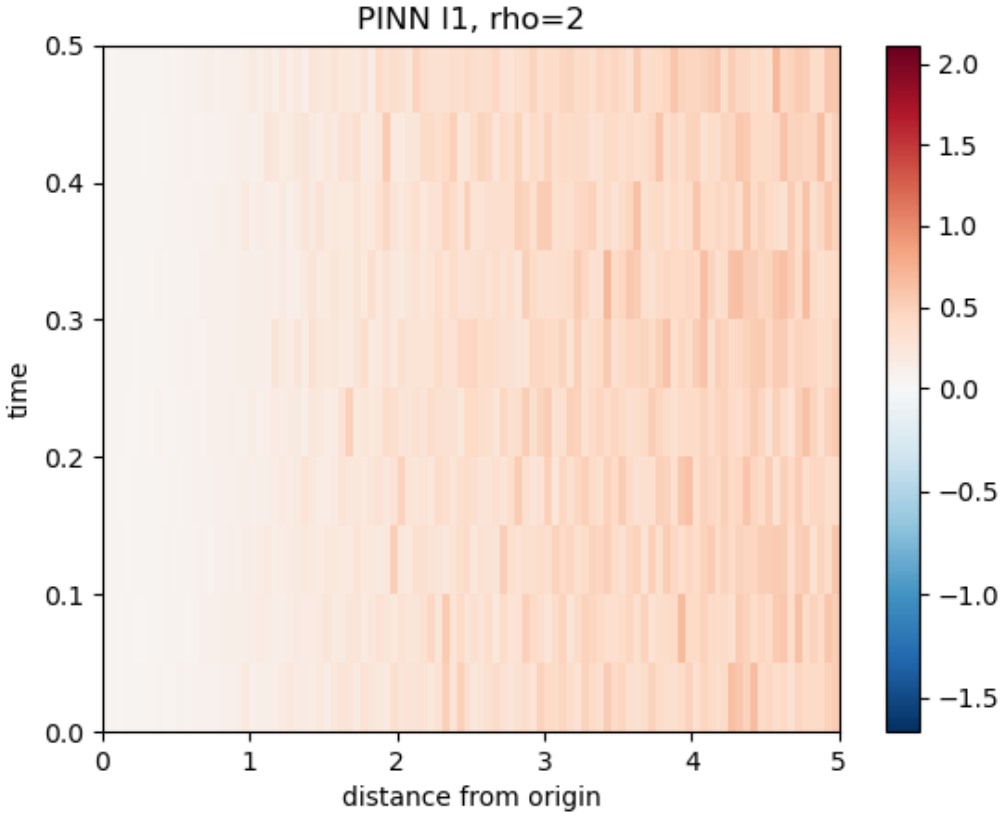


Neumann_Boundary

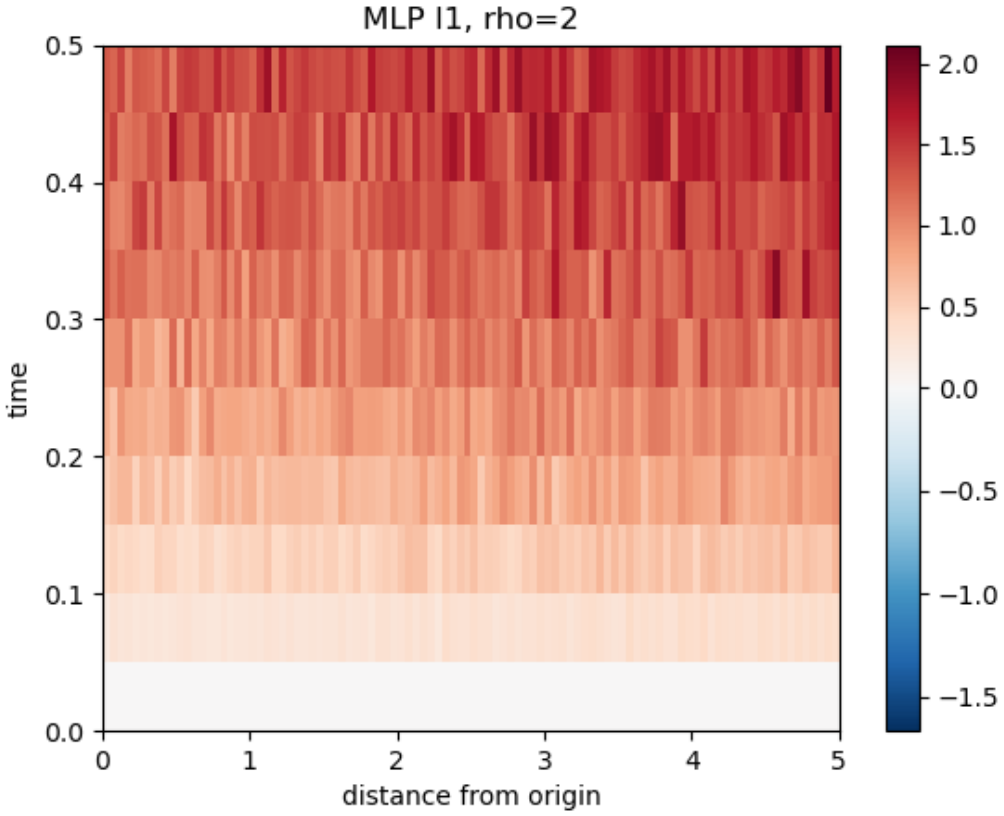
NormalSphere

100d

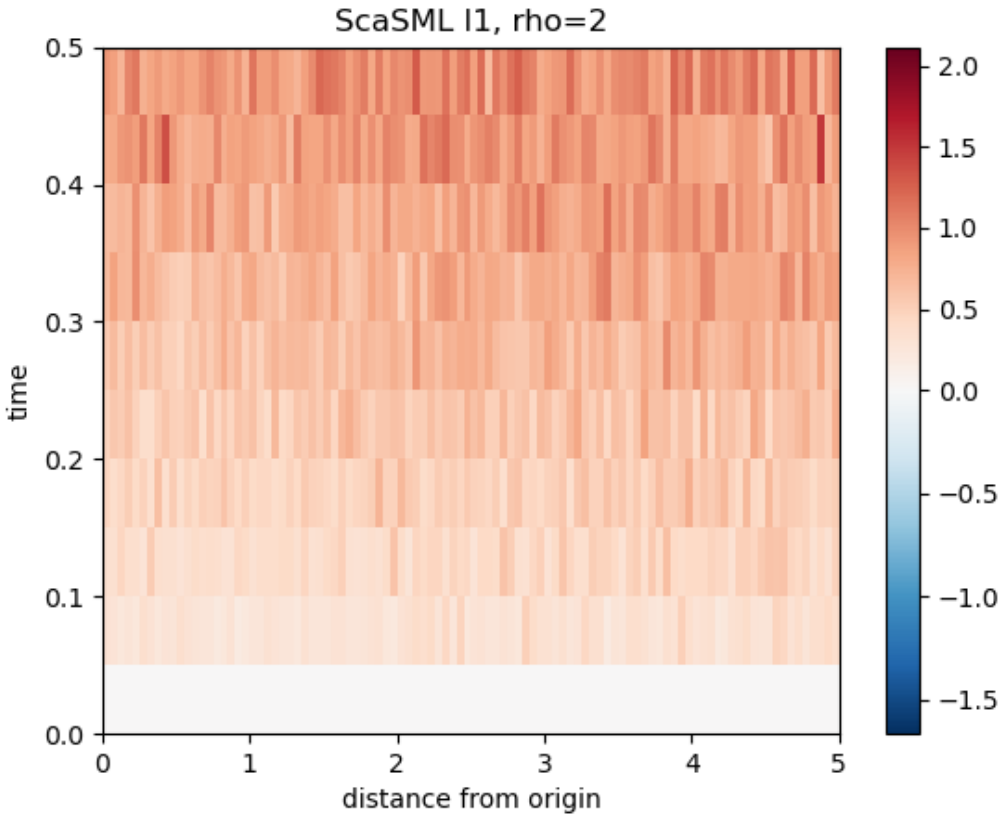
PINN


PINN I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5

MLP

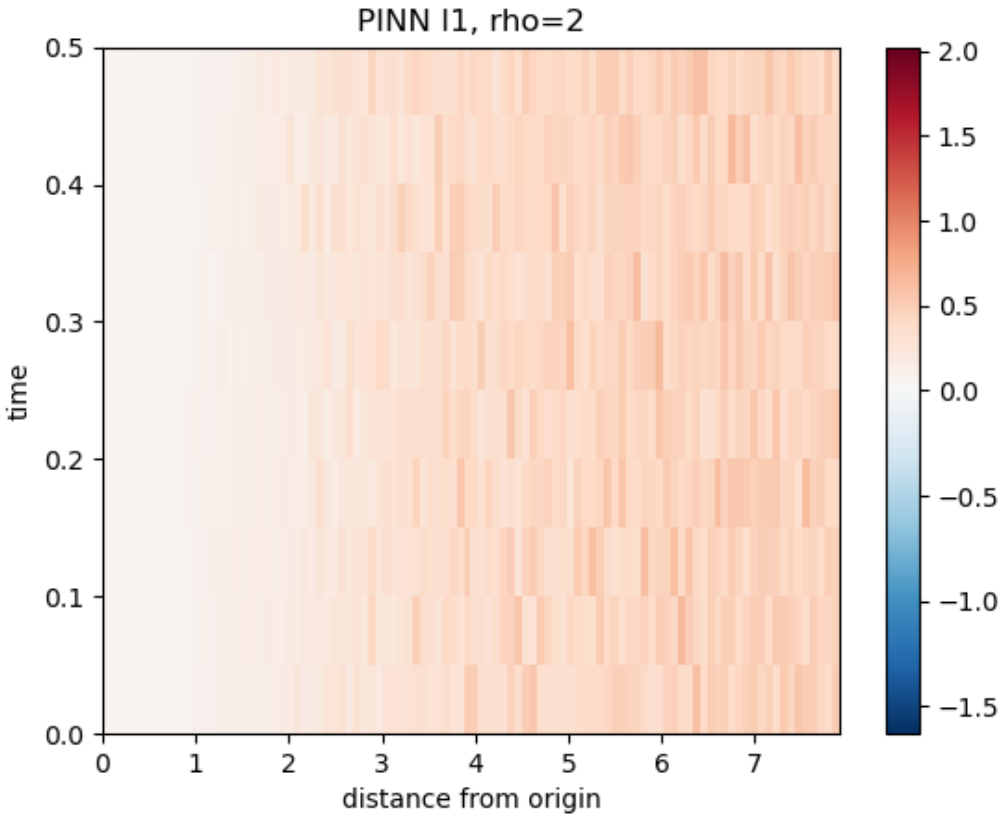

MLP I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5

SCaSML

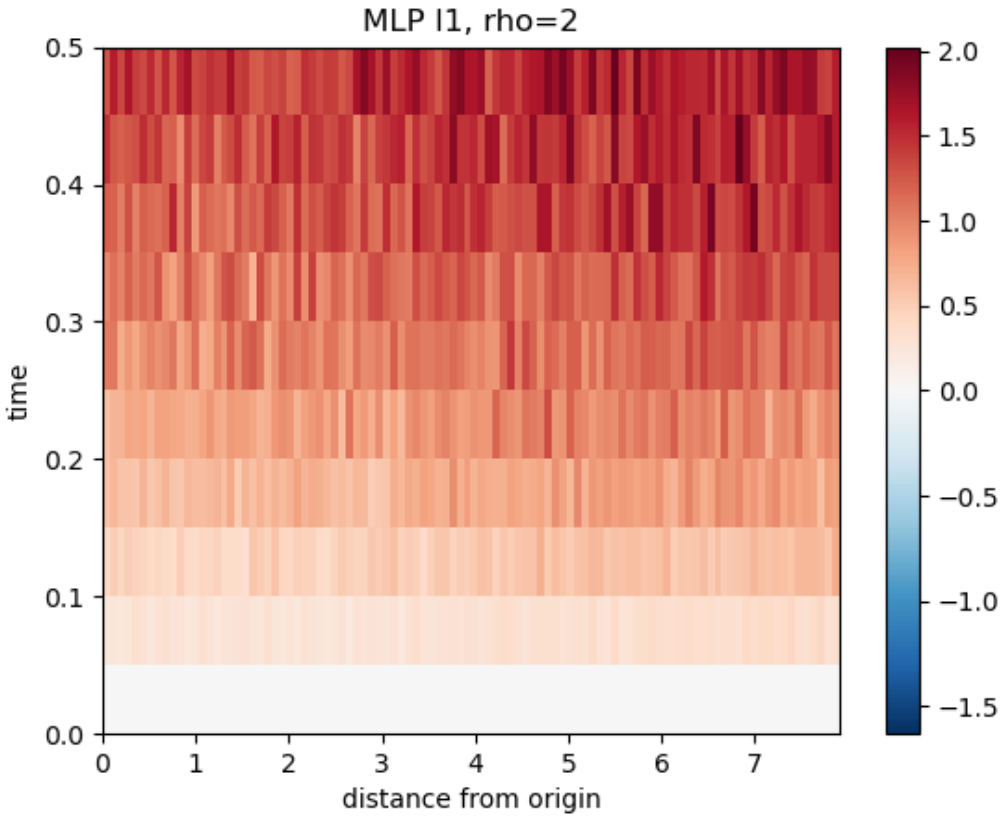


250d

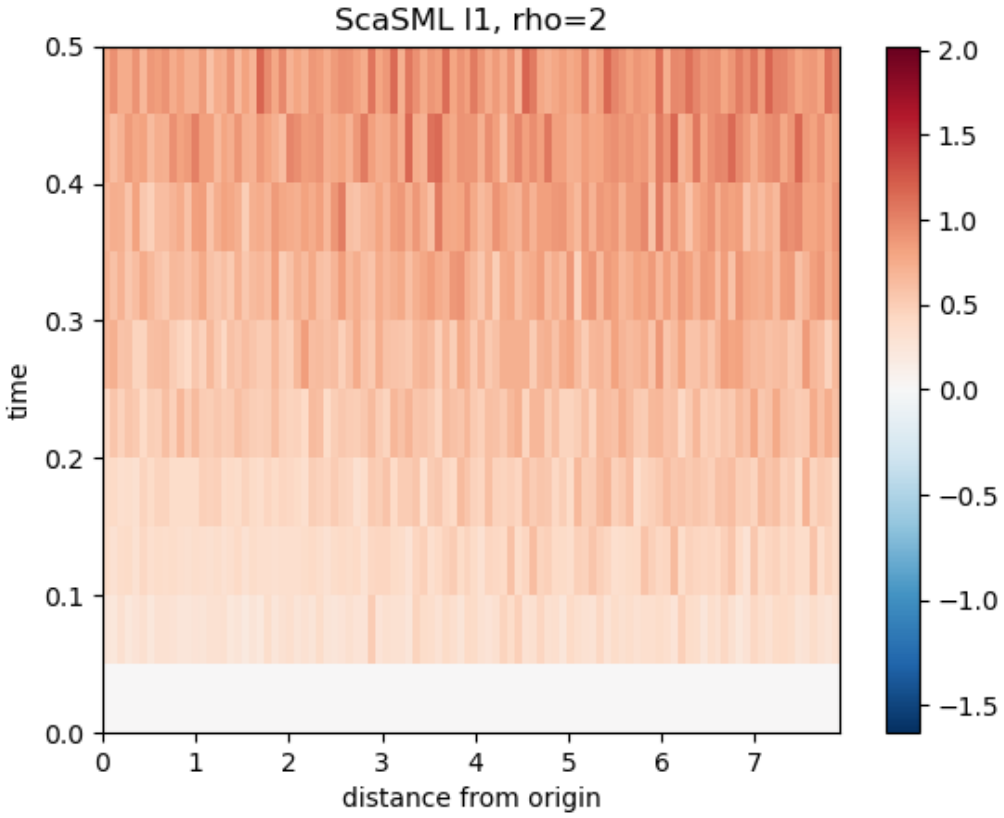
PINN



MLP


MLP I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6
7
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5

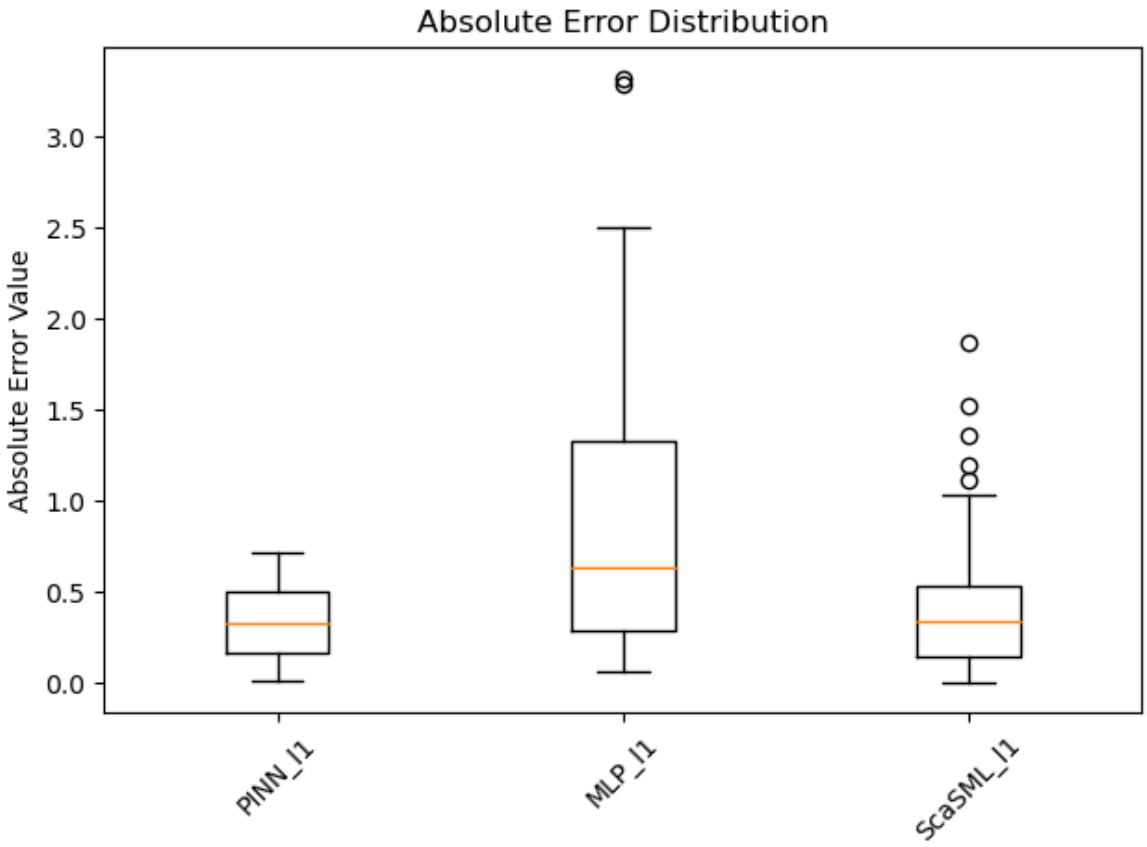
SCaSML


ScaSML I1, rho=2
time
distance from origin
0.0
0.1
0.2
0.3
0.4
0.5
0
1
2
3
4
5
6
7
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5

SimpleUniform

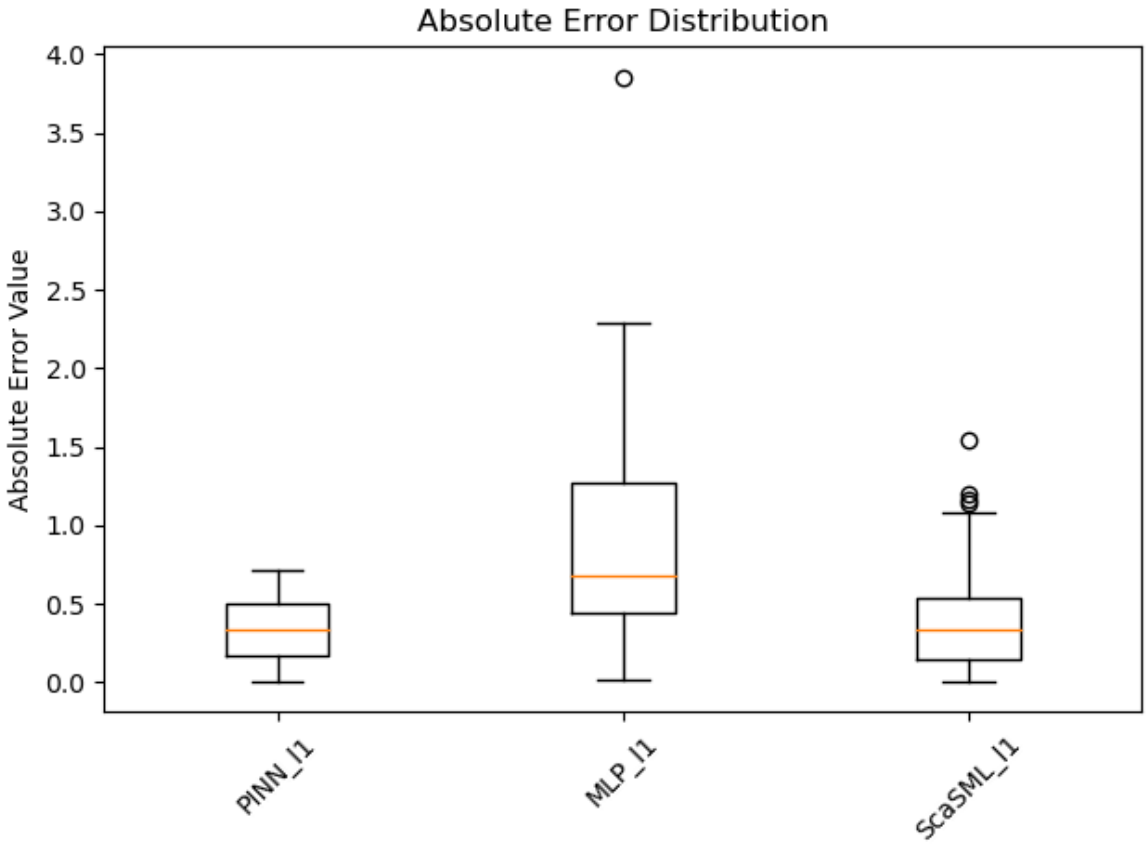
100d

Real Solution-> min: 0.0 max: 0.6700127 mean: 0.33071044
PINN l1, rho=2-> min: 0.0108173415 max: 0.72156096 mean: 0.33114284
MLP l1, rho=2-> min: 0.06014029962834201 max: 3.323620538566565 mean: 0.9010895304055028
ScaSML l1, rho=2-> min: 0.0008780561305484769 max: 1.8739295568279328 mean: 0.44091610712241114



250d

Real Solution-> min: 0.0 max: 0.6700127 mean: 0.33071044
PINN l1, rho=2-> min: 0.0073871575 max: 0.71813184 mean: 0.33100587
MLP l1, rho=2-> min: 0.010961447300344873 max: 3.8541148322681673 mean: 0.8835014103898221
ScaSML l1, rho=2-> min: 0.001643262105699761 max: 1.5398553768754413 mean: 0.4152441216040941



Conclusion

Complicated_HJB

Space-time distribution (NormalSphere):

The images show similar patterns to the Quadrature version, with PINN having a uniform error distribution, while MLP and SCaSML display more localized error patterns. The differences between MLP and SCaSML are subtle in the visualizations.

Statistical features (SimpleUniform):

- 100d:
  PINN: min 20.96, max 48.13, mean 31.84 (unchanged)
  MLP: min 0.039, max 8.00, mean 2.55 (improved from 0.058, 12.36, 2.95)
  SCaSML: min 0.068, max 10.01, mean 2.63 (slightly changed from 0.043, 10.53, 2.68)
- 250d:
  PINN: min 88.28, max 122.26, mean 102.61 (unchanged)
  MLP: min 0.16, max 15.49, mean 5.08 (slightly worse than 0.076, 14.60, 4.35)
  SCaSML: min 0.15, max 14.33, mean 4.20 (improved from 0.17, 17.97, 3.46)

In the Full_History version, MLP shows improvement in 100d, while SCaSML improves in 250d. Both still significantly outperform PINN.

Explicit_Solution_Example

Space-time distribution (NormalSphere):

The images remain similar to the Quadrature version, with PINN showing uniform error and MLP and SCaSML displaying near-zero errors throughout.

Statistical features (SimpleUniform):

- 100d:
  PINN: min 0.57, max 0.57, mean 0.57 (unchanged)
  MLP: min 0, max 1.02e-09, mean 9.22e-11 (slightly higher errors than 0, 1.08e-09, 7.37e-11)
  SCaSML: min 9.33e-15, max 1.20e-09, mean 8.66e-11 (slightly higher errors than 2.26e-14, 1.10e-09, 6.94e-11)
- 250d:
  PINN: min 0.55, max 0.55, mean 0.55 (unchanged)
  MLP: min 0, max 0, mean 0 (unchanged, perfect accuracy)
  SCaSML: min 3.18e-08, max 1.54e-06, mean 6.29e-07 (slightly improved from 1.05e-08, 1.62e-06, 6.59e-07)

The Full_History version shows minor changes in error magnitudes, but the overall performance remains similar, with both MLP and SCaSML vastly outperforming PINN.

Neumann_Boundary

Space-time distribution (NormalSphere):

The images show patterns similar to the Quadrature version, with PINN having a smooth error distribution and MLP and SCaSML displaying more complex patterns. SCaSML appears to have smaller error magnitudes compared to MLP.

Statistical features (SimpleUniform):

- 100d:
  Real Solution: min 0, max 0.67, mean 0.33 (unchanged)
  PINN: min 0.011, max 0.72, mean 0.33 (unchanged)
  MLP: min 0.060, max 3.32, mean 0.90 (worse than 0.0023, 2.45, 0.62)
  ScaSML: min 0.00088, max 1.87, mean 0.44 (changed from 0.024, 0.97, 0.28)
- 250d:
  Real Solution: min 0, max 0.67, mean 0.33 (unchanged)
  PINN: min 0.0074, max 0.72, mean 0.33 (unchanged)
  MLP: min 0.011, max 3.85, mean 0.88 (worse than 0.0016, 1.89, 0.65)
  ScaSML: min 0.0016, max 1.54, mean 0.42 (changed from 0.00069, 1.02, 0.30)

In the Full_History version, MLP's performance deteriorates, while SCaSML's results change but remain closer to the real solution mean compared to MLP.

Comparison with Quadrature Case:

1. Complicated_HJB: The Full_History version shows improvements for MLP in 100d and SCaSML in 250d. Both methods continue to outperform PINN significantly.
2. Explicit_Solution_Example: Results are very similar between versions, with only minor fluctuations in error magnitudes. MLP and SCaSML maintain their substantial advantage over PINN.
3. Neumann_Boundary: The Full_History version shows decreased performance for MLP, while SCaSML's results change but remain more accurate than MLP's. PINN's performance is unchanged.

Overall, the Full_History version demonstrates that the SCaSML algorithm still effectively calibrates the bias of PINN via MLP, although with some variations in performance compared to the Quadrature version:

1. Space-time distribution: The error patterns remain similar across versions, with SCaSML generally showing more localized and smaller error regions compared to PINN.
2. Statistical features: SCaSML maintains lower mean errors or closer approximations to the real solution mean in most cases, especially for complex equations like Complicated_HJB and Neumann_Boundary.
3. Relative performance: While there are some changes in absolute error values, SCaSML generally maintains its advantage over PINN and often outperforms MLP, particularly in higher dimensions.
4. Robustness: SCaSML demonstrates consistent performance across different equation types and dimensions, showcasing its ability to adapt to various problem scenarios.