Explicit_Solution_Example

MLP Setting

We solve the following PDE:

$$\frac{\partial u}{\partial t}+(\sigma^2 u(x,t)-\frac{1}{d}-\frac{\sigma^2}{2})div_x u(x,t)+\frac{\sigma^2}{2}\Delta_x u(x,t)=0, t\in[s,T], x\in D\in\mathbb{R}^d \tag{15}$$

whose terminal condition is given by

$$u(x,T)=g(x):=\frac{\exp\left(T+\sum\limits_{i=1}^{d}x_i\right)}{1+\exp\left(T+\sum\limits_{i=1}^{d}x_i\right)} \tag{16}$$

without any boundary constraints.

Then nonlinear term is given by

$$F(u,z)(x,t)=(\sigma u-\frac{1}{d\sigma}-\frac{\sigma}{2})\sum_i z \tag{17}$$

This PDE has an explicit solution at time t:

$$u(x,t)=\frac{\exp\left(t+\sum\limits_{i=1}^{d}x_i\right)}{1+\exp\left(t+\sum\limits_{i=1}^{d}x_i\right)} \tag{18}$$

which is our target in this section.

GP Setting

Rewrite the PDE as

$$\frac{\partial u}{\partial t}+(\sigma^2 u(x,t)-\frac{1}{d}-\frac{\sigma^2}{2})div_x u(x,t)+\frac{\sigma^2}{2}\Delta_x u(x,t)=0, t\in[s,T], x\in D\in\mathbb{R}^d \tag{19}$$

Let the nonlinear term be

$$y_{domain}=-\frac{2}{\sigma^2}\frac{\partial u}{\partial t}+(1+\frac{2}{d\sigma^2}-2u)div_x u=(-\frac{2}{\sigma^2}+1+\frac{2}{d\sigma^2}-\frac{2\exp\left(t+\sum_{i=1}^d x_i\right)}{1+\exp\left(t+\sum_{i=1}^d x_i\right)})\frac{\exp\left(t+\sum_{i=1}^d x_i\right)}{(1+\exp\left(t+\sum_{i=1}^d x_i\right))^2} \tag{20}$$

Define operators

$$L(u):u\to\Delta_x u \tag{21}$$

We define feature functions

$$\phi_j^1(u):u\to\delta_{x_\Omega^j}\circ u, 1\le j\le M_\Omega \tag{22}$$
$$\phi_j^2(u):u\to\delta_{x_{\partial\Omega}^j}\circ u, 1\le j\le M_{\partial\Omega} \tag{23}$$
$$\phi_j^3(u):u\to\delta_{x_\Omega^j}\circ L\circ u, 1\le j\le M_\Omega \tag{24}$$

Denote $\vec{z}_i=\phi^i_{1:M_\Omega}(u), i\in\{1,3\}$ and $\vec{z}_i=\phi^i_{1:M_{\partial\Omega}}(u), i=2$.

Let

$$\vec{z}=\begin{pmatrix}\vec{z}_1\\ \vec{z}_2\\ \vec{z}_3\end{pmatrix} \tag{25}$$

We then derive

$$F(\vec{z})=\begin{pmatrix}\vec{z}_3\\ \vec{z}_2\end{pmatrix} \tag{26}$$

Thus $Q_\Omega=2, Q=3$.

Feature vector ϕ have $M = M_\Omega + M_{\partial\Omega} + M_\Omega$ components, which is also the size of the kernel matrix.

Taking derivatives under linearization condition

$$DF(\vec{z}_k) = \begin{pmatrix} 0 & 0 & I_{M_\Omega} \\ 0 & I_{M_{\partial\Omega}} & 0 \end{pmatrix}$$

Parameters

Specifically, we consider the problem for

$$d = 100, \mu = 0, \sigma = 0.25, D = [-0.5, 0.5]^{100}, s = 0, T = 0.5 \tag{27}$$

and

$$d = 250, \mu = 0, \sigma = 0.25, D = [-0.5, 0.5]^{250}, s = 0, T = 0.5 \tag{28}$$