Complicated_HJB

MLP Setting

We solve the following PDE(labeled as Eq(100) in the original paper)

$$\frac{\partial u}{\partial t} - \frac{1}{d} div_x u(x,t) + 2 + \Delta_x u(x,t) = 0, t \in [s,T], x \in D \subset \mathbb{R}^d \tag{15}$$

whose terminal condition is given by

$$u(x,T) = g(x) := \sum_{i=1}^{d} x_i, \tag{16}$$

without any boundary constraints.

The nonlinear term is given by

$$F(u,z)(x,t) = 2 \tag{17}$$

This PDE has an explicit solution at time t

$$u(x,t) = \sum_{i=1}^{d} x_i + (T-t). \tag{18}$$

which is our target in this section.

GP Setting

Rewrite the PDE as

$$\frac{\partial u}{\partial t} - \frac{1}{d} div_x u(x,t) + \Delta_x u(x,t) = -2, t \in [s,T], x \in D \subset \mathbb{R}^d \tag{19}$$

Let the nonlinear term be

$$y_{domain} = -2 - \frac{\partial u}{\partial t} + \frac{1}{d} div_x u = -2 \tag{20}$$

Define operators

$$L(u) : u \to \Delta_x u \tag{21}$$

We define feature functions

$$\phi_j^1(u) : u \to \delta_{x_\Omega^j} \circ u, 1 \le j \le M_\Omega \tag{22}$$
$$\phi_j^2(u) : u \to \delta_{x_{\partial\Omega}^j} \circ u, 1 \le j \le M_{\partial\Omega} \tag{23}$$
$$\phi_j^3(u) : u \to \delta_{x_\Omega^j} \circ L \circ u, 1 \le j \le M_\Omega \tag{24}$$

Denote $\vec{z}_i = \phi^i_{1:M_\Omega}(u), i \in \{1,3\}$ and $\vec{z}_i = \phi^i_{1:M_{\partial\Omega}}(u), i = 2$.

Let

$$\vec{z} = \begin{pmatrix} \vec{z}_1 \\ \vec{z}_2 \\ \vec{z}_3 \end{pmatrix} \tag{25}$$

We then derive

$$F(\vec{z}) = \begin{pmatrix} \vec{z}_3 \\ \vec{z}_2 \end{pmatrix} \tag{26}$$

Thus $Q_\Omega = 2, Q = 3$.

Feature vector ϕ have $M = M_\Omega + M_{\partial\Omega} + M_\Omega$ components, which is also the size of the kernel matrix.

Taking derivatives under linearization condition

$$DF(\vec{z}_k) = \begin{pmatrix} 0 & 0 & I_{M_\Omega} \\ 0 & I_{M_{\partial\Omega}} & 0 \end{pmatrix}$$

Parameters

Specifically, we consider the problem for

$$d = 100, \mu = -1/d, \sigma = \sqrt{2}, D = [-0.5, 0.5]^{100}, s = 0, T = 0.5 \tag{27}$$

and

$$d = 250, \mu = -1/d, \sigma = \sqrt{2}, D = [-0.5, 0.5]^{250}, s = 0, T = 0.5 \tag{28}$$