Explicit_Solution_Example

MLP Setting

We solve the following PDE:

$$\frac{\partial u}{\partial t}+(\sigma^2u(x,t)-\frac{1}{d}-\frac{\sigma^2}{2})div_xu(x,t)+\frac{\sigma^2}{2}\Delta_xu(x,t)=0,t\in[s,T],x\in D\in\mathbb{R}^d \tag{19}$$

whose terminal condition is given by

$$u(x,T)=g(x):=\frac{\exp\left(T+\sum_{i=1}^{d}x_i\right)}{1+\exp\left(T+\sum_{i=1}^{d}x_i\right)} \tag{20}$$

without any boundary constraints.

Then nonlinear term is given by

$$F(u,z)(x,t)=(\sigma u-\frac{1}{d\sigma}-\frac{\sigma}{2})\sum_i z \tag{21}$$

This PDE has an explicit solution at time t:

$$u(x,t)=\frac{\exp\left(t+\sum_{i=1}^{d}x_i\right)}{1+\exp\left(t+\sum_{i=1}^{d}x_i\right)} \tag{22}$$

which is our target in this section.

GP Setting

Rewrite the PDE as

$$\frac{\partial u}{\partial t}+(\sigma^2u(x,t)-\frac{1}{d}-\frac{\sigma^2}{2})div_xu(x,t)+\frac{\sigma^2}{2}\Delta_xu(x,t)=0,t\in[s,T],x\in D\in\mathbb{R}^d \tag{23}$$

Let the nonlinear term be

$$\vec{y}_{domain}=0_{M_\Omega} \tag{24}$$

Define operators

$$L_1(u):u\to\Delta_xu \tag{25}$$
$$L_2(u):u\to\frac{\partial u}{\partial t} \tag{26}$$
$$L_3(u):u\to div_xu \tag{27}$$

We define feature functions

$$\phi^1_j(u):u\to\delta_{x^j_\Omega}\circ u,1\le j\le M_\Omega \tag{28}$$
$$\phi^2_j(u):u\to\delta_{x^j_{\partial\Omega}}\circ u,1\le j\le M_{\partial\Omega} \tag{29}$$

$$\phi_j^3(u): u \to \delta_{x_\Omega^j} \circ L_1 \circ u, 1 \le j \le M_\Omega \tag{30}$$
$$\phi_j^4(u): u \to \delta_{x_\Omega^j} \circ L_2 \circ u, 1 \le j \le M_\Omega \tag{31}$$
$$\phi_j^5(u): u \to \delta_{x_\Omega^j} \circ L_3 \circ u, 1 \le j \le M_\Omega \tag{32}$$

Denote $\vec{z}_i = \phi^i_{1:M_\Omega}(u), i \in \{1,3,4,5\}$ and $\vec{z}_i = \phi^i_{1:M_{\partial\Omega}}(u), i = 2$.

Let

$$\vec{z} = \begin{pmatrix} \vec{z}_1 \\ \vec{z}_2 \\ \vec{z}_3 \\ \vec{z}_4 \\ \vec{z}_5 \end{pmatrix} \tag{33}$$

We then derive

$$F(\vec{z}) = \begin{pmatrix} \vec{z}_4 + \sigma^2 \vec{z}_1 \odot \vec{z}_5 - (\frac{1}{d} + \frac{\sigma^2}{2})\vec{z}_5 + \frac{\sigma^2}{2}\vec{z}_3 \\ \vec{z}_2 \end{pmatrix} \tag{34}$$

Thus $Q_\Omega = 4, Q = 5$.

Feature vector ϕ have $M = 4M_{\partial\Omega} + M_\Omega$ components, which is also the size of the kernel matrix.

Taking derivatives under linearization condition

$$DF(\vec{z}_k) = \begin{pmatrix} \sigma^2 \mathrm{diag}(\vec{z}_5) & 0 & \frac{\sigma^2}{2} I_{M_\Omega} & I_{M_\Omega} & \sigma^2 \mathrm{diag}(\vec{z}_1) - (\frac{1}{d} + \frac{\sigma^2}{2}) I_{M_\Omega} \\ 0 & I_{M_{\partial\Omega}} & 0 & 0 & 0 \end{pmatrix}$$

Parameters

Specifically, we consider the problem for

$$d = 100, \mu = 0, \sigma = 0.25, D = [-0.5, 0.5]^{100}, s = 0, T = 0.5 \tag{35}$$

and

$$d = 250, \mu = 0, \sigma = 0.25, D = [-0.5, 0.5]^{250}, s = 0, T = 0.5 \tag{36}$$