Complicated_HJB

MLP Setting

We solve the following PDE(labeled as Eq(100) in the original paper)

$$\frac{\partial u}{\partial t} - \frac{1}{d} div_x u(x,t) + 2 + \Delta_x u(x,t) = 0, t \in [s,T], x \in D \subset \mathbb{R}^d \tag{1}$$

whose terminal condition is given by

$$u(x,T) = g(x) := \sum_{i=1}^{d} x_i, \tag{2}$$

without any boundary constraints.

The nonlinear term is given by

$$F(u,z)(x,t) = 2 \tag{3}$$

This PDE has an explicit solution at time t

$$u(x,t) = \sum_{i=1}^{d} x_i + (T-t). \tag{4}$$

which is our target in this section.

GP Setting

Rewrite the PDE as

$$\frac{\partial u}{\partial t} - \frac{1}{d} div_x u(x,t) + \Delta_x u(x,t) = -2, t \in [s,T], x \in D \subset \mathbb{R}^d \tag{5}$$

Let the nonlinear term be

$$\vec{y}_{domain} = -2 \cdot 1_{M_\Omega} \tag{6}$$

Define operators

$$L_1(u) : u \to \Delta_x u \tag{7}$$
$$L_2(u) : u \to \frac{\partial u}{\partial t} \tag{8}$$
$$L_3(u) : u \to div_x u \tag{9}$$

We define feature functions

$$\phi_j^1(u) : u \to \delta_{x_\Omega^j} \circ u, 1 \le j \le M_\Omega \tag{10}$$
$$\phi_j^2(u) : u \to \delta_{x_{\partial\Omega}^j} \circ u, 1 \le j \le M_{\partial\Omega} \tag{11}$$
$$\phi_j^3(u) : u \to \delta_{x_\Omega^j} \circ L_1 \circ u, 1 \le j \le M_\Omega \tag{12}$$
$$\phi_j^4(u) : u \to \delta_{x_\Omega^j} \circ L_2 \circ u, 1 \le j \le M_\Omega \tag{13}$$
$$\phi_j^5(u) : u \to \delta_{x_\Omega^j} \circ L_3 \circ u, 1 \le j \le M_\Omega \tag{14}$$

Denote $\vec{z}_i = \phi_{1:M_\Omega}^i(u), i \in \{1,3,4,5\}$ and $\vec{z}_i = \phi_{1:M_{\partial\Omega}}^i(u), i = 2$.

Let

$$\vec{z} = \begin{pmatrix} \vec{z}_1 \\ \vec{z}_2 \\ \vec{z}_3 \\ \vec{z}_4 \\ \vec{z}_5 \end{pmatrix} \tag{15}$$

We then derive

$$F(\vec{z}) = \begin{pmatrix} \vec{z}_4 - \frac{1}{d}\vec{z}_5 + \vec{z}_3 \\ \vec{z}_2 \end{pmatrix} \tag{16}$$

Thus $Q_\Omega = 4, Q = 3$.

Feature vector ϕ have $M = 4M_{\partial\Omega} + M_\Omega$ components, which is also the size of the kernel matrix.

Taking derivatives under linearization condition

$$DF(\vec{z}_k) = \begin{pmatrix} 0 & 0 & I_{M_\Omega} & I_{M_\Omega} & -\frac{1}{d} I_{M_\Omega} \\ 0 & I_{M_{\partial\Omega}} & 0 & 0 & 0 \end{pmatrix}$$

Parameters

Specifically, we consider the problem for

$$d = 100, \mu = -1/d, \sigma = \sqrt{2}, D = [-0.5, 0.5]^{100}, s = 0, T = 0.5 \tag{17}$$

and

$$d = 250, \mu = -1/d, \sigma = \sqrt{2}, D = [-0.5, 0.5]^{250}, s = 0, T = 0.5 \tag{18}$$