The Homotopy Test

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November 10, 2020 (revised April 1, 2025)

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A fundamental problem when dealing with curves on surfaces is to decide if a given closed curve can be contracted to a point, or more precisely to a constant curve. This is sometimes referred to as the **contractibility problem**. More generally, we can ask whether two closed curves on a surface are related by a continuous deformation. This question has two variants: we may or may not require the curves to share a given point that remains fixed during the deformation. Note that the problem with fixed point has an obvious reduction to the contractibility problem. Indeed, two curves c, d are homotopic with fixed point if and only if the concatenation $c \cdot d^{-1}$ is contractible. Without the fixed point requirement, that is when the curves are allowed to move *freely* on the surface, the problem is known as the **transformation problem** and can be expressed as a **conjugacy problem**. To see this, choose a point v on a surface S and suppose that c and d are homotopic[1]. We can deform c and d so that each of them passes through p. The resulting curves are still homotopic. In other words, there is a continuous mapping $h : \mathbb{S}^1 \times [0,1] \to S$ such that $h|_{\mathbb{S}^1\times\{0\}} = c$ and $h|_{\mathbb{S}^1\times\{1\}} = d$, and viewing $\mathbb{S}^1 \times [0,1]$ as an annulus, each boundary has a point sent on v by h. We connect these two points by a simple path a in the annulus. The map h sends this path to a closed path α. See Figure 1. Cutting the annulus through a we obtain a disk whose

[1] Homotopy without fixed point is often called *free* homotopy. For concision, we drop the term free. In general, it should be clear from the context whether we use free homotopy or homotopy with fixed point, and we will specify when necessary that the homotopy is with fixed point.

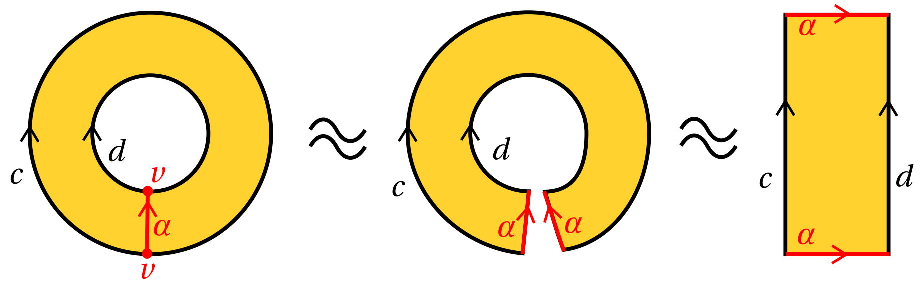

Figure 1: c and d are homotopic if and only if their homotopy classes are conjugate.

boundary is sent to $c \cdot \alpha \cdot d^{-1} \cdot \alpha^{-1}$ which is thus contractible. Hence, c is homotopic to $\alpha \cdot d \cdot \alpha^{-1}$ or, equivalently, the homotopy classes of c and d are conjugate in the fundamental group $\pi_1(S, v)$. For the reverse implication, if c and d have conjugate homotopy classes we can just read Figure 1 from right to left and conclude that c and d are indeed homotopic.

1 Dehn's Algorithm

Suppose that S is a **reduced** combinatorial surface, that is a map with a single vertex and a single face. Its graph G is thus composed of loop edges, each of which corresponds to a generator of the fundamental group of S. We can directly read the homotopy class of a closed path in G: the sequence of arcs of the path translates to the product of the corresponding generators and their inverses. This product is often viewed as a **word** on the generators and their inverses, so that the contractibility problem is the same as the **word problem** where we ask if a product of generators and their inverses is the trivial element in the fundamental group of S.

Max Dehn was among the first to establish and exploit the connection between Topology (the contractibility problem) and Algebra (the word problem). He proposed a solution to the word problem now known as **Dehn's algorithm** [Sti87, paper 5]. Dehn observed that the lift of G in the universal covering space of S induces a tessellation of the plane composed of copies of the unique polygonal face of G in S. This tessellation is actually the **Cayley complex** of $\pi_1(S, v)$ where v is the unique vertex of G. This complex $\tilde{S}$ is relative to the set of generators $\{\beta_i\}_i$ of $\pi_1(S, v)$ – the homotopy classes of the loop edges in G – and to their relation F obtained from the unique facial walk of G in S. The vertex set of $\tilde{S}$ are the elements of $\pi_1(S, v)$ and there is an (oriented) edge labelled β_i between every $\alpha \in \pi_1(S, v)$ and $\alpha \cdot \beta_i$. Finally, disks are glued along each closed path labelled by F in the resulting graph. If a closed path c in G is contractible in S, then any of its lifts is a closed path in $\tilde{S}$. Dehn further claims that

*any closed path in $\tilde{S}$ contains either a **spur**, i.e. an arc followed by its opposite arc, or more than half of F, i.e. a subpath labelled by some word U such that for some other V shorter than U, the concatenation UV is a cyclic permutation of F or its inverse.*

In both cases c is homotopic to a shorter closed path obtained by removing the spur in the former case and by replacing the path labelled by U with the complementary path labelled by V^{-1} in the latter case. This leads to an algorithm where we inductively search for spurs or large pieces of F until we obtain a word that we cannot reduce anymore. It then follows from Dehn's claim that c is contractible if and only if this word is empty.

In order to prove his claim, Dehn notes that the faces of the complex $\tilde{S}$ are arranged in rings of faces $R_1, R_2, \ldots$, where R_1 is the set of faces incident with a given vertex[2] v_0 of $\tilde{S}$ and R_{i+1} is the set of faces not in R_i sharing a vertex with the external boundary of R_i. Remark that a face of R_{i+1} has at most two vertices in R_i. Hence, if S is an orientable surface of genus $g \geq 2$, each face has $4g$ sides and a face of a ring has at least $4g-2 > 2g$ vertices on its external boundary. Consider now a closed path $\tilde{c}$ without spurs and passing through v_0. Let i be maximal such that $\tilde{c}$ contains a vertex of the external boundary of R_i. Figure 2 illustrates a factious case of a relation of length 6. Since $\tilde{c}$

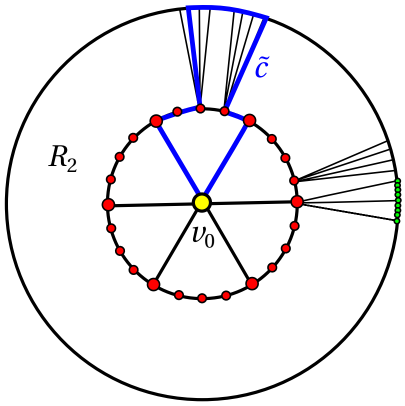

Figure 2: The faces of the complex $\tilde{S}$ are arranged in rings of faces.

has no spurs it is easily seen that it contains the whole intersection of a face with the external boundary of R_i. The previous remark allows to conclude the claim.

Dehn's algorithm has a simple implementation that runs in $O(g|c|)$ time where g is the genus of S. A more careful implementation with $O(g+|c|\log g)$ time complexity was described by Dey and Schipper [DS95]. Finally, optimal $O(g+|c|)$ algorithms were proposed [LR12, EW13]. We shall describe these last approaches to the contractibility and deformation problems, not so far from Dehn's original approach but including more recent techniques borrowed from geometric group theory.

2 van Kampen Diagrams

2.1 Disk diagrams

A useful tool concerning contractible curves is provided by the so called **van Kampen diagrams**. Such diagrams bear different names in the litterature, among which **disk diagrams** and **Dehn diagrams** are the most common. Intuitively, a disk diagram allows to express the combinatorial counterpart of the following characterization of contractible loops in a topological space X: a loop $\mathbb{S}^1 = \partial\mathbb{D}^2 \to X$ is contractible if and only if it extends to a continuous map $\mathbb{D}^2 \to X$, where $\mathbb{D}^2$ is the unit disk. Given a combinatorial map M with graph G, a **disk diagram over** M is a combinatorial sphere

[2] In his original work, Dehn defines R_1 as a single face.

D with a marked **outer face**, and a labelling of the arcs of D by the arcs of M such that opposite arcs are labelled by opposite arcs and such that every facial walk of D that is not the outer face is labelled by some facial walk of M. In other words, D is a gluing of faces and edges of M that is homeomorphic to the complement of an open disk in a sphere. For instance, this complement could be a tree. In general, it is a tree-like arrangement of topological closed disks connected by trees. The facial walk of the outer face of D is denoted by ∂D. The diagram is **reduced** if any two of its *inner* faces (i.e. not the outer one) sharing a vertex v are labelled by facial walks that are not inverse to each other when starting the facial walks at v.

Lemma 2.1 (van Kampen, 1933)**.** *A closed path c in M is contractible if and only if it is the label of the outer facial walk of a reduced disk diagram over M.*

The proof uses the intuitive fact that homotopic closed paths are **combinatorially homotopic**, where a combinatorial homotopy is a sequence of **elementary homotopies** that consist in either inserting or removing a spur, or replacing a subpath of a facial walk by the complementary subpath. See Theorem 4.7 in the previous lecture notes.

PROOF OF LEMMA 2.1. We first prove the existence of a not necessarily reduced disk diagram. Let $c_0 = 1 \to c_1 \to \cdots \to c_k = c$ be a sequence of k elementary homotopies attesting the contractibility of c, where 1 denotes a constant path. By induction on k, we may assume the existence of a disk diagram D such that ∂D is labelled by c_{k-1}. There are three cases to consider.

- If $c_{k-1} \to c_k$ consists in inserting a spur aa^{-1}, then we can form a disk diagram for c_k by attaching a pendant edge labelled with a to the boundary of D.
- If $c_{k-1} \to c_k$ consists in removing a spur, then either this spur corresponds to two consecutive arcs of ∂D with distinct edge support or it corresponds to the two arcs of a single pendant edge. In the former case, we form a disk diagram for c_k by gluing the two arcs along ∂D. In the latter case, we contract the pendant edge.
- Otherwise, $c_{k-1} \to c_k$ consists in the replacement of a subpath p by a subpath q such that pq^{-1} is a facial walk of M. We then perform a subdivision of the outer face of D, inserting a new edge between the extremities of p. The new outer face is chosen among the two new faces as the one not bounded by p. We next subdivide the new edge $k-1$ times, where k is the number of arcs of q. We finally extend the labelling trivially by sending the subdivided edge to the edges of q. This amounts to glue a face with facial walk pq^{-1} along p on D.

If the resulting diagram is not reduced, then there are two facial walks sharing a vertex v and labelled by opposite facial walks of M. We “open” D at v and identify the two facial walks according to the labels of their arcs. This produces a new diagram with two faces less and does not modify the outer face boundary. We repeat the procedure as long as the diagram is not reduced. By induction on the number of faces this procedure must end. Note that the final diagram may have no face, in which case its graph must be a tree corresponding to a closed path that can be reduced to a point by removing spurs only. □

Exercise 2.2. Relates the degree of an inner vertex in a reduced disk diagram over M with the degree of the corresponding vertex in M.

2.2 Annular diagrams

There is an analogous notion of **annular diagram** defined by a combinatorial sphere with two marked outer faces instead of one.

Lemma 2.3 (Schupp, 1968)**.** *Two closed paths c and d in M are homotopic if and only if there exists a reduced annular diagram over M such that the facial walks of its outer faces (oriented consistently) are labelled with c and d respectively.*

PROOF. By the introductory discussion there exists a path p such that $c \cdot p \cdot d^{-1} \cdot p^{-1}$ is contractible. By Lemma 2.1, there exists a disk diagram over M whose boundary is labelled with $c \cdot p \cdot d^{-1} \cdot p^{-1}$. We may identify the subpaths corresponding to p and p^{-1} respectively and get an annular diagram whose perforated faces are labelled with c and d. If the diagram is not reduced, we proceed as in the proof of Lemma 2.1. □

3 Gauss-Bonnet Formula

Another interesting tool is given by a combinatorial version of the famous Gauss-Bonnet theorem. This theorem relates the curvature of a Riemannian surface S (say a smooth surface embedded into $\mathbb{R}^3$) with its Euler characteristic χ, hence a local geometric quantity with a global topological one. If K is the Gauss curvature of S and k_g is the geodesic curvature along its (smooth) boundary ∂S then:

$$\int_S K \,\mathrm{d}s + \int_{\partial S} k_g \,\mathrm{d}\ell = 2\pi\chi \tag{1}$$

We can obtain a combinatorial version of this formula using some kind of angle structure over a combinatorial surface. Given an orientable combinatorial map $M = (A, \rho, \iota)$, we consider an angular assignment of its corners, that is a real function θ defined over the set of corners. Here, a **corner** is any pair $(a, \rho(a))$, for $a \in A$, of successive arcs around a vertex. We require that the sum of the angular assignments of the corners of any face f satisfies

$$\sum_{c \in f} \theta(c) = d_f/2 - 1, \tag{2}$$

where d_f is the degree of the face, i.e. the length of its facial walk. Intuitively, this condition amounts to assume that the faces are Euclidean polygons if we view an angular assignment as a normalized angle, measuring angles in terms of parts of a circle instead of radians. Indeed, the total angle of a Euclidean polygon with d_f sides is $(d_f - 2)\pi$, which is $d_f/2 - 1$ when normalized. We then define the curvature of an interior vertex v as

$$\kappa(v) = 1 - \sum_{c \in v} \theta(c), \tag{3}$$

where, $c \in v$ indicates that the corner $c = (a, \rho(a))$ is incident to the source vertex v of a. We also define the (geodesic) curvature of a boundary vertex[3] v as

$$\tau(v) = 1/2 - \sum_{c \in v} \theta(c) \tag{4}$$

Those curvatures thus measure the angle default with respect to the flat situation ($\kappa = 1$ and $\tau = 1/2$). They can be related to the Gauss curvature of the flat conic surface S_v with one singularity at v obtained by gluing small isocele triangles, one for each corner $c \in v$, with angle $2\pi\theta(c)$ at v. The boundary of S_v is a broken line so that Formula (1) should be corrected with the term $\sum_w (\pi - \alpha_w)$, where w runs over the boundary vertices of S_v and α_w is the interior angle at w. Since the geodesic curvature of a line segment is zero, Formula (1) becomes

$$\int_{S_v} K \, \mathrm{d}s + \sum_w (\pi - \alpha_w) = 2\pi\chi = 2\pi$$

Noting that with $\sum_w (\pi - \alpha_w)$ is the sum of the angles at the corners of v we obtain $\int_{S_v} K = 2\pi\kappa_v$.

Theorem 3.1 (Combinatorial Gauss-Bonnet —)**.** *Let M be a combinatorial map whose boundary is composed of disjoint simple cycles in the graph of M. Denote by χ the Euler characteristic of M and by $V^o \cup V^\partial = V$ its interior and boundary vertex sets. Then, for any angular assignment, we have*

$$\sum_{v \in V^o} \kappa(v) + \sum_{v \in V^\partial} \tau(v) = \chi$$

It is possible to drop the condition on the boundary of M using a slightly different notion of curvature, see Erickson and Whittlesey [EW13]. The above presentation is inspired by Gersten and Short [GS90] and makes the parallel with the differentiable version rather transparent.

Proof. By definition, we compute

$$\sum_{v \in V^o} \kappa(v) = |V^o| - \sum_{c \in v \in V^0} \theta(c) \quad \text{and} \quad \sum_{v \in V^\partial} \tau(v) = |V^\partial|/2 - \sum_{c \in v \in V^\partial} \theta(c)$$

It follows that $\sum_{v \in V^o} \kappa(v) + \sum_{v \in V^\partial} \tau(v) = |V| - |V^\partial|/2 - \sum_{c \in v \in V} \theta(c)$. By distributing the corners according to faces rather than vertices and by the angular assignment requirement (2), we see that

$$\sum_{c \in v \in V} \theta(c) = \sum_{c \in f \in F} \theta(c) = \sum_{f \in F} (\frac{d_f}{2} - 1) = \frac{1}{2} \sum_{f \in F} d_f - |F|$$

[3]Formally, a combinatorial surface with boundary is defined by marking some faces as perforated, and a boundary vertex is any vertex incident to a perforated face.

where F is the set of faces of M. Since every arc appears in exactly one facial walk, except for those on the boundary of M, we have: $\sum_{f\in F} d_f = 2|E|-|E^\partial|$ where E and E^∂ are the set of edges and boundary edges respectively. Since $|E^\partial| = |V^\partial|$, we conclude that

$$\begin{aligned}\sum_{v\in V^o}\kappa(v)+\sum_{v\in V^\partial}\tau(v) &= |V|-|V^\partial|/2-(|E|-|E^\partial|/2)-|F|) \\ &= |V|-|E|+|F|\end{aligned}$$

□

4 Quad Systems

From an algorithmic point of view it is more convenient to work with combinatorial surfaces all of whose faces are quadrilaterals. We call such a surface a **quadrangulation** or a **quad system**. Given a combinatorial surface without boundary, we easily get a quadrangulation of the same topological surface as follows. We insert a vertex inside each face and connect this vertex to all the corners of the face. Hence, if a facial walk has length k we introduce k new edges in the face. This subdivides each face into triangles. We then delete all the edges of the original graph, thus merging all the triangles by pairs to form quadrilaterals. In practice, we will also require that the vertices have a high degree, say at least 8. For a surface of genus $g \geq 2$ this is easily obtained by first reducing the combinatorial surface to a single vertex and a single face before applying the above quadrangulation process. The resulting quadrangulation has two vertices, $4g$ edges and $2g$ quadrilaterals. Figure 3 shows a reduced surface and its quadrangulation.

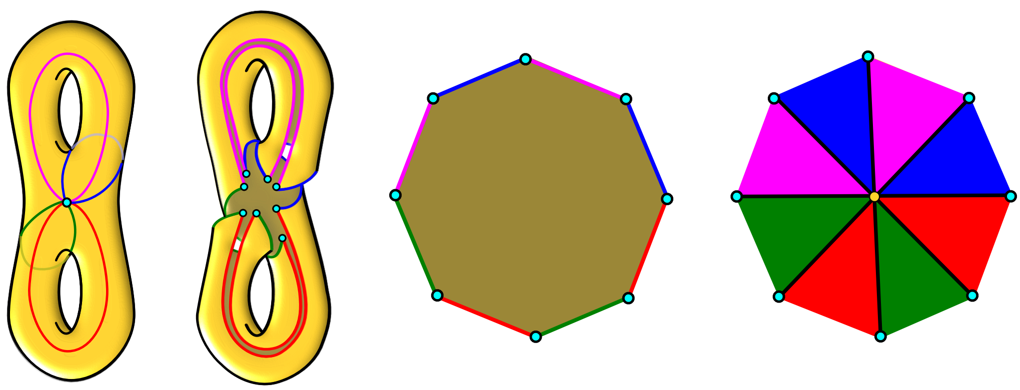

Figure 3: From left to right, a reduced surface is cut-opened and its unique face is triangulated by inserting a vertex in the center. Triangles of the same color are merged by deleting the original loop edges.

Lemma 4.1. *Let Q be a quadrangulation derived by the previous process from a given map M without boundary. We can preprocess M in linear time (proportional to its* *number of arcs) so that any closed walk c can be transformed in $O(|c|)$ time into a homotopic closed walk of size at most $2|c|$ in Q.*

To see this, consider a spanning tree T of the graph G of M. Contracting T gives a surface M' with graph G/T and with a single vertex. Next consider a spanning tree of the dual graph of M' and denote by L the corresponding set of primal edges. Deleting the edges in L leaves a reduced surface M'' and we construct Q by first inserting a new vertex z in the unique face of M'' together with all the edges from z to the corners of the face. We finally remove the remaining edges of G/T to get Q. Note that any edge e of G/T is homotopic to the path of length two in Q connecting z to the two endpoints of e. We can precompute and store these length two paths for each e in total linear time. Now, given any c, we contract all the occurrences of edges of T in c to obtain a homotopic closed walk c' in M'. We further replace every remaining edge by the corresponding length two path to obtain a homotopic closed walk as desired in Q. This transformation takes $O(|c|)$ time.

Exercise 4.2. Propose a construction of quadrangulation starting from a combinatorial surface with nonempty boundary. Can you extend Lemma 4.1 accordingly?

5 Reduction to Canonical Form

The last and most important ingredient of the homotopy test is the construction of a canonical representative in each free homotopy class. Given a closed walk in a quadrangulation, the idea is to shorten the walk as much as possible to obtain a combinatorial geodesic. As a homotopy class may contain several geodesics, we further consider the *rightmost* geodesic to define a canonical representative. Once a canonical representative has been computed for two given closed walks we can decide if the walks are homotopic by just checking if their representative are equal up to a circular permutation. The shortening process is based on successive simplifications of spurs and brackets as explained below.

5.1 The four-bracket lemma

Let (a_1, a_2) be a pair of arcs sharing their origin vertex v on a quadrangulation M. Following the terminology of Erickson and Whittlesey [EW13], we define the **turn** of (a_1, a_2) as the number of corners between a_1 and a_2 in counterclockwise order around v. Hence, if v is a vertex of degree d in M, the turn of (a_1, a_2) is an integer modulo d that is zero when $a_1 = a_2$. The **turn sequence** of a subpath $(a_i, a_{i+1}, \dots, a_{i+j-1})$ of a closed walk of length ℓ is the sequence of $j+1$ turns of $(a_{i+k}^{-1}, a_{i+k+1})$ for $-1 \le k < j$, where indices are taken modulo ℓ. The subpath may have length ℓ, thus leading to a sequence of $\ell+1$ turns. Note that the turn of $(a_{i+k}^{-1}, a_{i+k+1})$ is zero precisely when (a_{i+k}, a_{i+k+1}) is a spur. A **bracket** is any subpath whose turn sequence has the form 12^*1 or $\bar{1}\bar{2}^*\bar{1}$ where t^* stands for a possibly empty sequence of turns t and $\bar{x}$ stands for $-x$. Intuitively, if we imagine that every corner of M has a right angle, a bracket corresponds to a straight path ending with right angles. A quadrangulated disk is **non-singular** if its boundary is a simple cycle of its graph.

Lemma 5.1 (Four-bracket —, [GS90, EW13])**.** *Let D be a non-singular quadrangulated disk all of whose interior vertices have degree at least four. Then, the boundary of D contains at least four brackets.*

Figure 4 illustrates the Lemma.

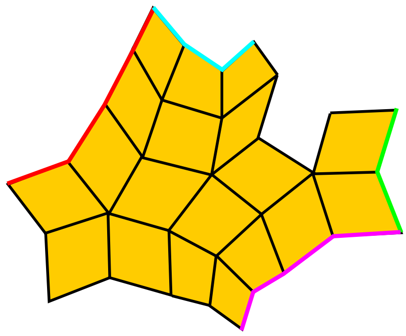

Figure 4: The quadrangulated disk has four highlighted brackets. Can you find them all?

PROOF. Consider the constant angular assignment $1/4$ over D. By the Gauss-Bonnet theorem 3.1, we have $\sum_{v \in int D} \kappa(v) + \sum_{v \in \partial D} \tau(v) = \chi(D) = 1$. By (3), every interior vertex has non-positive curvature. It follows that

$$\sum_{v \in \partial D} \tau(v) \geq 1 \tag{5}$$

Remark that $\tau(v) = (2 - c_v)/4$ where c_v is the number of corners incident to the boundary vertex v. Call v **convex, flat** or **concave** if $c_v = 1$, $c_v = 2$ or $c_v \geq 3$ respectively. In other words v is convex, flat or concave if its curvature is respectively $1/4$, zero or negative. Inequality (5) implies that the boundary of D contains at least four more convex vertices than concave vertices. The lemma easily follows. □

Corollary 5.2. *A nontrivial contractible closed walk in a quadrangulation all of whose interior vertices have degree at least four contains either a spur or a bracket.*

PROOF. Suppose that a nontrivial contractible closed walk c has no spurs. By the van Kampen Lemma 2.1, c is the label of the boundary of a reduced disk diagram D. Let H be the dual graph of D: it has one dual vertex per quadrilateral of D and one dual edge for each pair of quadrilaterals sharing an edge. If H is connected then D is non-singular. Indeed, if the boundary of its outer facial walk ∂D was not a cycle it would contain a degree one vertex, which would contradicts that c has no spurs. We can thus apply the four-brackets lemma 5.1 to conclude that ∂D has at least one bracket. However, the turn t at a vertex of ∂D is the same as the turn of the corresponding vertex in c (up to a multiple of the degree of that vertex in the quadrangulation). It follows that c has also a bracket. If H is not connected, then D consists of a tree-like arrangement of non-singular disks connected by trees through cut vertices. This arrangement has a “degree one” non-singular disk connected to the rest through a single cut vertex. By the previous Lemma 5.1 this disk has four brackets, two of which do not contain the cut vertex. These two brackets thus correspond to brackets in c. □

Exercise 5.3. Show that we can actually claim the existence of a spur or *four* brackets in Corollary 5.2.

5.2 Bracket flattening

A **bracket flattening** consists in replacing a bracket and the two incident edges with the "straight line" between their endpoints. Some care must be taken when the incident edges of the bracket share their endpoints or when these edges are part of the bracket. Figure 5 depicts the different cases. Corollary 5.2 provides a practical algorithm to test

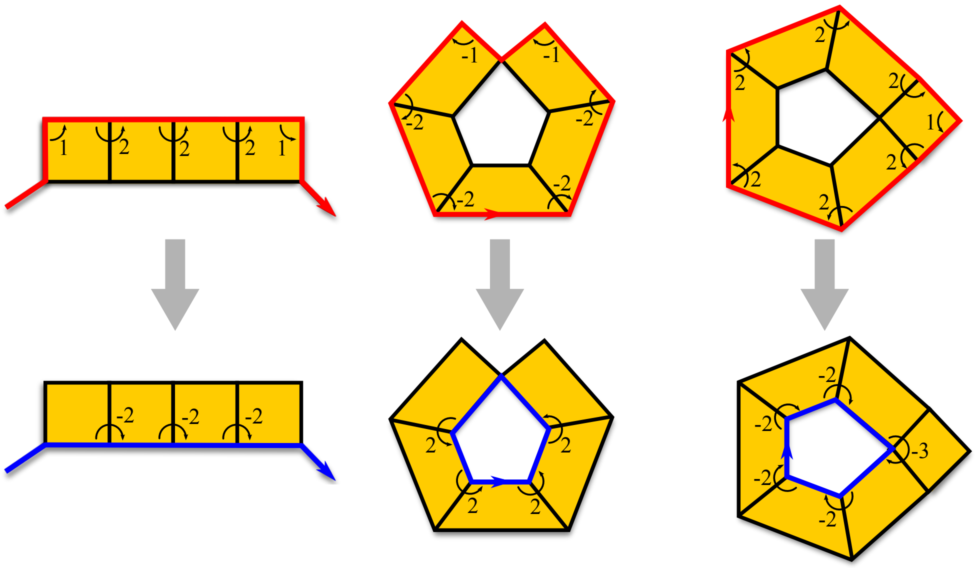

Figure 5: Left, a typical bracket flattening. Middle, the edges incident to the bracket share their endpoints. Right, the bracket covers the whole closed walk.

if a given closed walk c is contractible: remove the spurs and flatten the brackets until there is no more. Then c is contractible if and only if the resulting walk is reduced to a vertex. Note that the non-typical bracket flattening (Figure 5, Right) may only occur when c is non-contractible (why?).

5.3 Canonical representatives

A homotopy class may contain several closed walks without spurs and brackets. In order to get a canonical representative in each homotopy class we further push such reduced walks as much as possible "to their right". Say that a vertex of a walk is **convex** if its turn is 1 in the turn sequence of the walk. If a closed walk c contains a convex vertex v we consider the maximal subpath including v whose turning sequence has the form $x2^*12^*y$, where $x, y \neq 2$. This subpath, say p, bounds an L-shaped sequence of quadrilaterals that lies to its right. Replacing p by the complementary path bounding the sequence of quadrilaterals gives a closed walk homotopic to c with one less convex vertex. Note that this replacement does neither introduce a bracket nor a spur. Some care must again be taken when p covers c. See Figure 6 for all the possible typical and

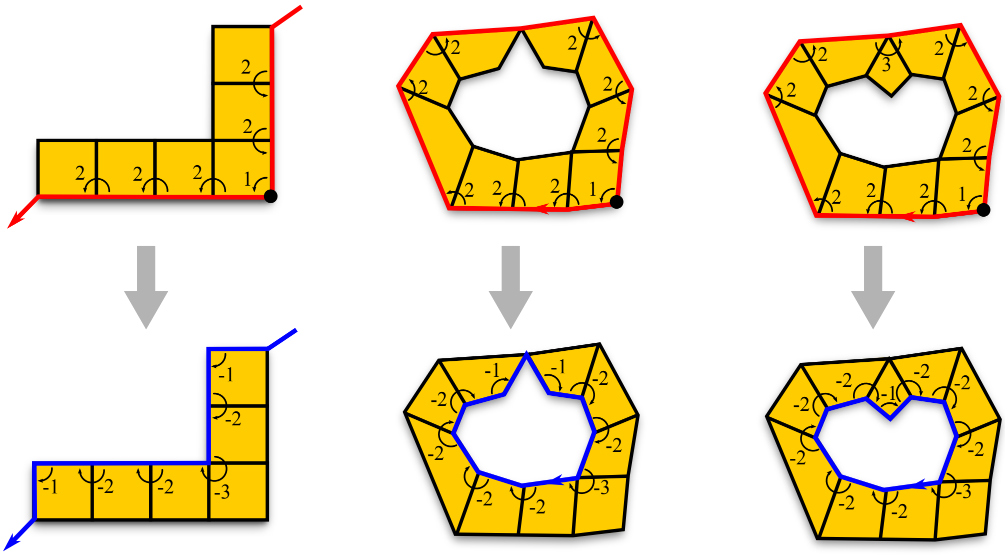

Figure 6: The different configurations for a right push.

non-typical configurations. A right push reduces the number of convex vertices by one, so that only a linear number of pushes can be applied. A last exceptional case occurs when the turn sequence of c is composed of 2's only. We also apply a right push in this case, which transforms the turn sequence into a sequence of $\bar{2}$ as on Figure 7. When no right pushes apply, the closed walk is said **reduced**, or in **canonical form**.

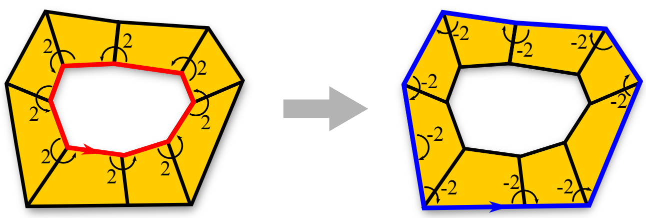

Figure 7: In case all the turns are equal to 2, we push the walk to the right to obtain a sequence $\bar{2}^*$ of turns.

Proposition 5.4. *Let M be a quadrangulation all of whose vertices have degree at least five. Then each homotopy class contains a unique reduced closed walk.*

PROOF. Let c and d be homotopic reduced closed walks. We need to show that $c = d$. Following Lemma 2.3 we consider a reduced annular diagram A for c and d. We first claim that the two boundaries of A are simple. Otherwise, one boundary has a cut vertex that separates A into a smaller annular part A' and a disk part D connected to A' through a single cut vertex. By the four-brackets lemma, the boundary of D has one (in fact at least two) bracket disjoint from this cut vertex. In turn, this bracket would appear in c or d, contradicting the hypothesis that c and d are reduced.

- If the two boundaries of A have a common vertex then cutting through that vertex gives a disk diagram D' bounded by (circular permutations of) c and d.

This diagram is a tree-like arrangement of non-singular disks connected by trees through cut vertices. For convenience, we also call cut vertices the two common endpoints of c and d. If a non-singular disk is incident to a single cut vertex, then it is bounded by a subpath of one of c or d. By the four-bracket lemma this subpath would contain a bracket, in contradiction with the reduction hypothesis. It follows that D' is a linear sequence of non-singular disks connected by simple paths (otherwise c or d would have a spur). We claim that none of those non-singular disks can have an interior vertex. Otherwise, considering the constant angular assignment $1/4$ over D', this interior vertex would have negative curvature. An argument similar to the proof of the four-bracket lemma 5.1 shows that the boundary of D' would contain five brackets, one of which not incident to any cut vertex. This would again lead to the contradiction that c or d has a bracket. The dual graph of each non-singular disk is thus a tree. However, no matter the shape of this tree and no matter how its boundary is split one of the resulting boundary paths would contain a bracket or a convex vertex. In both cases this would contradict the fact that c and d are reduced. It follows that D' has no non-singular disk, hence is a simple path, implying that $c = d$.

- Suppose now by way of contradiction that the two boundaries of A are disjoint. The proof of the four-bracket lemma applies to an annulus with a interior vertex of negative curvature to show the existence of a bracket. It follows that as above that A has no interior vertex. Indeed, we can mimic the proof of the four-bracket lemma for an annulus with a vertex of negative curvature to show the existence of a bracket.) The dual graph of A is thus a single cycle with some attached trees. It must actually be a cycle, since otherwise one of the boundaries of A would have a bracket. This cycle has to go straight without bending since otherwise c or d would have a convex vertex or a bracket. (This last case occurs even with a single bend as on Figure 5, Right.) It follows that one of the boundaries of A has 2-turns only as on right Figure 7, contradicting that c and d are reduced. In any case we have reached a contradiction, so that the boundaries of A cannot be disjoint.

□

A reduced closed walk can thus play the role of **canonical representative** for its homotopy class.

6 The Homotopy Test

We now have all the necessary ingredients to perform a homotopy test between two closed walks on a combinatorial surface. Thanks to Lemma 4.1, we can assume that the surface is quadrangulated. We compute the canonical form of each closed walk by first removing spurs and brackets as described in Section 5.2, then applying right pushes to remove convex vertices as explained in Section 5.3. We show below that this can be done for each closed walk in time proportional to its length. It remains to compare these canonical forms, which can be done in linear time.

Say that a walk, viewed as a path, is **geodesic** if it contains no spur or bracket.

Lemma 6.1. *Given a walk c in a quadrangulation, all of whose vertices have degree at least five, we can compute a homotopic geodesic in $O(|c|)$ time.*

PROOF. We "shorten" c as mush as possible by removing spurs and flattening brackets incrementally as we traverse c from its first vertex[4]. In order to facilitate the analysis, we consider that each spur removal amounts to delete two arcs from c and that each bracket flattening amounts to delete its two side arcs and to *move* the edges of the bracket by translating them by one quad. This way, every arc of the final walk can be traced back to an arc of c.

We use a stack to store the currently traversed subpath of c and maintain the invariant that this subpath is geodesic. We start with an empty stack and incrementally push the successive arcs of c. Each time an arc a is pushed on top of the stack we check in constant time[5] if a forms a spur or a bracket with the previous arcs on the stack and update the stack accordingly. We denote by TS the turn sequence of the subpath of c stored in the stack (we use a dummy turn for the first vertex of c).

- If a forms a spur with the previous arc, i.e. if TS has a suffix of length 3 of the form $x0y$, we simply pop a and the previous arc out of the stack. This amounts to replace $x0y$ by the suffix $x+y$ (of length one) in TS.

- If a closes a bracket of length $k+1$, i.e. if TS has a suffix of the form $x12^k1y$ or $x\bar{1}\bar{2}^k\bar{1}y$, we pop off the $k+1$ arcs of the (flat part of the) bracket as well as its two incident arcs and push the flat part, translated by one quad, again into the stack. See Figure 5, left. This replaces the suffix of TS by $(x-1)\bar{2}^k(y-1)$ or $(x+1)2^k(y+1)$, respectively.

After this update, the stack clearly contains a path without spurs or brackets. Call a bracket *positive* or *negative* if its turn sequence has the form 12^*1 or $\bar{1}\bar{2}^*\bar{1}$, respectively.

Claim 1. If an arc is moved twice because of two successive bracket flattenings, possibly separated by spur removals and bracket flattenings not involving that arc, then these brackets have the same sign.

PROOF OF THE CLAIM. Suppose otherwise and assume without loss of generality that a negative bracket succeeds a positive bracket, implying two moves of a same arc a. For some turns x, y and some possibly empty turn sequence X, we have TS $=$ $Xx12^k1y$ just before the positive bracket flattening and $X(x-1)\bar{2}^k(y-1)$ just after. As long as a is not moved some prefix $X(x-1)\bar{2}^{k'\leq k}$ of the turn sequence remains untouched. When the negative bracket occurs the turn sequence must then have the form $\mathrm{TS}' = X(x-1)\bar{2}^\ell\bar{1}y'$. Note that $x \neq 0$ since initially TS represents a path without spurs. Hence, in order for TS$'$ to contain a negative bracket it must be that

[4] In [EW13], a so-called run-length encoding of the turn sequence of c is used to perform the bracket flattening in linear time. The present algorithm does not use any specific encoding and only uses a stack as a data-structure. This simplification was suggested to us by Saul Schleimer in a Dagstuhl workshop.

[5] E.g., one can maintain a flag for every edge in the stack to record the fact that the turn sequence of the path ending at that edge has suffix of the form 12^* or $\bar{1}\bar{2}^*$.

$x-1=\bar{2}$, implying that $x=\bar{1}$ and that X has a suffix of the form $\bar{1}\bar{2}^*$. This would imply that $\mathrm{TS}=Y\bar{1}\bar{2}^*\bar{1}12^k1y$ contains the bracket $\bar{1}\bar{2}^*\bar{1}$. However, this negative bracket should have been flattened before the positive bracket, leading to a contradiction. See Figure 8. □

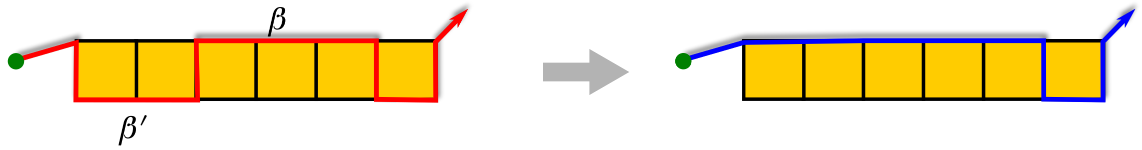

Figure 8: The flattening of the positive bracket β can not be part of a negative bracket, as the smaller bracket β' should first be flattened.

Recall that a bracket formally identifies with its flat part. Hence, when an arc is moved we may view that arc as part of the bracket that triggered the move.

Claim 2. If an arc a is moved twice by two consecutive bracket flattenings, possibly separated by spur removals and bracket flattenings not involving a, then a is the first arc of the first bracket.

PROOF OF THE CLAIM. Suppose by way of contradiction that a is part of the first bracket but not as the first arc. Let b be the edge preceding a. Then, as b is not moved before a, it must be part of the second bracket, or a side of it. Since by Claim 1 the two brackets have the same sign, the common vertex of b and a would have degree 4 in the first case and degree three in the second case. See Figure 9. This however contradicts the degree hypothesis in the lemma. □

We immediately infer from Claim 2 that

Claim 3. In all but at most one of the moves of any given arc, this arc is the first edge of the corresponding brackets.

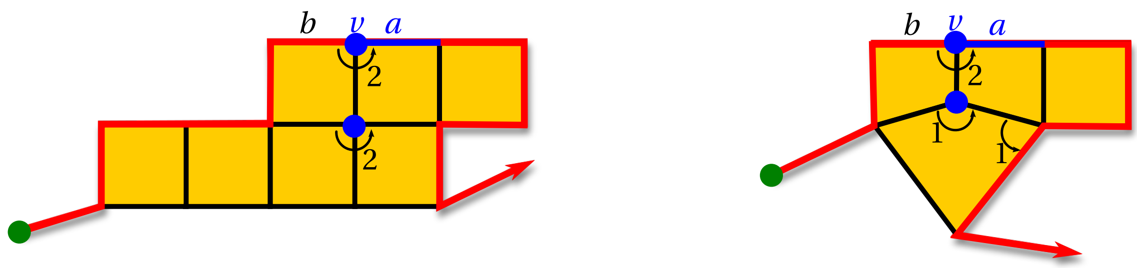

Figure 9: Left, arcs a and b are parts of a bracket of length 3 followed by a bracket of length 4. (There must be a spur removal in-between.) The common vertex v of a and b is moved to a vertex of degree four. Right, a and b are parts of a first bracket of length 3. Arc a is also part of a second bracket of length 1, while b is moved to a side of this bracket and v is moved to a vertex of degree three.

We can now analyse the time complexity of the shortening procedure. The time spent to remove the spurs and the brackets as we traverse the whole of c is proportional to the total number of spur removals and arc moves. We let R be the total number of spur removals, M_1 be the number of arc moves where the arc is the first in the corresponding bracket, and M_+ be the number of other arc moves. Note that M_1 trivially counts the total number of bracket flattenings since every bracket has exactly one first arc. As each bracket flattening involves the deletion of its sides, and as deleted arcs are never re-inserted, we infer that $R + M_1 \leq |c|/2$. Now, according to Claim 3, M_+ is bounded by $|c|$. The total time spent to remove the spurs and the brackets is thus proportional to $R + M_1 + M_+ \leq 3|c|/2$. The lemma follows. □

Lemma 6.2. *Given a closed walk c in a quadrangulation, all of whose vertices have degree at least five, we can compute a (freely) homotopic closed walk without spurs and brackets in $O(|c|)$ time.*

PROOF. Fix a basepoint v of c. By Lemma 6.1, we compute a homotopic path (with endpoints v) without spurs and brackets in $O(|c|)$ time. The resulting closed walk, that we again denote by c, may still have a spur or a bracket involving v. We traverse c from v in both directions until we discover a spur or a bracket in one of the following (excluding) situations, and process them as described.

1. v is incident to opposite arcs. In this case we remove the corresponding spur from c and move its basepoint to the other endpoint of the arcs. We obtain a homotopic closed walk with two arcs less.

2. v is a vertex of a bracket that covers c. See right Figure 5. We then flatten this non-typical bracket.

3. v is interior to a bracket that does not cover c. We then flatten the bracket.

4. v is the first or the last vertex (but not both) of a bracket, thus incident to a side of that bracket. We then flatten the bracket and move v to the other endpoint of the incident side. We immediately perform as many spur removals as we can, moving the basepoint accordingly.

It is easy to check that after processing any of these cases we obtain a (freely) homotopic closed walk whose new basepoint defines a path without spurs or brackets. We repeat the same search and processing until none of these cases occurs. Note that each case involves deletions of arcs, so that the search must end and c does not contain any more spur or bracket as desired.

Claim 4. As a cycle, c is free of spurs and brackets after processing cases 2 and 3.

Said differently, the search must end after processing cases 2 and 3.

PROOF OF THE CLAIM. This is obvious after processing case 2 since the resulting turn sequence has the form 2^*3 or $\bar{2}^*\bar{3}$. Case 3 occurs when the (cyclic) turn sequence of c has the form (i) $\mathrm{TS} = X\,x\,12^*\mathbf{2}2^*1\,y$ or (ii) $\mathrm{TS} = X\,x\,\bar{1}\bar{2}^*\bar{\mathbf{2}}\bar{2}^*\bar{1}\,y$, for some turns x, y, some possibly empty turn sequence X, and where the turn at v is in bold. (When X is empty it may be that x and y are turns at the same vertex.) After flattening we obtain in case (i) the turn sequence $\mathrm{TS}' = X(x-1)\bar{2}^*\bar{\mathbf{2}}\bar{2}^*(y-1)$. We have $x \neq 0, 1$ and $y \neq 0, 1$ since c is geodesic as a path. Equivalently, $x-1 \neq \bar{1}, 0$ and $y-1 \neq \bar{1}, 0$. Hence TS' does not contain spurs (i.e., 0's) and the only way it can contain a bracket is when $x-1 = y-1 = \bar{2}$ and $X = \bar{2}^*\bar{1}\,Y\,\bar{1}\bar{2}^*$ for some sequence Y. This would imply $\mathrm{TS} = \bar{2}^*\bar{1}\,Y\,\bar{1}\bar{2}^*\bar{1}12^*\mathbf{2}2^*1\bar{1}$, so that TS would contain two negative brackets, in contradiction with the fact that it does not contain any bracket as a path. Case (ii) can be dealt with similarly. □

Claim 5. If more than one edge of the bracket remains after processing case 4, then the cycle c is free of spurs and brackets.

PROOF OF THE CLAIM. Analogously to the proof of Claim 4 we may assume without loss of generality that the (cyclic) turn sequence of c has the form (i) $\mathrm{TS} = X\,x12^k\mathbf{1}\,y$ for some $k > 0$. After flattening we obtain the turn sequence $\mathrm{TS}' = X(x-1)\bar{2}^k(\mathbf{y-1})$, with the turn at the basepoint in bold. We must have $x \neq 0, 1$ and $y \neq 0$, since c is a geodesic path (it may be that $y = 1$, though). The same argument as for case 3 in Claim 4 shows that TS' does not contain a bracket. However TS' may contain a spur if $y = 1$. Assume the later, and write $X = x_1X_2$, so that $\mathrm{TS}' = x_1X_2(x-1)\bar{2}^k\mathbf{0}$. Removing the spur gives the new turn sequence $X_2(x-1)\bar{2}^{k-1}(\mathbf{x_1-2})$. Arguing as before, the corresponding cycle can not contain a bracket but may have a spur if $x_1 = 2$. By induction, we infer that in order to remove ℓ spurs, with $0 \leq \ell < k$, we must have $X = 2^\ell X_{\ell+1}$ for some turn sequence $X_{\ell+1}$. We then obtain a turn sequence without brackets of the form $X_{\ell+1}(x-1)\bar{2}^{k-\ell}(\mathbf{x_\ell-2})$. If $x_\ell \neq 2$, this sequence also contains no spurs. □

We can now bound the cost for removing all spurs and brackets from c. Since case 1 takes constant time and decreases the length of c by 2, it takes $O(|c|)$ time in total. According to Claim 4, cases 2 and 3 occur at most once, hence take $O(|c|)$ time in total. According to Claim 5, case 4 may occur at most once if more than one arc of the corresponding bracket remain after the spur removals. In every other occurrences of case 4, the number of subsequent spur removals is thus proportional to the length of the corresponding bracket (twice the number of spur removals is more than the length minus one). In other words the cost of handling case 4 is proportional to the number of arcs removed. We easily deduce that case 4 also takes linear time in total. □

Theorem 6.3. *The canonical representative of a closed walk c in a quadrangulation, all of whose vertices have degree at least five, can be computed in $O(|c|)$ time.*

PROOF. By Lemma 6.2, we may assume that c is a (cyclic) geodesic, i.e. has no spur or bracket. This geodesic needs to be pushed to its right in order to remove its convex vertices as described in Section 5.3. A right push transforms a subpath of the geodesic into another subpath of the same length without convex vertices. Morover, in the event where a 2-turn appears after a right push (e.g., if $\bar{3} = 2$), this turn must be adjacent to a $\bar{1}$-turn or a $\bar{2}$-turn in the resulting turn sequence. See Figure 6, left. It follows that none of the vertices of this subpath will be part of an L-shaped subpath, hence will not be pushed again. We can thus remove all the convex vertices by right pushes using a simple traversal of c without backtracking. (Again, some care must be taken to handle the non-typical cases as on Figure 6.) The total time needed to obtain a rightmost geodesic is thus linear. □

Corollary 6.4. *Given two closed walk of length at most ℓ in a combinatorial map of size n we can decide if they are homotopic in $O(n+\ell)$ time.*

PROOF. According to Lemma 4.1, we can reduce the combinatorial map to a quandrangulation in $O(n)$ time and get closed walks homotopic to the given one in $O(\ell)$ time. By Theorem 6.3 we can compute the canonical form of the walks in $O(\ell)$ time. Now these canonical forms, say c and d, are homotopic if and only if one is a circular permutation of the other. This can be tested in linear time by checking whether c is a substring of $d \cdot d$ thanks to the Knuth-Morris-Pratt string searching algorithm [KMP77] [CLRS02, Sec. 32.4]. □

This linear time homotopy test has been implemented and is available as a package of the C++ CGAL library.

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