Some occurrences of combinatorial properties of numbers

1. **Integer Carathéodory (Hilbert bases, normal semigroups)**
2. **Empty simplices (Integer Programming)**
3. **The Lonely Runner (View Obstructions)**

1. Integer Carathéodory

Applications :

- à des nœuds (Hass, Lagarias, Pippenger 1999)
- Rings, K-theory, toric varieties
  (Bruns, Gubeladze 2009, commutative algebra)

Linear Carathéodory

Given $A=\{a_1, \ldots, a_n\} \subseteq IR^d$

cone$(A) = \{ \lambda_1 a_{1.} + \ldots + \lambda_n a_{n.} : \lambda_1, \ldots, \lambda_n \geq 0 \}$

C is pointed, if x, -x $\in C \Rightarrow x=0$

$\Leftrightarrow$ no full line in it $\Leftrightarrow$ 0 nontriv nonneg comb $\Leftrightarrow$ $\exists$ c $\in Z^n$: $cx>0 \; \forall x \in C$

Thm: *Given* $A=\{a_1, \ldots, a_n\} \subseteq IR^d$, *for all* $b \in$ *cone(A)*
there exists $B \subseteq A$, $|B| \leq d$ *st.* $b \in$ cone(B)

Proof : $v = \lambda_1 a_{1.} + \ldots + \lambda_n a_{n.} : \lambda_1, \ldots, \lambda_n \geq 0$

If $D:=\{a_i : \lambda_i > 0\}$ dependent, eliminate some $a_i \in D$.

Hilbert basis

Given $A=\{a_1, \ldots, a_n\} \subseteq \boldsymbol{Z}^d$

int.cone$(A) = \{ \lambda_1 a_1 + \ldots + \lambda_n a_n : \lambda_1, \ldots, \lambda_n \in \boldsymbol{Z}_+ \}$

Hilbert basis : $H \subseteq Z^d$ s.t. int.cone(H) = cone (H) $\cap$ Z^d (Giles, Pulleyblank 1979)

~ ``*normal semigroup''*; cone (H) is then called a *Hilbert cone*

Exercise 1: For lin indep H: H is a Hb $\Leftrightarrow$ H is *unimodular*, that is, det(H)=1

Simple properties of Hilbert bases

(Gordan (1873), Hilbert (1890), van der Corput (1931))

Hilbert basis (Hb) : H tq int.cone(H)= cone (H) $\cap Z^d$

Exercise 2: If C is a pointed cone $\exists$ unique minimal Hb: i.e. H:

int.cone (H) = C $\cap Z^d$; H is *the (min) Hb of C* .

Hint : Chose c $\in Z^n$ such that $c^T x > 0$ for all x $\in$ C.

Exercise 3 : $A=\{a_1, \ldots, a_n\} \subseteq IR^d$ the minimal Hb of the pointed cone(*A*) *is finite.*

Hint: it is a subset of the parallelepiped

Examples

$\mathrm{par}(v_1,\dots,v_n) := \{\lambda_1 v_1 + \dots + \lambda_n v_n \in Z^n : \quad 0 \le (\lambda_1,\dots,\lambda_n) < 1\}$

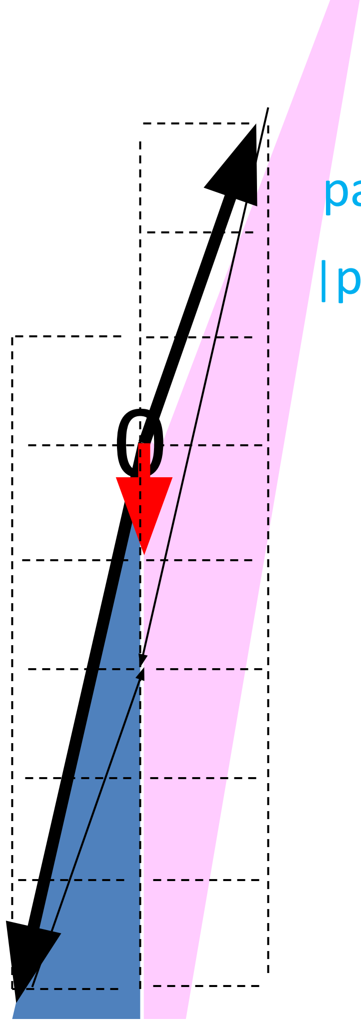
$|\mathrm{par}(v_1,\dots,v_n)| = \det(v_1,\dots,v_n)|$

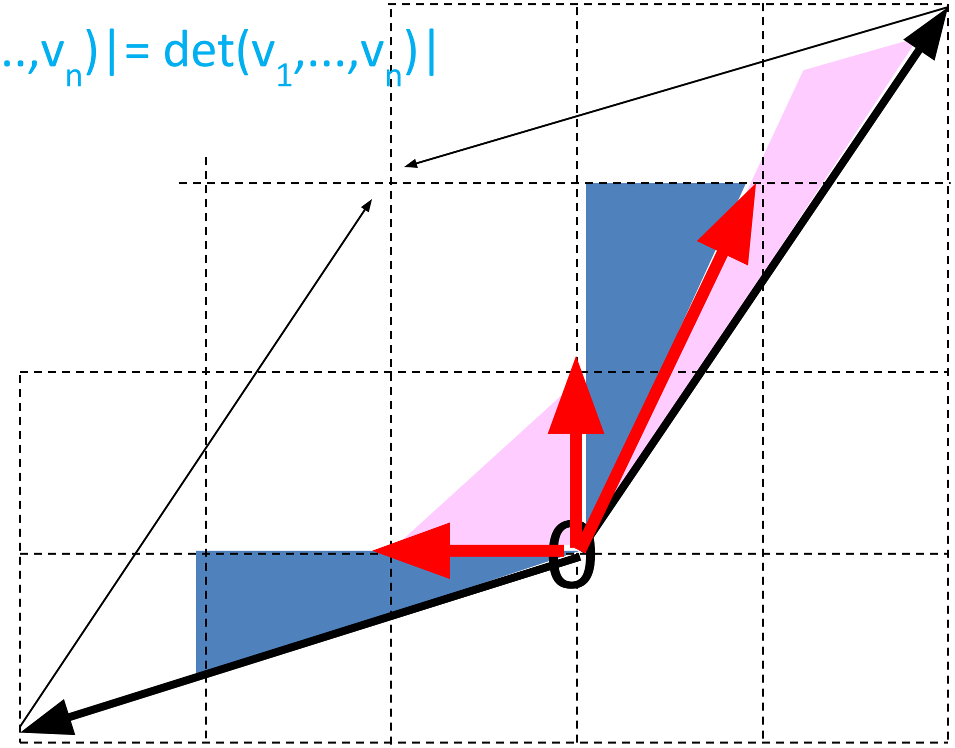
1 3

-1 -5

det=2

0 0 , 0 -1

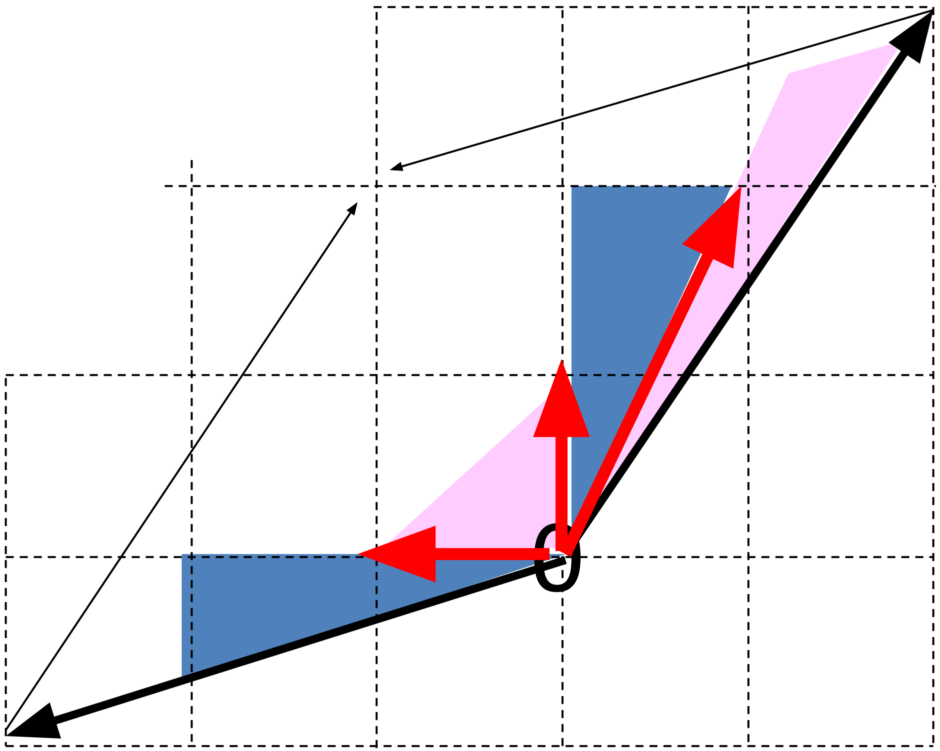




Hilbert basis : H tq int.cone(H)= cone (H) ∩ Z^d

(Many combinatorial examples : matchings in planar graphs, arborescences, matroid bases, constraints of TDI systems)

Advanced properties



Exercise 4*: If H is a 2 dim Hb it is Integer Caratheodory (IC).

Hint: delete an extreme ray, prove that it remains a Hb, and induction.

Exercise 5*: If H is a pointed Hilbert basis, for any n-1 element sub-Hb of H on a facet there exists an n-th of H so that det =1.

Integer Carathéodory Problem

Let $H=\{h_1, \ldots, h_n\} \subseteq IR^d$ Hb, that is,
int.cone$(H) = \{ \lambda_1 h_1 + \ldots + \lambda_n h_n : \lambda_1, \ldots, \lambda_n \in \mathbf{Z}_+ \}$ =cone (H) $\cap$ Z^d

Cook, Fonlupt, Schrijver, and 1 better in S. leading to more :

Thm: *If H is a Hilbert basis, then for all b* $\in$ *cone(H)*
there exists $B \subseteq H$, $|B| \leq 2d-2$ *st.* $b \in$ *int.*cone*(B)*

Proof : $v = \lambda_1 a_1 + \ldots + \lambda_d a_n \quad : \quad \lambda_1, \ldots, \lambda_d \geq 0,$
$\lambda_1 + \ldots + \lambda_d \longrightarrow \max$

Exercise* to finish it , Hint : later .

Parallelepipeds

Let $v_1,\ldots,v_n \in \mathbf{Z}^n$ linearly independent.

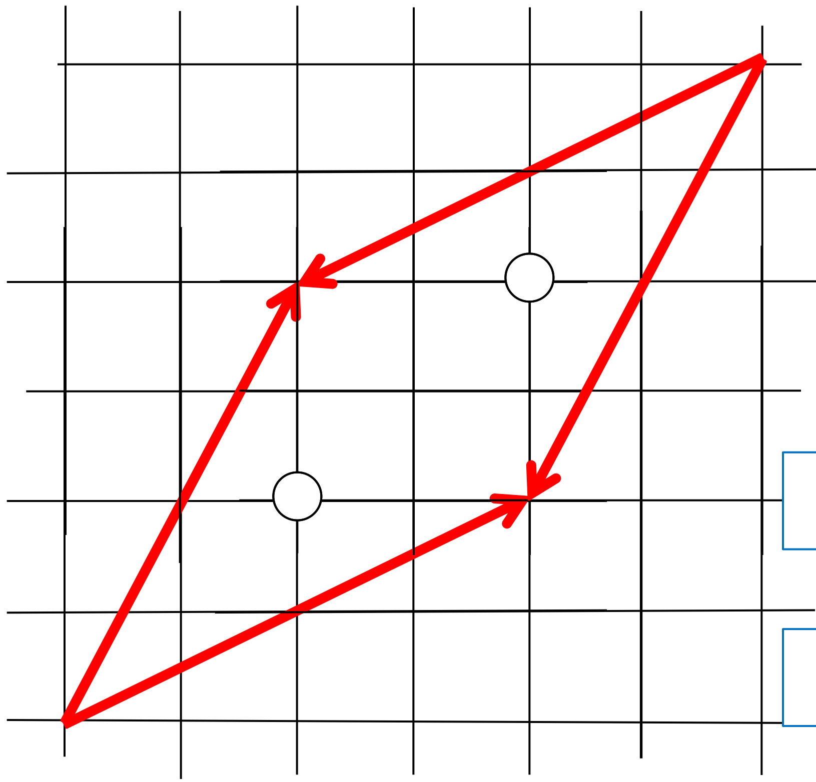
$\mathrm{par}(v_1,\ldots,v_n) := \{\lambda_1 v_1+\ldots+\lambda_n v_n \in \mathbf{Z}^n : \quad 0 \leq (\lambda_1,\ldots,\lambda_n) < 1\}$

$G(v_1,\ldots,v_n) := \{\ 0 \leq (\lambda_1,\ldots,\lambda_n) < 1 \ : \lambda_1 v_1+\ldots+\lambda_n v_n \in \mathbf{Z}^n\}$ *parallelepiped coeffs*

| par | = det ; *group* with *'+ mod 1' ;* denominators=det

$v_1,\ldots,v_n$ rows of a matrix, $G(v_1,\ldots,v_n)$ is left unchanged by:
column permutation; column + integer times another column

Example



A= $\begin{matrix} 1 & 2 \\ 2 & 1 \end{matrix}$

$|\det(A)| = 3$

par (A) = { 0, (1,1) , (2,2) }

G (A) = {(1/3,1/3) , (2/3, 2/3)}

Numerical example

G(V) :

1						5/14	10/14	1/14	
	1					9/14	4/14	13/14	
		1				3/14	6/14	9/14	
			1			8/14	2/14	10/14	$\{k\lambda_1\}+\{k\lambda_2\}=1$
				1		9/14	4/14	13/14	k=1,…, 13
9	5	11	6	5	14	1/14	2/14	3/14	…

If some face of par contains only 0: cyclic group with '+' mod 1

All coeffs nonzero $\Leftrightarrow \lambda_i = d_i/D$, $\gcd(d_i,D)=1$ $(i=1,\ldots,n)$

13 multiples but we have only log 14 time !

n=3

Theorem:(S.'90)For any Hilbert basis $H \subseteq IR^3$,cone (H) pointed
cone (H) is 'partitioned' by simplicial Hilbert cones

Proof: Let $b \in$ cone (H) and maximize the sum of variables for

$$Hx = b \ , \ x \geq 0 \ .$$

The simplex of the basic optimum is **empty** :

Lemma : $v_1, v_2, v_3 \in \mathbf{Z}^3$, for all $(\lambda_1, \lambda_2, \lambda_3) \in G(v_1, v_2, v_3) \setminus \{0\}$,

$$\det + 1 \leq (\lambda_1 + \lambda_2 + \lambda_3) \det \leq 2 \det - 1$$

This determines a particular parallelepiped structure, which allows to finish the proof , see « empty simplices » (nontrivial)

Carathéodory entier

$(7/6)n \leq$ Caratheodory(n) $\leq 2(n\text{-}1)$ **Reserrer !**

Test en dimension fixe (Cook, Lovász, Schrijver '84)
NP-complet en général (Júlia Pap 2008

Borne supérieure (Cook, Fonlupt, Schrijver '86, S. '90)
Base unimodulaire (Gerards, S. '87)
n=3 : partition unimodulaire (S. '90)
Appl: - à des nœuds (Hass, Lagarias, Pîppenger 1999)
- Rings and K-theory, toric varieties (Bruns, Gubeladze 2009)

n=4 : contre-exemple à la partition unimodulaire (Bouvier, Gr '94)
n=6 : contre-exemple à n (Bruns, Gubeladze '98)

Thm: (Gijswijt 2010) True for matroid bases.

Problem: For rooted arborescences ? Other comb objects ?

Couverture unimodulaire pour n= 3, 4, 5 ?

2. Empty Simplices

Applications :

- Programmation en nombres entiers (PLE)
- Preuves de Carathéodory pour n=3 (cf. 1.)

Définition

INPUT : $v_1,\dots,v_n \in Z^n$ linéairement indép.

QUESTION : $\exists$ point entier à part les sommets dans le simplexe **conv(0, $v_1,\dots,v_n$)** ?

NON : *vide*

CNS (bonne caract, NP$\cap$coNP) qui certifie quand il **est *vide*** ?

Ou **coNP-complet** ?