Linear Embeddings

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We already saw that every m-dimensional complex embeds linearly into $\mathbb{R}^{2m+1}$. What about the existence of linear embeddings into $\mathbb{R}^d$ with $d \leq 2m$? It turns out that independently of obstruction theories, like Whitney or van Kampen obstructions, this question is decidable. We first look at a combinatorial approach based on the notion of chirotope for a point configuration.

1 Affine point configurations

Given a set $\{p_1,\dots,p_n\}$ of n points in $\mathbb{R}^d$, its **chirotope** is the map $\{1,\dots,n\}^{d+1} \to \{-1,0,1\}$ defined by

$$(i_0,\dots,i_d) \mapsto \operatorname{sign}(\det\begin{pmatrix} 1 & \cdots & 1 \\ p_{i_0} & \cdots & p_{i_d} \end{pmatrix}) \tag{1}$$

In other words, the chirotope returns for every $(d+1)$-tuple of points the orientation of the d-simplex defined by those points. Here, the orientation is assumed to be zero if the points are affinely dependent. Intuitively, the chirotope records the relative positions of the points in a point configuration. For instance, it is easily seen that the chirotope

determines the combinatorial structure of the convex hull of a point configuration. Not every map $\{1,\dots,n\}^{d+1} \to \{-1,0,1\}$ can be the chirotope of a point configuration. The map has to satisfy certain conditions related to the Grassman-Plücker relations and to Radon partitions.

1.1 Grassman-Plücker relations

Let $V = (v_1,\dots,v_n)$ be a family of n vectors in $\mathbb{R}^d$. As usual, put $[n] := \{1,\dots,n\}$. For a sequence $I = (i_1,\dots,i_d)$ of d indices in $[n]$, we denote by

$$m_I := \det(v_{i_1},\dots,v_{i_d})$$

the determinant with respect to the canonical basis of $\mathbb{R}^d$ of the d-tuple of vectors of V indexed by I. The *minors* $(m_I)_{I\in\binom{[n]}{d}}$, where $\binom{[n]}{d} \subset [n]^d$ denotes the set of all increasing sequences of d indices in $[n]$, are the **homogeneous Plücker coordinates** associated to V.

Theorem 1.1. *The homogeneous Plücker coordinates associated to V satisfy the **Grassman-Plücker relations**:*

$$\forall I \in \binom{[n]}{d+1}, \forall J \in \binom{[n]}{d-1} : \sum_{s=0}^{d} (-1)^s m_{I-i_s} m_{J+i_s} = 0 \tag{2}$$

where $I - i_s$ is obtained by deleting i_s in $I = (i_0,\dots,i_d)$, and $J + i_s$ is obtained by appending i_s at the end of J. Note that $J + i_s$ is not necessarily increasing and that m_{J+i_s} cancels whenever $i_s \in J$.

PROOF. For $J = (j_0,\dots,j_{d-2})$ fixed, consider the $(d+1)$-linear map $f : (\mathbb{R}^d)^{n+1} \to \mathbb{R}$ given by

$$(u_0,\dots,u_d) \mapsto \sum_{s=0}^{d} (-1)^s \det(u_0,\dots,\widehat{u_s},\dots,u_d)\det(v_{j_0},\dots,v_{j_{d-2}},u_s)$$

We easily check that f is alternating. However, an alternating $(d+1)$-linear map over a d dimensional space must be zero. In particular, $f(v_{i_0},\dots,v_{i_d}) = 0$, which is precisely Equation (2). □

Exercise 1.2. Prove that the map f in the above proof is indeed alternating.

Cultural note: The homogeneous Plücker coordinates provide an embedding of the **Grassmannian** $\mathbf{Gr}(d,\mathbb{R}^n)$ into the projective space $\mathbb{P}\left(\bigwedge^d \mathbb{R}^n\right)$ of the d-fold exterior product $\bigwedge^d \mathbb{R}^n$ of $\mathbb{R}^n$. Let us briefly explain why. Recall that $\mathbf{Gr}(d,\mathbb{R}^n)$ is the set of d dimensional subspaces of $\mathbb{R}^n$. The exterior (or Grassmann) algebra $\bigwedge \mathbb{R}^n$ can be defined as the quotient of the tensor algebra $\bigotimes \mathbb{R}^n$ by the two-sided ideal generated by the tensor products $\{v \otimes v\}_{v\in\mathbb{R}^n}$. The exterior product $\wedge$ thus induced by the tensor product is antisymmetric as can be seen by expanding $(x+y)\otimes(x+y)$. As a vector space,

the d-fold exterior product $\bigwedge^d \mathbb{R}^n$ has a basis composed of the d-vectors $e_I = e_{i_1} \wedge \cdots \wedge e_{i_d}$, where $I = (i_1, \ldots, i_d) \in \binom{[n]}{d}$ and $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. Consider a family $W = (w_1, \ldots, w_d)$ of d vectors in $\mathbb{R}^n$ as a $d \times n$ matrix whose columns are the components of the v_i expressed in the canonical basis. Viewing the transpose matrix W^t as a family $V = (v_1, \ldots, v_n)$ of n vectors in $\mathbb{R}^d$, we compute $w_1 \wedge \cdots \wedge w_d = \sum_I m_I e_I$, where the m_I are the homogeneous Plücker coordinates associated to V. One can show that two families of d independent vectors have proportional wedge products if and only if they span the same vector space, whence the claimed embedding. In fact, the Grassman-Plücker relations (2) are a necessary and sufficient condition on the m_I to come from a wedge product of d vectors, the so-called *decomposable* d-vectors. The Grassmannian $\mathbf{Gr}(d, \mathbb{R}^n)$ is thus embedded in $\mathbb{P}\left(\bigwedge^d \mathbb{R}^n\right)$ as a projective algebraic variety determined by quadratic equations.

1.2 Radon partitions

Let $\mathscr{P}$ be a set of points in $\mathbb{R}^d$. Any partition $\mathscr{P} = \mathscr{P}' \cup \mathscr{P}''$ such that the convex hulls $\operatorname{Conv} \mathscr{P}'$ and $\operatorname{Conv} \mathscr{P}''$ have a nonempty intersection is called a **Radon partition** of $\mathscr{P}$. Recall that $\mathscr{P}$ is in **general position** if no affine hyperplane contains more than d points of $\mathscr{P}$.

Lemma 1.3. *Let $\mathscr{P} = \mathscr{P}' \cup \mathscr{P}''$ be a partition of a set of $d+1$ points in general position in $\mathbb{R}^d$. Then $\operatorname{Conv} \mathscr{P}'$ and $\operatorname{Conv} \mathscr{P}''$ are disjoint.*

Note that the lemma just says that two faces of a d-simplex with disjoint vertex sets are indeed disjoint.

PROOF. Write $\mathscr{P} = \{p_i\}_{i \in I}$, $\mathscr{P}' = \{p_i\}_{i \in I'}$ and $\mathscr{P}'' = \{p_i\}_{k \in I''}$ with $I = I' \cup I''$. Suppose by way of contradiction that $\operatorname{Conv} \mathscr{P}' \cap \operatorname{Conv} \mathscr{P}''$ contains a point p. Then we can write p as two convex combinations $\sum_{i \in I'} \alpha_i p_i$ and $\sum_{i \in I''} \alpha_i p_i$ with $\sum_{i \in I'} \alpha_i = \sum_{i \in I''} \alpha_i = 1$. It follows that $\sum_{i \in I'} \alpha_i p_i - \sum_{i \in I''} \alpha_i p_i = 0$. This provides an affine dependency between the points of $\mathscr{P}$ in contradiction with the general position assumption. □

Theorem 1.4 (Radon, 1921)**.** *Any set of $d+2$ points in $\mathbb{R}^d$ admits a Radon partition. Moreover, if the $d+2$ points are in general position any two of them are in the same part if and only if they are separated by the hyperplane spanned by the remaining d points. In particular, the Radon partition is unique.*

PROOF. Any $d+2$ points, say $\mathscr{P} = \{p_1, p_2, \ldots, p_{d+2}\}$, must be affinely dependent in $\mathbb{R}^d$. We can thus find real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{d+2}$, not all zero, such that $\sum_{i=1}^{d+2} \alpha_i = 0$ and $\sum_{i=1}^{d+2} \alpha_i p_i = 0$. Let $I_+ := \{i \in [d+2] \mid \alpha_i \geq 0\}$ and $I_- := \{i \in [d+2] \mid \alpha_i < 0\}$. Then, $\sum_{I_+} \alpha_i = \sum_{I_-} -\alpha_i$ and denoting the common sum by A we derive the two convex combinations $\sum_{I_+} (\alpha_i / A) p_i = \sum_{I_-} (-\alpha_i / A) p_i$. It follows that $\mathscr{P} = \{p_i\}_{i \in I_+} \cup \{p_i\}_{i \in I_-}$ is a Radon partition of $\{p_i\}_{i \in [d+2]}$.

Suppose that $\mathscr{P}$ is in general position and consider a Radon partition $\mathscr{P} = \mathscr{P}' \cup \mathscr{P}''$. Let $p, q \in \mathscr{P}$ and let H be the affine hull of the remaining points $\mathscr{P} \setminus \{p, q\}$. By general position, H is a hyperplane that does not contain p nor q. By Lemma 1.3 applied to

$\mathscr{P}\setminus\{p,q\}$ in H, the convex hulls $\mathrm{Conv}(\mathscr{P}'\setminus\{p,q\})$ and $\mathrm{Conv}(\mathscr{P}''\setminus\{p,q\})$ are disjoint. If p and q are in a same part, then they must be separated by H. Otherwise, $\mathrm{Conv}\,\mathscr{P}'$ and $\mathrm{Conv}\,\mathscr{P}''$ would also be disjoint, contradicting that $\mathscr{P}'\cup\mathscr{P}''$ is a Radon partition. Conversely, if $p\in\mathscr{P}'$ and $q\in\mathscr{P}''$, then they must lie on the same side of H since otherwise $\mathrm{Conv}\,\mathscr{P}'$ and $\mathrm{Conv}\,\mathscr{P}''$ would be disjoint, again contradicting that $\mathscr{P}'\cup\mathscr{P}''$ is a Radon partition. □

Corollary 1.5. *Let $\mathscr{P}$ be a set of $d+2$ points in $\mathbb{R}^d$, not all on a same hyperplane. There exists a hyperplane H that contains d of the points in $\mathscr{P}$ and such that the two remaining points are on the same side of H, i.e. contained in the same component of $\mathbb{R}^d\setminus H$.*

PROOF. By induction on the dimension d. The base case $d=1$ is trivial and left to the reader. If $d>1$, first suppose that $\mathscr{P}$ is in general position. By the previous theorem, $\mathscr{P}$ has a (unique) Radon partition. Choose one point in each part and take for H the affine hull of the remaining points. Then H has the required properties by the same previous theorem.

If $\mathscr{P}$ is not in general position, there must be a hyperplane K that contains a subset $\mathscr{Q}$ of $d+1$ points of $\mathscr{P}$. Let p be the remaining point in $\mathscr{P}\setminus\mathscr{Q}$. Note that the points in $\mathscr{Q}$ cannot lie on a same $(d-1)$-plane. For otherwise, $\mathscr{P}$ would be contained in a hyperplane. By induction applied to $\mathscr{Q}$ in K, there is a $(d-1)$-plane L in K that contains $d-1$ of the points in $\mathscr{Q}$ such that the two remaining points of $\mathscr{Q}$ are on the same side of L. Taking for H the affine hull of $L\cup\{p\}$, we obtain a hyperplane with the required properties. □

1.3 From chirotopes to oriented matroids

Let $\mathscr{P}=\{p_1,\dots,p_n\}$ be a set of n points in $\mathbb{R}^d$. Recall that its chirotope χ returns for every $(d+1)$-tuple $I\in[n]^{d+1}$ the orientation $\chi(I)\in\{-1,0,1\}$ of the d-simplex spanned by the vertices of $\mathscr{P}$ indexed by I. The fact that $\mathscr{P}$ is a subset of a d dimensional affine space imposes some relations between the signs of its chirotope.

Theorem 1.6. *The chirotope χ of a set of n points in $\mathbb{R}^d$ is alternating and satisfies the following conditions.*

- **C-GP:** *For all $I=(i_0,\dots,i_{d+1})\in\binom{[n]}{d+2}$ and $J\in\binom{[n]}{d}$ the set of signs*

$$\{(-1)^s\chi(I-i_s)\chi(J+i_s)\}_{s=0,\dots,d+1}$$

  either contains $\{-1,1\}$, or is reduced to $\{0\}$.

- **C-R:** *For all $I=(i_0,\dots,i_{d+1})\in\binom{[n]}{d+2}$ the set of signs*

$$\{(-1)^s\chi(I-i_s)\}_{s=0,\dots,d+1}$$

  either contains $\{-1,1\}$, or is reduced to $\{0\}$.

PROOF. Let $\mathscr{P} = \{p_1, \dots, p_n\}$ be a set of n points in $\mathbb{R}^d$. The definition of their chirotope as the sign of a determinant shows that it is indeed alternating. Put $v_i = \begin{pmatrix} 1 \\ p_i \end{pmatrix} \in \mathbb{R}^{d+1}$ and let $V = (v_1, \dots, v_n)$. From the very definitions we see that the chirotope χ of $\mathscr{P}$ coincides with the signs of the homogeneous Plücker coordinates $(m_K)_{K \in \binom{[n]}{d+1}}$ of V:

$$\forall K \in \binom{[n]}{d+1} : \chi(K) = \operatorname{sign}(m_K)$$

The Grassman-Plücker relations (2) in Theorem 1.1 implies that either all terms in $\sum_{s=0}^{d+1} (-1)^s m_{I-i_s} m_{J+i_s}$ are zero or two terms are non-zero with opposite signs. Condition **C-GP** follows.

For Condition **C-R**, we first remark that when $\mathscr{P}_I = \{p_{i_0}, \dots, p_{i_{d+1}}\}$ is contained in a hyperplane, then the chirotope cancels on all $(d+1)$-tuples of indices in I and thus satisfies **C-R**. Otherwise, we may apply Corollary 1.5 to find two points p_{i_j}, p_{i_k} in $\mathscr{P}_I$ such that

$$\det(v_{i_j}, v_{i_0}, \dots, \widehat{v_{i_j}}, \dots, \widehat{v_{i_k}}, \dots, v_{i_{d+1}}) = \det(v_{i_k}, v_{i_0}, \dots, \widehat{v_{i_j}}, \dots, \widehat{v_{i_k}}, \dots, v_{i_{d+1}}) \tag{3}$$

and this quantity is nonzero. By the alternating property of the determinant we have

$$\det(v_{i_j}, v_{i_0}, \dots, \widehat{v_{i_j}}, \dots, \widehat{v_{i_k}}, \dots, v_{i_{d+1}}) = (-1)^j \det(v_{i_0}, \dots, v_{i_j}, \dots, \widehat{v_{i_k}}, \dots, v_{i_{d+1}})$$

and

$$\det(v_{i_k}, v_{i_0}, \dots, \widehat{v_{i_j}}, \dots, \widehat{v_{i_k}}, \dots, v_{i_{d+1}}) = (-1)^{k-1} \det(v_{i_0}, \dots, \widehat{v_{i_j}}, \dots, v_{i_k}, \dots, v_{i_{d+1}})$$

reporting in (3), we get that

$$(-1)^j \det(v_{i_0}, \dots, v_{i_j}, \dots, \widehat{v_{i_k}}, \dots, v_{i_{d+1}}) = -(-1)^k \det(v_{i_0}, \dots, \widehat{v_{i_j}}, \dots, v_{i_k}, \dots, v_y i_{d+1})$$

It ensues that $(-1)^j \chi(I - i_j)$ and $(-1)^k \chi(I - i_k)$ have opposite signs and are both nonzero, so that **C-R** holds in all cases. □

The pair $([n], \chi)$, where $\chi : [n]^{d+1} \to \{-1, 0, 1\}$ is an alternating map satisfying the condition of Theorem 1.6, is called an **affine oriented matroid** of rank $d+1$. χ is the chirotope of this oriented matroid. Any set of points in $\mathbb{R}^d$ whose chirotope coincides with χ is a *realization* of χ.

2 Linear embeddings and immersions

Recall that a linear mapping of a simplicial complex K into $\mathbb{R}^d$ is entirely determined by the image of the vertices of K. It is an embedding if it induces an injective map $|K| \hookrightarrow \mathbb{R}^d$. For an **immersion** we only require that this map is locally injective, which amounts to ask that the restriction of the map to the star of each vertex is injective. Here, the **star** of a vertex of K is the subcomplex comprising all the simplices containing that vertex and all their faces.

Lemma 2.1. *A linear map $f : K \to \mathbb{R}^d$ is an embedding if and only if every pair of disjoint simplices in K is sent to disjoint simplices in $\mathbb{R}^d$. It is an immersion if and only if the previous condition holds locally, i.e., f sends every pair of disjoint simplices in the star of a vertex to disjoint simplices in $\mathbb{R}^d$.*

PROOF. The conditions in the lemma are trivially necessary. Suppose that the condition for f to be an embedding holds. We first claim that the restriction of f to each simplex $[v_0,\dots,v_k] \in K$ is injective. Otherwise, $f(v_0),\dots,f(v_k)$ must span a flat (affine subspace) of dimension at most $k-1$. By Radon's theorem 1.4 we can partition the $f(v_i)$ in two subsets whose convex hulls intersect. The corresponding subsets of v_i define two disjoint faces of $[v_0,\dots,v_k]$ whose images have a common intersection. This is however in contradiction with the embedding condition in the lemma.

Now, by way of contradiction, consider two points $x \neq y$ in $|K|$ such that $f(x) = f(y)$. Let $\sigma,\tau \in K$ be the supporting simplices of x and y, respectively. By the previous claim and the embedding condition, σ and τ must have a common face different from both σ and τ. Let $\{u_i\}_{i\in I}$ be the vertices of that face, and let $\{v_j\}_{j\in J}$ and $\{w_k\}_{k\in K}$ be the remaining vertices of σ and τ, respectively. We have $x = \sum_I \alpha_i u_i + \sum_J \beta_j v_j$ and $y = \sum_I \alpha'_i u_i + \sum_K \gamma_k w_k$ for some positive coefficients $\alpha_i,\beta_j,\alpha'_i,\gamma_k$ with $\sum_I \alpha_i + \sum_J \beta_j = \sum_I \alpha'_i + \sum_K \gamma_k = 1$. Set $I_+ = \{i \in I \mid \alpha_i > \alpha'_i\}$ and $I_- = \{i \in I \mid \alpha_i < \alpha'_i\}$. We deduce from $f(x) = f(y)$ that $\sum_{I_+}(\alpha_i - \alpha'_i)f(u_i) + \sum_J \beta_j f(v_j) = \sum_{I_-}(\alpha'_i - \alpha_i)f(u_i) + \sum_K \gamma_k f(w_k)$. Remarking that $\sum_{I_+}(\alpha_i - \alpha'_i) + \sum_J \beta_j = \sum_{I_-}(\alpha'_i - \alpha_i) + \sum_K \gamma_k$ and denoting by A the common positive sum, we obtain after dividing by A two convex combinations of $\{u_i\}_{i\in I_+} \cup \{v_j\}_{j\in J}$ on one side and of $\{u_i\}_{i\in I_-} \cup \{w_k\}_{k\in K}$ on the other side whose image by f coincide. This again contradicts the embedding condition. It follows that the linear extension of f is indeed injective. The second part of the lemma is proved similarly, working separately in the star of each vertex. □

Lemma 2.2. *If a simplicial complex K has a linear embedding into $\mathbb{R}^d$, then it has a linear embedding sending the vertices to a pointset in general position in $\mathbb{R}^d$. The same holds, replacing embedding by immersion. Moreover, one may enforce that the image vertices have rational coordinates.*

PROOF. By the previous lemma, being an embedding or an immersion is ensured by a finite set of open conditions, namely the existence of a separating hyperplane for the images of pairs of disjoint simplices. It ensues that any sufficiently small perturbation of the vertex images preserves the property of being an embedding or an immersion. In particular, one may require that the image vertices are in general position and that all their coordinates are rational. □

2.1 A certificate of non-embeddability

Suppose that a simplicial complex K has a linear embedding $f : K \to \mathbb{R}^d$. Let $V = \{v_i\}_{i\in I}$ be the vertices of K. By Lemma 2.2, we can assume that $f(V)$ is in general position. In other words, the chirotope $\chi : I^{d+1} \to \{-1,0,1\}$ of $f(V)$ does not cancel on $\binom{I}{d+1}$. An oriented matroid with such a chirotope is said **uniform**. We shall also say

that the chirotope itself is uniform. Lemma 2.1 provides a simple criterion for f to be an embedding. This criterion turns out to be encoded in the chirotope of $f(V)$ as stated in the next Corollary 2.4.

Lemma 2.3. *Let σ, τ be two intersecting simplices in $\mathbb{R}^d$ such that $\dim\sigma + \dim\tau > d$. Then, we can find a face of σ and a face of τ that intersect and whose dimensions add up to exactly d.*

PROOF. We first make two simple observations.

1. Let H be a flat intersecting a set S in a Euclidean space. Then, the boundary points of $H \cap S$ in H (it is all of $H \cap S$ if its interior in H is empty) are contained in the boundary of S.

2. If two sets intersect in a Euclidean space, then one of the two intersects the boundary of the other one.

Let $k = \dim\sigma$ and $\ell = \dim\tau$. We prove the lemma by induction on $k+\ell$. Denote by H the intersection of the affine hulls of σ and τ. Then, $\sigma \cap H$ and $\tau \cap H$ are two intersecting convexes in H. If one of them, say $\sigma \cap H$, has empty interior in H, then by observation (1) applied in the affine hull of σ, it is included in the boundary of σ. It follows that a proper face σ' of σ intersects τ. Replacing σ' by a larger face of σ if necessary, we may assume that $\dim\sigma + \ell > \dim\sigma' + \ell \geq d$. We can thus invoke the induction to conclude. If both $\sigma \cap H$ and $\tau \cap H$ have nonempty interior in H, then their intersection contains a boundary point of one of them, say $\sigma \cap H$, by observation (2). By observation (1) this boundary point is also in the boundary of σ and we may conclude as in the previous case. □

Corollary 2.4. *Let K be a simplicial complex of dimension at most d with vertex set $[n]$. Consider a map $f : [n] \to \mathbb{R}^d$ such that $f([n])$ is in general position and denote its chirotope by $\chi : [n]^{d+1} \to \{-1, 0, 1\}$. Then f linearly extends to an embedding $f : |K| \to \mathbb{R}^d$ if and only if the following condition is satisfied.*

- **C-E:** *for all $I \in \binom{[n]}{d+2}$, the subsets*

$$I_+ := \{i \in I \mid (-1)^i \chi(I - i)\} = 1 \quad \text{and} \quad I_- := \{i \in I \mid (-1)^i \chi(I - i) = -1\}$$

  are not the vertex sets of a pair of simplices in K.

A similar condition **C-I** *characterizes immersions, where we only ask that I^+, I^- are not the vertex sets of a pair of simplices in the star of some vertex in K.*

PROOF. From Radon's theorem 1.4 and looking at the proof of Condition **C-R** in Theorem 1.6, it is easily seen that $f(I_+) U f(I_-)$ defines the unique Radon partition of $f(I)$. In particular, Conv $f(I_+)$ and Conv $f(I_-)$ intersect. Condition **C-E** is thus necessary for the extension of f to be an embedding. Conversely, assume that **C-E** holds. Consider two disjoint simplices $\sigma, \tau \in K$. If $\dim\sigma + \dim\tau < d$ then $f(\sigma)$ and

$f(\tau)$ are disjoint by the general position hypothesis. If $\dim\sigma + \dim\tau \geq d$ we also claim that $f(\sigma)$ and $f(\tau)$ are disjoint. Otherwise, by Lemma 2.3 we can assume that $\dim\sigma + \dim\tau = d$. Let I be the concatenation of the vertices of σ and τ. Then, $I \in \binom{[n]}{d+2}$ and the uniqueness of the Radon partition for $f(I)$ implies that $\{I_+, I_-\} = \{\sigma, \tau\}$. This would however be in contradiction with condition **C-E**. It follows that every pair of disjoint simplices in K is sent by f to disjoint simplices in R^d. Lemma 2.1 implies that f indeed defines an embedding. A similar proof holds for Condition **C-I** on immersions. □

The previous theorem, together with Lemma 2.2 and Theorem 1.6 have the following consequence. If K has a linear embedding in $\mathbb{R}^d$, then there should exist a uniform chirotope *admissible* for the embedding of K, *i.e.*, satisfying conditions **C-GP**, **C-R** and **C-E**. The existence of an admissible chirotope is purely combinatorial and only depends on d and K. It can thus be checked by a computer. If no admissible chirotope is found then we can claim that K has no linear embedding in $\mathbb{R}^d$. A brute force algorithm would try all maps $\binom{[n]}{d+1} \to \{-1, 1\}$ to see if one satisfies conditions **C-GP**, **C-R** and **C-E**. The number of possible maps, $2^{\binom{n}{d+1}}$, is already far too large, not to mention the tests for conditions **C-GP**, **C-R** and **C-E**, to be tractable in practice, except for very small complexes.

2.2 Linear embedding of surfaces

A finite simplicial surface is a simplicial complex S whose carrier $|S|$ is a compact two dimensional manifold. Equivalently, every simplex of S should be a face of a triangle in $|S|$ and every edge should be a face of at most two triangles. One says that S *triangulates* $|S|$, or is a *triangulation* of $|S|$. Recall that every simplicial surface embeds linearly in $\mathbb{R}^5$. It follows from their classification that all orientable surfaces can be obtained from the connected sum of a sphere, possibly with boundary, with a certain number of tori. In particular, all orientable surfaces have a topological embedding into $\mathbb{R}^3$. In fact, the method of Burago and Zalgaller described in the first lecture shows that all orientable surfaces have a PL embedding in $\mathbb{R}^3$. The answer becomes less trivial if one asks for the linear embedding into $\mathbb{R}^3$ of a specific triangulation of a surface. Until a counterexample was found in 2000, it was not known whether all simplicial surfaces could be linearly embedded in 3-space. Here are some known facts.

- It follows from a celebrated theorem of Steinitz (1922) that all triangulations of a sphere have a linear embedding into $\mathbb{R}^3$. In fact, each such triangulation is the boundary complex of a convex polyhedron in $\mathbb{R}^3$. See [Zie95, Chap. 4] for a proof.

- Archdeacon et al. [ABEM07] proved that all triangulations of the torus can be linearly embedded into $\mathbb{R}^3$. In particular, the toroidal triangulation with the smallest number of vertices, the so-called **Möbius torus**, has many linear embeddings. The 1-skeleton of this triangulation is the complete graph K_7 on 7 vertices. The first known linear embedding of the Möbius torus, due to Császár (1949), is shown Figure 1. Bokowski and Eggert [BE91] have listed all the 72 admissible uniform chirotopes of the Möbius torus (up to an automorphism of the triangulation) and they were able to exhibit realizations for each of them.

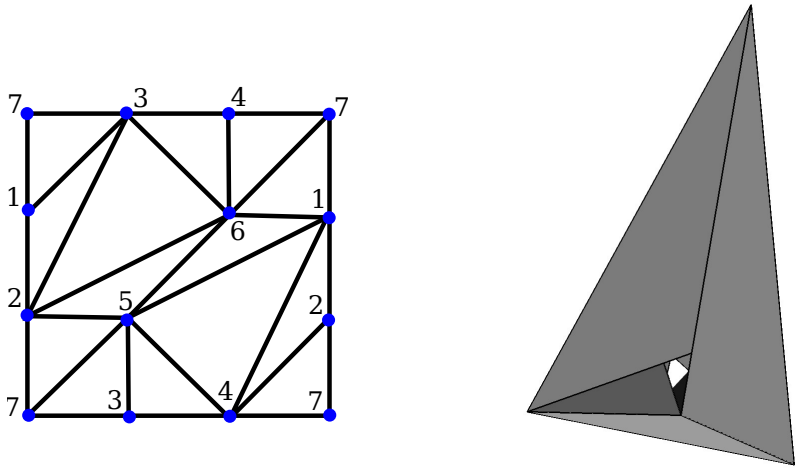

Figure 1: Left, layout of the Möbius torus. Right, Császár's linear embedding. The vertex coordinates are, in order :
$(3,-3,0),(-3,3,0),(-3,-3,1),(3,3,1),(-1,-2,3),(1,2,3),(0,0,15)$

- For higher genus, there exists simplicial orientable surfaces without any linear embedding in $\mathbb{R}^3$. All the results in this direction were proved with the help of a computer to check that some specific triangulation had no admissible chirotope. For instance, Altshuler et al. [ABS96] proved that a certain simplicial surface of genus 6 with 12 vertices has no admissible chirotope. Using a more efficient heuristic to explore the set of chirotopes Schewe [Sch10] proved that none of the 59 genus 6 triangulations with 12 vertices has an admissible chirotope. He proved a similar result for a triangulation of genus 5 with one triangle removed. As a consequence, any triangulation obtained from a connected sum along this triangle cannot be realized into $\mathbb{R}^3$. Similar nonrealizability results were obtained only asking for immersions rather than embeddings.

3 Deciding linear embeddability

The preceding approach, based on chirotopes, does not always allow to decide when a simplicial complex K is linearly embeddable in some $\mathbb{E}^d$. Even if K has an admissible chirotope, we still have to exhibit an actual embedding, or prove that no such embedding exists in order to conclude. The conditions for this existence happens to be dictated by a set of polynomials inequalities. Indeed, assuming that K has an admissible chirotope χ all what we need to find is a set of points in $\mathbb{E}^d$, one for each vertex of K, such that the corresponding chirotope is equal to χ. Now, the chirotope of the set of points is given by sign conditions on determinants (see (1)) which are polynomials in the coordinates of the points.

In fact, it is not necessary to know in advance an admissible chirotope to express that K has a linear embedding. By Lemma 2.1, it is equivalent to look for a set of points $\{p_1, p_2, \ldots, p_n\}$, corresponding to the vertices $i \in [n]$ of K, such that every pair of disjoint simplices $([i_0, \ldots, i_k], [j_0, \ldots, j_\ell])$ in K is sent to non-intersecting simplices in $\mathbb{E}^d$. This condition can be rephrased as the existence of a hyperplane separating

$[p_{i_0},\dots,p_{i_k}]$ and $[p_{j_0},\dots,p_{j_\ell}]$. In other words, there should exist coefficients $c_0,\dots,c_d$ such that the hyperplane equation $c_0+\sum_{i=1}^{d} c_i x_i$ evaluates positively on $p_{i_0},\dots,p_{i_k}$ and negatively on $p_{j_0},\dots,p_{j_\ell}$. Hence, by introducing new variables c_i, we are again reduced to the satisfiability of a set of polynomials inequalities.

A subset of $\mathbb{R}^d$ defined by polynomials inequalities is said **real semi-algebraic**. Deciding linear embeddability thus reduces to decide whether a real semi-algebraic set is nonempty. Decision problems that reduce (in a sens to be defined) to the (non)vacuity of a real semi-algebraic[1] set are known as decision problems for the **existential theory of the reals**. The existential theory of the reals thus defines a complexity class that turns out to lie somewhere between the classes **NP** and **PSPACE**. In particular, the existential theory of the reals is decidable. In order to make sense out of these claims we need to recall some basic definitions from the theory of computation.

3.1 Turing machines and complexity

This section is intended to be a crash introduction to computational complexity. The following notes are greatly inspired by Avi Widgerson [Wig06].

3.1.1 Turing machines

The most popular model of computation was introduced by Alan Turing in 1936. It was proved equivalent to other notions of computation such as recursive functions or λ-calculus. Formally, a Turing machine is a triple $(\mathscr{A},\mathscr{Q},\mathscr{T})$, where $\mathscr{A}$ is a finite alphabet including a special **blank** character denoted by $\emptyset$, $\mathscr{Q}$ is a finite set of **states**, and[2] $\mathscr{T}\subset\mathscr{A}\times\mathscr{Q}\times\mathscr{A}\times\mathscr{Q}\times\{R,L\}$ is a **transition table** specifying how the machine operates on **configurations**. Those are words of the form $uqv\in\mathscr{A}^*\times\mathscr{Q}\times\mathscr{A}^*$, where $\mathscr{A}^*$ denotes the set of words (*i.e.*, finite sequences) over $\mathscr{A}$. Intuitively, the machine can be represented by a linear **tape** composed of a bi-infinite sequence of **cells** that each contains one alphabet symbol, and by a read/write head pointing to one cell and containing the machine state. Configuration uqv then corresponds to a tape marked with the word uv and otherwise with blanks and whose read/write head points to the first letter in v (the empty word is interpreted as a blank). Transition $aqbpD\in\mathscr{T}$ applies to any configuration uqv such that a is the first letter in v. It transforms uqv replacing a with b, the state q by p, and moves the head one step to the left or right according to whether D equals L or R, respectively.

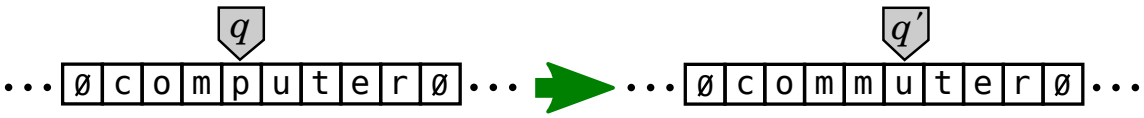

Figure 2: Illustration of the transition pqm$q'R$ applied to configuration com$q$puter on a Turing machine operating on the Latin alphabet.

A Turing machine is **deterministic** if at most one transition applies to a given configuration: $aqbpD\in\mathscr{T}$ and $aqb'p'D'\in\mathscr{T}$ implies $b'=b$, $p'=p$ and $D'=D$. The

[1] Here, we are only interested in systems of polynomials with integer (equivalently, rational) coefficients.

[2] In these notes, we use the symbol $\subset$ to indicate the subset relation, not necessarily proper.

machine is **halting** in a given configuration when no transition applies. Usually, a Turing machine has two special halting states interpreted as *accepting* and *rejecting*. As opposed to a deterministic machine, a nondeterministic Turing machine may lead to several computations starting from a same configuration.

3.1.2 Complexity classes

In computer science a **decision problem** refers to a subset of words over a fixed alphabet $\mathscr{A}$. Words in the subset are the *YES instances* of the problem. Intuitively, the YES instances correspond to the encoding of objects – such as numbers, graphs, or Boolean formulas – satisfying a certain property. For instance, one may consider the problem of primality testing where the YES instances are the binary encoding of prime integers over the alphabet $\mathscr{A} = \{0,1\}$. In full generality, a decision problem can be any subset $I \subset \mathscr{A}^*$. Such a subset is also called a **language**. A Turing machine is said to solve or decide[3] problem I if given any word $w \in \mathscr{A}^*$ as input, *i.e.*, starting with a configuration of the form $q_0 w$, where q_0 is a chosen initial state, it halts in the accepting state whenever $w \in I$ and halts in the rejecting state otherwise. An *algorithm* for problem I is just another name for a Turing machine solving I. The **time complexity** of the computation on input w is the number of transitions needed to reach a halting state. The **space complexity** is the maximum length of a configuration during the computation.

Polynomial and exponential classes. An algorithm has **polynomial time complexity** if for every $n \in \mathbb{N}$ and every input of length n the computation on this input has time complexity at most $p(n)$, where p is a polynomial that only depends on the algorithm. The set of problems admitting algorithms of polynomial time complexity is denoted by **P**. Replacing $p(n)$ by $2^{p(n)}$ we obtain the class **EXP** of problems with exponential time complexity. Analogously, the set of problems solved by Turing machines whose space complexity is polynomial is denoted by **PSPACE**. It is believed, but not known, that **EXP** $\not\subset$ **PSPACE**.

Exercise 3.1. Show that **PSPACE** $\subset$ **EXP**.

The class NP. The acronym **NP** stands for the class of *nondeterministic polynomial time* algorithms. A problem I is in **NP** if there is a nondeterministic Turing machine such that (1) given any $w \in I$ as input at least one computation leads to an accepting state in polynomial time and (2) no computation leads to an accepting state whenever $w \notin I$. Case (2) leaves the possibility that the machine runs forever, but computations that take more than polynomial time may be discarded without affecting the functionality of the machine, so that we can always assume that the computation takes polynomial time in both cases (1) and (2). However, the two cases are highly asymmetric since a computation leading to a rejecting state does not say anything about the input. There is another useful definition of the class **NP** in terms of efficiently verifiable certificate. A problem I is in **NP** if there is a deterministic Turing machine with polynomial time complexity, the *verifier*, such that (a) for every $w \in I$ there exists

[3] or, referring to the language terminology, to *recognize*.

$c \in \mathscr{A}^*$ so that the verifier accepts wc in polynomial time and (b) if $w \notin I$ the verifier rejects wc whichever c we choose. Hence, c acts as a **certificate**, or efficiently verifiable proof for being a YES instance.

Theorem 3.2. *The two definitions of the class* **NP** *by means of nondeterministic machines or in terms of certificates and deterministic verifiers are equivalent.*

PROOF. Suppose that a language I is recognized by a nondeterministic machine M in polynomial time. An input word w determines a directed rooted tree of computations where each node corresponds to a configuration of M and the children of a configuration node correspond to the various transitions that apply to that configuration. The degree of a node is bounded by a constant, namely the size of the transition table of M. A computation path in this tree is easily encoded as the list ℓ of branching choices at the nodes along the path. By assumption, ℓ has polynomial size and may serve as a certificate. We can define a verifier V that takes the concatenation $w\ell$ (with some predefined separator) as input and essentially simulates the computation of M on w guided by ℓ. The successive branching choices in ℓ allow V to maintain the current configuration of M determined by those choices. The main task of the verifier is thus to check that each branching choice corresponds to an actual transition of M that applies to the current configuration. Clearly, V operates in polynomial time and $w \in I$ if and only if we can choose ℓ so that the simulation leads to an accepting state of M. We have thus proved that I is in **NP** according to the second definition.

Conversely, suppose that every word in I has a certificate verifiable by a polynomial time Turing machine V. We define a nondeterministic machine M operating in two stages. In the first stage, M guesses a certificate with polynomial length. In the second stage, M simulates V deterministically on the input word concatenated with the guessed certificate. The nondeterminism of M is thus concentrated in the first stage. It is easily seen that M recognize I as a member of **NP** in the sense of the first definition. □

Exercise 3.3. Show that **NP** $\subset$ **PSPACE**.

3.1.3 Reduction and completeness

The notion of reduction allows to compare the difficulty of different problems. Given two problems $I, J \subset \mathscr{A}^*$, we say that I **reduces** (in polynomial time) to J, written $I \leq J$, if there is a function $r : \mathscr{A}^* \to \mathscr{A}^*$, computable by a Turing machine with polynomial time complexity, such that $I = r^{-1}(J)$. In other words, r transforms YES and NO instances of the first problem to, respectively, YES and NO instances of the second problem[4]. Hence, $I \leq J$ and $J \in \mathbf{P}$ implies $I \in \mathbf{P}$. This is obviously true replacing **P** by any other larger complexity class. If I reduces to J and J to K, it is easily seen I reduces to K. The reduction relation is thus a preorder (*i.e.*, a reflexive and transitive relation).

[4]This type of reduction is called *many-one*, or *Karp reduction*. *Polynomial-time Turing reduction*, also known as *Cook reduction*, is another common notion of reduction, where I reduces to J if I can be solved in polynomial time by a Turing machine with an oracle for J, meaning that the machine is allowed to call a subroutine for problem J at anytime during the computation, in constant time for each call.

Any problem which is an upper bound for a complexity class C is said **C-hard**. It is said **C-complete** if it furthermore belongs to **C**. A **C**-complete problem is thus a hardest representative in **C**. It is a priori not clear whether a complexity class has complete problems.

Exercise 3.4. Show that every non-trivial problem (proper subset of $\mathscr{A}^*$) in **P** is **P**-complete.

It turns out that the class **NP** has complete problems, among which the satisfiability problem. A Boolean formula is a logical expression over Boolean variables connected by the usual $\wedge, \vee, \neg$ operators. A formula is *satisfiable* if there is an assignment of its variables that makes the formula evaluate to true. The problem **SAT** is the set of satisfiable formulas encoded, say, over the alphabet $\{0, 1, \wedge, \vee, \neg, (,)\}$.

Theorem 3.5 (Cook'71 - Levin'73)**.** *SAT is* **NP***-complete.*

PROOF. Any truth assignment of a formula in SAT provides a certificate that is easily checkable in polynomial time. It follows that SAT $\in$ **NP**. It remains to show that every problem $I \in$ **NP** reduces to SAT. Let $M = (\mathscr{A}, \mathscr{Q}, \mathscr{T})$ be a nondeterministic machine solving I in polynomial time. For every instance w, we need to construct a formula Φ_w so that $w \in I$ if and only if Φ_w is satisfiable.

Number the cells of the tape once for all from left to right so that at the initial step the tape contains $w = w_1 w_2 \dots w_n$ with cell 1 containing w_1. By assumption on M, the number of computation steps given w as input is bounded by $p(n)$ for some polynomial p, where $n := |w|$ is the length of w. By convention, we consider that M stays in the same configuration once in a halting state. This way we can assume that the number of computation steps is exactly $p(n)$. It follows that the head of M can only point to a cell with index in the range $J := [-p(n), p(n)]$. In particular, cells with index outside this range must contain the empty symbol. The whole computation is thus entirely described by the content of the jth cell at the ith step (configuration) of the computation, with $1 \leq i \leq p(n)$ and $j \in J$, and the sequence of $p(n)$ states and head positions during the computation. In accordance with this description, we introduce Boolean variables $C_{i,j,s}, Q_{i,q}, H_{i,j}$ with $1 \leq i \leq p(n)$, $j \in J$, $s \in \mathscr{A}$ and $q \in \mathscr{Q}$. The variable $C_{i,j,s}$ is intended to be true whenever the jth cell at the ith step contains s and false otherwise. Similarly, $Q_{i,q}$ and $H_{i,j}$ are intended to be true exactly when M is in state q at step i with the head pointing to the jth cell.

We next consider the following Boolean formulas. We recall that $A \implies B$ is a shorthand for $\neg A \vee B$.

- $\phi_{i,j} = \bigvee_{s \in \mathscr{A}} (C_{i,j,s} \wedge (\bigwedge_{t \neq s} \neg C_{i,j,t}))$ expresses that the jth cell at the ith step takes one and only one value.
- $\phi_i = \big(\bigvee_{q \in \mathscr{Q}} (Q_{i,q} \wedge (\bigwedge_{r \neq q} \neg Q_{i,r}))\big) \wedge \big(\bigvee_{j \in J} (H_{i,j} \wedge (\bigwedge_{k \neq j} \neg H_{i,k}))\big)$ expresses that the state and head position each take exactly one value at the ith step.
- $\phi_b = \bigwedge_{1 \leq j \leq n} C_{1,j,w_j} \wedge \bigwedge_{j \notin [1,n]} C_{1,j,\emptyset} \wedge Q_{1,q_0} \wedge H_{1,1}$ expresses that the initial tape contains the input w and that M is in the initial state q_0 with the head pointing to the first symbol of w.

- $\phi_e = Q_{p(n),q_a}$, where q_a is the accepting state, expresses that M accepts w.
- $\psi_i = \bigwedge_{\substack{j\in J\\ s\neq t}} \big((C_{i,j,s} \wedge C_{i+1,j,t}) \implies H_{i,j}\big)$ expresses that only the cell pointed by the head may change from step i to $i+1$.
- $\psi_{i,j,q,s} = (Q_{i,q} \wedge H_{i,j} \wedge C_{i,j,s}) \implies \bigvee_{sqtrD\in\mathcal{T}} (Q_{i+1,r} \wedge H_{i+1,j+D} \wedge C_{i+1,j,t})$ expresses that when M is in state q at step i with the head pointing to the jth cell containing s, only the relevant transitions may apply. Here, $j+D$ is $j-1$ or $j+1$ depending on whether $D=L$ or $D=R$.

We finally set $\Phi_w = \bigwedge_{i,j} \phi_{i,j} \wedge \phi_b \wedge \phi_e \wedge \bigwedge_i \psi_i \wedge \bigwedge_{i,j,q,s} \psi_{i,j,q,s}$. To conclude, it remains to notice that the description of the formula Φ_w can be computed in polynomial time (with respect to n) and that Φ_w is satisfiable if and only if M recognizes w, *i.e.* $w \in I$. □

3.2 Existential theory of the reals

We are now ready to characterize the complexity of the linear embedding problem. Given as input an abstract simplicial complex K and a dimension d, the problem is to decide if K has a linear embedding into $\mathbb{R}^d$. As we shall see this problem can be reduced in polynomial time to test the non-emptiness of a semi-algebraic set defined by polynomials with integer coefficients.

Semi-algebraic set. An *atomic formula* may have one of two forms $\{p=0\}$ or $\{p>0\}$, where p is a polynomial in a finite number of variables, with integer coefficients. A predicate $\Phi(X_1,\dots,X_d)$ in the language of fields with integer coefficients is a Boolean predicate applied to atomic formulas using the free variables $X_1,\dots,X_d$. In other words, $\Phi(X_1,\dots,X_d)$ can be obtained recursively from atomic formulas using the logical connectors $\wedge, \vee$ and $\neg$. A **semi-algebraic set** over the integers is any set of the form

$$\{x = (x_1,\dots,x_d) \in \mathbb{R}^d \mid \Phi(x)\}$$

with Φ a predicate as above. An **existential formula** is a proposition of the form

$$\exists x \in \mathbb{R}^d \mid \Phi(x)$$

Deciding the falsity or truth of an existential formula is thus the same as deciding if a semi-algebraic set is empty or not. The set of problems that reduces in polynomial time to deciding the status of existential formulas has been gathered under the name of **existential theory of the reals**. This complexity class is denoted by $\exists\mathbb{R}$.

Lemma 3.6. **NP** $\subset \exists\mathbb{R}$.

PROOF. By Theorem 3.5 of Cook and Levin, it is enough to prove that SAT reduces to $\exists\mathbb{R}$. Let $\Phi(X)$ be a Boolean formula with variables $X = (X_1,\dots,X_d)$. Using the distributivity rules of negation over disjunction and conjunction we can assume that the

negations in Φ may only apply to the atomic variables X_i. Let $Y = (Y_1, \ldots, Y_d)$ be free variables, we recursively define a polynomial $P_\Phi(Y)$ using the formulas: $P_{X_i}(Y) = Y_i$, $P_{\neg X_i}(Y) = 1 - Y_i$, $P_{\Phi_1 \wedge \Phi_2} = P_{\Phi_1}(Y) \times P_{\Phi_2}(Y)$ and $P_{\Phi_1 \vee \Phi_2} = P_{\Phi_1}(Y) + P_{\Phi_2}(Y)$. We now consider the existential formula defined by the conjunction of the following predicates.

- $Y_i^2 - Y_i = 0$, $i = 1 \ldots d$.
- $P_\Phi(Y) > 0$.

Noting that $Y_i^2 - Y_i = 0$ implies $Y_i \in \{0, 1\}$, it is easily checked that $\Phi(X)$ can be satisfied if and only if the above existential formula defines a nonempty semi-algebraic set. Moreover, a description of the existential formula can be obtained in time proportional to the length of the description of Φ, thus providing the required reduction. □

A much more challenging task is to provide an upper bound on the complexity of $\exists\mathbb{R}$. The first approach to decide the vacuity of a system of polynomials (in)equations used the cylindrical decomposition of Collins (1975). This cylindrical decomposition includes a decomposition of $\mathbb{R}^k$, where k is the number of variables in the system, into semi-algebraic cells such that each polynomial in the system has a constant sign $-, 0$ or $+$ over each cell. Hence, the system has at least one solution if we can find one cell in the decomposition such that the sign of each polynomial agrees with the corresponding (in)equality in the system. The best known computation of such an adapted decomposition takes time $O(s d^{2^k})$ where s is the number of polynomials in the system and d is their maximal degree. The cylindrical decomposition approach thus leads to a doubly exponential time algorithm. It was eventually shown that $\exists\mathbb{R}$ could be solved using polynomial space only [Can88, Ren88].

Theorem 3.7 (Canny'88)**.** $\exists\mathbb{R} \subset$ **PSPACE**

The proof of this result is far beyond the purpose of this lecture. Describing all the details takes a whole thick book [BPR06]. There are excellent surveys [Bas14, RRSED00] that can serve as introductory lectures. An important step is to decide the (non)emptiness of a real algebraic set defined by a system $\mathscr{S}$ of polynomial equations. The main idea is to augment $\mathscr{S}$ with other polynomial conditions so that the new system has only a finite number of solutions, a so-called *zero-dimensional system*, with at least one solution in each (semi-algebraically) connected component defined by $\mathscr{S}$. Those solutions can even be returned implicitly using *rational univariate representations*. This is done by searching for the critical points of a given functional (e.g. the squared distance to a fixed point) over the algebraic set. For this method to work it is required that the critical points are non-singular and that the components are bounded. One way to enforce these conditions is to use symbolic perturbations. They are obtained by introducing new variables playing the role of infinitesimals, replacing equations of the form $P = 0$ by $P = \varepsilon$, for ε an infinitesimal. Other modifications may be introduced to take care of the unbounded components leading to a new system of polynomial equations whose coefficients are now polynomials in the infitesimals. After solving the modified system, it remains to substitute zero for the infinitesimals to obtain real solutions. A huge amount of techniques from real algebraic geometry are necessary, such as the use of resultants, root counting, Gröbner basis computations,

etc. In the end, it can be proved that the emptiness of a semi-algebraic set defined by a system of polynomial (in)equations can be decided using polynomial space, in terms of the size of the encoding of the system. Only a few implementations seems to exist and are hardly able to deal with more than a dozen variables with polynomials of relatively low degree.

Exercise 3.8. Show that the emptiness of a semi-algebraic set defined by polynomial (in)equations can be reduced to the emptiness of an algebraic set defined by polynomial equations. Show that you can furthermore impose that the algebraic set is defined by a single polynomial equation.

3.2.1 Linear embeddability belongs to $\exists\mathbb{R}$

In the introduction to Section 3 we already observed that the embeddability of a simplicial complex K could be reduced to the satisfiability of a set of polynomial inequalities. We still need to check that this reduction takes polynomial time. Recall that we have to encode the conditions that pairs of disjoint simplices are sent to non-intersecting simplices in $\mathbb{R}^d$. The transcription into polynomials of those conditions for each pair of simplices just claims the existence of a separating hyperplane and clearly takes polynomial time. There still remains the potential problem that the number of simplices, hence the number of polynomials conditions, is very large compared to the encoding of K. A reasonable encoding should indeed only records the **maximal simplices** of K — those that are not a face of larger simplices — the other simplices being implicitly encoded as faces of the maximal ones. For instance, if $|K|$ is an m-dimensional simplex, its total number of faces is 2^{m+1} while its encoding is essentially the single set $[m+1]$. Nonetheless, since $m \leq d$ is an obvious condition for embeddability in $\mathbb{R}^d$, we are led to conclude that

Theorem 3.9. *The linear embedding problem into $\mathbb{R}^d$ is in $\exists\mathbb{R}$ for any fixed dimension d.*

The question raised by the potentially large number of polynomial conditions can be dealt with at the expense of getting larger polynomials. We can indeed replace the conditions in Lemma 2.1 by a smaller number of conditions. To see this, we first make a simple observation.

Lemma 3.10. *Let σ, τ be two simplices in $\mathbb{R}^d$ intersecting along a common face. There exists a hyperplane intersecting each of σ, τ along their common face and otherwise separating them.*

PROOF. Let $\{u_i\}_{i\in I}$ be the vertices of the common face, and let $\{v_j\}_{j\in J}$ and $\{w_\ell\}_{\ell\in L}$ be the remaining vertices of σ and τ, respectively. Let $u_i^{\sigma,\varepsilon} := (1-\varepsilon)u_i + \varepsilon v_{j_0}$ for some fixed $j_0 \in J$. Likewise, let $u_i^{\tau,\varepsilon} := (1-\varepsilon)u_i + \varepsilon w_{\ell_0}$ for some $\ell_0 \in L$. For $0 < \varepsilon < 1$, $\sigma_\varepsilon := \mathrm{Conv}(\{u_i^{\sigma,\varepsilon}\}_{i\in I} \cup \{v_j\}_{j\in J})$ and $\tau_\varepsilon := \mathrm{Conv}(\{u_i^{\tau,\varepsilon}\}_{i\in I} \cup \{w_\ell\}_{\ell\in L})$ are disjoint compact convexes, hence separated by a hyperplane H_ε defined by a unit normal vector ν_ε and a point u_ε, say between $u_1^{\sigma,\varepsilon}$ and $u_1^{\tau,\varepsilon}$. As ε tends to zero, ν_ε and u_ε converge toward a vector ν_0 and a point u_0 defining a hyperplane H_0. It is easily seen that H_0 has the required property. □

With some abuse of terminology we still call the hyperplane as in Lemma 3.10 a *separating hyperplane* for (σ, τ).

Corollary 3.11. *The embedding conditions in Lemma 2.1 can be replaced by the following: (1) the vertices of each maximal simplex of K are sent to affinely independent points in $\mathbb{R}^d$ and (2) for every pair of distinct maximal simplices of K, there exists a separating hyperplane in the sense of the previous lemma.*

PROOF. Condition (1) is trivially necessary for any linear embedding. Lemma 3.10 implies that condition (2) is also necessary. Conversely, suppose that a linear map $f : K \to \mathbb{R}^d$ satisfies (1) and (2). Let σ, τ be to disjoint simplices of K. σ and τ are faces of two maximal simplices, say σ' and τ' respectively. On the one hand, if $\sigma' = \tau'$, condition (1) implies that σ, τ are sent to disjoint faces of a non-degenerate simplex in $\mathbb{R}^d$. On the other hand, if $\sigma' \neq \tau'$, condition (2) implies the existence of a separating hyperplane for (σ', τ') providing a separating hyperplane for (σ, τ). In any case, σ and τ are sent to non-intersecting simplices in $\mathbb{R}^d$, showing that f is an embedding by Lemma 2.1. □

By Corollary 3.11 we just need a number of polynomial conditions that is quadratic in the number of maximal simplices. Beware, though, that condition (1) is expressed by the non-cancellation of determinants that may contain up to $d!$ terms. The potential benefit of this approach in terms of the number of polynomials should thus be balanced with the increase in the size of the polynomials.

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