Discrete metric spaces

Victor Chepoi

LIF, Université d'Aix-Marseille II

Purpose of this minicourse: to present the most important classical and recent results about metric spaces.

Part 0: Preliminaries and definitions

Part I: Isometric embeddings into l_2-, l_∞, and l_1-spaces

Part II: Median spaces, hyperconvex spaces, CAT(0) spaces

Part III: Approximate isometric embeddings into l_1 and l_2

Part IV: Graph classes defined by metric properties

#1 Definitions I: metric spaces

metric space: (X, d), where X is a set and $d: X \times X \longrightarrow \mathbb{R}_+$ a function called distance such that

(i) $d(x,y) = d(y,x) \quad \forall x,y \in X$;

(ii) $d(x,y) = 0 \iff x = y$;

(iii) $d(x,y) \leq d(x,z) + d(z,y)$
(triangle inequality)

<u>Examples</u>:

(i) norm metrics (Minkowski metrics):

$(\mathbb{R}^m, d_{\|\cdot\|})$, where $d_{\|\cdot\|}(x,y) = \|x-y\| \quad \forall x,y \in \mathbb{R}^m$

ℓ_p-metrics ($p \geq 1$):

$$d_{\ell_p}(x,y) = \left(\sum_{k=1}^{m} |x_k - y_k|^p\right)^{\frac{1}{p}}, \qquad \begin{array}{l} x = (x_1, \dots, x_m) \\ y = (y_1, \dots, y_m) \end{array}$$

ℓ_p^m denotes the metric space $(\mathbb{R}^m, d_{\ell_p})$

Three basic host ℓ_p-metric spaces:

$\ell_2^m \qquad d_{\ell_2}(x,y) = \left(\sum_{k=1}^{m} |x_k - y_k|^2\right)^{\frac{1}{2}}$ – Euclidean distance

$\ell_1^m \qquad d_{\ell_1}(x,y) = \sum_{k=1}^{m} |x_k - y_k|$ – ℓ_1-distance or rectilinear distance

$\ell_\infty^m \qquad d_{\ell_\infty}(x,y) = \max\{|x_k - y_k| : 1 \leq k \leq m\}$ – Chebyshev distance

Definitions I: metric spaces (cont.)

(ii) Hamming distance d_H:

$$d_H(x,y) = |\{i \in \{1,\dots,m\} : x_i \neq y_i\}| \text{ for } \forall x,y \in \mathbb{R}^m.$$

For binary vectors $x,y \in \{0,1\}^m$, the Hamming distance $d_H(x,y)$, the ℓ_1-distance $d_{\ell_1}(x,y)$, and the graph distance $d(x,y)$ of the m-cube coincide.

(iii) standard graph distance d_G:

$G = (V, E)$ - connected, not necessarily finite, undirected and unweighted graph

length of a path – number of edges in this path

$d_G(x,y)$ = the length of a shortest path connecting two vertices x,y of G

(iv) finite metric spaces:

(V_n, d), where $V_n = \{1,\dots,n\}$ (for convenience) and d is a metric on V_n

Distance matrix: the $n \times n$ symmetric matrix D whose (i,j)-th entry is $d(i,j)$ for all $i,j \in V_n$.

Metric cone: set $E_n := \{ij : i,j \in V_n, i \neq j\}$; then any distance d on V_n can be viewed as a vector $(d_{ij})_{1 \le i < j \le n} \in \mathbb{R}^{E_n}$.

The triangle inequalities $d(i,j) + d(i,k) \geq d(k,j)$ for all $i,j,k \in V_n$ define a convex cone in the space $\mathbb{R}^{E_n}$ called the semimetric cone and denoted by MET_n.

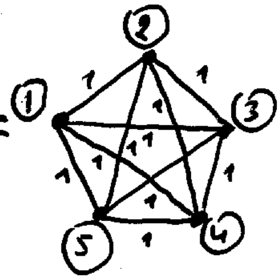
Particular finite metric spaces:

(i) planar metrics (V_n, d)

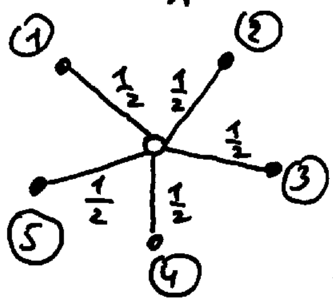
(ii) tree metrics (V_n, d)

one can construct a planar graph $G=(V,E)$ or a tree $T=(V,E)$ such that $V_n \subseteq V$ and $d = d_G|_{V_n}$

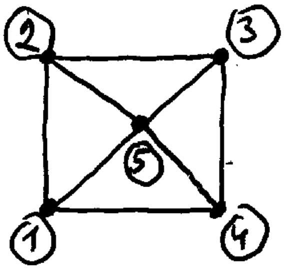
$$D_n = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & & \ddots & & 1 \\ \vdots & & & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} =$$



$=$



$$D = \begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ & 0 & 1 & 2 & 1 \\ & & 0 & 1 & 1 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} =$$



not a tree metric

Definitions II: isometric embeddings

A metric space (X, d) is isometrically embeddable into a metric space (X', d') if there exists a mapping φ (the isometric embedding) from X to X' such that

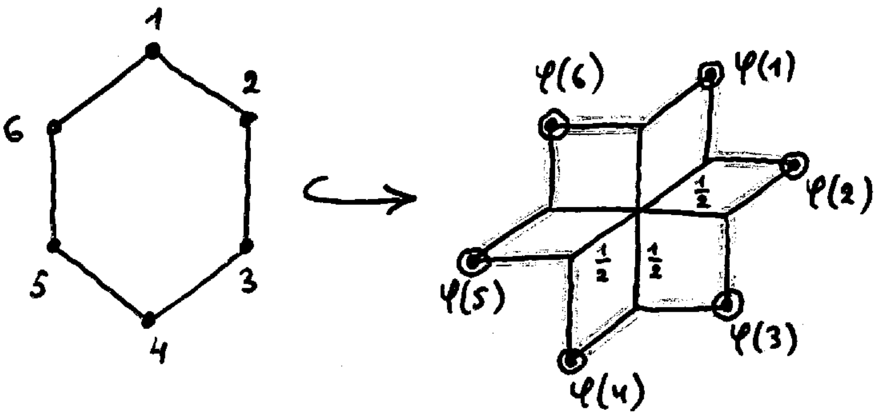
$$d(x,y) = d'(\varphi(x), \varphi(y)) \quad \forall x,y \in X$$

Then (X,d) is said to be an isometric subspace of (X',d').

(X,d) is ℓ_p-embeddable if (X,d) is isometrically embeddable into the space ℓ_p^m for some $m \geq 1$.

<u>Example</u>: the ℓ_1-embedding of the metric of the 6-cycle

$$D = \begin{pmatrix} 0 & 1 & 2 & 3 & 2 & 1 \\ & 0 & 1 & 2 & 3 & 2 \\ & & 0 & 1 & 2 & 3 \\ & & & 0 & 1 & 2 \\ & & & & 0 & 1 \\ & & & & & \end{pmatrix}$$



A metric space (X,d) is distortion β embeddable into a metric space (X',d') if there exists a mapping φ (the β-embedding) from X to X' such that

$$d(x,y) \leq d'(\varphi(x), \varphi(y)) \leq \beta \cdot d(x,y) \quad \forall x,y \in X.$$

(or equivalently – for ℓ_p-embeddings – if there exists a mapping $\psi : X \to X'$ such that

$$\frac{1}{\beta} \cdot d(x,y) \leq d'(\psi(x), \psi(y)) \leq d(x,y) \quad \forall x,y \in X)$$

Definitions II: isometric embeddings

The host space (X', d') is said to have order of congruence p if, for every metric space (X, d),

(Y, d) embeds isometrically into (X', d') for every $Y \subset X$, $|Y| \leq p$

$$\Downarrow$$

(X, d) embeds into (X', d')

and p is the smallest such integer (possibly infinite).

Questions

For a given host space or class of host spaces:

(i) is the order of congruence finite?

(ii) is the order of congruence bounded by a constant?

(iii) can the isometric subspaces be effectively characterized?

(iv) is the decision question "Is an input finite metric space isometrically embeddable into the host space?" polynomial or NP-complete?

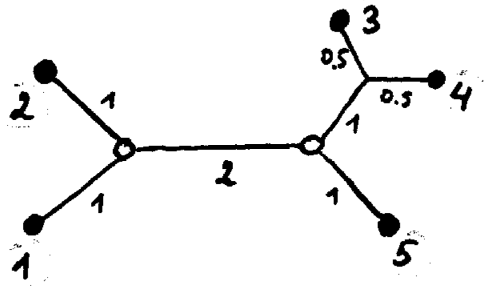
(v) find small distortion embeddings of finite metric spaces into the host space(s).

(vi) find small distortion embeddings into host spaces of small dimension.

Tree metrics

Tree metric a finite metric space isometrically embeddable into a weighted tree

$$D = \begin{pmatrix} 0 & 2 & 4.5 & 4.5 & 4 \\ & 0 & 4.5 & 4.5 & 4 \\ & & 0 & 1 & 2.5 \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$



Theorem (Zaretskii, 1965; Buneman, 1971) For a finite metric space (V_n, d) the following conditions are equivalent:

(i) (V_n, d) is a tree metric;

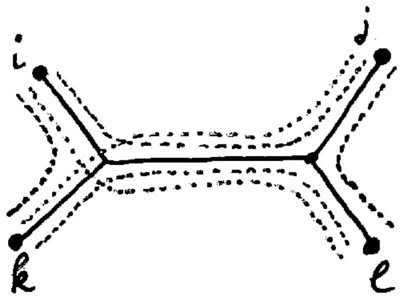
(ii) every quadruplet of V_n isometrically embeds into a tree;

(iii) (V_n, d) satisfies the following four-point condition:

$$d(i,j)+d(\ell,k) \le \max\{d(i,\ell)+d(j,k),\ d(i,k)+d(j,\ell)\}.$$

Idea of the proof:

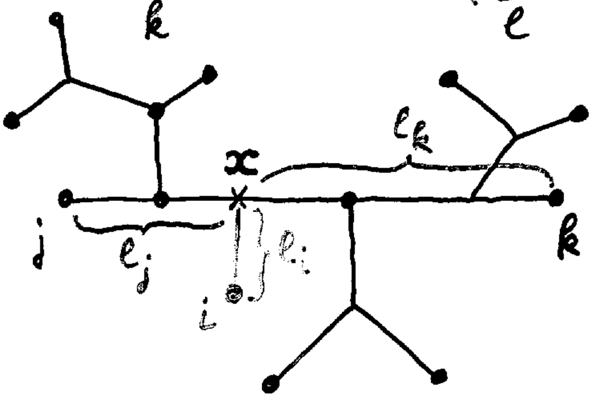
$\Rightarrow$



$\Leftarrow$ By induction. For a current point i find the points $j<i$ and $k<i$ // ℓ_i minimizing $\frac{1}{2}(d(i,j)+d(i,k)-d(j,k))$

On the path of the current tree pick the point x such that $d(x,j)=\frac{1}{2}(d(j,i)+d(j,k)-d(i,k))=\ell_j$

To the current tree add the edge xi of length ℓ_i so that i becomes a leaf and x an inner vertex.



Isometric embeddings into ℓ_p-spaces

Theorem 1 (Bretagnolle, Dacunha-Castelle, Krivine, 1966)
A metric space (X,d) is L_p-embeddable if and only if every finite subspace of (X,d) is L_p-embeddable.

Theorem 2 (Malitz, Malitz, 1992) Let $p, m \geq 1$ be integers. Then a metric space (X,d) is ℓ_p^m-embeddable if and only if every finite subspace of (X,d) is ℓ_p^m-embeddable.

Theorem 3 (Frechet) Any n point metric space (V_n, d) can be isometrically embedded into ℓ_∞^n.

Proof: For each point $i \in V_n$ define a cooridate $\varphi_i : V_n \to \mathbb{R}_+$ by setting $\varphi_i(j) = d(i,j)$ and let $\varphi(j) = (\varphi_1(j), \dots, \varphi_n(j))$. We claim that φ is an isometric embedding into ℓ_∞^n.

Indeed:

$$\begin{aligned}\|\varphi(j)-\varphi(k)\|_\infty &= \max_{i\in V} |\varphi_i(j)-\varphi_i(k)| \\ &= \max_{i \in V} |d(i,j)-d(i,k)| \\ &\leq d(j,k) \text{ by triangle inequality.}\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\varphi(j)-\varphi(k)\|_\infty &\geq |\varphi_k(j)-\varphi_k(k)| \\ &= |d(j,k)-d(k,k)| \\ &= d(j,k) \qquad \square\end{aligned}$$

Theorem 4 (Menger, 1928) Given $m \geq 1$, a metric space (X, d) can be isometrically embedded in ℓ_2^m if and only if for every $Y \subseteq X$ with $|Y| = m+3$, (Y, d) can be isometrically embedded in ℓ_2^m.

Proof sketch:

Stronger assertion: (X, d) can be isometrically embedded in ℓ_2^m and not into ℓ_2^{m-1} if and only if there exist a subset $Y = \{x_0, x_1, \ldots, x_m\}$ of X such that

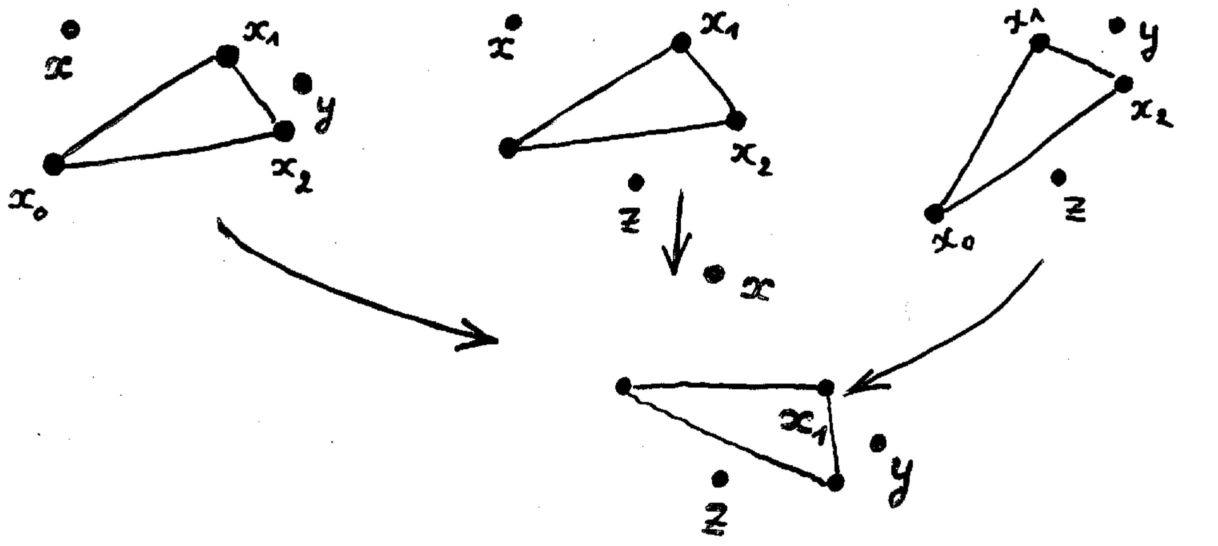
(i) (Y, d) can be embedded into ℓ_2^m but not in ℓ_2^{m-1};

(ii) for every $x, y \in X$, the metric space $(Y \cup \{x, y\}, d)$ can be isometrically embedded in ℓ_2^m.

Notice that:

(i) if φ is an isometric embedding of Y in ℓ_2^m, then $\varphi(Y)$ has full affine rank $m+1$;

(ii) if φ', φ'' are two isometric embeddings of Y in ℓ_2^m, then we can find an orthogonal transformation mapping every $\varphi'(x_i)$ into $\varphi''(x_i)$, $x_i \in Y$. donc, une isométrie



Question: Given a finite metric space (V_n, d), can we check in polynomial time if (V_n, d) embeds in ℓ_2?

Answer: Let φ be a mapping from V_n to ℓ_2.
Let $\varphi(i) = v_i$ and assume that $\varphi(1) = v_1 = \vec{0}$.

Then

$$\|v_i - v_j\|^2 = d_{ij}^2 \quad \forall i,j \in V \iff$$

$$\|v_i\|^2 + \|v_j\|^2 - 2\langle v_i, v_j\rangle = d_{ij}^2 \iff$$

$$\langle v_i, v_j\rangle = \frac{1}{2}\left(d_{1i}^2 + d_{1j}^2 - d_{ij}^2\right),$$

because $\|v_i\|^2 = \|v_i - v_1\|^2 = d_{1i}^2$, $\|v_j\|^2 = \|v_j - v_1\|^2 = d_{1j}^2$

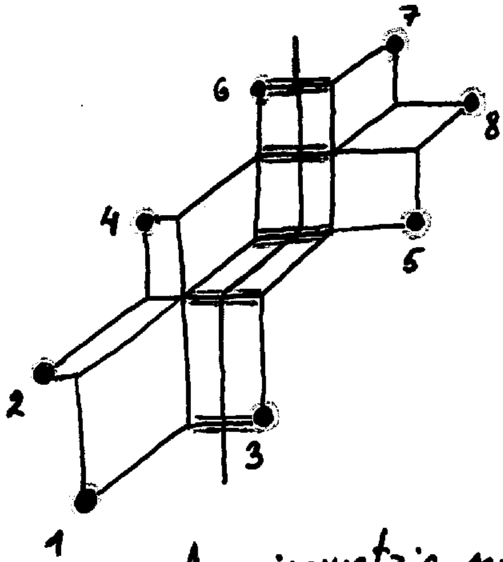
Denote $A_{ij} = \frac{1}{2}\left(d_{1i}^2 + d_{1j}^2 - d_{ij}^2\right)$ and consider the matrix A. Then φ is an isometric embedding in ℓ_2 if and only if $A_{ij} = \langle v_i, v_j\rangle$, i.e. A can be written as $B^T B$, or, in other words, A is a positive semi-definite matrix (Schoenberg, 1938)

recognition of positive semi-definiteness can be done in polynomial time using an algorithm based on Gaussian elimination.

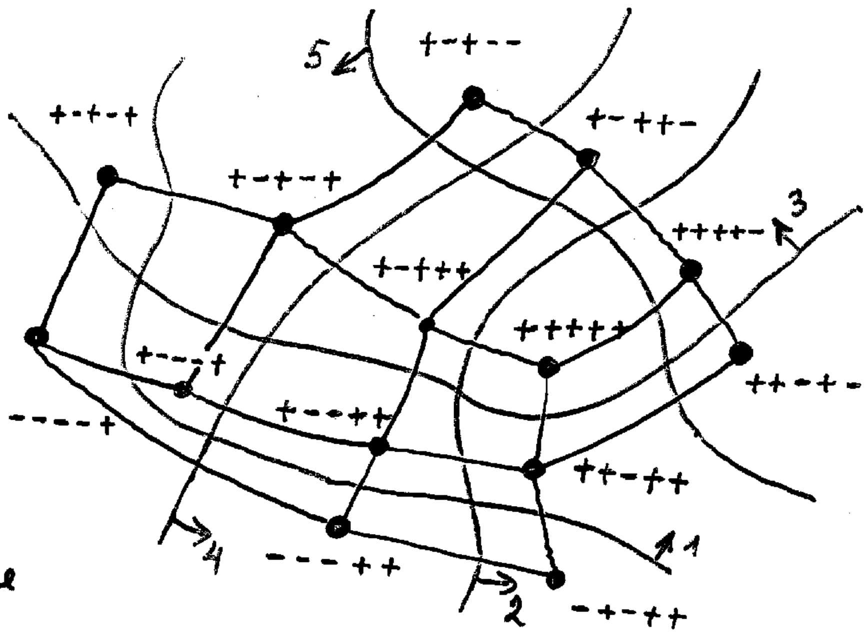
Isometric embeddings into ℓ_1

Theorem 5 (Karzanov, 1986) Deciding if a finite metric space (V, d) ℓ_1-embeds is NP-complete.

Reduction from MAX CUT



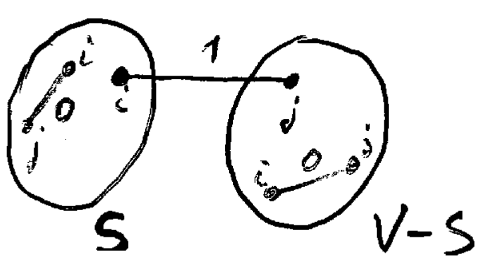
An isometric subspace of ℓ_1^3



An arrangement of five pseudoline and the isometric embedding of its graph of regions into the cube $\{\pm 1\}^5$.

Cut semimetric: for $S \subseteq V_n = \{1, \dots, n\}$

$$\delta(S)_{ij} = \begin{cases} 1 & \text{if } |S \cap \{i,j\}| = 1 \\ 0 & \text{otherwise} \end{cases}$$



<u>Theorem 6</u>: For a finite metric space (V_n, d) the following conditions are equivalent:

(i) (V_n, d) is l_1-embeddable, i.e., there exist n vectors $u_1, \dots, u_n \in \mathbb{R}^m$ for some m such that $\|u_i - u_j\|_1 = d_{ij}$ $\forall i,j \in V_n$;

(ii) $d = \sum\limits_{S \subseteq V_n} \lambda_S \delta(S)$ for some nonnegative λ_S, i.e. d belongs to the cut cone $CUT_n = \{\sum\limits_{S \subseteq V_n} \lambda_S \delta(S) : \lambda_S \geq 0 \ \forall S \subseteq V_n\}$

(iii) there exist a measure space $(\Omega, \mathcal{A}, \mu)$ and events $A_1, \dots, A_n \in \mathcal{A}$ such that $d_{ij} = \mu(A_i \Delta A_j)$ $\forall i,j \in V_n$.

<u>Remark</u>: In case of embedding into hypercubes or Hamming metric spaces, λ_S is a nonnegative integer and $d_{ij} = |A_i \Delta A_j|$.

<u>Remark</u> (Ball, 1990): The dimension m of $\mathbb{R}^m$ in (i) can be as large as $\frac{(n-3)(n-2)}{2}$ for $n \geq 4$! Fichet (1988) showed that $m \leq \frac{n(n-1)}{2} - 1$.

<u>Remark</u>: The difficulty to find an l_1-embedding via (ii) consists in finding the cuts $(S, V_n - S)$ such that $\lambda_S > 0$ (by Carathéodory's theorem their number is $\leq \frac{n(n-1)}{2}$), i.e. those cuts which define the embedding.

<u>Theorem 7</u> (Bandelt & Chepoi, 1995) A metric space (X, d) isometrically embeds into $(\mathbb{R}^2, d_{\ell_1})$ if and only if (Y, d) embeds for any $Y \subseteq X$, $|Y| \leq 6$.

The idea of the proof will be given below.

<u>Remark</u>: From Theorem 7 follows that the congruence orders of ℓ_1^2 and ℓ_∞^2 is 6.

<u>Remark</u> (Bandelt, Chepoi, Laurent, 1997) The congruence order of ℓ_1^m is at least m^2 for $m \geq 3$ odd and at least m^2-1 for $m \geq 4$ even.

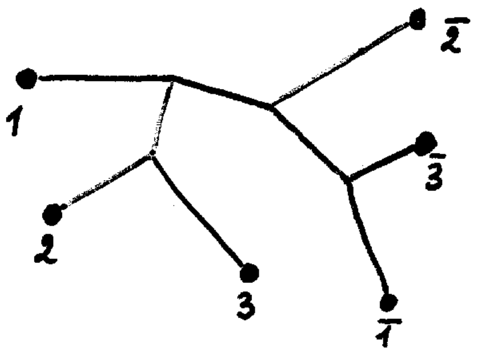
<u>Open question</u>: Find the congruence order of ℓ_1^3. Is it finite? (It is only known to be ≥ 10).

<u>Remark</u> (Jeff Erikson, 2004) The congruence order of ℓ_∞^3 is not bounded!

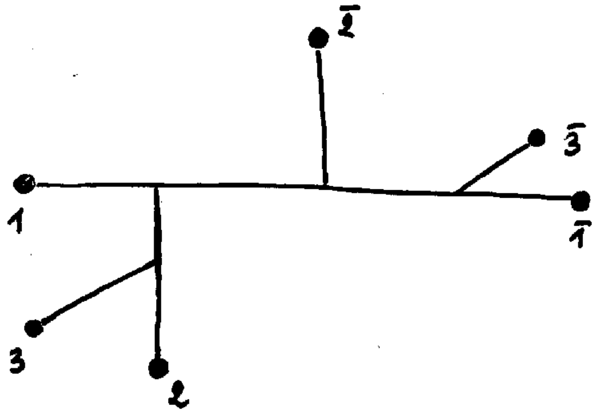
<u>Open question</u>: What is the largest set of an equilateral set of ℓ_1^m? It is conjectured to be $2m$, but it is known to be $\leq m \log_2 m$ (Alon, Pudlak, 2003)

Examples of ℓ_1-embeddable spaces

(i) Tree metrics (folklore): any tree with m leaves isometrically embeds into $\ell_1^{\lfloor \frac{m}{2} \rfloor}$.



$\Longrightarrow$



(ii) Spherical metrics (Kelly, 1970)

$S_m = \{x \in \mathbb{R}^{m+1} : \sum_{i=1}^{m+1} x_i^2 = 1\}$ - m-dimensional unit sphere

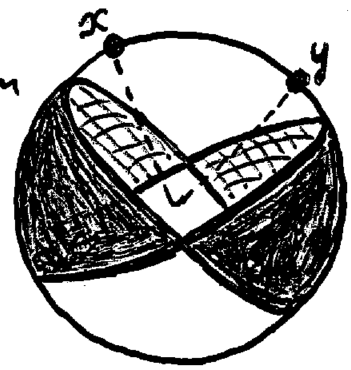
$d_S(x,y) := \arccos(x^T y) \quad \forall x,y \in S_m$ - spherical distance

the geodesic distance on the sphere S_m between the points x and y (great circle metric)

For $x \in S_m$, let $H(x) = \{y \in S_m : d_S(x,y) \le \frac{\pi}{2}\}$ be the hemisphere containing x.

Consider the measure μ on S_m defined by

$$\mu(A) = \frac{vol(A)}{vol(S_m)} \quad \text{for } A \subseteq S_m$$



Theorem 8 (Kelly, 1970)

$$\mu(H(x) \Delta H(y)) = \frac{1}{\pi}\arccos(x^T y) = \frac{1}{\pi} d_S(x,y)$$
$$\forall x,y \in S_m$$

(iii) ℓ_2-distances

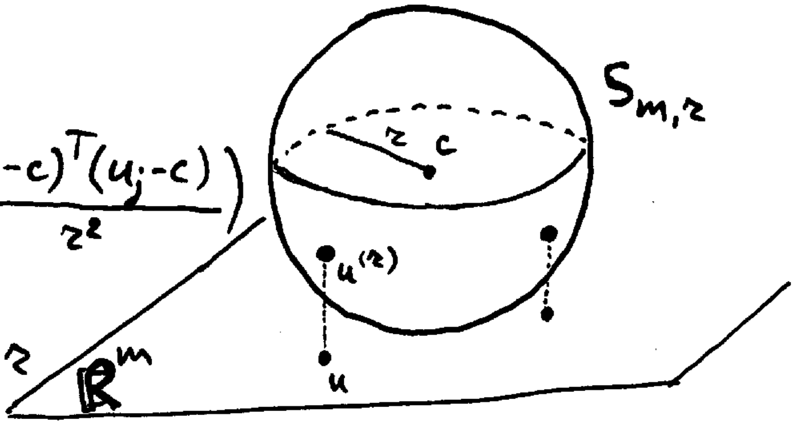
Theorem 9 (Schoenberg, 1935, Kelly, 1975) For a finite metric space (V_n, d), d is isometrically ℓ_2-embeddable implies that d is isometrically ℓ_1-embeddable.

Idea of the proof: Let $u_1, \dots, u_n \in \mathbb{R}^m$, $d_{ij} = \|u_i - u_j\|_2 \ \forall i,j$

Express d as a limit of distances $d^{(r)}$ ($r \to \infty$), where each $(V_n, d^{(r)})$ can be isometrically embeddable into the sphere $S_{m,r}$ of radius r and apply Theorem 8.

Let $S_{m,r}$ be the sphere of $\mathbb{R}^{m+1}$ of radius r and center $c = (0, \dots, 0, r)$. Lift every $u \in \mathbb{R}^m$ with $\|u\|_2 \leq r$ to a point $u^{(r)} \in S_{m,r}$ by setting $u^{(r)} := \left(u, r - \sqrt{r^2 - (\|u\|_2)^2}\right)$

Let $r \geq \max_{i=1}^{n} \|u_i\|_2$.



Set $d^{(r)}(i,j) = r \cdot \arccos\left(\dfrac{(u_i - c)^T (u_j - c)}{r^2}\right)$

the spherical distance in $S_{m,r}$ between the points $u_i^{(r)}$ and $u_j^{(r)}$

$\lim_{r \to \infty} d^{(r)}(i,j) = \|u_i - u_j\|_2$ because

$$d^{(r)}(i,j) \approx r \arccos\left(1 - \frac{(\|u_i - u_j\|_2)^2}{2r^2}\right) \approx \|u_i - u_j\|_2$$

(iv) projective metrics in the plane

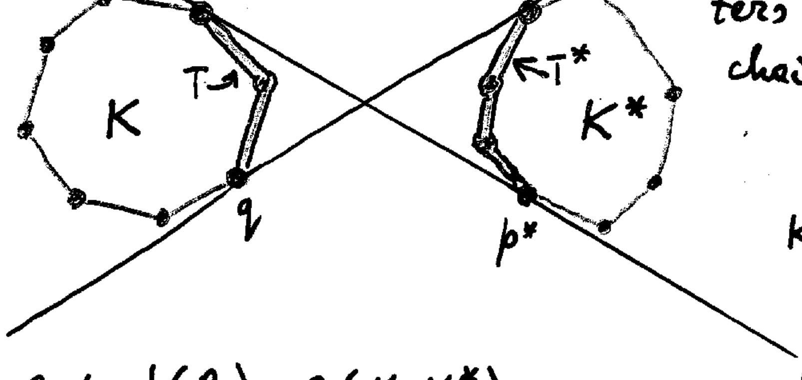
A continuous metric d on $\mathbb{R}^2$ is called a projective metric if it satisfies $d(x,z) = d(x,y) + d(y,z)$ for any collinear points x, y, z lying in that order on a common line

The following theorem proved independently by Alexander'78 and Ambartzumian'77 gives a simple solution to the Hilbert's fourth problem in the plane.

Theorem 10 (Alexander, 1978; Ambartzumian, 1977) Let d be a projective metric on $\mathbb{R}^2$. Then $(\mathbb{R}^2, d)$ is L_1-embeddable, namely there exists a positive Borel measure μ on the lines of $\mathbb{R}^m$ satisfying $2d(x,y) = \mu([[x,y]])$ for $x, y \in \mathbb{R}^2$, where $[[x,y]]$ denotes the set of lines crossing the segment $[x,y]$

The main step in its proof is to define explicitly the measure on lines crossing the segments $[p_i, p_j]$ for any finite set $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$.

$\sigma(K, K^*) = d(p,p^*) + d(q,q^*) - d(T) - d(T^*)$, where $d(T)$, $d(T^*)$ are the perimeters (with respect to d) of the chains T and T^*.



For any line ℓ, denote by K the convex hull of points of P left from ℓ, and by K^* the convex hull of points of P right from ℓ.

Set $d(\ell) = \sigma(K, K^*)$

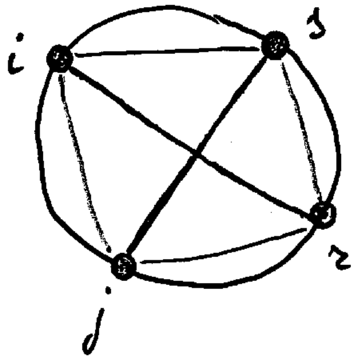
Theorem 11: $\forall p_i, p_j \in P$, $2d(p_i, p_j) = \sum\{d(\ell) : \ell \cap [p_i, p_j] \neq \emptyset\}$.

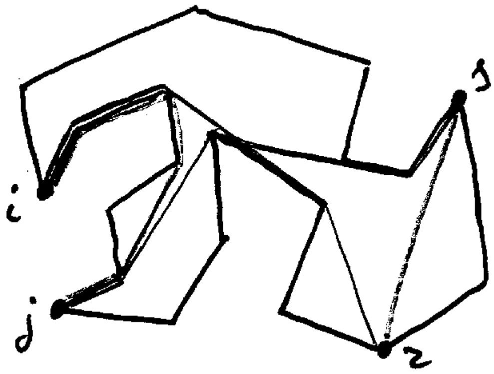
(V) Kalmanson distances

A distance d on V_n is called a Kalmanson distance there exists a (circular) ordering $1,\dots,n$ of the points of V_n such that

$$\max\{d_{ij}+d_{rs},\ d_{is}+d_{jr}\}\leq d_{ir}+d_{js}$$

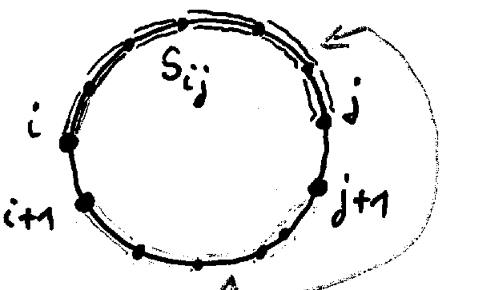
for all $i<j<r<s$ in the circular order





Theorem 12 (Chepoi & Fichet, 1996) A distance d is a Kalmanson distance if and only if it is circular decomposable. In particular, Kalmanson distances are l_1-embeddable.

Idea of the proof:



Circular cut $\{S_{ij},\overline{S_{ij}}\}$

$$\alpha_{ij}=d_{ij+1}+d_{i+1j}-d_{ij}-d_{i+1\,j+1}$$

Notice that $\alpha_{ij}\geq 0$ for any circular order compatible with a Kalmanson distance d, but in general we can have $\alpha_{ij}<0$

Lemma: For any finite metric space (V_n,d) and any circular ordering of V_n, we have

$$2\,d(u,v)=\sum\{\alpha_{ij}:\{S_{ij},\overline{S_{ij}}\}\text{ separates } u \text{ and } v\}\ \forall u,v\in$$

(vi) Totally decomposable metrics

Recall that d is l_1-embeddable iff $d = \sum_{S \subseteq V_n} \lambda_S \delta(S)$.

In general, (i) it is difficult to find $\lambda_S \geq 0 \ \forall S \subseteq V_n$;

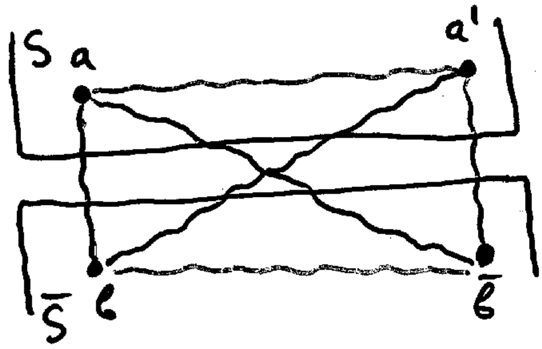
(ii) the l_1-decomposition is not unique;

(iii) there is an exponential number of cuts $(S, \bar{S})$ participating in the decomposition.

Bandelt & Dress (1992) a canonical decomposition of every finite metric into a sum of $O(n^2)$ cut metrics plus a residue.

For a cut $\{S, \bar{S}\}$ of V_n define its isolation index by

$$\alpha_S = \frac{1}{2} \min_{\substack{a,a' \in S \\ b,\bar{b} \in \bar{S}}} \Big\{ \max\{ d(a,b)+d(a',b'), d(a,b')+d(a',b), d(a,a')+d(b,b')\} - d(a,a') - d(b,b') \Big\}$$



$\{S, \bar{S}\}$ is a d-cut if $\alpha_S > 0$

Theorem 13 (Bandelt & Dress, 1992) For any metric d on V_n

$$d = \sum_{\substack{\{S,\bar{S}\} \text{ is} \\ \text{a d-cut}}} \alpha_S \delta(S) + d'$$

prime residue

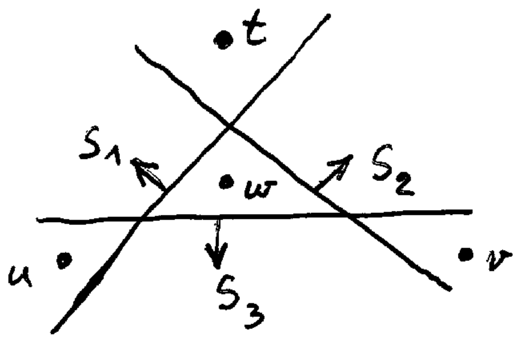
The metric d is called totally decomposable if $d' = 0$, i.e. if $d = \sum_{\{S,\bar{S}\}-\text{d-cut}} \alpha_S \delta(S)$.

Bandelt & Dress (1992) established the following properties of totally decomposable metrics and d-cuts:

(i) there exists only $O(n^2)$ d-cuts (in fact, their (0,1)-vectors are linearly independent);

(ii) the d-cuts can be constructed in polynomial time;

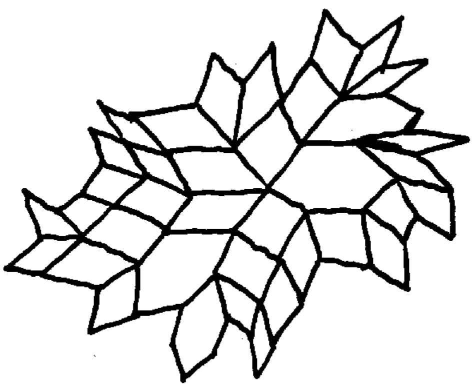
(iii) the d-cut are weakly compatible, i.e. for any three d-cuts $\{S_1, \bar{S}_1\}, \{S_2, \bar{S}_2\}, \{S_3, \bar{S}_3\}$ if $S_1 \cap S_2 \cap S_3 \neq \emptyset$ implies $\bar{S}_1 \cap \bar{S}_2 \cap \bar{S}_3 = \bar{S}_i \cap \bar{S}_j$ for some i, j.

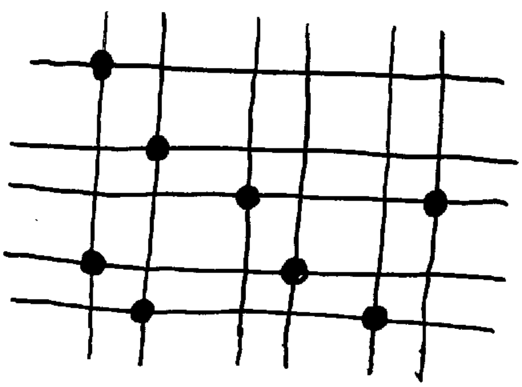


not weakly compatible

<u>Theorem 14</u> (Bandelt & Dress, 1992) A finite metric space (V_n, d) is totally decomposable if and only if every 5-point subspace of V_n is totally decomposable.

Examples of totally decomposable metrics: Kalmanson metrics, tree metrics, finite subspaces of $(\mathbb{R}^2, d_{\ell_1})$ and Cartesian products of two trees, metric of some plane graphs, ...





Theorem 7 (Bandelt & Chepoi, 1995) A metric space (X, d) isometrically embeds into $(\mathbb{R}^2, d_{\ell_1})$ if and only if (Y, d) embeds for any $Y \subseteq X$, $|Y| \leq 6$.

Idea of the proof:

(i) By compactness theorem (Theorem 2) it suffices to consider finite X;

(ii) Every finite isometric subspace of $(\mathbb{R}^2, d_{\ell_1})$ is totally decomposably, so using Theorem 14 one can test if (X,d) is totally decomposable by inspecting all (Y,d) $Y \subseteq X$, $|Y| \leq 5$.

(iii) So (X,d) is totally decomposable, and the d-cuts should define the embedding into $\mathbb{R}^2$. For this, the d-cuts should be represented into two chains:

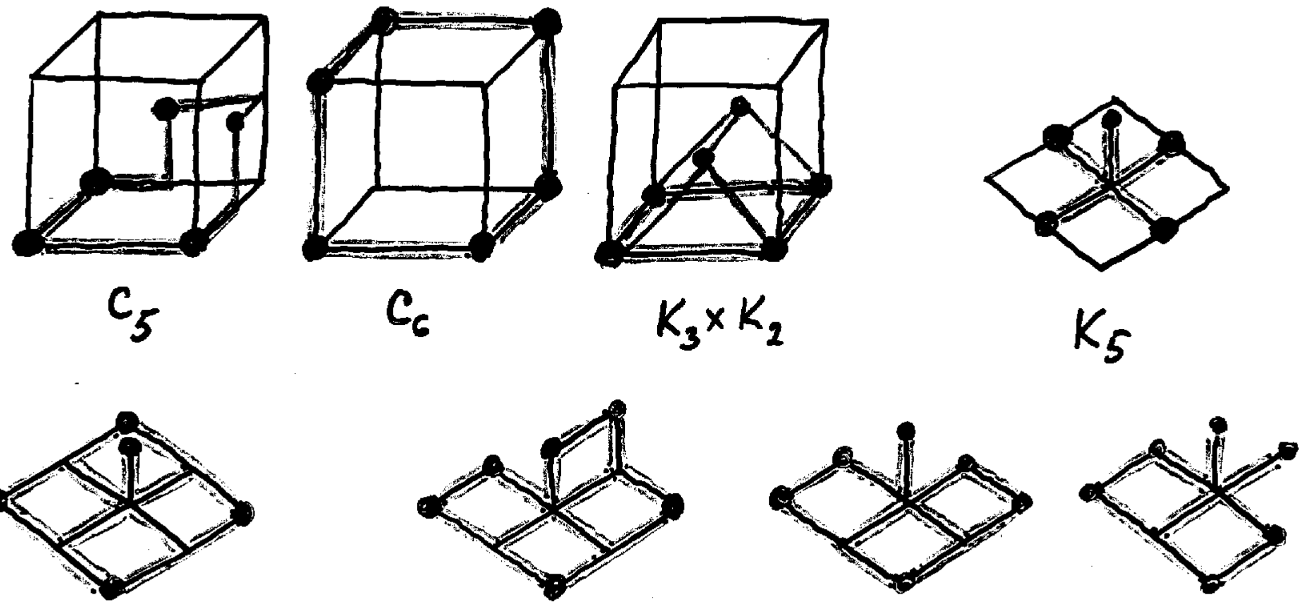
$$S_1' \subseteq S_2' \subseteq \ldots \subseteq S_p', \quad \bar{S}_1' \supseteq \bar{S}_2' \supseteq \ldots \supseteq \bar{S}_p'$$
$$S_1'' \subseteq S_2'' \subseteq \ldots \subseteq S_q'', \quad \bar{S}_1'' \supseteq \bar{S}_2'' \supseteq \ldots \supseteq \bar{S}_q''.$$

To establish this, we show the following theorem

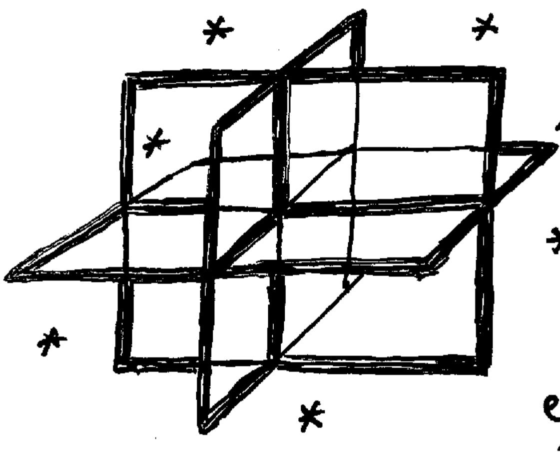
Theorem A totally decomposable space (X,d) embeds in $(\mathbb{R}^2, d_{\ell_1})$ if and only if for any d-cuts $\{S_1, \bar{S}_1\}, \ldots \{S_k, \bar{S}_k\}$ ($k \leq 5$ the ordered set of halfs $S_1, \ldots, S_k, \bar{S}_1, \ldots, \bar{S}_k$ has at most four minimal members (equivalently, the d-cuts can be partitionned into two chains if and only if any subset of at most 5 d-cuts can be partitionned).

(iv) Assume that $\forall Y \subset X$, $|Y| \leq 6$, we have that (Y, d) is embeddable into $(\mathbb{R}^2, d_{e_1})$, however X contains $k \leq$ d-cuts $\{S_1, \bar{S}_1\}, \ldots, \{S_k, \bar{S}_k\}$ that violate the condition of previous Theorem.

We consider the cases $k=3$, $k=4$, $k=5$, and in each case we derive a 6-point subspace of X which is not embeddable into $(\mathbb{R}^2, d_{e_1})$. Those critical subspaces (the minors) are:



For example, if $k=3$, then $S_1, S_2, S_3, \bar{S}_1, \bar{S}_2, \bar{S}_3$ are all minimal by inclusion. Then $S_i \cap S_j \neq \emptyset$, $S_i \cap \bar{S}_j \neq \emptyset$ $\forall i,j$ and we get C_6



If the intersection of every two non-complementary halfspace intersect X, then

* then there is a point of X in every orthant except maybe two opposite orthant