3. Le coureur solitaire

Applications :

- Diophantine approximation (Wills 1967)

- View obstructions (Cusick 1973)
- Distance graphs, Eggleton, Erdős, Skillton (1985)
- Nowhere zero flows (BGGST '98):
- Regular colorings (Zhu 2001)

Conjecture de Wills (1967) et Cusick (1973)

Conjecture du coureur solitaire, Lonely Runner Conjecture(LRC):

k coureurs partent en même temps de START
piste circulaire de longueur 1, vitesses nonzéros, constantes,

=>

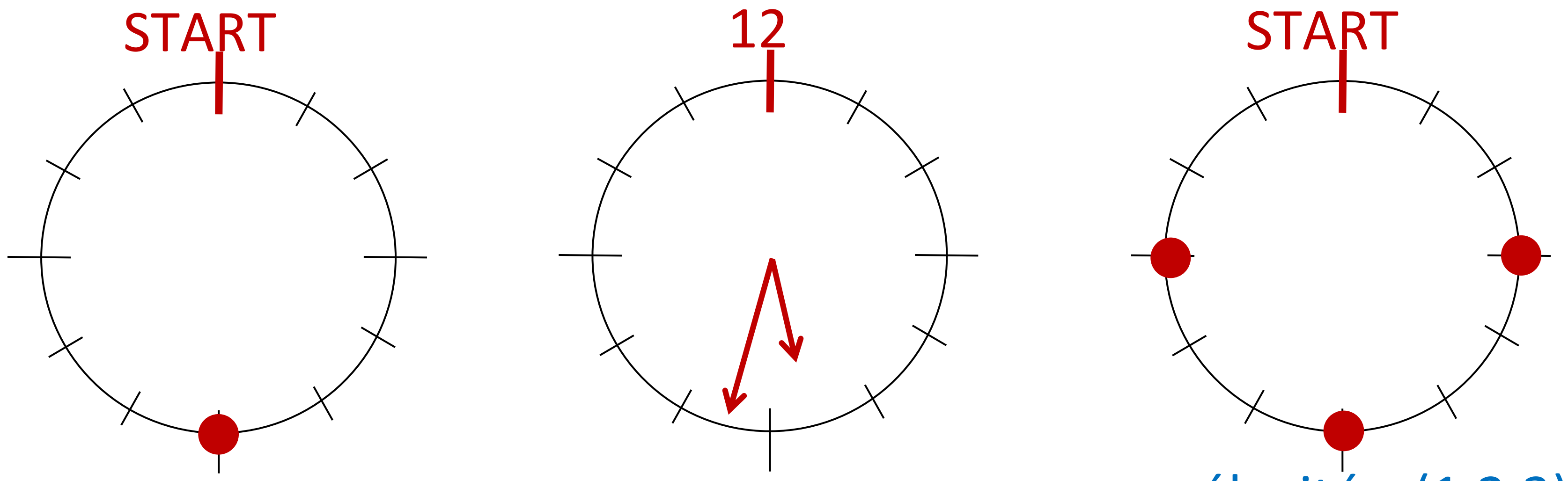
$\exists$ moment quand la distance de tous de START $\geq 1 / (k+1)$

$||x|| := \min\{x - \lfloor x \rfloor , \lceil x \rceil - x\}$, T=1, vélocités entiers

Conjecture: $\forall\, v \in \mathbb{N}^k \;\exists\, t \in (0,1): ||vt|| \geq 1 / (k+1)$

Exemples

Le solitaire c'est START:



k=1

exemple connu k=2 : (1,12)

vélocités (1,2,3) à ¼ - parcours

Examples for which the equality holds:

(1,2, … , k) : the optimum is reached in 1/(k+1)

Not unique: (1,3,4,7) ; (1,3,4,5,9) ; (1,4,5,6,7,11,13)

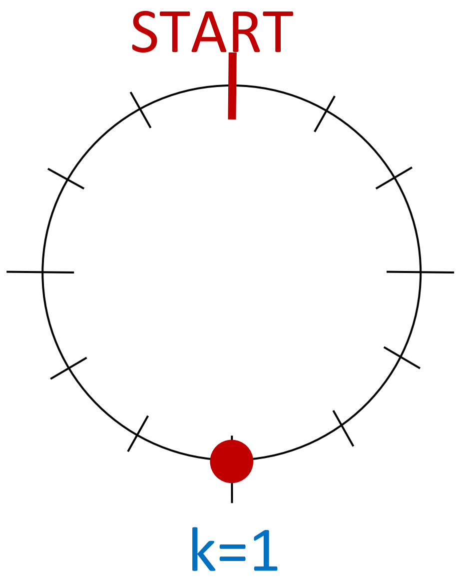
Formulation Solitaire

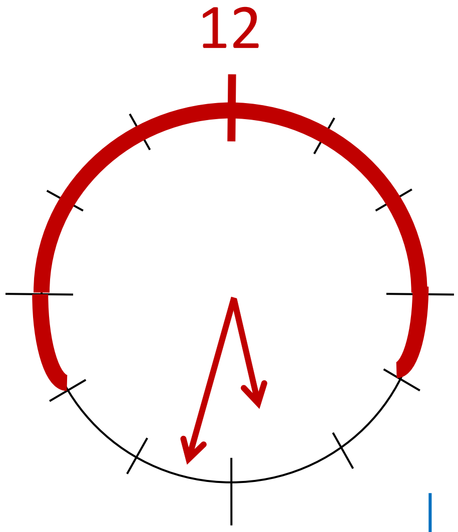
Formulation solitaire: Si $k+1$ coureurs partent en même temps du même endroit avec des vitesses différentes, constantes sur une piste circulaire de longueur 1, alors :

$\forall$ coureur $\exists$ moment quand il est « solitaire », cad:

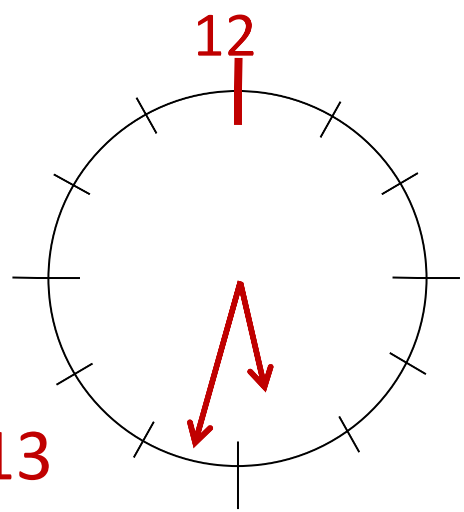
à une distance $\geq 1 / (k+1)$ de son prédécesseur, et de son successeur.

k=1, k=2

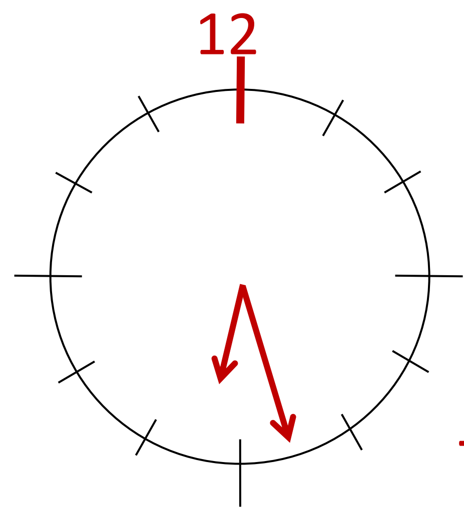




Exercice : k= 2 exact opt = $\frac{\lfloor (v_1+v_2)/2 \rfloor}{v_1+v_2}$



t=6/13

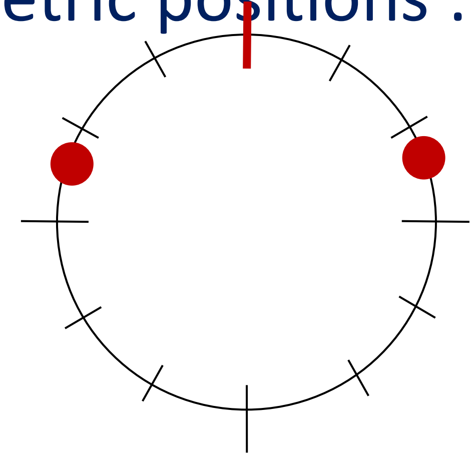


t=7/13

Le problème est discret et rationnel

In the optimum, two runners, i, j, are in symmetric positions :

$v_i t = z - v_j t$, $t = \frac{z}{v_i + v_j}$ $(z=1, \ldots, v_i + v_j - 1)$



Small changes induce only small changes:
sufficient to prove LRC for rationals.

Sufficient to consider multiples of v_i mod

$N := \mathrm{lcm}\left\{ \frac{v_i + v_j}{\gcd(v_i, v_j)} : i \neq j = 1, \ldots, k \right\}$

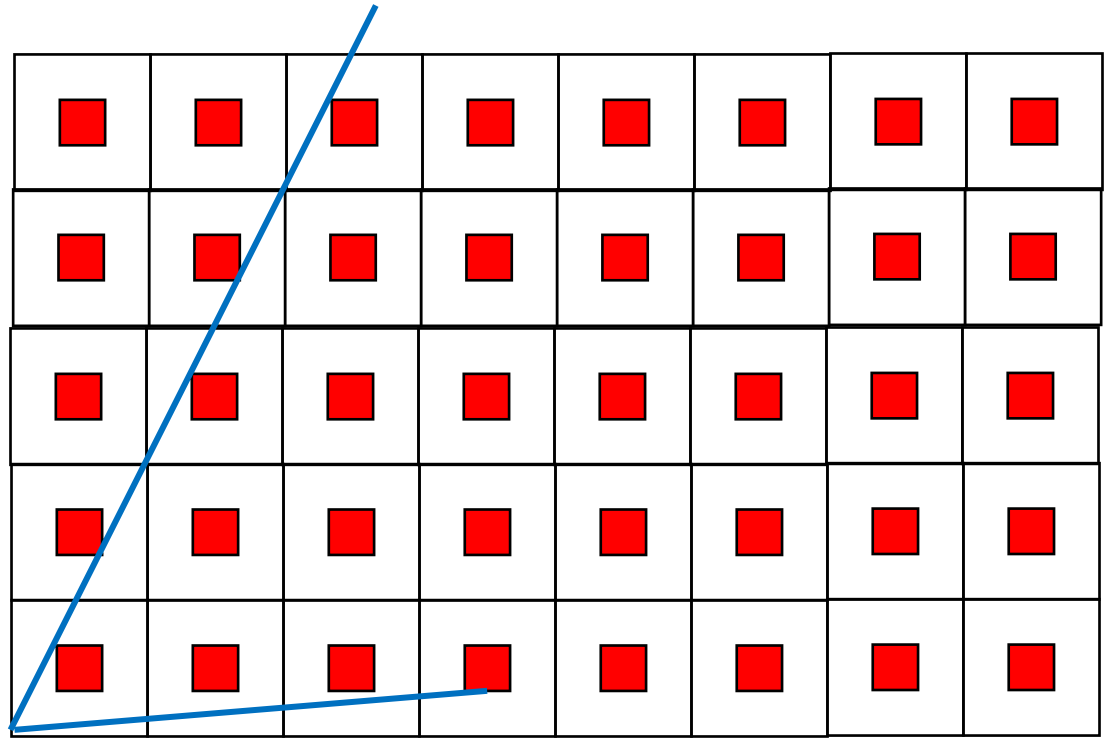
position : $x := vt$

Applications

View obstruction:

(Cusick 1973)

In 3D cubes of size 1/4

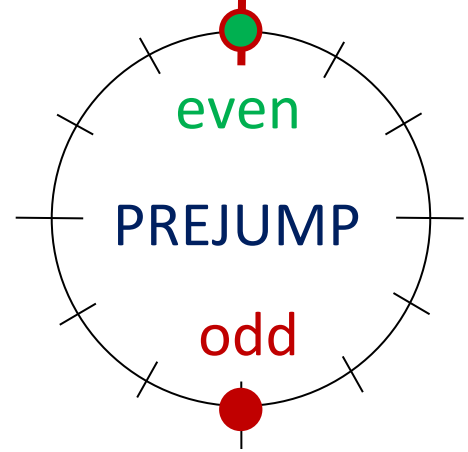


Nowhere zero flows (BGGST ‘98):

Si ∃ NZF avec k valeurs alors aussi avec {1,…, k} …

Distance graphs (Zhu 2002)

Prejumps, k=2

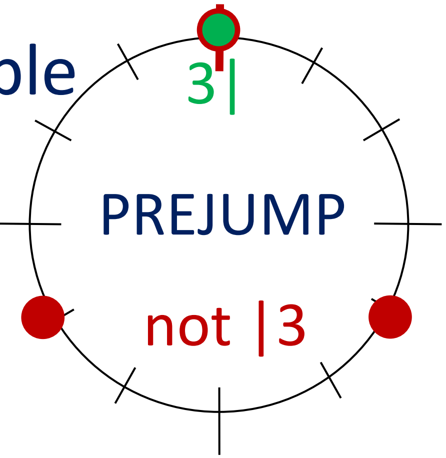


If $x_1 = vt_1$, $x_2 = vt_2$ are positions, then

$x_1 + x_2$ too (defined by $t_1 + t_2$ mod 1)

Examples: Jump ½ **for all odd runners** is possible

Jump 1/3 **for all runners not divisible by 3 -"-**

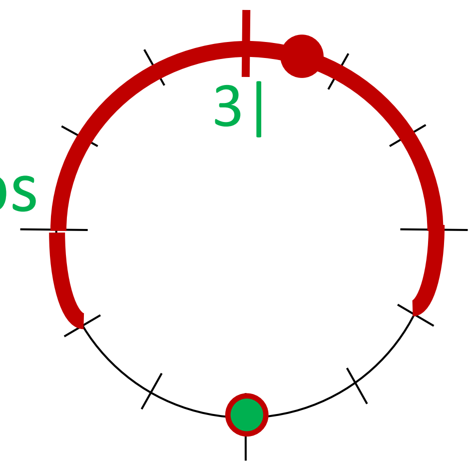


Proof for k=2

0 runner divisible by 3 : 1/3 is good

1 runner divisible by 3 : to ½ , the other jumps

2 runners divisible by 3 : not possible



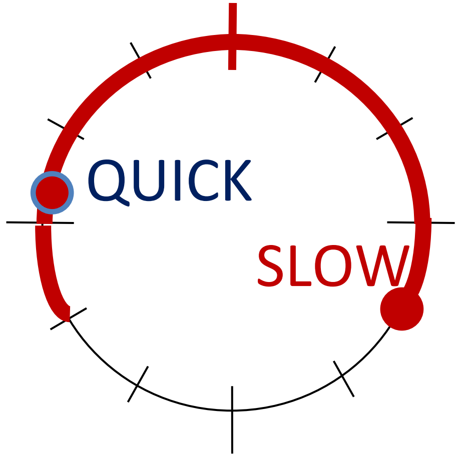
Divisibility or inequalities ?

Another proof for k=2 : « SLOW » and « QUICK »

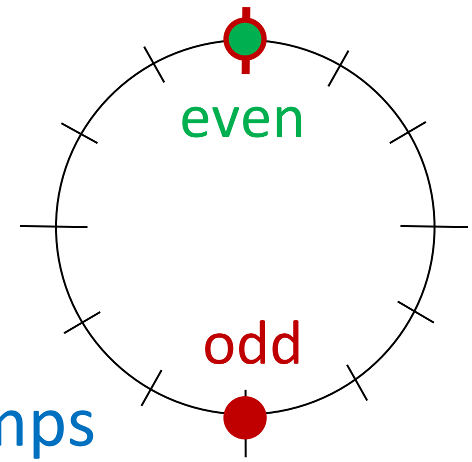
Let SLOW reach 1/3

If not done, QUICK is then not distant. But then its speed at least the double.

The bad region is twice as long as the good one, so QUICK leaves it earlier than SLOW enters



k=3 (originally Betke, Wills, Cusick) with prejumps,



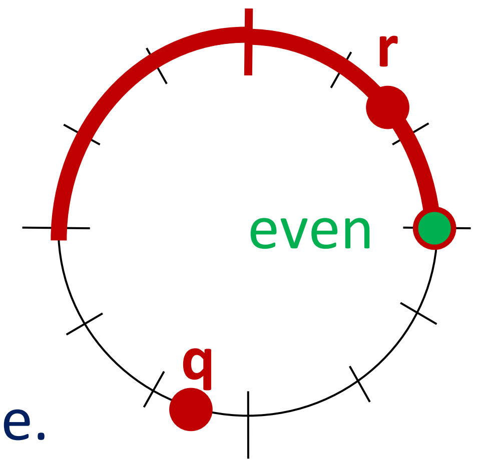
Proof for k=3

0 even runner (all odd) : t=1/2 is good

1 even runner : see below

2 even runners : induction & odd one jumps

3 even runners : not possible



1 even : When this even first arrives to ¼,
if both odd are in the upper half : they jump, so
one, **q**, must be in the lower half => quicker,
another, **r**, in the upper half, otherwise we are done.

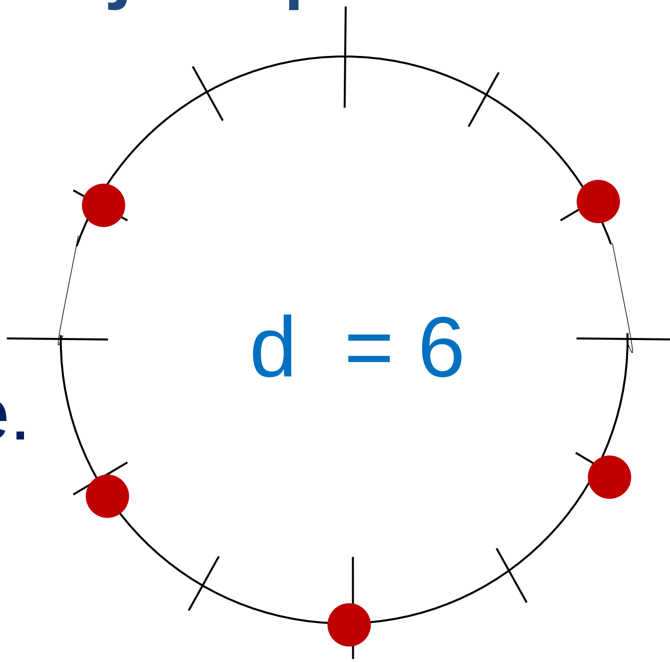
But **q** leaves lower half before even, so jump, or **r** enters before, DONE.

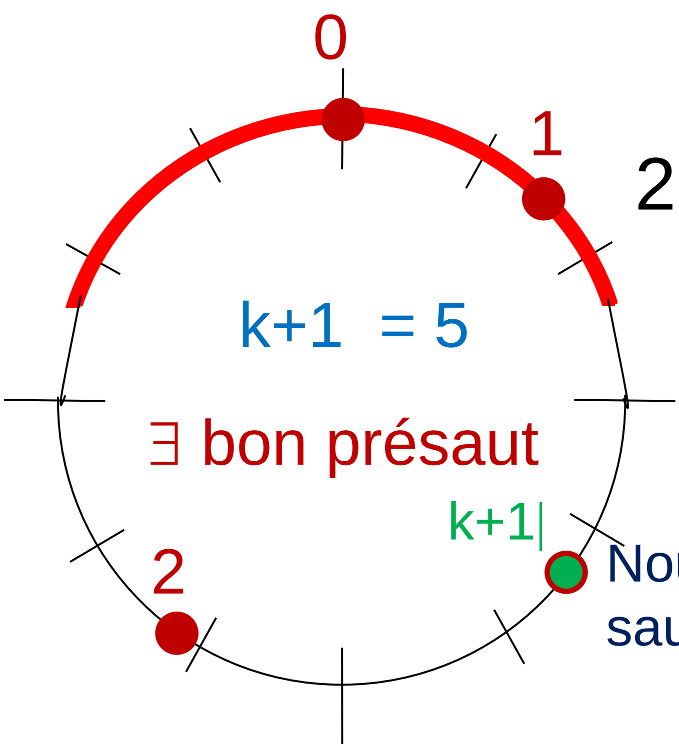
Some more general prejumps

1.) d:=gcd(v_1,...,v_{k-1}) ne divise pas v_k

présauts de v_k: 1/d, 2/d, … , (d-1)/d

∀ k on peut finir LRC par récurrence.





2.) k+1 premier,

un seul divisible par k+1,

présauts par multiples de 1/ (k+1)

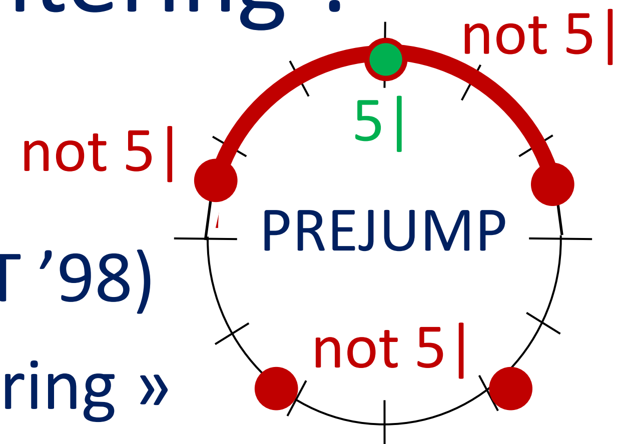
Nous avons indiqué en rouge le nombre de mauvais sauts de chaque coureur, parmi t=q/k+1 (q=1,..k) ,

Prejumps k=4, prime filtering ?

Cusick, Pomerance '84

Bienia, Goddyn, Gwozdjak, S., Tarsi (BGGST '98)

Barajas, Serra (2007) simpler « prime filtering »

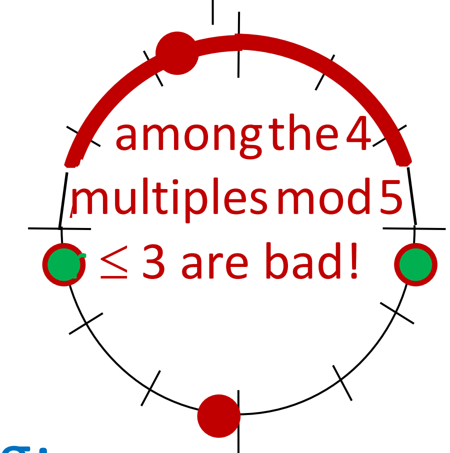


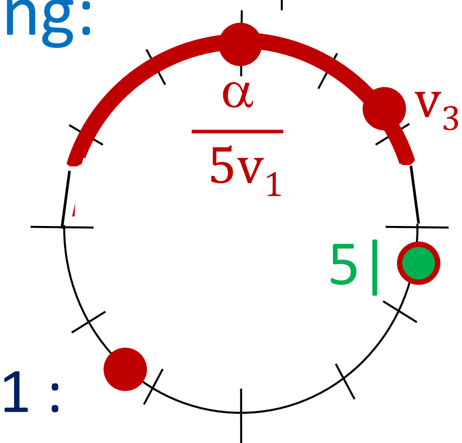
Proof for k=4 (BGGST)

0 runner divisible by 5 : 1/5 is good

1 runner divisible by 5 : see below

2 runners divisible by 5 : induction and counting:





Only one divisible by 5 : If it is the quickest, DONE

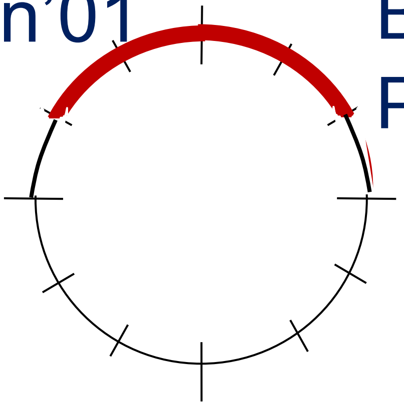
Let the quickest be v_1 , and $v_3 \neq \pm v_2$ mod v_1

If gcd $(v_1,v_3) > 1$ DONE (with some prejumps), and if =1 :

k=4, 5, k=6, prime filtering !

k=5

Bohman, Holzman,Kleitman’01
Renault ‘04



k=6

Barajas, Serra ’07
Prime filtering

Eg. Generalizing (BGGST ‘98) ? : k+1 premier divise une seul,
d   divise au moins la moitié des autres. Alors QED.

Applicable à   k+1 = 5 simplifié ; systématique et plus général :

Pour k > 6 ouvert !

Coureurs et graphes de distance

For $D \subseteq \mathbf{Z}$ (fini) : $G(\mathbf{Z}, D) := (\mathbf{Z}, E)$

E:= paires de sommets à distance D.

Eggleton, Erdős, Skillton (1985) … périodique

$\chi(D) := \chi(G(\mathbf{Z},D))$ $\leq |D| + 1$

Zhu (2001): k+1-coloration régulière de IR avec $\lambda \in \mathrm{IR}$:

couleur de x: $c_{\lambda,k+1}(x) := \lambda x \mod k+1$

Si $\lambda d \geq 1\ \forall d \in D$, then $\lfloor \lambda d \rfloor$ is a coloring of $G(\mathbf{Z},D)$

Cyclic chromatic number and runners

k+1

(circular) k+1 –coloring of graph G= (V,E):

c: V(G) → [0, k+1)

$1 \le |c(u) - c(v)| \le k$

for all uv $\in$ E

$\lfloor c(d) \rfloor$ or $\lceil \lambda d \rceil$

$\lambda \in [1,k]$; The regular coloring

$c_{\lambda,k+1}(x) := \lambda x \mod k+1$

is a circular coloring,

$\Leftrightarrow$

$1 \le \lambda d \le k \mod k+1 \le k$

for all $d \in D$

Dividing by k+ 1 :

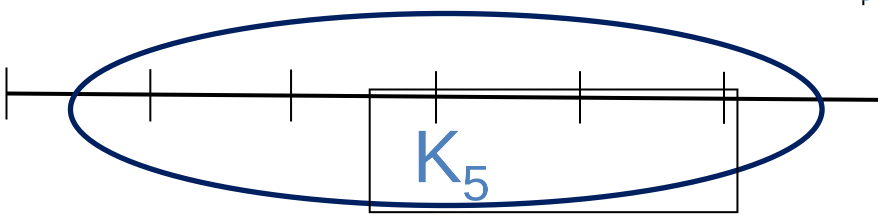
$\Leftrightarrow$ t= λ is the solution of the lonely runner conj for D

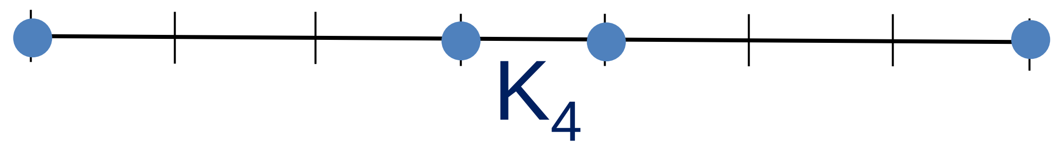
Exemples

5-coloration régulière de G(**Z**, D) :

D: = 1, 2, 3, 4

D: = 1, 3, 4, 7





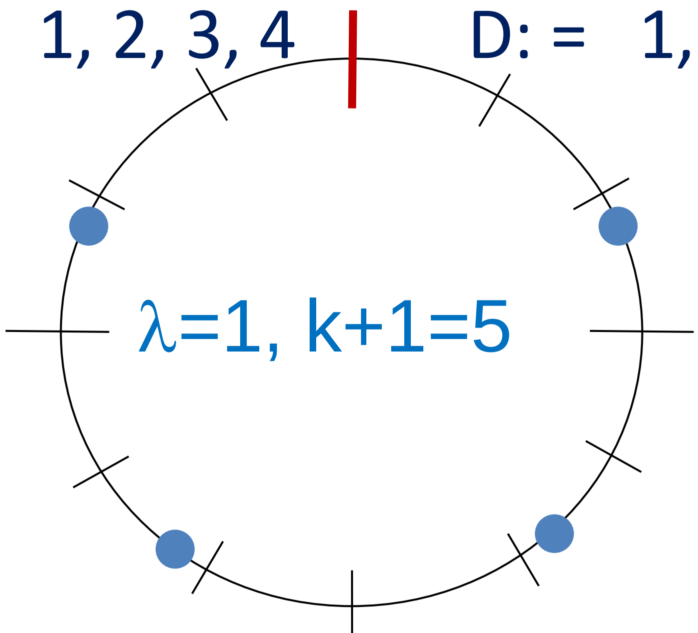
$c_{\lambda,5}$: 5-coloration

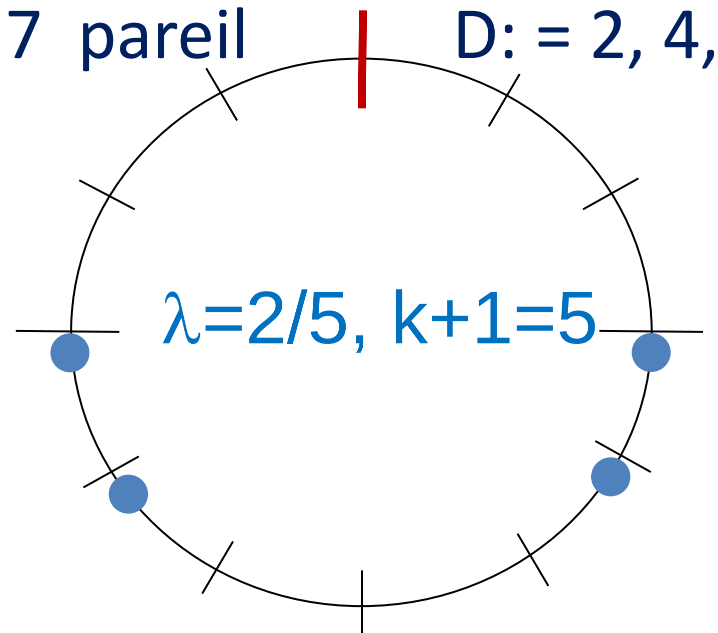
5-coloration régulière de G(**Z**, D) :

D: = 1, 2, 3, 4

D: = 1, 3, 4, 7 pareil

D: = 2, 4, 7, 9





Conclusion: défi des conjectures, nouvel élan des méthodes ? Intérêts des applications

Nombres de Carathéodory entiers pour cônes normaux (Bruns, Gubeladze, apparences combinatoires de TDI, ID, Hb) …

Preuve élémentaire du Théorème de Reid

Conjecture de Wills – filtrage, distances (Zhu, Barajas & Serra) …