Numerical example

1					
	1				
		1			
			1		
				1	
9	5	11	6	5	14

G(V) :

5/14	10/14	1/14
9/14	4/14	13/14
3/14	6/14	9/14
8/14	2/14	10/14
9/14	4/14	13/14
1/14	2/14	3/14

$\{k\lambda_1\}+\{k\lambda_2\}=1$

$k=1,\ldots, 13$

…

If some face of par contains only 0: cyclic group with '+' mod 1

All coeffs nonzero $\Leftrightarrow \lambda_i = d_i/D$, $\gcd(d_i,D)=1$ $(i=1,\ldots,n)$

13 multiples but we have only log 14 time !

n=3

Theorem:(S.'90)For any Hilbert basis H $\subseteq$ IR^3 ,cone (H) pointed: cone (H) is 'partitioned' by simplicial Hilbert cones

Proof: Let b $\in$ cone (H) and maximize the sum of variables for

$$Hx = b\ ,\ x \geq 0\ .$$

The simplex of the basic optimum is **empty** :

Lemma : $v_1, v_2, v_3 \in \mathbf{Z}^3$, for all $(\lambda_1, \lambda_2, \lambda_3) \in G(v_1, v_2, v_3) \setminus \{0\}$,

$$\det + 1 \leq (\lambda_1 + \lambda_2 + \lambda_3) \det \leq 2 \det - 1$$

This determines a particular parallelepiped structure, which allows to finish the proof , see « empty simplices » (nontrivial)

Carathéodory entier

$(7/6)n \leq$ Caratheodory$(n) \leq 2(n-1)$ **Reserrer !**

Test en dimension fixe (Cook, Lovász, Schrijver '84)
NP-complet en général (Júlia Pap 2008

Borne supérieure (Cook, Fonlupt, Schrijver '86, S. '90)
Base unimodulaire (Gerards, S. '87)
n=3 : partition unimodulaire (S. '90)
Appl: - à des nœuds (Hass, Lagarias, Pîppenger 1999)
- Rings and K-theory, toric varieties (Bruns, Gubeladze 2009)

n=4 : contre-exemple à la partition unimodulaire (Bouvier, Gr '94)
n=6 : contre-exemple à n (Bruns, Gubeladze '98)

Thm: (Gijswijt 2010) True for matroid bases.

Problem: For rooted arborescences ? Other comb objects ?

Couverture unimodulaire pour n= 3, 4, 5 ?

2. Empty Simplices

Applications :

- Programmation en nombres entiers (PLE)
- Preuves de Carathéodory pour n=3 (cf. 1.)

Définition

INPUT : $v_1,\ldots,v_n \in Z^n$ linéairement indép.
QUESTION : $\exists$ point entier à part les sommets dans le simplexe **conv(0, $v_1,\ldots,v_n$)** ?

NON : *vide*

CNS (bonne caract, NP$\cap$coNP) qui certifie quand il **est *vide*** ?
Ou **coNP-complet** ?

En dimension fix ILP donc $\in$P ;
Un thm general pourrait etre profond et utile

d=3

Cas spéciaux résolus (S. IPCO '99): sac à dos, n=3:

conv(0, v_1,v_2,v_3)

Thm vide $\Leftrightarrow$ det=1, ou $\exists$ $v=\lambda_1 v_1+\lambda_2 v_2+\lambda_3 v_3 \in \mathbf{Z}^3$, $0<(\lambda_1, \lambda_2, \lambda_3)<1$, tq (après permutation possible)

$\lambda_1+\lambda_2=1$, $\lambda_i=a_i/\det$, gcd(a_i,det)=1, i=1,2,3. ($\Leftrightarrow \forall$)

($\Leftrightarrow$ thms géométriques of White, Reeves, Scarf, Reznik.)

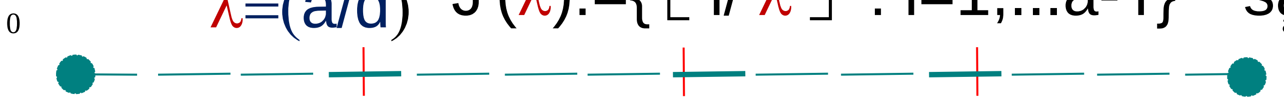
Idée de preuve: parallelepipèdes et sauts

$\mathrm{par}(v_1,\ldots,v_n) := \{\lambda_1 v_1+\ldots+\lambda_n v_n \in \mathbb{Z}^n : \quad 0 \le (\lambda_1,\ldots,\lambda_n) < 1\}$

$G(v_1,\ldots,v_n) := \{\, 0 \le (\lambda_1,\ldots,\lambda_n) < 1 \quad : \lambda_1 v_1+\ldots+\lambda_n v_n \in \mathbb{Z}^n\}$

d:=| par | = det ; groupe avec '+ mod 1' ; denominators=det

$\lambda=(a/d)$ $J(\lambda) := \{\lfloor i/\lambda \rfloor : i=1,\ldots a-1\}$ 'saut'



Quand 'saute' $\{k\lambda\}$ 'en arrière' ? : $\{(k+1)\lambda\} < \{k\lambda\}$

Quand $k = \lfloor i/\lambda \rfloor$ $i=1, \ldots, d-2$!

λ saute $\Leftrightarrow$ $1-\lambda$ ne saute pas (Exercice du Résumé)

Triv: $a|b$ ou $d-b|d-a \Rightarrow J(a/d) \subseteq J(d)$ Pour simplex vide:

Lemma : Si $a,b \le d/2$, $J(a/d) \subseteq J(b/d) \Leftrightarrow a|b$

Theorem de Reid

Conjecture (S. 1998) : $\{kx_1\}+ \ldots+\{kx_n\} \geq n/2$

(or 0) $\forall$ k $\Leftrightarrow$ $x_1 < \ldots < x_n$

ont des dénominateurs réduits égaux et

$\{x_i\} + \{x_{n-i}\} = 1 \ \forall$ i.

Théorème : Vrai pour $n \leq 4$.

Exemple: 4/15, 11/15, 2/15, 13/15,

Théorème (Miles Reid 1987): " "

Young Persons Guide To Canonical Singularities

Reid's Young Person's guide

Proceedings of Symposia in Pure Mathematics
Volume 46 (1987)

Young Person's Guide to Canonical Singularities

MILES REID

In memory of Oscar Zariski

This article aims to do three things: (I) to give a tutorial introduction to canonical varieties and singularities, with some of the motivating examples; (II) to provide a skeleton key to the results of my two papers on canonical singularities [**C3-f**], [**Pagoda**], and those of [**Morrison-Stevens**] and [**Mori**, Terminal singularities]; and (III) to explain the recent "exact plurigenus formula." The expository intention is reflected in explanations of some well-known standard technical points (well-known to experts but maybe not to the algebraic geometer in the street), and also worked examples, exercises, and deliberate mistakes to entertain the reader; I apologise if any secrets of the priesthood are divulged despite my best efforts.

After §4, most of the material is new: §§5, 6 and 7 contain the material of [**Morrison-Stevens**] and [**Mori**] in substantially laundered form, and the results on equivariant RR and the plurigenus formula of Chapter III appear here for the first time. The juxtaposition of these two topics reveals quite amazing relations between the cyclotomic sums appearing traditionally in connection with equivariant RR and toric geometry; of course, these are linked in a primary way by the fact that quotient singularities make contributions (for example, to $H^0(\mathbf{P}, \mathcal{O}(k))$ for a weighted projective space $\mathbf{P}$) which can be computed either by equivariant RR or as the number of lattice points of a polyhedron. However, it was something of a shock to discover how intricately cyclotomy relates to the combinatorics of the Newton polyhedron at the heart of the classification of terminal singularities.

My contribution to the subject has mainly been concerned with the study of 3-fold singularities. It should be noted that most of the recent work on varieties of dimension ≥ 4 (in particular the two circles of ideas, cone, contraction, nonvanishing theorems of Kawamata and Shokurov, and positivity of $f_*\omega, C^+_{n,m}$ of Fujita, Viehweg, Kawamata and Kollár) uses only the definitions of canonical and terminal singularities (and their log generalisations), together with general properties such as rationality and behaviour in codimension 2, but does not

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14E30; Secondary 14B05.

PROOF. The points $P_k = \frac{1}{r}(\overline{ak}, \overline{bk}, \overline{ck})$ are just the points of $N \cap \square$. The two sides are congruent mod r, and equality holds if and only if the left-hand side is in the interval $(r, 2r)$.

(5.4) The lemma reduces Theorem 5.2 to the case $n = 3$, $m = 1$ of the following more general result.

MANUEL

THEOREM (TERMINAL LEMMA). *Let n and m be integers with $n \equiv m$ mod 2, and suppose that $\frac{1}{r}(a_1, \dots, a_n; b_1, \dots, b_m)$ is an $(n+m)$-tuple of rational numbers with denominator r.*

(A) *Suppose each a_i and b_j is coprime to r; then the following two conditions are equivalent:*

(i) $$\sum_{i=1}^{n} \overline{a_i k} = \sum_{j=1}^{m} \overline{b_j k} + \frac{n-m}{2} \cdot r \quad \textit{for } k = 1, \dots, r-1.$$

(ii) *The $n+m$ elements $\{a_i, -b_j\}$ can be split up into $(n+m)/2$ disjoint pairs of the form $(a_i, a_{i'})$ or $(b_j, b_{j'})$ or $(a_i, -b_j)$ which add to 0 mod r. (That is, each a_i is either paired with another $a_{i'}$ such that $a_{i'} \equiv -a_i$ mod r, or with one of the b_j such that $b_j \equiv a_i$ mod r, and similarly for the b_j's.)*

(B) *More generally (without the coprime condition), (i) is equivalent to (ii) plus the following condition:*

(iii) *For every divisor q of r,*

$$\#\{\text{pairs}(a_i, a_{i'}) \mid q = hcf(a_i, r)\} = \#\{\text{pairs}(b_j, b_{j'}) \mid q = hcf(b_j, r)\}.$$

Note that (ii) $\Rightarrow$ (i) is trivial, since

$$\overline{ak} + \overline{(r-a)k} = \begin{cases} r & \text{if } \overline{ak} \neq 0, \\ 0 & \text{if } \overline{ak} = 0; \end{cases}$$

the implication (i) $\Rightarrow$ (iii) in (B) is similar and I leave it as an easy exercise.

(5.5) REMARK. For (5.2) I only need the case $n = 3$, $m = 1$ with a_i and b_j coprime; the case $n = 4$, $m = 2$ will be used in §6 in connection with terminal hyperquotient singularities in the form of Corollary 5.6. The tuple might more generally correspond to an action of μ_r on a complete intersection singularity $Q \in Y \subset \mathbf{A}^n$, for example with $\frac{1}{r}(a_1, \dots, a_n)$ specifying the type of the action on the coordinates, $\frac{1}{r}(b_1, \dots, b_{m-1})$ that on the defining equations, and $\frac{1}{r}(b_m)$ corresponding to a choice of generator of the class group of the singularity $P \in X = Y/\mu_r$ (a "polarisation" of the singularity).

(5.6) COROLLARY. *Let $\frac{1}{r}(a_1, \dots, a_4; e, 1)$ be a 6-tuple of rational numbers with denominator r such that*

$$q = \text{hcf}(e, r) = \text{hcf}(a_4, r) \quad \textit{and} \quad a_1, a_2, a_3 \textit{ are coprime to } r;$$

assume that

$$\sum_{i=1}^{4} \overline{a_i k} = \overline{ek} + k + r \quad \textit{for } k = 1, \dots, r-1.$$

3. Le coureur solitaire

Applications :

- Diophantine approximation (Wills 1967)

- View obstructions (Cusick 1973)
- Distance graphs, Eggleton, Erdős, Skillton (1985)
- Nowhere zero flows (BGGST '98):
- Regular colorings (Zhu 2001)

Conjecture de Wills (1967) et Cusick (1973)

Conjecture du coureur solitaire, Lonely Runner Conjecture(LRC):

k coureurs partent en même temps de START
piste circulaire de longueur 1, vitesses nonzéros, constantes,

=>

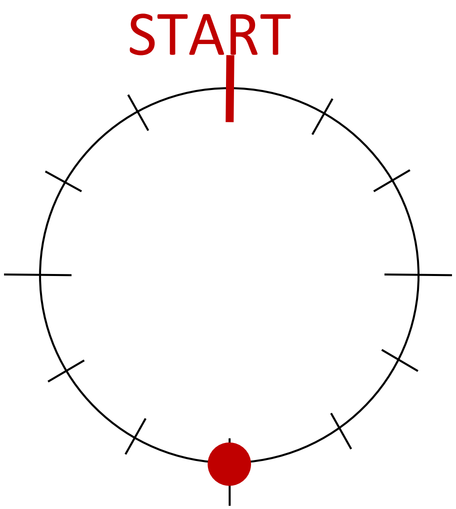
$\exists$ moment quand la distance de tous de START $\geq 1 / (k+1)$

$||x|| := \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}$, T=1, vélocités entiers

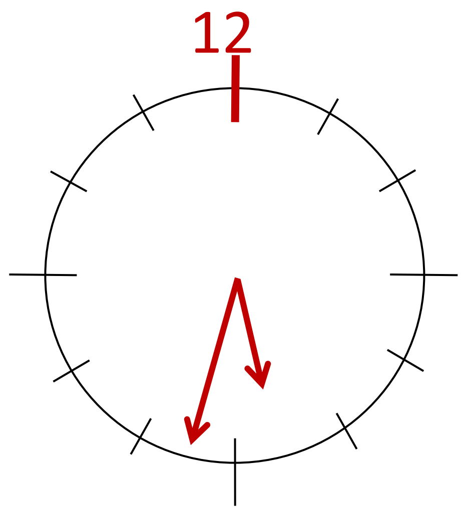
Conjecture: $\forall\, v \in \mathbb{N}^k \;\exists\, t \in (0,1): ||vt|| \geq 1 / (k+1)$

Exemples

Le solitaire c'est START:

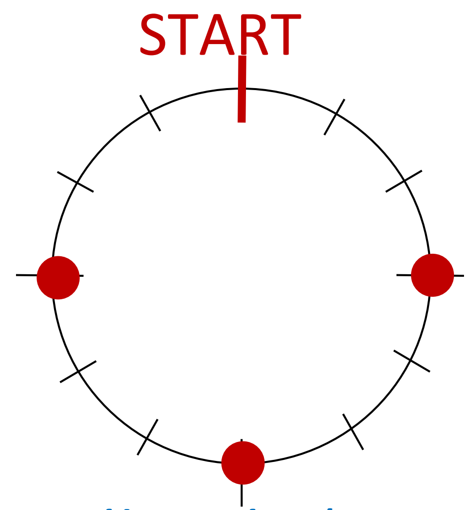


k=1



exemple connu k=2 :

(1,12)



vélocités (1,2,3)

à ¼ - parcours

Examples for which the equality holds:

(1,2, … , k) : the optimum is reached in 1/(k+1)

Not unique: (1,3,4,7) ; (1,3,4,5,9) ; (1,4,5,6,7,11,13)

Formulation Solitaire

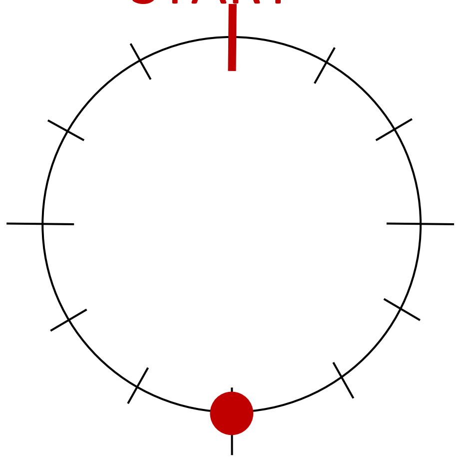
Formulation solitaire: Si k+1 coureurs partent en même temps du même endroit avec des vitesses différentes, constantes sur une piste circulaire de longueur 1, alors :

$\forall$ coureur $\exists$ moment quand il est « solitaire », cad:

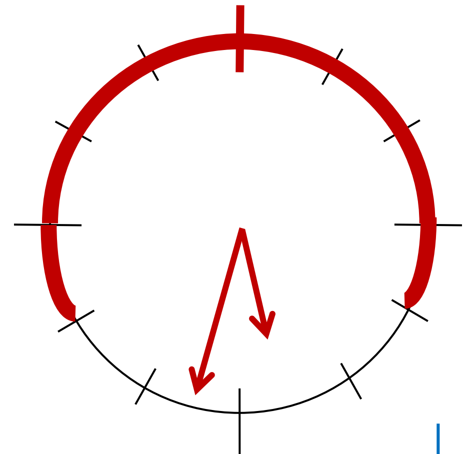
à une distance $\geq 1 / (k+1)$ de son prédécesseur, et de son successeur.

k=1, k=2

START



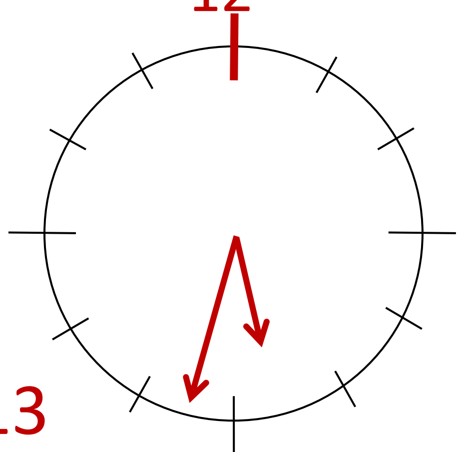
k=1



12

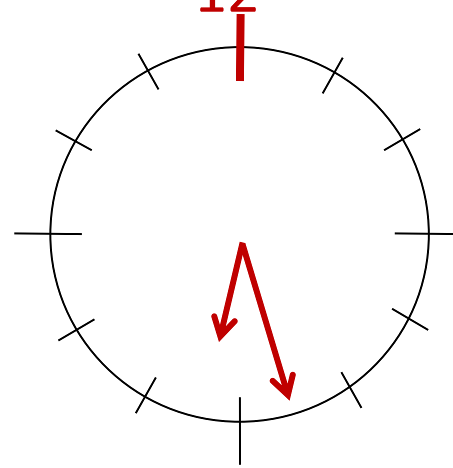
Exercice : k= 2 exact opt = $\frac{\lfloor (v_1+v_2)/2 \rfloor}{v_1+v_2}$

12



t=6/13

12

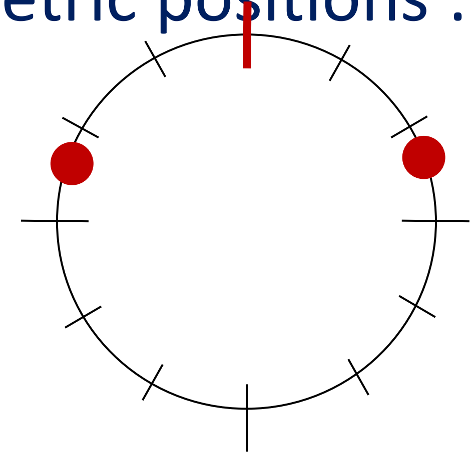


t=7/13

Le problème est discret et rationnel

In the optimum, two runners, i, j, are in symmetric positions :

$v_i t = z - v_j t$, $t = \frac{z}{v_i + v_j}$ $(z=1, \dots, v_i + v_j - 1)$



Small changes induce only small changes:
sufficient to prove LRC for rationals.

Sufficient to consider multiples of v_i mod

$N := \mathrm{lcm} \left\{ \frac{v_i + v_j}{\gcd(v_i, v_j)} : i \neq j = 1, \dots, k \right\}$

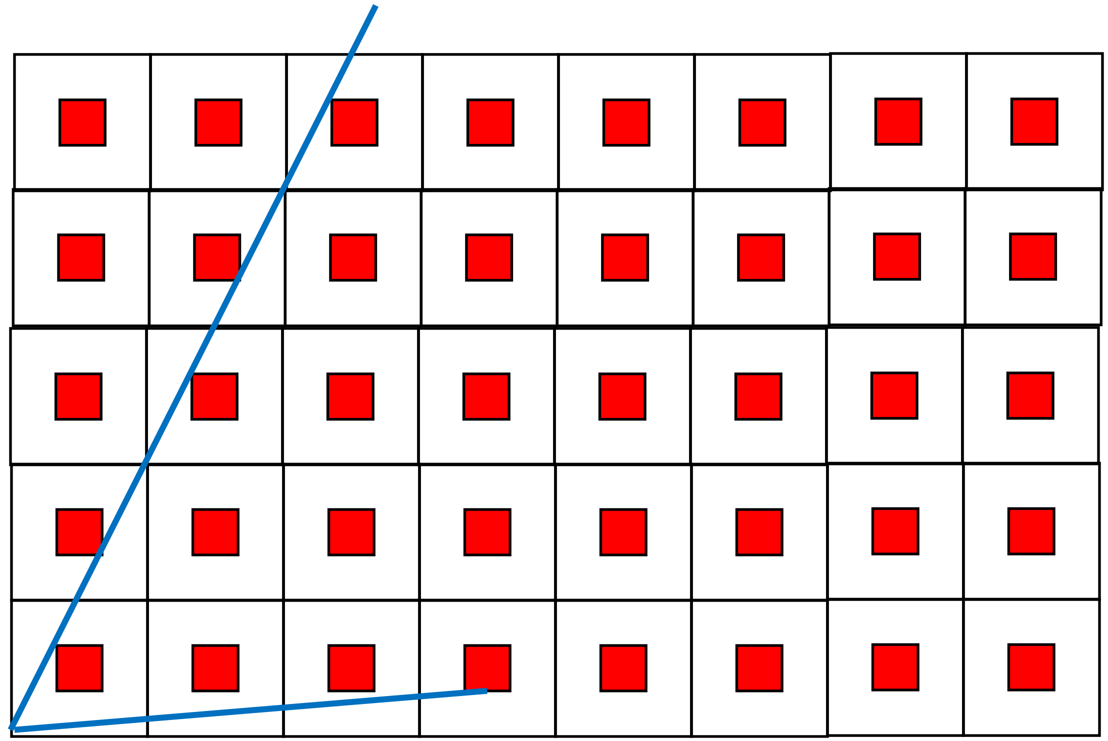
position : $x := vt$

Applications

View obstruction:

(Cusick 1973)

In 3D cubes of size 1/4



Nowhere zero flows (BGGST '98):

Si ∃ NZF avec k valeurs alors aussi avec {1,..., k} ...

Distance graphs (Zhu 2002)