#3 Graph classes defined by distance properties

Purpose: introduce and characterize main graph classes interesting from the metric point of view and related to l_1, l_2, l_∞-metrics and Hamming distance

Classes of graphs:

(i) median graphs;

(ii) Helly graphs;

(iii) bridged graphs;

(iv) weakly median graphs;

(v) isometric subgraphs of hypercubes and Hamming graphs

(vi) l_1-graphs;

(vii) superconnected (lopsided) set systems and graphs;

(viii) basis graphs of matroids and Δ-matroids.

Main generalizations of l_2, l_∞, and l_1:

(1) $l_2 \longrightarrow$ CAT(0) metric spaces;

(2) $l_\infty \longrightarrow$ hyperconvex (injective) metric spaces;

(3) $l_1 \longrightarrow$ median metric spaces.

Bridged graphs can be viewed as discrete analogous of CAT(0) spaces

Helly graphs are discrete analogous of hyperconvex!

Graph classes defined by distance properties

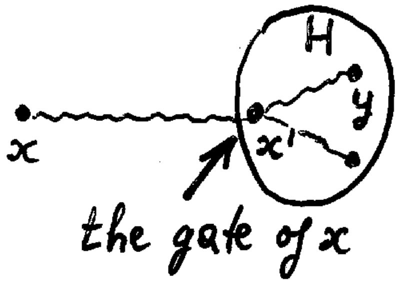
$G=(V,E)$ - connected not necessarily finite, undirected and unweighted graph endowed with the standard graph distance $d(u,v):=d_G(u,v)$

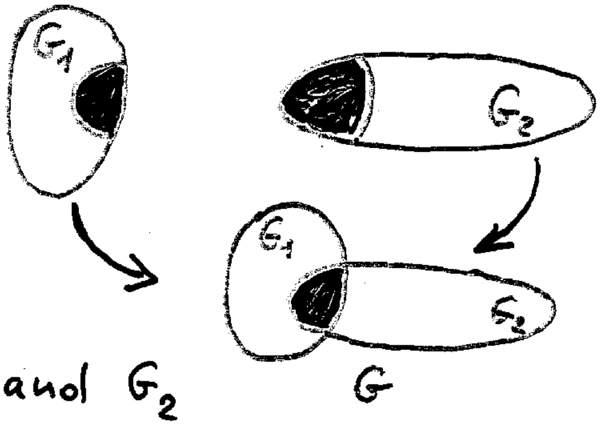
Interval $I(u,v)=\{x\in V: d(u,v)=d(u,x)+d(x,v)\}$

Convex set $S\subseteq V$: $I(u,v)\subseteq S \quad \forall\, u,v\in S$

Halfspace $H\subseteq V$: convex set with a convex complement $V-H$

Gated subgraph (set) H: $\forall x\notin H\ \exists x'\in H: x'\in I(x,y)\ \forall y\in H$



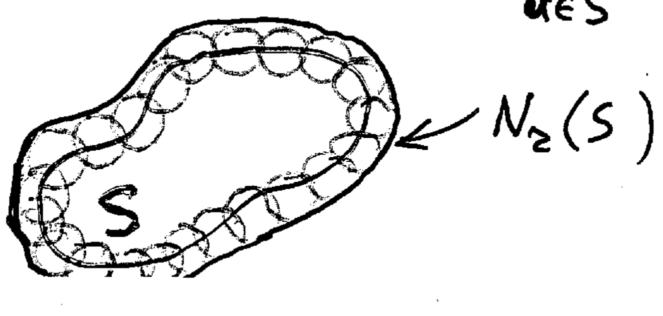


Gated amalgam G of G_1 and G_2

Ball (or r-neighborhood) $B(u,r)=N_r(u)=\{x\in V: d(u,x)\le r\}$

r-neighborhood of a set S:

$$N_r(S)=\{x\in V: d(x,S)\le r\}=\bigcup_{u\in S} N_r(u)$$



Definitions (cont.)

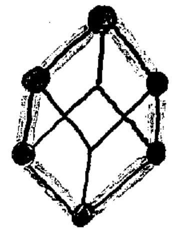
isometric subgraph: an induced subgraph $H=(Y,F)$ of a graph $G=(X,E)$ such that $d_H(u,v)=d_G(u,v)\ \forall u,v\in Y$

isometric embedding $\varphi: H \to G$: $\forall u,v\in Y$, $d_G(\varphi(u),\varphi(v)) = d_H(u,v)$

scale k embedding $\varphi: H\to G$: $\forall u,v\in Y$, $d_G(\varphi(u),\varphi(v)) = k\, d_H(u,v)$

retract: a subgraph $H=(Y,F)$ of $G=(X,E)$ such that there exists an idempotent nonexpansive mapping ψ from G to H, i.e $\psi(y)=y\ \forall y\in Y$ and $d_G(\psi(x),\psi(y))\le d_G(x,y)\ \forall x,y\in X$

Remark: retracts are isometric subgraphs of the host graph, but not the converse:
the 6-cycle C_6 is an isometric subgraph but not a retract of the 3-cube Q_3



Remark: in previous definitions, the host graph $G=(X,E)$ can be replaced by an arbitrary metric space (X,d)

Main host spaces for isometric embedding of graphs
geometric: l_1- and l_∞-spaces
graphic: hypercubes, Hamming graphs, half-cubes, Johnson graphs, l_1- and l_∞-grids

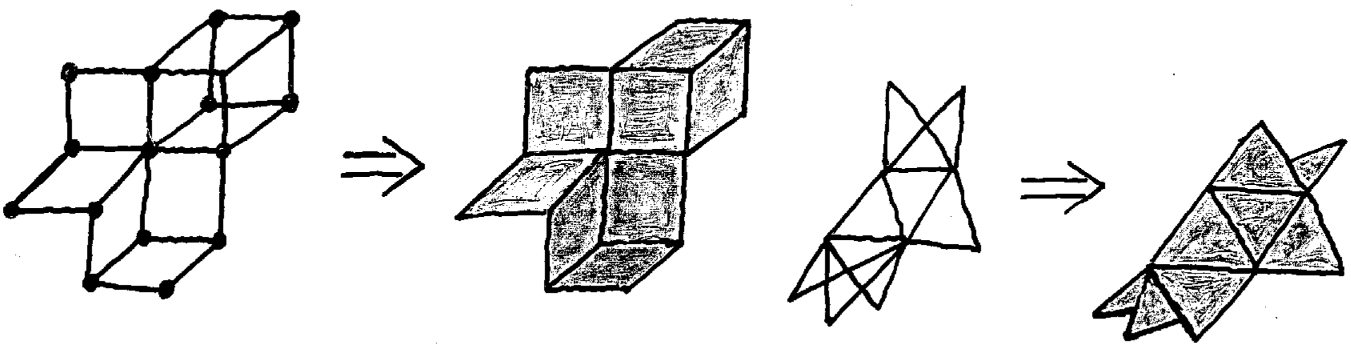
Definitions (cont.)

How to derive cell complexes from graphs?

(a) cubical complexes: replace every graphic cube by a unit solid cube;

(b) simplicial complexes: replace every clique (complete subgraph) by a simplex;

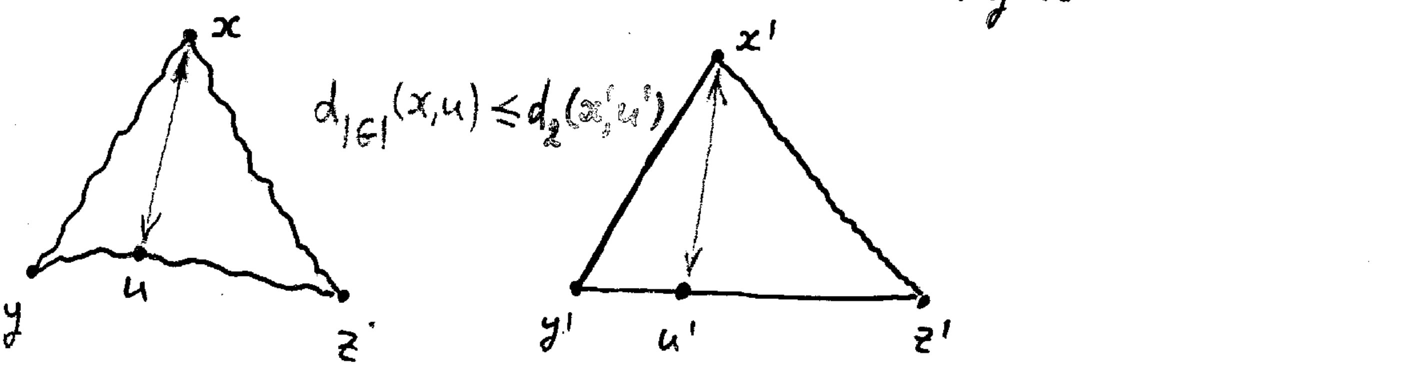
(c) cell complexes from planar graphs: replace every interior face by a regular polygon with unit side.



$G \longrightarrow |G|$ $\qquad$ $G \longrightarrow |G|$

Remark: $|G|$ can be endowed with an intrinsic l_1-, l_2-, or l_∞-metric.

CAT(0) complexes: geodesic triangles in $|G|$ are thinner than the comparison euclidean triangle



CAT(0) complexes have many nice properties, some of which characterize them:

(i) any two points can be joined by a unique geodesic (shortest path);

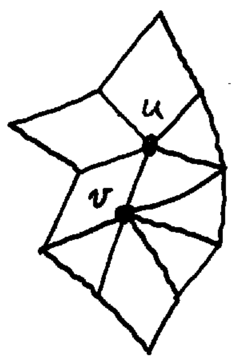
(ii) r-neighborhoods of convex sets are convex;

(iii) do not contain isometrically embedded cycles;

(iv) if α and β are geodesics in $|G|$, then the function $f:[0,1] \to |G|$ given by $f(t) = d(\alpha(t), \beta(t))$ is convex;

(v) global nonpositive curvature.

For our case (c), the condition (v) can be read as: the sum of angles around any interior vertex is at least 2π.



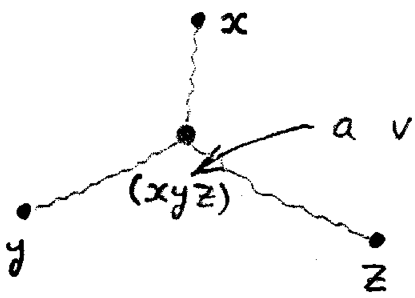
$$\sum(u) = 3 \times \frac{\pi}{2} + 2 \times \frac{\pi}{3} > 2\pi$$

$$\sum(v) = 2 \times \frac{\pi}{2} + 4 \times \frac{\pi}{3} > 2\pi$$

For more details on CAT(0) spaces see the book by Bridson and Haefliger.

Median graphs

median

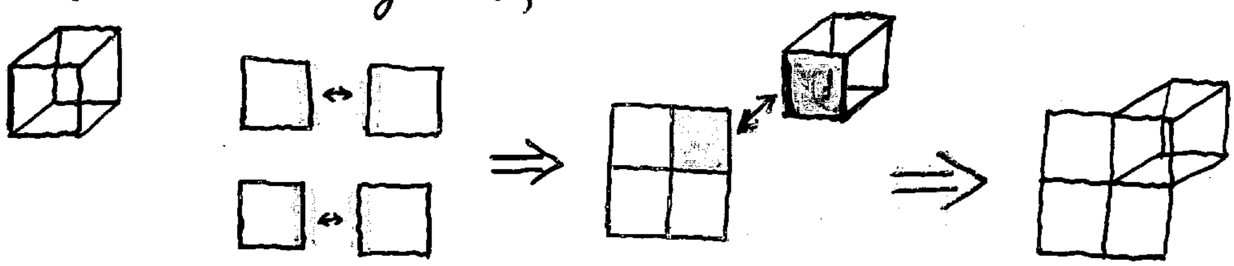


a vertex in $I(x,y) \cap I(y,z) \cap I(z,x)$

median graphs: graphs in which every triplet x, y, z has a unique median denoted (xyz)

Characterizations of median graphs:

(i) (Isbell) Median graphs are precisely the graphs which are obtained from cubes via successive gated amalgams;



resulting median graph

(ii) (Bandelt) Median graphs are precisely the retracts of hypercubes;

(iii) (Schaefer) Median graphs are precisely the connected components of solutions of 2SAT instances;

Median graphs (cont.)

(iv) (Avann) The median operator of a median graph satisfies the following equations:

(1) $(aab) = a$ (majority)

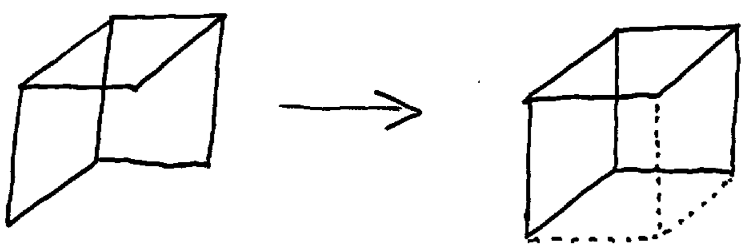
(2) $(\sigma(a)\sigma(b)\sigma(c)) = (abc)$ $\forall$ permutation σ (symmetry)

(3) $((abc)dc) = (a(bcd)c)$ (associativity)

Conversely, every ternary algebra satisfying (1), (2), and (3) comes from a median graph;

(v) (~~Ch.~~ Roller) G is a median graph if and only if the cubical complex $|G|$ is CAT(0);

(vi) (Gromov) a cubical complex $|G|$ is CAT(0) if and only if $|G|$ is simply connected and satisfies the following combinatorial condition: if three $(k+2)$-cubes intersect in a k-cube and pairwise intersect in $(k+1)$-cubes, then they are contained in a $(k+3)$-cube;



Other properties:

(vii) (van de Vel) $(|G|, \ell_1)$ is an ℓ_1-subspace;

(viii) (Mai & Tang) $(|G|, \ell_\infty)$ is an absolute retract, i.e. a retract of every space in which it embeds isometrically

Bridged graphs

bridged graph: a graph in which every isometric cycle has length 3

<u>Characterizations of bridged graphs</u>:

(i) (Ch. & Soltan, Farber & Jamison) Bridged graphs are precisely the graphs in which the neighborhoods $N_r(S)$ of convex sets S are convex;

(ii) (Chepoi) G is bridged if and only if the simplicial complex $|G|$ is simply connected and for every vertex v, $N_1(v)$ does not contain induced 4-cycles and 5-cycles;

(iii) (Anstee & Farber) Bridged graphs are precisely the dismantlable graphs without induced 4- and 5-cycles.
(Chepoi) The dismantling scheme is provided by BFS.

<u>Dismantling scheme</u>: ordering $v_1, \ldots, v_n$ of vertices of G such that $\forall v_i \; \exists v_j \in N_1(v_i), j>i$ such that all neighbors $v_k, k>i$, of v_i are also neighbors of v_j.

<u>Examples</u>: (a) Chordal graphs

(b) graphs for which the simplicial complex $|G|$ is 2-dimensional and CAT(0);

(c) planar graphs in which all inner faces are triangles and all inner vertices have degree ≥ 6

Hyperconvex spaces and Helly graphs

Hyperconvex space: a geodesic (Menger-convex) metric space in which every family of pairwise intersecting balls has a point in common (Helly property).

Helly graphs: the graphs in which the balls have the Helly property.

Remark: Helly graphs are the discrete analogies of hyperconvex spaces.

Theorem (Aronszajn, Panitchpakdi, 1959)

(i) Hyperconvex spaces are exactly the absolute retracts in the category of metric spaces, or equivalently, they are the retracts of l_∞-spaces;

(ii) Helly graphs are exactly the absolute retracts in the category of (reflexive) graphs.

Examples of Helly graphs: l_∞-grid, and, more generally take a median graph G and replace every maximal cube by a clique; the resulting graph G^∇ is Helly.

Theorem (Isbell, 1964; Dress, 1984) For every finite metric space (V_n, d) there exists the smallest hyperconvex space containing (V_n, d) as an isometric subspace (the tight span or the injective hull of d);

The same holds for graphs.

Weakly median graphs

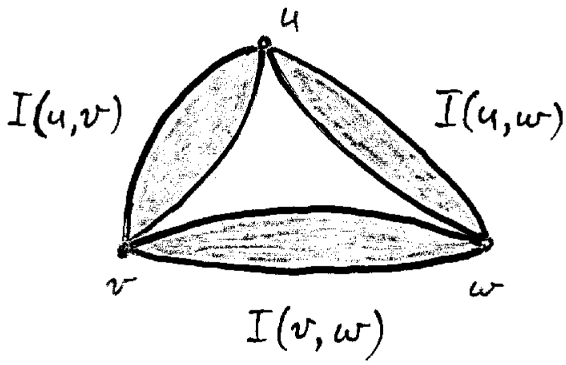
<u>Question</u>: How to extend notions like "median" and "median graph"?

metric triangle uvw:

$I(u,v) \cap I(v,w) = \{v\}$

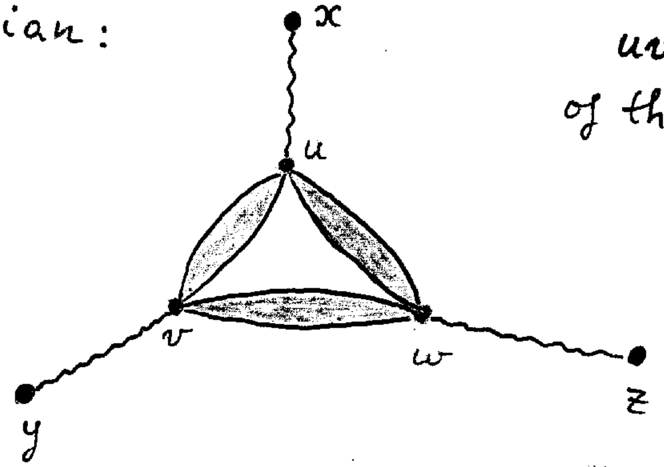
$I(v,w) \cap I(w,u) = \{w\}$

$I(w,u) \cap I(u,v) = \{u\}$



strongly equilateral metric triangle uvw: $d(u,x) \equiv \text{const} \quad \forall x \in I(v,w)$

quasi-median:



uvw is a quasi-median of the triplet x, y, z

Remark: every triplet of vertices admits at least one quasi-median

apex: u is called an apex of x with respect to y, z and is denoted by (xyz)

Analogously are defined the apices (yxz) and (zxy)

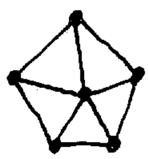
Weakly median graphs (cont.)

weakly modular graphs: graphs in which all metric triangles are strongly equilateral;

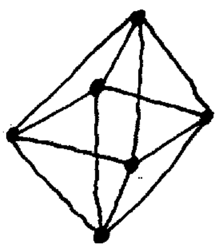
weakly median graphs: weakly modular graphs in which every triplet of vertices has a unique quasi-median

Characterisation:

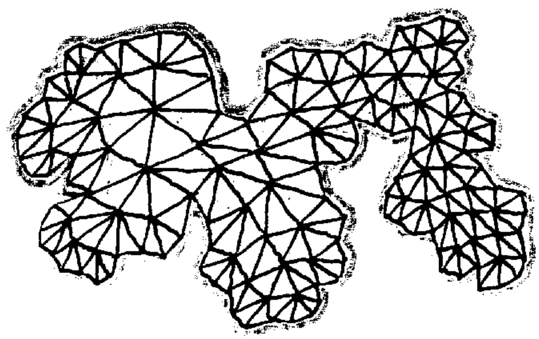
(i) (Bandelt & Ch.) Finite weakly median graphs are precisely the graphs obtained by successive applications of gated amalgamations from Cartesian products of the following prime graphs: 5-wheels, subhyperoctahedra, and two-connected plane graphs such that all inner faces are triangles and all inner vertices have degrees ≥ 6.



5-wheel



3-octahedron



bridged triangulation

(ii) (Bandelt & Ch.) Every finite weakly median graph is a retract of a Cartesian product of prime weakly median graphs and vice versa.

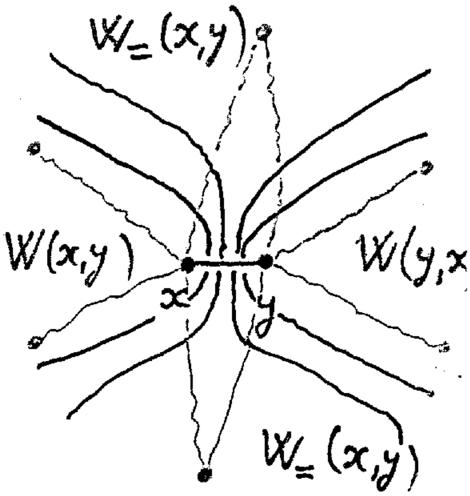
(iii) (Bandelt & Ch.) Every weakly median graph G is l_1-embeddable. G has a scale 2 embedding in a hypercube iff it does not contain an induced K_6 minus an edge.

(iv) (Bandelt & Ch.) Apex algebras of weakly median graphs are characterized by a set of 5 axioms among discrete ternary algebras

Isometric subgraphs of hypercubes and Hamming graphs

For an edge xy of a graph G set:

$$W(x,y)=\{z:\ d(x,z)<d(y,z)\}$$
$$W(y,x)=\{z:\ d(y,z)<d(x,z)\}$$
$$W_=(x,y)=\{z:\ d(x,z)=d(y,z)\}$$



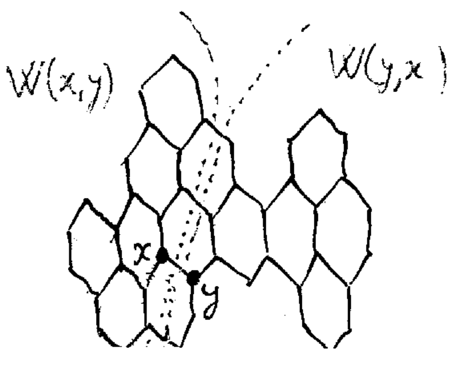
Djokovic: G is isometrically embeddable into a hypercube iff it is bipartite and for every edge xy the sets $W(x,y)$ and $W(y,x)$ are convex (i.e., they are complementary halfspaces).

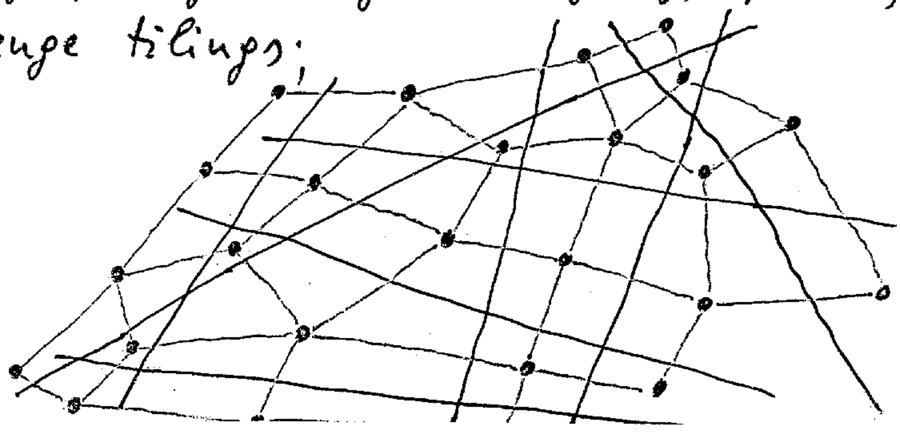
Ch. (answering a question by Winkler): G is isometrically embeddable into a Hamming graph (Cartesian product of complete graphs) iff for every edge xy the sets $W(x,y)$, $W(y,x)$, $W(x,y)\cup W_=(x,y)$, and $W(y,x)\cup W_=(x,y)$ are convex.

Examples: (i) benzenoids: planar graphs in which all inner faces are hexagons and all inner vertices have degree 3;

(ii) tope graphs of arrangements of hyperplanes;

(iii) lozenge tilings;





l_1-graphs

Remark: G is an l_1-graph iff it admits a scale embedding into a hypercube.

Ch., Deza, Grishukhin: A planar graph G is an l_1-graph iff it admits a scale 2 embedding into a hypercube (i.e., an isometric embedding into a halfcube).

Shpectorov: G is an l_1-graph iff it admits an isometric embedding into a Cartesian product of halfcubes and octahedra.

Remark: Shpectorov's result yields a polynomial recognition of l_1-graphs (in contrast to l_1-metrics)

Question: Provide a Djokovic-like characterization of isometric subgraphs of halfcubes.

Some classes of planar l_1-graphs (Chepoi, Dragan, Vaxès)

(4,4)-graphs, i.e. plane graphs in which all inner faces have length ≥ 4 and all inner vertices have degree ≥ 4;

(6,3)-graphs, i.e. plane graphs in which all inner faces have length ≥ 6 and all inner vertices have degree ≥ 3;

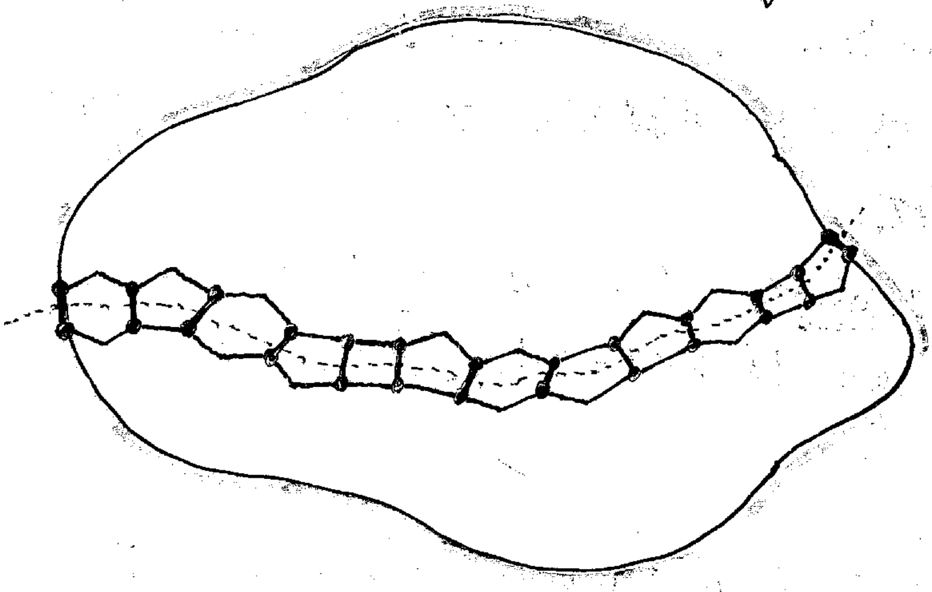
(3,6)-graphs, i.e. plane graphs in which all inner faces have length ≥ 3 and all inner vertices have degree ≥ 6.

l_1-graphs (cont.)

<u>Remark</u>: For every planar graph G of type $(4,4)$, $(3,6)$, or $(6,3)$, the cell complex $|G|$ is CAT(0).

<u>Remark</u>: It turn out that the planar graphs of types $(4,4)$, $(3,6)$, and $(6,3)$ have been investigated in combinatorial group theory, in particular by R. Lyndon who established the following maximality principle: if S is a subgraph of G bounded by a simple cycle ∂S and v is a vertex of S, then all furthest from v vertices of S are located on ∂S.

<u>Idea of the l_1-embedding</u>: use the alternating cuts of G



(i) the union of faces cut by an alternating cut is a strip consisting by the edges of the cut and two paths whose lengths differ by at most 1;

(ii) any alternating cut split the vertices of G into two convex sets

(iii) via every edge of G pass two alternating cuts.

Superconnected subsets of hypercubes

(following Bandelt, Ch., Dress & Koolen)

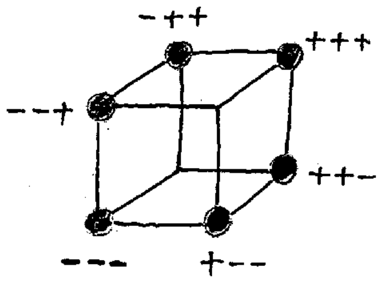
$\mathcal{S} \subseteq \{\pm 1\}^I$: set of maps from a finite set I to $\{\pm 1\}$;

For $Y \subseteq I$, let

$$\mathcal{S}_Y := \{ t \in \{\pm 1\}^{I-Y} : \text{some extension } s \in \{\pm 1\}^I \text{ of } t \text{ belongs to } \mathcal{S} \}$$

$$\mathcal{S}^Y := \{ t \in \{\pm 1\}^{I-Y} : \text{every extension } s \in \{\pm 1\}^I \text{ of } t \text{ belongs to } \mathcal{S} \}$$

Example: $I = \{1, 2, 3\}$



$$\mathcal{S}_{\{3\}} = \{ \{--\}, \{-+\}, \{+-\}, \{++\} \}$$

$$\mathcal{S}^{\{3\}} = \{ \{--\}, \{++\} \}$$

$$\mathcal{S}_{\{1,2\}} = \{ \{-\}, \{+\} \}, \quad \mathcal{S}^{\{1,2\}} = \emptyset$$

Two ways to derive an abstract simplicial complex from $\mathcal{S}$:

$$\overline{\mathcal{X}}(\mathcal{S}) := \{ Y \subseteq I : \mathcal{S}_{I-Y} = \{\pm 1\}^Y \}$$

$$\underline{\mathcal{X}}(\mathcal{S}) := \{ Y \subseteq I : \mathcal{S}^Y \neq \emptyset \}$$

In previous example,

$$\overline{\mathcal{X}}(\mathcal{S}) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\} \}$$

$$\underline{\mathcal{X}}(\mathcal{S}) = \{ \emptyset, \{1\}, \{2\}, \{3\} \}$$

Superconnected subsets of hypercubes (cont.)

$\forall\, \mathcal{S} \subseteq \{\pm 1\}^I, \quad \#\underline{\mathcal{X}}(\mathcal{S}) \leq \#\mathcal{S} \leq \#\overline{\mathcal{X}}(\mathcal{S})$ holds

Bandelt, Ch, Dress, Koolen: For a set $\mathcal{S} \subseteq \{\pm 1\}^I$ of sign maps the following conditions are equivalent:

(i) superconnectivity: $\mathcal{S}^Y$ is connected for all $Y \subseteq I$;

(ii) superisometry: $\mathcal{S}^Y$ is isometric for all $Y \subseteq I$;

(iii) commutativity: $(\mathcal{S}^Y)_Z = (\mathcal{S}_Z)^Y$ for all disjoint subsets Y, Z of I;

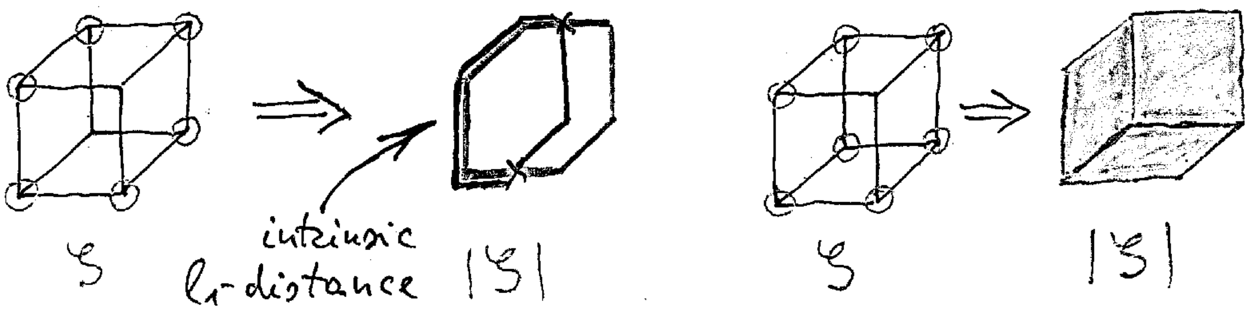
(iv) ampleness I: $\#\mathcal{S} = \#\overline{\mathcal{X}}(\mathcal{S})$;

(v) ampleness II: $\underline{\mathcal{X}}(\mathcal{S}) = \overline{\mathcal{X}}(\mathcal{S})$;

(vi) $\mathcal{S}$ is isometric and both $\mathcal{S}^e$ and $\mathcal{S}_e$ are superconnected for some $e \in I$;

(vii) $\mathcal{S}$ is connected, and $\mathcal{S}^e$ is superconnected for every $e \in I$;

(viii) l_1-isometry: the cubical complex $|\mathcal{S}|$ endowed with the intrinsic l_1-metric is an isometric subspace of $(\mathbb{R}^I, \|\cdot\|_1)$.



Superconnected subsets of hypercubes (cont.)

Examples: (i) vertex-sets of median graphs;

(ii) signed maps of convex sets of antimatroids (convex geometries);

(iii) maximum set systems of a given Vapnik-Chervonenskis dimension;

(iv) lozenge tilings;

(v) signed maps of regions of simple affine arrangements of hyperplanes;

(vi) (J. Lawrence) signed maps of orthants intersecting a given convex (in the usual sense) set.

Remark: Superconnected sets are equivalent to lopsided sets introduced and characterized in a different way by J. Lawrence.

Open question: Is it true that for every proper nonempty superconnected subset $\mathcal{S} \subset \{\pm 1\}^I$ there exist $s \in \mathcal{S}$ and $t \in \{\pm 1\}^I \setminus \mathcal{S}$ such that $\mathcal{S} \cup \{t\}$ and $\mathcal{S} \setminus \{s\}$ are superconnected?

Basis graphs of matroids and even Δ-matroids

matroid: a collection $\mathcal{B}$ of subsets of a finite set I, called bases, which satisfy the following exchange property;

(EP) for all $A, B \in \mathcal{B}$ and $i \in A \setminus B$ there exists $j \in B \setminus A$ such that $A \setminus \{i\} \cup \{j\} \in \mathcal{B}$.

The base $A \setminus \{i\} \cup \{j\}$ is obtained from A by an elementary exchange;

basis graph $G = G(\mathcal{B})$ of a matroid $\mathcal{B}$ is the graph whose vertices are the bases of $\mathcal{B}$ and edges are the pairs A, B of bases differing by an elementary exchange

Remark: Basis graphs faithfully represent their matroids

Remark: Since all bases of a matroid $\mathcal{B}$ have the same cardinality, (EP) implies that $G(\mathcal{B})$ is an isometric subgraph of a Johnson graph (one slice of a halfcube).

Remark: A characterization of basis graphs of matroids emloying distance properties was provided by S. Maurer. We simplified and generalized this result to basis graphs of even Δ-matroids

Basis graphs of matroids and even Δ-matroids (cont.)

Δ-matroid (Bouchet; Chandrasekaran & Kabadi; Dress & Havel)
a collection $\mathcal{B}$ of subset of a finite set I, called bases, not necessarily equicardinal, satisfying the following symmetric exchange property:

(SEP) for all $A, B \in \mathcal{B}$ and $i \in A \Delta B$, there exists $j \in B \Delta A$ such that $A \Delta \{i,j\} \in \mathcal{B}$.
The base $A \Delta \{i,j\}$ is obtained from A by an elementary exchange;

even Δ-matroid: all bases have the same cardinality modulo 2

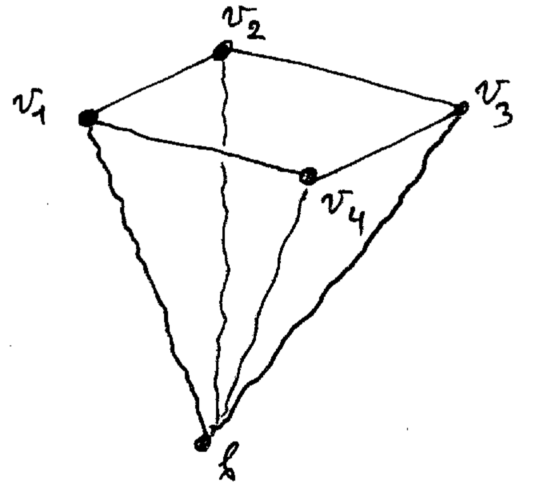
basis graph $G = G(\mathcal{B})$ of an even Δ-matroid $\mathcal{B}$: the graph whose vertices are the bases of $\mathcal{B}$ and edges are the pairs A, B of bases differing by a single exchange, i.e. $|A \Delta B| = 2$.

Axiom (SEP) implies that $G(\mathcal{B})$ is an isometric subgraph the halfcube, i.e. $|A \Delta B| = 2 d_{G(\mathcal{B})}(A,B)$ $\forall A, B \in \mathcal{B}$

interval condition
(ICm) if $d(u,v) = 2$, then $I(u,v)$ contain an induced 4-cycle and itself is an induced subgraph of the m-octahedron

positioning condition
(PC) for each vertex b and each induced 4-cycle $v_1 v_2 v_3 v_4$
$$d(b, v_1) + d(b, v_3) = d(b, v_2) + d(b, v_4)$$

Basis graphs of matroids and even Δ-matroids (cont.)



$$d(v_1,b)+d(v_3,b)=d(v_2,b)+d(v_4,b)$$

Ch.: G is a basis graph of an even Δ-matroid iff it satisfies the positioning condition (PC), the interval condition (IC4), and the neighborhood $N(b)$ of some vertex is the line graph of some graph Γ.

Maurer: G is a basis graph of a matroid iff it satisfies the positioning condition (PC), the interval condition (IC3) and the neighborhood $N(b)$ of some vertex is the line graph of some bipartite graph $\Gamma=(A\dot\cup B, F)$.

Idea of the proof: define a mapping $\varphi: V \to 2^I$ in the following way:

(a) $\varphi(b)=\emptyset$;

(b) $\forall x \in N(b)$ encodes some edge ij of Γ; put $\varphi(x)=\{i,j\}$;

(c) $\forall v \notin N(b)\cup\{b\}$, let $\varphi(v)=\bigcup\{\varphi(x): x\in I(v,b)\cap N(b)\}$.

Properties of φ: φ is injective; all sets $\varphi(v)$ have even cardinality; φ is an isometric embedding of G into a halfcube. This implies that $\mathcal{B}_\varphi=\{\varphi(v): v\in V\}$ is an even Δ-matroid. If Γ is bipartite with $I=A\dot\cup B$, then $\mathcal{B}_\varphi \Delta A=\{\varphi(v)\Delta A: v\in V\}$ is a matroid of rank $|A|$.