Homework
Due December 9th, 2016

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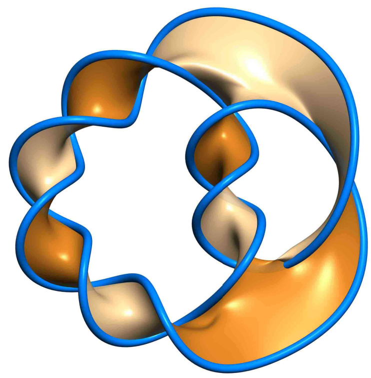
Exercise 1

What is the genus, orientability and number of boundaries[1] of the following surfaces? (Please write some reasoning, not only the flat out answer)

1. The surface obtained from the polygonal scheme $a_1 a_2 \ldots a_{n-1} a_n \bar{a}_1 \bar{a}_2 \ldots \bar{a}_{n-1} a_n$
2. The surface in this picture :

3. The surface in this picture :



4. (This is a hard one! You may skip is if you really don't see how to do it) The surface in this picture (sculpture by Bathseba Grossman):

[1]We have not covered surfaces with boundaries in class but they are not very different from usual surfaces : take a look at the lecture notes.

Exercise 2

An **independent subset** of a graph $G = (V, E)$ is a subset of vertices S so that no two vertices S are linked by an edge of G. The **Euler genus** $\bar{g}$ of a surface $\mathscr{S}$ is equal to $2 - \chi(\mathscr{S})$, where χ is the Euler characteristic.

1. Find constants $\alpha > 1$ and $\beta > 1$ such that, for a simple planar graph with n vertices, there are at most n/α vertices of degree β or more.
2. Find constants $\alpha' > 1$ and $\beta' > 1$ such that every simple planar graph with n vertices has an independent subset of at least $\frac{n}{\alpha'}$ vertices, each with degree less than β'. What are the best values of α' and β' you can obtain?
3. Find constants $\alpha'' > 1$, $\beta'' > 1$, and $\gamma > 1$ such that for all non-negative integers n and $\bar{g}$ such that $n \geq \gamma \bar{g}$, every n-vertex graph embedded on a surface of Euler genus $\bar{g}$ has an independent set of size $\frac{n}{\alpha''}$, in which every vertex has degree at most β''.
4. Design a linear-time algorithm to find such an independent set.

Exercise 3

When S is a combinatorial surface and nonempty boundary, it can be "retracted" onto a graph by a series of collapses, where a collapse consists in deleting an edge contained in a single face as well as this face. By the same arguments as the ones used in the proof of Theorem 4.7 in the notes on homotopy, the fundamental group of S is isomorphic to the fundamental group of this graph. Propose a simple contractibility test in this case and analyze its complexity.

Exercise 4

Let $\mathscr{S}$ be an orientable surface, a **system of loops** is a graph cellularly embedded on S with one face and one vertex, and a **canonical system of loops** is a graph embedded on $\mathscr{S}$ having as polygonal scheme the one used in the classification of surfaces, i.e., $a_1 b_1 \bar{a}_1 \bar{b}_1 \dots a_g b_g \bar{a}_g \bar{b}_g$. The **dual graph** of a cellularly embedded graph G is the graph obtained by putting a vertex in each face and an edge between two vertices corresponding to faces f_1 and f_2 (possibly the same) for each edge e of G between the faces f_1 and f_2 (possibly the same).

1. Let G be a cellularly embedded graph with n vertices on the orientable surface $\mathscr{S}$ of genus g, and G^* its dual graph. Let x^* be a vertex of G^*, T^* be a spanning tree of G^*, and e^* be an edge of G^* not in T^*. Let $\gamma_{e^*}^{T^*}$ denote the loop based at x^* obtained by connecting e^* to x^* using paths in T^* (as in Section 4 of the lecture notes). How many times does $\gamma_{e^*}^{T^*}$ cross the edges of G?

2. Let G be a cellularly embedded graph on $\mathscr{S}$ with n vertices. Show that there exists a canonical system of loops avoiding the vertices of G and meeting the edges of G at a finite number of points, such that each loop of the system has at most $O(gn)$ intersections with the edges of G.

   Hint: Apply the techniques of the proof of the classification of surfaces to the dual graph of G and use the previous question.

3. Design an algorithm running in time $O(g^2n)$ to compute such a canonical system of loops.

Cultural note 1: canonical systems of loops can be used to compute parameterizations of surfaces, i.e., explicit homeomorphisms between surfaces, which is useful in various applied settings, for example for texture mapping.

Cultural note 2: one can improve the bounds in this exercise to design an $O(gn)$ algorithm for this problem. For non-orientable surfaces, the techniques of this exercise work similarly, but such an improvement is an open problem.