Homework
Due December 9th, 2016

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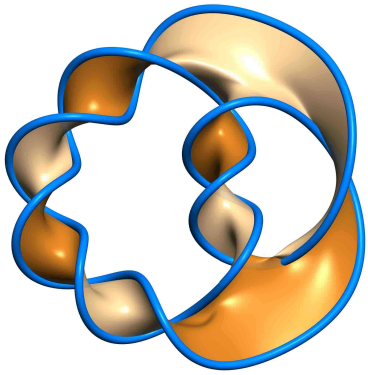
Exercise 1

What is the genus, orientability and number of boundaries[1] of the following surfaces? (Please write some reasoning, not only the flat out answer)

1. The surface obtained from the polygonal scheme $a_1 a_2 \dots a_{n-1} a_n \bar{a}_1 \bar{a}_2 \dots \bar{a}_{n-1} a_n$
2. The surface in this picture :



3. The surface in this picture (sculpture by Bathseba Grossman):

4. The surface in this picture :

[1] We have not covered surfaces with boundaries in class but they are not very different from usual surfaces : take a look at the lecture notes.

Correction:

(a) One way to do this is to draw the scheme, count the number of faces (1), the number of edges (n) and the number of vertices (1). Since there is a symbol occurring twice with the same sign (a_n), the surface is non-orientable, and its Euler characteristic is $2-n$, thus it has genus n. Another way to do this is to use the tools involved in the proof of the classification of surfaces, in particular the move $aPaQ \to bbP\bar{Q}$. We denote equivalent polygonal schemes with the $\equiv$ sign.

We have

$$\begin{aligned} a_1 \dots a_n \bar{a}_1 \dots \bar{a}_{n-1} a_n &\equiv a_n \bar{a}_1 \dots \bar{a}_{n-1} a_n a_1 \dots a_{n-1} \\ &\equiv b_n b_n \bar{a}_1 \dots \bar{a}_{n-1} \bar{a}_{n-1} \dots \bar{a}_1 \\ &\equiv \bar{a}_1 \dots \bar{a}_{n-1} \bar{a}_{n-1} \bar{a}_1 b_n b_n \\ &\equiv b_1 b_1 \bar{a}_2 \dots \bar{a}_{n-1} \bar{a}_{n-1} \dots \bar{a}_2 \bar{b}_n \bar{b}_n \end{aligned}$$

and thus by induction this gives the scheme $b_1 b_1 \dots b_n b_n$. Thus the surface has genus n and is non-orientable.

(b) Let us denote the number of twisted bands by n. The surface is non-orientable since one can get on both sides by following a twisted band. We first follow the boundaries to observe that there is a single boundary. Then, considering the points where the bands twist as edges (as in Figure 1), one obtains a cellularly embedded graph with a single face, $3n$ edges and $2n$ vertices, and thus the Euler characteristic is $1-3n = 2-b-g$, and the genus is n.

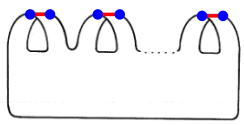


Figure 1: Adding edges where the surface twists.

(c) There is a single boundary and the surface is orientable (as indicated by the colors). For the genus, we can use the same trick as before: putting an edge where the surface twists in the projection (see Figure 2, we obtain a cellularly embedded graph with 8 faces, 27 edges and 18 vertices. Computing the Euler characteristic gives genus 1.

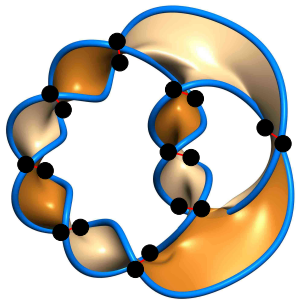


Figure 2: Adding edges where the surface twists redux.

Nota bene: Some students used the cut and paste tools from the classification of surfaces here, and obtained genus 2. The error is not in the approach, which is totally fine, but in the lecture notes (sorry!), where we have erroneously put the reduction $aPbQ\bar{a}R\bar{b}S \to dc\bar{d}\bar{c}PSQR$ while it should have been $aPbQ\bar{a}R\bar{b}S \to dc\bar{d}\bar{c}PSRQ$. We gave full credits to students who gave a wrong answer because of our mistake.

(d) Our idea with this exercise was that the small holes are avatars of the sculpture and should not be counted as such – but of course this was ambiguous and we gave credit to the students who took them into account. In the following we do not. There are now 3 boundaries and the surface is orientable (one can easily distinguish two sides). It is hard to find a cellularly embedded graph straight away because the perspective is ill-chosen: there are some place where there are two layers of the surface (which did not happen in the previous example). However, by tilting a bit the sculpture upwards as in Figure 3, left, or equivalently, by doing type III Reidemeister moves on the two triangles indicated by the blue arrows in Figure 3 right, and following what happens to the surface, one can see that the surface above is the same as the one in Figure 4. You can play with the applet on the sculptor's website : `https://www.bathsheba.com/math/borromean/` (click the rotating arrow) to move the sculpture in 3d and convince yourself that it works.

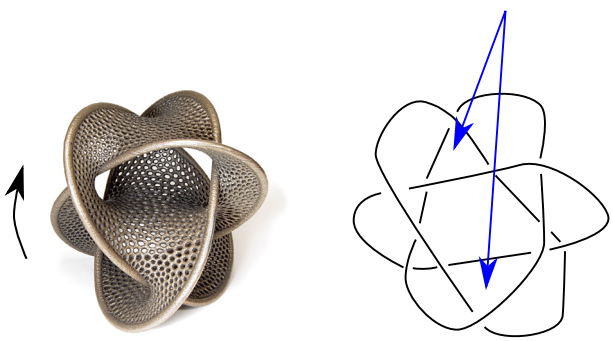


Figure 3: Tilting the sculpture upwards, or equivalently, doing type III Reidemeister moves on the two indicated triangles.

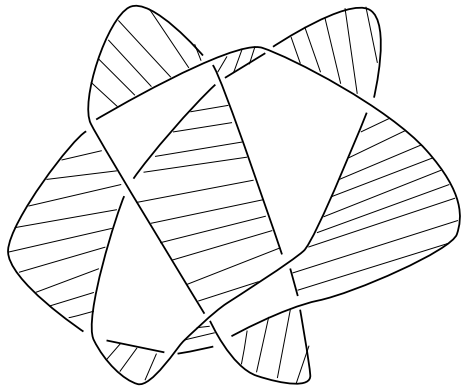


Figure 4: The surface in question is isotopic to the one pictured by the dashes here, where there are half-twists at the vertices of the projection.

Then, putting edges at the on the half-twists as before, we get a cellularly embedded graph with 9 faces, 36 edges and 24 vertices, thus the Euler characteristic is -3, which gives genus 1 since there are 3 boundaries.

Exercise 2

An **independent subset** of a graph $G = (V, E)$ is a subset of vertices S so that no two vertices S are linked by an edge of G. The **Euler genus** $\bar{g}$ of a surface $\mathscr{S}$ is equal to $2 - \chi(\mathscr{S})$, where χ is the Euler characteristic.

1. Find constants $\alpha > 1$ and $\beta > 1$ such that, for a simple planar graph with n vertices, there are at most n/α vertices of degree β or more.

2. Find constants $\alpha' > 1$ and $\beta' > 1$ such that every simple planar graph with n vertices has an independent subset of at least $\frac{n}{\alpha'}$ vertices, each with degree less than β'. What are the best values of α' and β' you can obtain?

3. Find constants $\alpha'' > 1$, $\beta'' > 1$, and $\gamma > 1$ such that for all non-negative integers n and $\bar{g}$ such that $n \geq \gamma \bar{g}$, every n-vertex graph embedded on a surface of Euler genus $\bar{g}$ has an independent set of size $\frac{n}{\alpha''}$, in which every vertex has degree at most β''.

4. Design a linear-time algorithm to find such an independent set.

Correction:

1. Suppose that G is planar. By Euler's formula: $n - |E| + f = 2$ where f is the number of faces of some plane embedding. Since G is simple, every face has at least three sides and it follows by the handshaking lemma that $2|E| \geq 3f$. Reporting in Euler's formula gives: (i) $|E| \leq 3n - 6$. Let $n_{<\beta}$ and $n_{\geq\beta}$ be the numbers of vertices of degree respectively less than β and at least β. By the handshaking

lemma again: $2|E| \geq n_{<\beta} + \beta n_{\geq\beta} = n + (\beta-1)n_{\geq\beta}$. Together with (i), we obtain (ii) $n_{\geq\beta} \leq \frac{5n-12}{\beta-1} \leq \frac{5n}{\beta-1}$. Hence, for

$$\alpha = \frac{\beta-1}{5} \tag{1}$$

at most n/α vertices have degree β or more.

2. Let I_β an (inclusionwise) maximal independent subset of vertices of degree less that β. Since I_β is maximal, each vertex of degree less than β is either in I_β or has a neighbor in V_β. Whence, $n_{<\beta} \leq \beta|I_\beta|$. By (1), $n - n_{<\beta} = n_{\geq\beta} \leq n/\alpha$. We conclude that
$$|I_\beta| \geq \frac{\alpha-1}{\alpha\beta}n = \frac{\beta-6}{\beta(\beta-1)}n.$$
Choosing $\beta' = \beta$ and $\alpha' = \frac{\beta(\beta-1)}{\beta-6}$ gives the result. The best values for α' and β' depends on your criteria. Say you want to minimize α', then by differentiating $\alpha'(\beta)$, we obtain $\beta' = 11$ and $\alpha' = 22$.

3. Euler's formula becomes $n - |E| + f = 2 - \bar{g}$ and Equation (i) is replaced by $|E| \leq 3n - 6 + 3\bar{g}$, whereas (ii) is replaced by
$$n_{\geq\beta} \leq \frac{5n + 6\bar{g} - 12}{\beta-1} \leq \frac{5}{\beta-1}n + \frac{6\bar{g}}{\beta-1}.$$
Assuming $\bar{g} \leq n\gamma^{-1}$, we get $n_{\geq\beta} \leq n/\alpha''$ with
$$\alpha'' = \frac{\beta-1}{5+6\gamma^{-1}}. \tag{2}$$
As in question 2, we derive
$$|I_\beta| \geq \frac{\alpha''-1}{\alpha''\beta}n = \frac{\beta-6-6\gamma^{-1}}{\beta(\beta-1)}n.$$
Choosing $\beta'' = \beta$ and $\alpha'' = \frac{\beta(\beta-1)}{\beta-6-6\gamma^{-1}}$ gives the result.

4. The following greedy strategy returns a maximal independent subset I_β of vertices of degree less than β: Initialize I_β to the empty set. Pick a vertex of degree less than β and put it in I_β. Mark the chosen vertex and all its neighbors. Repeat as long as there remains an unmarked vertex of degree less than β. The algorithm can be easily implemented to run in linear time.

Note that the constants α', β' in Question 2 can be improved. According to Snoeyink and van Kreveld, one can choose $\beta' = 10$ and $\alpha' = 6$ using an appropriate (non-greedy) approach. See Section 4 of this paper.

Exercise 3

When S is a combinatorial surface and nonempty boundary, it can be "retracted" onto a graph by a series of collapses, where a collapse consists in deleting an edge contained in a single face as well as this face. By the same arguments as the ones used in the proof of Theorem 4.7 in the notes on homotopy, the fundamental group of S is isomorphic to the fundamental group of this graph. Propose a simple contractibility test in this case and analyze its complexity.

Correction:

We view S as a combinatorial surface of Euler genus $\bar{g}$ with $b > 0$ faces marked as perforated. For concision, a perforated face is called a **hole**, and a non-perforated face is simply called a **face**. By contracting a spanning tree we can assume that S has a single vertex. Any walk in the uncontracted S gives rise to a homotopic curve in the contracted surface by simply discarding the edges of the spanning tree.

Let G be the adjacency graph of the faces of S: G has one node per face and one dual edge per pair of faces sharing an edge of S. Consider a spanning tree of a connected component of G. One of the nodes in this tree must correspond to a face sharing an edge e with a hole h. Connect this node to a root node h^* with a dual edge e^*. The resulting rooted tree T may be contracted to a single node; this amounts to delete the corresponding primal edges, or equivalently, to collapse all the faces corresponding to the nodes of T, starting with the face incident with e. The effect is to enlarge h by extending the hole. Let D be the tree of faces corresponding to $T - h^*$. The endpoints of each deleted edge a cuts the boundary of D into two paths and we denote by $p[a]$ the path that does not contain e. We store $p[a]$ in a table. Note that a, as a path, is homotopic to $p[a]$. Moreover, after contracting T, every edge in the boundary of D is incident to holes only. Hence, repeating the same operation for all the connected components of G will not affect the edges of $p[a]$. In the end, S is collapsed to a surface C with holes only. Since a collapse does not change the Euler characteristic, we have $\chi(C) = \chi(S)$. Applying Euler's formula to C, we obtain

$$1 - |E(C)| + 0 = \chi(S) = 2 - \bar{g} - b,$$

whence $|E(C)| = O(\bar{g} + b)$. Since all the edges of $p[a]$ are edges of C and occur at most twice in $p[a]$, we also have $p[a] = O(\bar{g} + b)$. Clearly, the computation of all the $p[a]$'s takes time proportional to their size, *i.e.*, $O((\bar{g} + b)|E(S)|)$.

Given a closed walk γ in S, we can now replace each edge a of γ by $p[a]$ to obtain a homotopic curve γ' in C of size $|\gamma'| = O((\bar{g} + b)|\gamma|)$. We can further reduce γ' in time proportional to its size by repeatedly removing all of its spurs. Let γ'' be the resulting reduced curve. Because C has no faces, there is no other elementary homotopy to apply. It follows that γ is contractible if and only if γ'' is the trivial path. Said differently C retracts onto a graph and its fundamental group is thus a free group in which every element (homotopy class) has a unique reduced form. See Section 4.2. of the lecture notes. To summarize, after $O((\bar{g} + b)|E(S)|)$ time preprocessing, the contractibility test can be performed in $O((\bar{g} + b)|\gamma|)$ time.

By working a little bit, the preprocessing and contractibility test can be reduced to respectively $O(|E(S)|)$ and $O(|\gamma|)$ time. See this short paper for a solution.

Exercise 4

Let $\mathscr{S}$ be an orientable surface, a **system of loops** is a graph cellularly embedded on S with one face and one vertex, and a **canonical system of loops** is a graph embedded on $\mathscr{S}$ having as polygonal scheme the one used in the classification of surfaces, i.e., $a_1 b_1 \bar{a}_1 \bar{b}_1 \dots a_g b_g \bar{a}_g \bar{b}_g$. The **dual graph** of a cellularly embedded graph G is the graph obtained by putting a vertex in each face and an edge between two vertices corresponding to faces f_1 and f_2 (possibly the same) for each edge e of G between the faces f_1 and f_2 (possibly the same).

1. Let G be a cellularly embedded graph with n vertices on the orientable surface $\mathscr{S}$ of genus g, and G^* its dual graph. Let x^* be a vertex of G^*, T^* be a spanning tree of G^* containing x^*, and e^* be an edge of G^* not in T^*. Let $\gamma_{e^*}^{T^*}$ denote the loop based at x^* obtained by connecting e^* to x^* using paths in T^* (as in Section 4 of the lecture notes). How many time does $\gamma_{e^*}^{T^*}$ cross the edges of G?

2. Let G be a cellularly embedded graph on $\mathscr{S}$ with n vertices. Show that there exists a canonical system of loops avoiding the vertices of G and meeting the edges of G at a finite number of points, such that each loop of the system has at most $O(gn)$ intersections with the edges of G.

   Hint: Apply the techniques of the proof of the classification of surfaces to the dual graph of G and use the previous question.

3. Design an algorithm running in time $O(g^2 n)$ to compute such a canonical system of loops.

Cultural note 1: canonical systems of loops can be used to compute parameterizations of surfaces, i.e., explicit homeomorphisms between surfaces, which is useful in various applied settings, for example for texture mapping.

Cultural note 2: one can improve the bounds in this exercise to design an $O(gn)$ algorithm for this problem. For non-orientable surfaces, the techniques of this exercise work similarly, but such an improvement is an open problem.

Correction:

Foreword: There was a slight inaccuracy in the exercise: n should be the number of edges of the graph instead of the number of vertices, otherwise the results are wrong in the case of a single vertex graph for example. Given the low success rate of the exercise, we accounted for this in the correction, but it did not change much.

1. The loop $\gamma_{e^*}^{T^*}$ uses by construction at most twice an edge of G^* (it might need to follow twice the same edge when going back to x^*), and thus crosses at most twice each edge of G (by definition of the dual graph G^*).

2. As in the proof of the classification of surfaces, the first step here is to choose a spanning tree of G^* and contract it. We are left with a single vertex, and merging adjacent distinct faces (or equivalently contracting a spanning tree of the dual of G^*, hence of G), this gives a system of loops $\mathscr{L}$. However, contrarily to what we saw in many homeworks, we are not done at this step since this system of loops may not be canonical! For example, the loops associated with the scheme

$a_1 \dots a_{2g} b_1 \dots b_{2g}$ correspond to an orientable genus g surface, but this is not canonical. So one needs to turn this system of loops into a canonical one.

To do that, we once again follow the proof of the classification of surfaces which shows how to turn $\mathscr{L}$ into a canonical system of loops. One of the corresponding moves is pictured in Figure 5, where the loops $\gamma_c^{T^*}$ and $\gamma_d^{T^*}$ are added to replace $\gamma_a^{T^*}$ and $\gamma_b^{T^*}$.

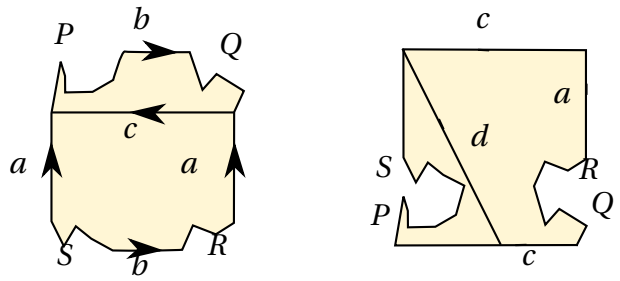


Figure 5: Transforming a system of loops into a canonical one.

The remaining issue to solve this question (and the whole point of the exercise) is to bound the number of times these new loops intersect G. At the first iteration, it seems pretty obvious that they cross each edge of G at most once inside the disk (and maybe twice in the remaining part because of the contraction, see question 1.), but as we proceed in the induction, things get complicated and this is not obvious anymore. A solution is as follows. We first observe that in the transformations pictured in Figure 5, one can always assume that the edges in P, Q and R are loops of the original system of loops $\mathscr{L}$, i.e., all the new loops get put into S. Indeed, they can all be put in a consecutive order, and at each step of the induction, it is just a matter of choosing a to be the first original loop after the consecutive set of new loops. Then, at each step of the induction, we view each new loop $\gamma_c^{T^*}$ as the loop $\gamma_{\bar{Q}\bar{b}\bar{P}}^{T^*}$. Similarly, one can write $\gamma_d^{T^*}$ as the loop $\gamma_{\bar{c}\bar{a}\bar{R}\bar{Q}c}^{T^*}$. Since each loop of P, Q, R and b is of the form $\gamma_{e^*}^{T^*}$, by the question 1, it crosses G at most $O(n)$ times, and thus each of the new loops crosses G at most $O(gn)$ times, since R, P and Q might have length $O(g)$.

3. The proof above is algorithmic. Each loop of the canonical system of loops is described as a concatenation of $O(g)$ loops of the form $\gamma_{e^*}^{T^*}$, which are all made of $O(n)$ edges of G. Since there are $O(g)$ loops, the total complexity is $O(g^2 n)$.