1. **Merton's 1969 Portfolio Problem**

   (a) Problem Statement and Notation

   - 1 riskless asset R_t with fixed annual return r, $dR_t = r \cdot R_t \cdot dt$
   - 1 risky asset S_t following Geometric Brownian Motion, $dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dz_t$, where the first term is used to model deterministic trends($\mu > r$), the second term is used to model a set of unpredictable events occurring during this motion.
   Applying Ito's Lemma with $f(S_t) = \log(S_t)$ gives:

$$\begin{aligned} d\log S_t &= \left(\frac{\partial f}{\partial t} + \mu S_t \frac{\partial f}{\partial s} + \frac{1}{2} S_t^2 \sigma^2 \frac{\partial^2 f}{\partial s^2}\right) dt + \sigma S_t \frac{\partial f}{\partial s} dz_t \qquad \text{(Ito's Lemma)} \\ &= \left(\mu S_t f'(S_t) + \frac{1}{2} S_t^2 \sigma^2 f''(S_t)\right) dt + \sigma S_t f'(S_t) dz_t \\ &= f'(S_t) dS_t + \frac{1}{2} f''(S_t) S_t^2 \sigma^2 dt \\ &= \frac{1}{S_t}\left(\mu S_t dt + \sigma S_t dz_t\right) - \frac{1}{2}\sigma^2 dt \\ &= \sigma dz_t + (\mu - \frac{1}{2}\sigma^2) dt \end{aligned}$$

   We can see $f(S_t)$ follows the generalized Wiener process. Take integral on both sides we get:

$$\log(S_t) = \log(S_0) + \sigma z_t + (\mu - \frac{1}{2}\sigma^2)t \quad \Leftrightarrow \quad S_t = S_0 e^{\sigma z_t + (\mu - \frac{1}{2}\sigma^2)t}$$

   - wealth W_t with consumption rate c_t, initial wealth W_0
   - fraction of wealth allocated to risky asset π_t

$$\begin{aligned} dW_t &= d\left(R_t = (1-\pi_t)W_t\right) + d(S_t = \pi_t W_t) - c_t d_t \\ &= r \cdot (1-\pi_t) W_t \cdot dt + \mu \cdot \pi_t W_t \cdot dt + \sigma \cdot \pi_t W_t \cdot dz_t - c_t dt \\ &= \left((\pi_t \cdot (\mu - r) + r) \cdot W_t \cdot - c_t\right) dt + \sigma \cdot \pi_t W_t \cdot dz_t \end{aligned}$$

   In a special case where $\forall t \quad c_t = 0 \quad \text{and} \quad \pi_t = \pi$, we can combine the expression of $d\log S_t$ and dW_t which yields:

$$d\log W_t = \sigma\pi dz_t + (\pi(\mu - r) + r - \frac{1}{2}\sigma^2\pi^2)dt$$

   We notice $\log(W_t)$ still follows the generalized Wiener process, with drift rate $(\pi(\mu - r) + r - \frac{1}{2}\sigma^2\pi^2)$ and variance $\sigma^2\pi^2$. This is saying that W_t is log-normally distributed:

$$\log W \sim \mathcal{N}(\pi(\mu - r) + r - \frac{1}{2}\sigma^2\pi^2, \sigma^2\pi^2)$$

- **Goal:** $\max_{(\pi_t,c_t),t\in[0,T]} \mathbb{E}[U(W_T \mid W_0]$, assuming CRRA on utility $U(w)$. In the simplified case mentioned above(no consumption, constant distribution ratio), knowing that W_t being log-normally distributed, we can solve for π^* in close form:

$$\max_\pi \mathbb{E}[U(W_T)] \Rightarrow \max_\pi \left[\pi(\mu - r) + r - \frac{1}{2}\sigma^2\pi^2 + \frac{\sigma^2\pi^2}{2}(1-\gamma)\right]$$
$$\pi^* = \frac{\mu - r}{\gamma\sigma^2}$$

Non-surprisingly the same π^* holds for the general case. In the next section we show how to formulate Model Merton's Portfolio problem as an MDP and solve it.

It's worth mentioning that the expression of dW_t we derived above holds only under the assumption π_t is unrestricted, meaning borrowing or shorting at any time is allowed. **That said, it's really interesting to think about the restricted case**. I believe this will make the MDP more complicated for several reasons: first the actions π_t and c_t are sequential now since there's an extra constraint $c_t + \text{total investment}_t = W_t$, second the optimal policy may be stochastic, moreover all the nice properties we have here may not hold anymore.

(b) MDP

This is a continuous-time stochastic control problem where we want to find optimal policy(mapping: $(t, W_t) \to [\pi_t, c_t]$) for all $t \in [0, T]$ so that following this policy maximizes $\mathbb{E}[U(W_T) \mid W_0]$

- **state space** $\mathcal{S}$: (t, W_t) for $t \in [0, T]$
- **action space** $\mathcal{A}$: $[\pi_t, c_t]$ for $t \in [0, T)$
- **transition**:

$$dW_t = ((\pi_t \cdot (\mu - r) + r) \cdot W_t \cdot - c_t)\, dt + \sigma \cdot \pi_t W_t \cdot dz_t$$

- **reward** $R_t = U(c_t)$
- **(discounted) value function**: Define $\rho \geq 0$ as the utility discount rate, consider bequest function $B(T) = \epsilon^\gamma$, with ϵ parameterizing the desired level of bequest.

$$V(W_t, t) = \mathbb{E}\left[\int_t^T e^{-\rho\tau} U(c_\tau) d\tau + \epsilon^\gamma e^{-\rho T} U(W_T)\right]$$

$$\begin{aligned} V^*(W_t, t) &= \max_{\pi_t, c_t} \mathbb{E}\left[\int_t^T e^{-\rho\tau} \frac{c_\tau^{1-\gamma}}{1-\gamma} d\tau + \epsilon^\gamma e^{-\rho T} \frac{W_T^{1-\gamma}}{1-\gamma}\right] \\ &= \max_{\pi_t, c_t} \mathbb{E}\left[\int_t^{t_1} e^{-\rho\tau} \frac{c_\tau^{1-\gamma}}{1-\gamma} d\tau + V^*(t_1, W_{t_1})\right] \end{aligned}$$

(Bellman Optimal Eqn)