1. **Understanding Risk-Aversion through Utility Theory**

Some concepts:

- risk
- utility function(utility of consumption) $U(x)$ (x : the uncertain outcome being consumed, e.g. good, service)
- law of diminishing marginal utility $\Rightarrow U''(x) < 0 \Rightarrow \mathbb{E}[U(x)] < U(\mathbb{E}[x])$
- certainty-equivalent value $x_{CE} = U^{-1}(\mathbb{E}[U(x)])$
  - $x_{CE} < \mathbb{E}[x]$
  - interpretation 1: the guaranteed amount of cash a person would consider as having the same amount of desirability as a risky asset
  - interpretation 2: the guaranteed return a person would accept now, rather than taking a chance on a higher, but uncertain, return in the future.
  - interpretation 3: the certain amount we'd pay to consume an uncertain outcome
- risk-aversion
  - absolute $A(x) \doteq -\frac{U''(x)}{U'(x)}$
  - relative $R(x) \doteq -\frac{U''(x)\cdot x}{U'(x)}$
- risk-premium
  - absolute $\pi_A \doteq \mathbb{E}[x] - x_{CE} \approx \frac{1}{2} \cdot A(\bar{x}) \cdot \sigma_x^2$
  - relative $\pi_R \doteq 1 - \frac{x_{CE}}{\mathbb{E}[x]} \approx \frac{1}{2} \cdot R(\bar{x}) \cdot \sigma_{\frac{x}{\bar{x}}}^2$
  - determined given d_x and $U(\cdot)$, can be expressed w.r.t [extent of risk-aversion at $\bar{x}$] $\times$ [extent of uncertainty of outcome at $\bar{x}$]

(a) Constant Absolute Risk-Aversion (CARA)

$$U(x) \doteq \begin{cases} \frac{1-e^{-ax}}{a} & a \neq 0 \\ x & a = 0 \quad \text{risk-neutral} \end{cases}$$

Intuitively from the name, absolute risk-aversion $A(x) \doteq -\frac{U''(x)}{U'(x)} = a$.
Assume $x \sim \mathcal{N}(\mu, \sigma^2)$, for the risk-averse case($a > 0$)

$$\mathbb{E}[U(x)] = \int_{-\infty}^{\infty} f(x)U(x)dx = \int_{-\infty}^{\infty} \frac{1-e^{-ax}}{a} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx = \frac{1-e^{-a\mu+\frac{a^2\mu^2}{2}}}{a}$$

(directly using the conclusion of Gaussian integral $\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}$)
It's obvious that for the risk-neutral case $\mathbb{E}[U(x)] = \mathbb{E}[X] = \mu$

$$x_{CE} = U^{-1}(\mathbb{E}[U(x)]) = \mu - \frac{a\sigma^2}{2} \quad \forall a \geq 0$$

$$\pi_A = \mathbb{E}[x] - x_{CE} = \frac{a\sigma^2}{2}$$

$$\boxed{\max \mathbb{E}[U(x)] \Leftrightarrow \max x_{CE} \Leftrightarrow \max \mu - \frac{a\sigma^2}{2}}$$

(b) Portfolio Application of CARA
Here our utility is function of wealth W. Given the settings where the annual return for risky and riskless asset are $r_1 \sim \mathcal{N}(\mu, \sigma^2)$ and r_2 respectively, $w = (1-\pi)(1+r_2) + \pi r_1 \sim \mathcal{N}(1+r2+\pi(\mu-r2), \pi^2\sigma^2)$. Since it still follows normal distribution we can directly use the conclusion above,

$$\text{argmax}_{\pi} \left[\mu_W - \frac{a\sigma_W^2}{2}\right] \Rightarrow \pi^* = \frac{\mu - r}{a\sigma^2}$$

(c) Constant Relative Risk-Aversion (CRRA)

$$U(x) \doteq \begin{cases} \frac{x^{1-\gamma}-1}{1-\gamma} & \gamma \neq 1 \\ \log(x) & \gamma = 1 \end{cases}$$

Note that the risk-neutral case is when $\gamma = 0$.
Intuitively from the name, relative risk-aversion $R(x) \doteq -\frac{U''(x)\cdot x}{U'(x)} = \gamma$.
Assume $\log x \sim \mathcal{N}(\mu, \sigma^2)$, when $\gamma \neq 1$,

$$\begin{aligned} \mathbb{E}[U(x)] &= \int_{-\infty}^{\infty} f(x)U(x)dx \\ &= \int_{-\infty}^{\infty} \frac{1}{x} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} \frac{x^{1-\gamma}-1}{1-\gamma} dx \\ &= \frac{e^{\mu(1-\gamma)+\frac{\sigma^2}{2}(1-\gamma)^2}-1}{1-\gamma} \end{aligned}$$

(The derivation of the pdf of lognormal distribution can be found here)
It's obvious that when $\gamma = 1$ $\mathbb{E}[U(x)] = \mu$.

$$x_{CE} = U^{-1}(\mathbb{E}[U(x)]) = e^{\mu+\frac{\sigma^2}{2}(1-\gamma)} \quad \forall\gamma$$

$$\pi_R = 1 - \frac{x_{CE}}{\mathbb{E}[x]} = 1 - \frac{e^{\mu+\frac{\sigma^2}{2}(1-\gamma)}}{e^{\mu+\frac{\sigma^2}{2}}} = 1 - e^{-\frac{\sigma^2\gamma}{2}}$$

$$\boxed{\max \mathbb{E}[U(x)] \Leftrightarrow \max x_{CE} \Leftrightarrow \max \mu + \frac{\sigma^2}{2}(1-\gamma)}$$

(d) Portfolio Application of CRRA (Merton's Problem)
Detailed description and solution will be provided in the next written assignment.